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# On the Absence of Poincaré Lemma in Tangential Cauchy-Riemann Complexes.

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This study originated from the example of H. Lewy of a differential equation with variable coefficients without solutions [9]. The equation of H. Lewy is the following one

$$Lu \equiv \frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} - i(x_1 + ix_2)\frac{\partial u}{\partial x_3} = f$$

where $x_1, x_2, x_3$ are cartesian coordinates in $\mathbb{R}^3$ and where $f$ is a given $C^\infty$ function.

One realizes that the operator $L$ of H. Lewy has the following geometric meaning. On $\mathbb{C}^2$ we consider the hypersurface $S$: $\operatorname{Im} z_2 = z_1\bar{z}_1$, where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ are holomorphic coordinates in $\mathbb{C}^2$. A necessary condition for a $C^\infty$ function $u$ on $S$ (where $x_1, x_2, x_3$ are taken as coordinates) to be the restriction of a holomorphic function in $\mathbb{C}^2$ is given by $Lu = 0$, (cf. [0], [3]).

Here we consider the following general situation. We consider on a complex manifold $X$ the Dolbeault complex of the $\bar{\partial}$-operator (exterior differentiation with respect to local antiholomorphic coordinates). We consider a real smooth submanifold $S \subset X$. One can then define an associated complex on $S$ by a general procedure that is explained in section 1 and 2. Let

$$(*) \qquad Q^{(0)} \xrightarrow{\bar{\partial}_S} Q^{(1)} \xrightarrow{\bar{\partial}_S} Q^{(2)} \xrightarrow{\bar{\partial}_S} \dots$$

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be the complex obtained on $S$ considered as a complex of sheaves. In section 3 we prove that if $S$ is « generic », then the above complex is a complex of differential operators whose cohomology is isomorphic to the cohomology of the Dolbeault complex of Whitney functions on $S$ (proposition 3).

For instance for $S \subset \mathbf{C}^2$ given by $\operatorname{Im} z_2 = z_1 \bar{z}_1$ we deduce $Q^{(0)} = Q^{(1)} = \mathcal{E}(S) =$ the sheaf of $C^\infty$ functions on $S$, and $Q^{(j)} = 0$ for $j \geqslant 2$. The complex $(*)$ reduces to $\mathcal{E}(S) \xrightarrow{L} \mathcal{E}(S) \to 0$ $(L = \bar{\partial}_S)$ and the meaning of H. Lewy example is that the $C^\infty$ Poincaré lemma is not valid for this complex in dimension one.

In the case $S$ is a hypersurface and its Levi form is nondegenerate with $p$ positive and $q$ negative $(p + q = \dim_{\mathbf{C}} X - 1)$ eigenvalues at a point $x_0 \in S$, then the following result was established in [3]: the Poincaré lemma for the complex $(*)$ fails in dimensions $p$ and $q$ (and zero) at the point $x_0$. This case showed that the H. Lewy phenomenon of equations without solution is present also for overdetermined and underdetermined systems (even if the data satisfy all possible integrability conditions).

In this paper we continue the investigation also for the case $S$ is no longer a hypersurface.

We first define at each point $x_0 \in S$ the « Levi form » of $S$. This appears to be the following object. Let $H(S)_{x_0}$ be the maximal complex space contained in the real tangent space to $S$ at $x_0$. Let $N_{x_0}(S)$ be the fiber of the real normal bundle to $S$ at $x_0$ and let herm $(H(S)_{x_0})$ be the linear space of hermitian quadratic forms on $H(S)_{x_0}$. The Levi form is a linear map

$$\mathcal{L}\colon N_{x_0}(S) \to \operatorname{herm}\left(H(S)_{x_0}\right).$$

For each $\lambda \in N_{x_0}(S) - \{0\}$ let $e_{x_0}(\lambda) = (p, q)$ be the signature of $\mathcal{L}(\lambda)$ (i.e. $p =$ number of positive eigenvalues, $q =$ number of negative eigenvalues).

Here we prove the following theorem (theorem 3): let $S$ be generic at $x_0$ with $e_{x_0}(\lambda) = (p, q)$ for some $\lambda \in N_{x_0}(S) - \{0\}$ and with $p + q = \dim_{\mathbf{C}} H(S)_{x_0}$. Then in the complex $(*)$ the Poincaré lemma is not valid at the point $x_0$ in dimensions $p, q$ (and zero).

The proof of this theorem is obtained with an adaptation of the argument used by Hörmander in obtaining a necessary condition for solvability in the case of a single operator. ([7], ch. VI).

Heuristically stated for a single operator $P(x, D)$ the condition of Hörmander could be formulated by saying that the adjoint homogeneous equation ${}^tPw = 0$ has no short wave solutions whose amplitude decreases exponentially with the reciprocal of the wave length. This condition is very reminiscent of Sommerfeld's radiation condition (cf. [12] and [16]).

Returning to the above stated theorem we show that under the specified assumptions the local cohomology groups

$$\mathcal{H}^q_{x_0} = \frac{\operatorname{Ker}\{Q^{(q)}_{x_0} \xrightarrow{\bar\partial_S} Q^{(q+1)}_{x_0}\}}{\operatorname{Im}\{Q^{(q-1)}_{x_0} \xrightarrow{\bar\partial_S} Q^{(q)}_{x_0}\}}, \qquad \mathcal{H}^p_{x_0} = \frac{\operatorname{Ker}\{Q^{(p)}_{x_0} \xrightarrow{\bar\partial_S} Q^{(p+1)}_{x_0}\}}{\operatorname{Im}\{Q^{(p-1)}_{x_0} \xrightarrow{\bar\partial_S} Q^{(p)}_{x_0}\}}$$

(and of course $\mathcal{H}^0_{x_0}$) are infinite dimensional.

At the end of the paper we establish a spectral sequence connecting the cohomology of $S$ with values in $\mathcal{O}_S = \operatorname{Ker}\{Q^{(0)} \xrightarrow{\bar\partial_S} Q^{(1)}\}$ with the cohomology of the tangential complex $(*)$, and we treat some special cases by means of a Mayer-Vietoris sequence proved at the beginning of this paper.

In relation with this type of results one should also refer to global results contained in the papers of Kohn [8], to the microfunction cohomology of Sato-Kaway-Kashiwara [15], to a paper of Boutet de Monvel [5] for cohomology « modulo the $C^\infty$ functions » and to results of Trèves [17] for a pseudodifferential approach.

A last remark. If $S \subset X$ is real analytic and one considers for $S$ generic the analogue of the complex $(*)$ in the real analytic category (i.e. for real analytic « forms ») then the Poincaré lemma is always valid (Proposition 7).

## 1. – Rough Mayer-Vietoris sequence.

*a*) Let $X$ be a $C^\infty$ differentiable manifold, let $E$ be a $C^\infty$ differentiable vector bundle on $X$ and let $A$ be a *closed* subset of $X$. We set

$$\mathcal{E}(X) = \mathcal{E}(X, E) = \text{space of } C^\infty \text{ sections of the bundle } E \text{ on } X;$$

$$\boldsymbol{F}_A(X) = \boldsymbol{F}_A(X, E) = \{s \in \mathcal{E}(X) |\ s \text{ is flat at each point of } A\}.$$

A section $s$ of $E$ is « flat » at a point $a$ if with respect to a system of local coordinates at $a$ on $X$ the components of $s$ have all their partial derivatives zero at $a$. This notion is independent of the choice of local coordinates at $a$ on $X$ and of the trivialization of $E$ near $a$.

We then define the space $W(A) = W(A, E)$ of Whitney sections of $E$ on $A$ by the exact sequence

$$0 \to \boldsymbol{F}_A(X) \to \mathcal{E}(X) \to W(A) \to 0 .$$

*b*) Given two closed sets $A$, $B$ in $X$ we will say that they are ***regularly situated*** if they satisfy the following condition of Łojasiewicz.

For any point $a \in A \cap B$ we can find a relatively compact coordinate patch $U \ni a$ in $X$ and constants $c > 0$, $\alpha > 0$ such that, for any point $x \in A \cap U$ we have

$$\operatorname{dist}(x, B \cap U) \geqslant c \operatorname{dist}(x, A \cap B \cap U)^{\alpha}$$

where « dist » means the euclidean distance in the patch $U$.

Note that if $A \cap B = \emptyset$ the condition is empty, thus verified.

Let us fix a complete Riemannian metric on $X$ and let $d(x, y)$ denote the geodesic distance. The condition of Łojasiewicz can also be stated in the following global form

either $A \cap B = \emptyset$;

or given $K \subset A$ compact we can find constants $c > 0$, $\alpha > 0$ such that for any $x \in K$ we have

$$d(x, B) \geqslant c \operatorname{dist}(x, A \cap B)^{\alpha} .$$

The verification of this fact is omitted.

We consider now the following sequence of linear maps

$$0 \to W(A \cup B) \xrightarrow{\alpha} W(A) \oplus W(B) \xrightarrow{\beta} W(A \cap B) \to 0 \tag{1}$$

where

$$\alpha(f) = f|A \oplus f|B , \quad f \in W(A \cup B);$$

$$\beta(f \oplus g) = f|A \cap B - g|A \cap B , \quad f \oplus g \in W(A) \oplus W(B) .$$

Clearly $\alpha$ is injective, $\beta$ is surjective and $\beta \circ \alpha = 0$.

The following is a theorem of Łojasiewicz

THEOREM 1. *The necessary and sufficient condition that the sequence* (1) *be an exact sequence is that* $A$ *and* $B$ *be regularly situated.*

*c*) A subset $C$ closed in $X$ will be called locally semianalytic if for any point $c \in C$ we can find a coordinate patch $U \ni c$ and a finite number of real analytic functions $u_{ij}$: $U \to \mathbb{R}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, such that

$$C \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in U | u_{ij}(x) \geqslant 0\} .$$

Given two closed sets $A$ and $B$ in $X$ we will say that they are (simultaneously) *locally semianalytic* if for any point $c \in A \cap B$ we can find a coordinate patch $U \ni c$ and real analytic functions $u_{ij}: U \to \mathbf{R}$, $v_{rs}: U \to \mathbf{R}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$; $1 \leqslant r \leqslant \alpha$, $1 \leqslant s \leqslant \beta$ such that

$$A \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in U | u_{ij}(x) \geqslant 0\}$$

$$B \cap U = \bigcup_{r=1}^{\alpha} \bigcap_{s=1}^{\beta} \{x \in U | v_{rs}(x) \geqslant 0\} .$$

The following is a useful criterion also due to Łojasiewicz.

THEOREM 2. *If the closed sets $A$ and $B$ are simultaneously locally semianalytic then $A$ and $B$ are regularly situated.*

*d*) We give now on $X$ a sequence $E^j$, $j = 0, 1, 2, \ldots$ of vector bundles and for each one we define the spaces

$$\begin{aligned} \mathcal{E}^{(j)}(X) &= \mathcal{E}(X, E^j) \\ \boldsymbol{F}_A^{(j)}(X) &= \boldsymbol{F}_A(X, E^j) \\ W^{(j)}(A) &= W(A, E^j) . \end{aligned}$$

We assume that we have given differential operators

$$A_j: \mathcal{E}^{(j)}(X) \to \mathcal{E}^{(j+1)}(X) , \quad j = 0, 1, 2, \ldots$$

so that the sequence

$$\mathcal{E}^{(0)}(X) \xrightarrow{A_0} \mathcal{E}^{(1)}(X) \xrightarrow{A_1} \mathcal{E}^{(2)}(X) \xrightarrow{A_2} \ldots \tag{2}$$

is a complex.

For any closed set $A \subset X$ we have $A_j \boldsymbol{F}_A^{(j)}(X) \subset \boldsymbol{F}_A^{(j+1)}(X)$ and therefore the sequence

$$\boldsymbol{F}_A^{(0)}(X) \xrightarrow{A_0} \boldsymbol{F}_A^{(1)}(X) \xrightarrow{A_1} \boldsymbol{F}_A^{(2)}(X) \xrightarrow{A_2} \ldots$$

is a subcomplex of the complex (2). The quotient complex will be the complex

$$W^{(0)}(A) \xrightarrow{A_0} W^{(1)}(A) \xrightarrow{A_1} W^{(2)}(A) \xrightarrow{A_2} \ldots \tag{3}$$

where $A_j$ here means the differential operator $A_j$ applied to the Whitney functions.

We will denote by

$$H^j(A)\,, \qquad j = 0, 1, 2, \dots$$

the cohomology groups of the complex (3).

PROPOSITION 1. *Let $A$ and $B$ be closed sets regularly situated in $X$. We have an exact cohomology sequence* (*rough Mayer-Vietoris sequence*)

$$0 \to H^0(A \cup B) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \to H^1(A \cap B) \to \dots .$$

PROOF. Since $A$ and $B$ are regularly situated we do have exact sequences

$$0 \longrightarrow W^{(j)}(A \cup B) \xrightarrow{\alpha} W^{(j)}(A) \oplus W^{(j)}(B) \xrightarrow{\beta} W^{(j)}(A \cap B) \longrightarrow 0$$

for any $j \geqslant 0$. These give an exact sequence of complexes and therefore an exact cohomology sequence, the one in the statement of the proposition.

REMARK. Let $\phi$ be any paracompactifying family of supports on $X$. We can then define the spaces

$$\mathcal{E}^{(j)}_{\phi}(X) \;= \{s \in \mathcal{E}_{(j)}(X) \mid \operatorname{supp} s \in \phi\}$$
$${}^{\phi}\boldsymbol{F}^{(j)}_A(X) = \{s \in \boldsymbol{F}^{(j)}_A(X) \mid \operatorname{supp} s \in \phi\}$$

and

$$W^{(j)}_{\phi}(A) = \mathcal{E}^{(j)}_{\phi}(X)/{}^{\phi}\boldsymbol{F}^{(j)}_A .$$

If $A$ and $B$ are regularly situated we have exact sequences

$$0 \longrightarrow W^{(j)}_{\phi}(A \cup B) \xrightarrow{\alpha} W^{(j)}_{\phi}(A) \oplus W^{(j)}_{\phi}(B) \xrightarrow{\beta} W^{(j)}_{\phi}(A \cap B) \longrightarrow 0$$

therefore setting for any $A$ closed $H^j_{\phi}(A)$ to be the cohomology of the complex

$$W^{(0)}_{\phi}(A) \xrightarrow{A_0} W^{(1)}_{\phi}(A) \xrightarrow{A_1} W^{(2)}_{\varphi}(A) \xrightarrow{A_2} \dots$$

we deduce the rough Mayer-Vietoris sequence with supports in $\phi$

$$0 \to H^0_{\phi}(A \cup B) \to H^0_{\phi}(A) \oplus H^0_{\phi}(B) \to H^0_{\phi}(A \cap B) \to$$
$$\to H^1_{\phi}(A \cup B) \to \dots .$$

## 2. – Mayer-Vietoris sequence (proper).

*a*) Let $S$ be a smooth submanifold of $X$ on which we make the following restrictive hypothesis: $S$ is intersection of smooth hypersurfaces $F_\alpha$ of $X$: $S = \bigcap_{F_\alpha \supset S} F_\alpha$.

Given a complex (2) of differential operators on $X$ (and a family $\phi$ of supports) we can define for every smooth hypersurface $F$ of $X$ the subspaces (cf. [1])

$$\boldsymbol{I}_{A_j}(F, X) \quad (\text{resp } {}^\phi\boldsymbol{I}_{A_j}(F, X))$$

and we know that $A^j \boldsymbol{I}_{A_j}(F, X) \subset \boldsymbol{I}_{A_{j+1}}(F, X)$ so that

$$\boldsymbol{I}_{A_0}(F, X) \to \boldsymbol{I}_{A_1}(F, X) \to \boldsymbol{I}_{A_2}(F, X) \to \dots \tag{4}$$

is a subcomplex of (2). Similarly if we use the family $\phi$ of supports.

Let $\{F_\alpha\}_{\alpha \in A}$ be the family of all smooth hypersurfaces in $X$ containing $S$. We define

$${}^\phi\boldsymbol{I}_{A_j}(S, X) = \sum_{F_\alpha \supset S} {}^\phi\boldsymbol{I}_{A_j}(F_\alpha, X)\,.$$

In other words the space ${}^\phi\boldsymbol{I}_{A_j}(S, X)$ is the subspace of $\mathcal{E}^{(j)}(X)$ generated by the subspaces ${}^\phi\boldsymbol{I}_{A_j}(F_\alpha, X)$.

By the previous remarks we have that $A_j {}^\phi\boldsymbol{I}_{A_j}(S, X) \subset {}^\phi\boldsymbol{I}_{A_{j+1}}(S, X)$ so that we have a subcomplex of the complex

$$\mathcal{E}_\phi^{(0)}(X) \xrightarrow{A_0} \mathcal{E}_\phi^{(1)}(X) \xrightarrow{A_1} \mathcal{E}_\phi^{(2)}(X) \xrightarrow{A_2} \dots \tag{$2_\phi$}$$

in the complex

$${}^\phi\boldsymbol{I}(S, X) \xrightarrow{A_0} {}^\phi\boldsymbol{I}_{A_1}(S, X) \xrightarrow{A_1} {}^\phi\boldsymbol{I}_{A_2}(S, X) \xrightarrow{A_2} \dots\,. \tag{$4_{\phi,S}$}$$

We set

$${}^\phi\boldsymbol{F}_S^{(j)}(X) = \{s \in \mathcal{E}_\phi^{(j)}(X) | s \text{ is «flat» on } S\}$$

and we have another subcomplex of $(2)_\phi$ in the complex

$${}^\phi\boldsymbol{F}_S^{(0)}(X) \xrightarrow{A_0} {}^\phi\boldsymbol{F}_S^{(1)}(X) \xrightarrow{A_1} {}^\phi\boldsymbol{F}_S^{(2)}(X) \xrightarrow{A_2} \dots\,. \tag{$5_{\phi,S}$}$$

This is also a subcomplex of $(4)_{\phi,S}$.

We define

$$Q_{\phi}^{(j)}(S) = \mathcal{E}_{\phi}^{(j)}(X)/{}^{\phi}\boldsymbol{I}_{A_j}(S, X)\,.$$

These are the spaces of the quotient complex of $(2)_{\phi}$ by $(4)_{\phi,S}$ i.e.

$$(6)_{\phi,S} \qquad Q_{\phi}^{(0)}(S) \xrightarrow{A_{0S}} Q_{\phi}^{(1)}(S) \xrightarrow{A_{1S}} Q_{\phi}^{(2)}(S) \xrightarrow{A_{2S}} \ldots$$

where $A_{jS}$ are the induced linear maps by the differential operators $A_j$. Its cohomology will be denoted by $H_{\phi}^{j}(S, A_S^*)$.

As in the case of a hypersurface we introduce the following definition: the submanifold $S$ is called *formally noncharacteristic* (with respect to the family of supports $\phi$) if the sequence

$$(7)_{\phi,S} \qquad 0 \longrightarrow \frac{{}^{\phi}\boldsymbol{I}_{A_0}(S, X)}{{}^{\phi}\boldsymbol{F}_S^{(0)}(X)} \xrightarrow{A_0} \frac{{}^{\phi}\boldsymbol{I}_{A_1}(S, X)}{{}^{\phi}\boldsymbol{F}_S^{(1)}(X)} \xrightarrow{A_1} \frac{{}^{\phi}\boldsymbol{I}_{A_2}(S, X)}{{}^{\phi}\boldsymbol{F}_S^{(2)}(X)} \xrightarrow{A_2} \ldots$$

is an exact sequence.

*b*) Let us consider now the following situation.

We give a finite set of orientable hypersurfaces $F_1, \ldots, F_k$ in $X$ by their equations

$$\varrho_j \colon X \to \mathbb{R}, \; 1 \leqslant j \leqslant k$$

where $\varrho_j$ is $C^\infty$ and $d\varrho_j \neq 0$ on $F_j = \{\varrho_j = 0\}$.

Let

$$S = \{x \in X | \varrho_1(X) = \ldots = \varrho_k(X) = 0\} = F_1 \cap \ldots \cap F_k$$

$$\Omega^+ = \{x \in X | \varrho_1(X) \geqslant 0, \ldots, \varrho_k(X) \geqslant 0\}$$

$$\Omega^- = \{x \in X | \varrho_1(X) \leqslant 0, \ldots, \varrho_k(X) \leqslant 0\}\,.$$

We assume moreover that $d\varrho_1 \wedge \ldots \wedge d\varrho_k \neq 0$ at each point of $S$ so that $S$ is a smooth manifold of codimension $k$ in $X$.

Then $\Omega^+$ and $\Omega^-$ are regularly situated and $S = \Omega^+ \cap \Omega^-$. We set

$$\Omega = \Omega^+ \cup \Omega^-\,.$$

PROPOSITION 2. *Let $S$ be as above and assume that $S$ is formally noncharacteristic with respect to the family of supports $\phi$. Then we have a Mayer-Vietoris sequence*

$$0 \to H_{\phi}^0(\Omega) \to H_{\phi}^0(\Omega^+) \oplus H_{\phi}^0(\Omega^-) \to H_{\phi}^0(S, A_S^*) \to$$
$$\to H_{\phi}^1(\Omega) \to H_{\phi}^1(\Omega^+) \oplus H_{\phi}^1(\Omega^-) \to H_{\phi}^1(S, A_S^*) \to \ldots\,.$$

PROOF. We have an exact sequence

$$0 \to \frac{{}^{\phi}I_{A_j}(S, X)}{{}^{\phi}F_S^{(j)}(X)} \to W_{\phi}^{(j)}(S) \to Q_{\phi}^{(j)}(S) \to 0 .$$

Because $(7)_{\phi,S}$ is exact we have

$$H_{\phi}^j(S) \simeq H_{\phi}^j(S, A_S^*) , \quad \forall j \geq 0 .$$

The Mayer-Vietoris sequence follows from the remark to proposition 1.

## 3. – The case of the Dolbeault complex.

*a*) We assume now that $X$ is a complex manifold of pure complex dimension $n$.

Let $F_j = \{z \in X | \varrho_j(z) = 0\}$ be smooth hypersurfaces in $X$ as explained above $(1 \leq j \leq k)$ and let $S = F_1 \cap \ldots \cap F_k$ be their intersection. We assume as before that $d\varrho_1 \wedge \ldots \wedge d\varrho_k | S \neq 0$.

We consider on $X$ the Dolbeault complex

$$\mathcal{E}^{(0)}(X) \xrightarrow{\bar\partial} \mathcal{E}^{(1)}(X) \xrightarrow{\bar\partial} \mathcal{E}^{(2)}(X) \xrightarrow{\bar\partial} \ldots \tag{8}$$

where

$\mathcal{E}^{(j)}(X) =$ space of $C^\infty$ forms of type $o$, $j$ with values in a holomorphic vector bundle $E$ ($E$ independent of $j$)

and where $\bar\partial$ is the exterior differentiation with respect to antiholomorphic coordinates.

Replacing $X$ by an open set $U \subset X$ we define the spaces $\mathcal{E}^{(j)}(U)$ and thus the fine sheaves $U \to \mathcal{E}^{(j)}(U)$, $j = 0, 1, \ldots$.

Given a hypersurface $F = \{\varrho = 0\}$ on $X$ we can consider for $j = 0, 1, \ldots$ the space $I^{(j)}(F \cap U, U)$, and the sheaf $U \to I^{(j)}(F \cap U, U)$.

We have

$$I^{(j)}(F \cap U, U) = \varrho\mathcal{E}^{(j)}(U) + \bar\partial\varrho \wedge \mathcal{E}^{(j-1)}(U)$$

(cf. [3], part I). Therefore $U \to I^{(j)}(F \cap U, U)$ is a fine sheaf.

Let $S$ be as above; we can now consider the sheaf (for $j = 0, 1, \ldots$)

$$U \to I^{(j)}(S \cap U, U) .$$

From the above remark we deduce that

$$I^{(j)}(S \cap U, U) = \sum_{l=1}^{k} \varrho_l \mathcal{E}^{(j)}(U) + \sum_{l=1}^{k} \bar{\partial}\varrho_l \wedge \mathcal{E}^{(j-1)}(U)$$

so that the above sheaf is a fine sheaf for every $j \geq 0$.

Note that $I^{(0)}(S \cap U, U)$ is just the space of $C^\infty$ sections of $E$ over $U$ vanishing on $S \cap U$: $I^{(0)}(S \cap U, U) = \sum_{l=1}^{k} \varrho_l \mathcal{E}^{(0)}(U)$.

*b*) We recall the following definition. The submanifold $S$ is called a *generic submanifold* of $X$ if at each point of $S$

$$\bar{\partial}\varrho_1 \wedge \ldots \wedge \bar{\partial}\varrho_k \neq 0 .$$

If $S$ is a generic submanifold then the spaces (for $j = 0, 1, \ldots$)

$$Q^{(j)}(S \cap U) = \mathcal{E}^{(j)}(U)/I^{(j)}(S \cap U, U)$$

are free $\mathcal{E}(S \cap U)$ modules where $\mathcal{E}(S \cap U)$ denotes the space of $C^\infty$ functions on $S \cap U$. It follows then that the linear operators

$$\bar{\partial}_S \colon Q^{(j)}(S \cap U) \to Q^{(j+1)}(S \cap U)$$

are differential operators.

For every $j \geq 0$ let us consider the sheaf

$$U \to \frac{I^{(j)}(S \cap U, U)}{F^{(j)}_{S \cap U}(U)}$$

that we denote briefely by $I^{(j)}(S)/F^{(j)}_S$. These sheaves are fine sheaves. We have a complex of sheaves

$$0 \to \frac{I^{(0)}(S)}{F^{(0)}_S} \xrightarrow{\bar{\partial}} \frac{I^{(1)}(S)}{F^{(1)}_S} \xrightarrow{\bar{\partial}} \frac{I^{(2)}(S)}{F^{(2)}_S} \xrightarrow{\bar{\partial}} \ldots . \tag{9}$$

PROPOSITION 3. *Let $S$ be a generic submanifold of $X$. Then the complex of sheaves* (9) *is acyclic* (*i.e. the sequence* (9) *is exact*).

PROOF. Let $z_0 \in S$ be fixed. We have to prove the following: given $u \in I^{(j)}(S)_{z_0}$ such that $\bar{\partial}u \in F^{(j+1)}_{Sz_0}$ we can find $v \in I^{(j-1)}(S)_{z_0}$ so that

$$u - \bar{\partial}v \in F^{(j)}_{Sz_0} .$$

Here we have set $I^{(-1)}(S) = 0$.

Given a form $u \in \mathcal{E}^{(j)}_{z_0}$ by $u|S$ we denote the same form with the coefficients evaluated on $S$.

$\alpha$) Let $j = 0$. Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{N}^k$ be a multi-index and set

$$\varrho^\alpha = \varrho_1^{\alpha_1} \dots \varrho_k^{\alpha_k}, \qquad |\varrho| = \Big\{\sum_1^k \varrho_j^2\Big\}^{\frac{1}{2}}, \ |\alpha| = \alpha_1 + \dots + \alpha_k .$$

« Let $u = \sum_{|\alpha|=m} u_\alpha \varrho^\alpha \in \boldsymbol{I}^{(0)}(S)_{z_0}$ and assume that

$$\bar\partial\big(\sum u_\alpha \varrho^\alpha\big) = O(|\varrho|^m) .$$

Then for all $\alpha$ with $|\alpha| = m$ we can find functions $l_{\alpha j} \in \mathcal{E}^{(0)}_{z_0}$ such that

$$u_\alpha = \sum_{j=1}^k l_{\alpha j} \varrho_j \text{ »} .$$

Let $(i_1, \dots, i_m)$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_m \leqslant k$ be the sequence of $m$ integers in which the first $\alpha_1$ are equal to 1, the successive $\alpha_2$ are equal to $2, \dots,$ the last $\alpha_k$ are equal to $k$. We can write

$$\sum_{|\alpha|=m} u_\alpha \varrho^\alpha = \sum_{i_1 \leqslant \dots \leqslant i_m} u_{i_1 \dots i_m} \varrho_{i_1} \varrho_{i_2} \cdots \varrho_{i_m} .$$

The assumption gives

$$\sum u_{i_1 \dots i_m} (\bar\partial \varrho_{i_1} \varrho_{i_2} \cdots \varrho_{i_m} + \dots + \varrho_{i_1} \cdots \varrho_{i_{m-1}} \bar\partial \varrho_{i_m}) = O(|\varrho|^m)$$

which we can write as

$$\sum_{j=1}^k \sum_{|\alpha'|=m-1} u_{\alpha' j} \bar\partial \varrho_j \varrho^{\alpha'} = O(|\varrho|^m) .$$

From this, since the $\varrho$ can be taken as local coordinates, we derive that

$$\sum_{j=1}^k u_{\alpha' j} \bar\partial \varrho_j | S = 0$$

for any $\alpha'$ with $|\alpha'| = m - 1$. But this implies that $u_{\alpha' j}|S = 0$ i.e. $u_\alpha|S = 0$, $\forall \alpha$ with $|\alpha| = m$. Therefore

$$u_\alpha = \sum_{j=1}^k l_{\alpha j} \varrho_j$$

with some $l_{\alpha j} \in \mathcal{E}^{(0)}_{z_0}$. This proves our statement.

Now let $u \in \boldsymbol{I}^{(0)}(S)_{z_0}$. Then $u = \sum u_j \varrho_j$. Assume that $\bar\partial u \in \boldsymbol{F}^{(1)}_{S,z_0}$. By the above statement we must have $u_j = \sum_1^k l_{jh}\varrho_h$ and therefore $u = \sum u_{jh}\varrho_j\varrho_h$ is $O(|\varrho|^2)$. Again by the assumption we must have $u_{jh} = \sum l_{jhs}\varrho_s$ and therefore $u = \sum u_{jhs}\varrho_j\varrho_h\varrho_s$ is $O(|\varrho|^3)$. In this way we prove that $u$ is $O(|\varrho|^m)$ for every $m > 0$ and therefore that $u \in \boldsymbol{F}^{(0)}_{S,z_0}$ as we wanted.

$\beta$) Let $j \geqslant 1$. Let

$$u = \sum_1^k A_s\varrho_s + \sum_1^k B_s\,\bar\partial\varrho_s \in \boldsymbol{I}^{(j)}(S)_{z_0}; \qquad A_s \in \mathcal{E}^{(j)}_{z_0},\ B_s \in \mathcal{E}^{(j-1)}_{z_0},\ 1 \leqslant s \leqslant k\,.$$

We can write

$$u = \bar\partial\Big((-1)^{j-1}\sum_1^k B_s\varrho_s\Big) + \sum\big(A_s - (-1)^{j-1}\bar\partial B_s\big)\varrho_s\,.$$

Since $\sum B_s\varrho_s \in \boldsymbol{I}^{(j-1)}(S)_{z_0}$ we realize that, for our purpose, it is not restrictive to assume that $u$ has the form

$$u = \sum u_s\varrho_s; \qquad u_s \in \mathcal{E}^{(j)}_{z_0},\ 1 \leqslant s \leqslant k\,.$$

Assume now that $\bar\partial u \in \boldsymbol{F}^{(j+1)}_{S,z_0}$. Then

$$\sum_1^k u_s\,\bar\partial\varrho_s = O(|\varrho|)$$

and therefore

$$\sum_1^k u_s\,\bar\partial\varrho_s|S = 0\,.$$

LEMMA 1. *Let $j \geqslant 1$ and assume that for $u_s \in \mathcal{E}^{(j)}_{z_0}$, $1 \leqslant s \leqslant k$,*

$$\sum_1^k u_s\,\bar\partial\varrho_s|S = 0\,.$$

*Then we can find $l_{sh} \in \mathcal{E}^{(j-1)}_{z_0}$ with $l_{sh} = l_{hs}$ and*

$$u_s|S = \sum_1^k l_{sh}\,\bar\partial\varrho_h|S\,.$$

PROOF OF THE LEMMA. For $k = 1$ the statement is easy to establish as the symmetry condition on the $l$'s is void. By induction on $k$. We do have

$$u_1\,\bar\partial\varrho_1\wedge\bar\partial\varrho_2\wedge\ldots\wedge\bar\partial\varrho_k|S=0\,.$$

This gives (for $j\leqslant n-k$ and trivially for $j>n-k$).

$$(*)\qquad u_1=l_{11}\,\bar\partial\varrho_1+\ldots+l_{1k}\,\bar\partial\varrho_k\quad\text{on } S\,.$$

Hence

$$(u_2-l_{12}\,\bar\partial\varrho_1)\wedge\bar\partial\varrho_2+\ldots+(u_k-l_{1k}\,\bar\partial\varrho_1)\wedge\bar\partial\varrho_k=0\qquad\text{on } S\,.$$

By the inductive hypothesis we deduce that

$$(**)\qquad\begin{cases} u_h-l_{1h}\,\bar\partial\varrho_1=\sum\limits_2^k l_{hs}\,\bar\partial\varrho_s\quad\text{on } S\\ 2\leqslant h\leqslant k\end{cases}$$

with $l_{hs}=l_{sh}$. Relations (*) and (**) prove the statement of the lemma.

We deduce then, using lemma 1 that

$$u_s=\sum_1^k l_{sh}\,\bar\partial\varrho_h+\sum_1^k\nu_{sh}\varrho_h\,,\qquad\text{with } l_{sh}=l_{hs}\,,$$

and therefore, with $\nu_{sh}$ and $w_{sh}$ in $\mathcal{E}^{(j)}_{z_0}$,

$$\begin{aligned} u&=\sum u_s\varrho_s=\sum l_{sh}\,\bar\partial_h\varrho_s+\sum\nu_{sh}\varrho_h\varrho_s\\ &=\bar\partial(\tfrac12\sum l_{sh}\varrho_h\varrho_s)-\sum w_{sh}\varrho_h\varrho_s\,.\end{aligned}$$

To proceed in the proof we now need the following

LEMMA 2. *Let* $u=\sum\limits_{|\alpha|=m}u_\alpha\varrho^\alpha$ *with* $u_\alpha\in\mathcal{E}^{(j)}_{z_0}$ *and let* $m\geqslant1$. *Assume that*

$$\bar\partial\Big(\sum_{|\alpha|=m}u_\alpha\varrho^\alpha\Big)=O(|\varrho|^m)\,.$$

*Then for* $\beta\in\mathbb{N}^k$, $|\beta|=m+1$ *we can find* $\mathcal{L}_\beta\in\mathcal{E}^{(j-1)}_{z_0}$, *symmetric in the indices, such that*

$$\begin{cases} u_\alpha=\sum\limits_{j=1}^k\mathcal{L}_{\alpha j}\bar\partial\varrho_j\quad\textit{on } S\,.\\ |\alpha|=m\end{cases}$$

When we say that $\mathcal{L}_\beta$ is symmetric in the indices we mean the following; given $\beta \in \mathbb{N}^k$, $\beta = (\beta_1, \dots, \beta_k)$ we identify this multi-index with the sequence of $|\beta| = m + 1$ indices $i_1, \dots, i_{m+1}$ with $1 \leqslant i_1 \leqslant \dots \leqslant i_{m+1} \leqslant k$ in which the first $\beta_1$ indices equal $1, \dots,$ the last $\beta_k$ indices equal $k$.

Then $\mathcal{L}_\beta = \mathcal{L}_{i_1 \dots i_{m+1}}$ and its definition is extended to all sequences of $m+1$ indices postulating symmetry in the indices.

We proceed to the *proof of the lemma.* We first note that the lemma reduces to lemma 1 if $m = 1$ or $k = 1$. We can thus proceed by induction on $m$ and $k$ assuming the lemma proved up to $m - 1$ and $k - 1$ respectively. With $u_{i_1 \dots i_m} \in \mathcal{E}^{(j)}_{z_0}$ we can write

$$\sum u_\alpha \varrho^\alpha = \sum u_{i_1 \dots i_m} \varrho_{i_1} \cdots \varrho_{i_m} .$$

As in the case $j = 0$ the assumption yields

$$\sum_{j=1}^{k} \sum_{|\alpha'| = m-1} u_{\alpha' j} \bar{\partial} \varrho_j \varrho^{\alpha'} = O(|\varrho|^m)$$

and therefore for any $\alpha' \in \mathbb{N}^k$ with $|\alpha'| = m - 1$ we get

$$\sum_{j=1}^{k} u_{\alpha' j} \bar{\partial} \varrho_j | S = 0 .$$

We may assume $m \geqslant 2$. For every $\alpha'' \in \mathbb{N}^k$ with $|\alpha''| = m - 2$ we must have

$$\sum_{j=1}^{k} u_{1\alpha'' j} \bar{\partial} \varrho_j | S = 0$$

$$\sum_{j=1}^{k} u_{2\alpha'' j} \bar{\partial} \varrho_j | S = 0$$

$$\dots$$

$$\sum u_{k\alpha'' j} \bar{\partial} \varrho_j | S = 0 .$$

From the first set of equations we derive by the inductive assumption

$$(*) \qquad u_{1\alpha'' s} = \sum_{h=1}^{k} \mathcal{L}_{1\alpha'' sh} \bar{\partial} \varrho_h \quad \text{on } S$$

with $\mathcal{L}_\beta$, $|\beta| = m + 1$ defined for all sets of indices containing a 1 and symmetric in the indices.

Substituting in the second set of equations we obtain

$$(u_{2\alpha''2} - \mathfrak{L}_{1\alpha''22}\,\bar{\partial}\varrho_1)\,\bar{\partial}\varrho_2 + \ldots + (u_{2\alpha''k} - \mathfrak{L}_{1\alpha''2k}\,\bar{\partial}\varrho_1)\,\bar{\partial}\varrho_k = 0 \quad \text{on } S\,.$$

By the induction on $k$ we obtain

$$u_{2\alpha''s} - \mathfrak{L}_{1\alpha''2s}\,\bar{\partial}\varrho_1 = \sum_{h=2}^{k} \mathfrak{L}_{2\alpha''sh}\,\bar{\partial}\varrho_h \quad \text{on } S$$

with the new $\mathfrak{L}_\beta$ introduced (with a set of indices containing a 2) symmetric in the indices. From (*) and these last relations we get

$$(**) \qquad u_{2\alpha''s} = \sum_{h=1}^{k} \mathfrak{L}_{2\alpha''sh}\,\bar{\partial}\varrho_h \quad \text{on } S$$

with $\mathfrak{L}$ symmetric in their indices.

Substituting (*) and (**) in the third set of relations we obtain after a simplification

$$(u_{3\alpha''3} - \mathfrak{L}_{3\alpha''13}\,\bar{\partial}\varrho_1 - \mathfrak{L}_{3\alpha''23}\,\bar{\partial}\varrho_2)\,\bar{\partial}\varrho_3 + \ldots$$
$$+ (u_{3\alpha''k} - \mathfrak{L}_{3\alpha''1k}\,\bar{\partial}\varrho_1 - \mathfrak{L}_{3\alpha''2k}\,\bar{\partial}\varrho_2) \wedge \bar{\partial}\varrho_k = 0 \quad \text{on } S\,.$$

Arguing as before we derive relations

$$(***) \qquad u_{3\alpha''s} = \sum_{h=1}^{k} \mathfrak{L}_{3\alpha''sh}\,\bar{\partial}\varrho_h \quad \text{on } S$$

with the new $\mathfrak{L}_\beta$ introduced (with a set of indices containing a 3) symmetric in the indices.

The general argument is now clear and after $k$ steps we conclude with the statement of lemma 2.

Now for $u \in \boldsymbol{I}^{(j)}(S)_{z_0}$ we can write successively, with obvious notations

$$\begin{aligned}
u &= \bar{\partial}\Big(\sum \mathfrak{L}_s\varrho_s\Big) + \sum u_s\varrho_s \\
&= \bar{\partial}\Big(\sum \mathfrak{L}_s\varrho_s\Big) + \sum \mathfrak{L}_{sh}\varrho_s\,\bar{\partial}\varrho_h + \sum v_{sh}\varrho_h\,\varrho_s \\
&= \bar{\partial}\Big(\sum \mathfrak{L}_s\varrho_s + \tfrac{1}{2}\sum \mathfrak{L}_{sh}\varrho_s\varrho_h\Big) + \sum_{|\alpha|=2} u_\alpha\varrho^\alpha \\
&= \bar{\partial}\Big(\sum \mathfrak{L}_s\varrho_s + \tfrac{1}{2}\sum \mathfrak{L}_{sh}\varrho_s\varrho_h\Big) + \sum_{|\alpha|=2} \mathfrak{L}_{\alpha h}\varrho^\alpha\,\bar{\partial}\varrho_h + \sum_{|\beta|=3} v_\beta\varrho^\beta \\
&= \bar{\partial}\Big(\sum \mathfrak{L}_s\varrho_s + \tfrac{1}{2}\sum \mathfrak{L}_{sh}\varrho_s\varrho_h + \tfrac{1}{3}\sum_{|\beta|=3} \mathfrak{L}_\beta\varrho^\beta\Big) + \sum_{|\alpha|=3} u_\alpha\varrho^\alpha
\end{aligned}$$

. . . . . . . .

The $\mathcal{L}$'s being symmetric in their indices. We construct in this way a formal power series in $\varrho$

$$v = \sum_{\substack{\beta\in\mathbf{N}^k\\|\beta|>0}} \mathcal{L}_\beta \varrho^\beta \qquad \text{with } \mathcal{L}_\beta \in \mathcal{E}_{z_0}^{(j-1)}$$

and therefore an element of $\boldsymbol{I}^{(j-1)}(S)/\boldsymbol{F}_S^{(j-1)}$ such that $u - \bar\partial v$ represents the zero element of $\boldsymbol{I}^{(j)}(S)/\boldsymbol{F}_S^{(j)}$. This achieves the proof of proposition 3.

COROLLARY. *If $\phi$ is any paracompactifying family of supports we do have an exact sequence, for $S$ generic,*

$$0 \to \frac{{}^\phi\boldsymbol{I}^{(0)}(S, X)}{{}^\phi\boldsymbol{F}_S^{(0)}(X)} \to \frac{{}^\phi\boldsymbol{I}^{(1)}(S, X)}{{}^\phi\boldsymbol{F}_S^{(1)}(X)} \to \dots$$

*(i.e. $S$ is formally noncharacteristic).*

This follows from the fact that the sequence (9) is an exact sequence of fine sheaves and thus on them the functor $\Gamma_\phi$ is exact.

*c*) We set, as usual,

$$\Omega^+ = \{z \in X | \varrho_1(z) \geqslant 0, \dots, \varrho_k(z) \geqslant 0\}$$

$$\Omega^- = \{z \in X | \varrho_1(z) \leqslant 0, \dots, \varrho_k(z) \leqslant 0\}$$

and we assume that on $S = \Omega^+ \cap \Omega^-$ we have $\bar\partial\varrho_1 \wedge \dots \wedge \bar\partial\varrho_k \neq 0$ (thus $1 \leqslant k \leqslant n$ and $S$ is a generic submanifold of $X$). Because of theorem 2 the sets $\Omega^+$ and $\Omega^-$ are regularly situated.

Let $\phi$ be a paracompactifying family of supports. We denote by $\Omega = \Omega^+ \cup \Omega^-$ and by

$$H^j_\phi(\Omega, \bar\partial)\,, \quad H^j_\phi(\Omega^+, \bar\partial)\,, \quad H^j_\phi(\Omega^-, \bar\partial)$$

the cohomology groups of the complexes of Whitney forms

$$W_\phi^{(0)}(\Omega^*) \xrightarrow{\bar\partial} W_\phi^{(1)}(\Omega^*) \xrightarrow{\bar\partial} W_\phi^{(2)}(\Omega^*) \xrightarrow{\bar\partial} \dots$$

where $W_\phi^{(j)}(\Omega^*) = \mathcal{E}_\phi^{(j)}(X)/{}^\phi\boldsymbol{F}_{\Omega^*}^{(j)}(X)$, $j = 0, 1, \dots,$ and where $*$ denotes the void symbol or « $+$ » or « $-$ ».

Similarly we can consider the group $H^j_\phi(S, \bar\partial_S)$ introduced above.

From the previous corollary we deduce then the following

PROPOSITION 4. *Under the above specified assumptions we have an exact sequence (Mayer-Vietoris sequence)*

$$0 \to H^0_\phi(\Omega, \bar\partial) \to H^0_\phi(\Omega^+, \bar\partial) \oplus H^0_\phi(\Omega^-, \bar\partial) \to H^0_\phi(S, \bar\partial_S) \to$$
$$\to H^1_\phi(\Omega, \bar\partial) \to H^1_\phi(\Omega^+, \bar\partial) \oplus H^1_\phi(\Omega^-, \bar\partial) \to H^1_\phi(S, \bar\partial_S) \to \dots .$$

## 4. – Failure of the Poincaré lemma on generic submanifolds for $\bar\partial_S$.

*a*) We want to establish a criterium to ensure that Poincaré lemma is *not* valid for the $\bar\partial_S$-complex on a generic submanifold $S$ at a given point $z_0 \in S$.

First of all we need to introduce for a submanifold $S$ of $X$ at a point $z_0 \in S$ the Levi form of $S$; the submanifold $S$ need not be generic for this definition. The question being of local nature we may assume that $X$ is an open neighborhood $U$ of the origin $0 \in \mathbf{C}^n$, and that $S$ is defined by the equations $\varrho_j(z) = 0$, $1 \leqslant j \leqslant k$;

$$S = \{z \in U | \varrho_1(z) = \dots = \varrho_k(z) = 0\} ,$$

with $\varrho_j(0) = 0$, $1 \leqslant j \leqslant k$, so that $0 \in S$, and $(d\varrho_1 \wedge \dots \wedge d\varrho_k)_0 \neq 0$.

The analytic tangent space to $S$ at 0 is then defined by the equations

$$(\partial\varrho_j)_0 = \sum_{\alpha=1}^{n} \left(\frac{\partial\varrho_j}{\partial z_\alpha}\right)_0 u_\alpha = 0 , \qquad 1 \leqslant j \leqslant k$$

where $u_\alpha = dz_\alpha$ are taken as holomorphic coordinates. These define a complex linear space of dimension $l \geqslant n - k$, and exactly of dimension $l = n - k$ if $S$ is generic at 0.

We denote this analytic tangent space to $S$ at 0 by $H(S)_0$.

Now let $(\lambda_1, \dots, \lambda_k) = \lambda \in \mathbb{R}^k$ where $\mathbb{R}^k$ is identified with $N(S)_0$ the real normal space to $S$ at 0. ($S$ has real dimension $2n - k$ and real codimension $k$ in $\mathbf{C}^n$.)

For every $\lambda$ we consider the hermitian form on $H(S)_0$

$$\left(\sum_{j=1}^{k} \sum_{\alpha,\beta=1}^{n} \lambda_j \left\{\frac{\partial^2 \varrho_j}{\partial z_\alpha \partial \bar z_\beta}\right\}_0 u_\alpha \bar u_\beta\right) | H(S)_0 .$$

We obtain in this way a linear map

$$\mathfrak{L}: N(S)_0 \to \operatorname{herm}\big(H(S)_0\big)$$

where herm denotes the linear space of hermitian forms on the complex space in parenthesis. This linear map $\mathcal{L}$ will be called the *Levi form* of $S$ at 0. It is defined up to a real linear automorphism of the normal space $N(S)_0$ of $S$ at 0, up to a complex linear automorphism of the analytic tangent space $H(S)_0$ of $S$ at 0. A direct verification shows that the Levi form of $S$ at 0 is independent of the choice of holomorphic coordinates $z_1, \ldots, z_n$ of $U$ at 0 and of the choice of the equations $\varrho_1 = 0, \ldots, \varrho_k = 0$ for $S$ near 0 in $U$.

In particular we can consider the unit sphere $\Sigma \subset N(S)_0$

$$\Sigma = \left\{\lambda \in \mathbb{R}^k \mid \sum_{j=1}^{k} \lambda_j^2 = 1\right\}$$

and for each vector $\lambda \in \Sigma$ the integers $p(\lambda)$ and $q(\lambda)$ denoting the number of eigenvalues of $\mathcal{L}(\lambda)$ that are strictly $> 0$ or, respectively, strictly $< 0$. This gives a map

$$e_0 : \Sigma \to \Delta$$

where

$$\Delta = \{(p, q) \in \mathbf{N}^2 \mid p + q \leqslant l\} .$$

This finite valued function $e_0$ on $\Sigma$ is therefore an invariant of $S$ at 0 with respect to all complex germs of automorphisms of $U$ at 0. We will call $e_0$ the *partition function associated to the Levi form.*

*b*) Let us consider now on $S$ near $0 \in S$ the tangential Cauchy-Riemann complex that we write at the sheaf level

$$Q^{(0)}(S) \xrightarrow{\bar{\partial}_S} Q^{(1)}(S) \xrightarrow{\bar{\partial}_S} Q^{(2)}(S) \xrightarrow{\bar{\partial}_S} \ldots . \tag{10}$$

If we *assume that $S$ is generic* at 0 (and thus near by) then the sheaves $Q^{(j)}(S)$ are locally free sheaves as sheaves of $\mathcal{E}(S)$-modules, $\mathcal{E}(S)$ denoting the sheaf of rings of $C^\infty$ functions on $S$, and the linear operators $\bar{\partial}_S$ are differential operators.

We will say that the Poincaré lemma is valid in dimension $j \geqslant 1$ for the complex (10) at the point $0 \in S$ if the sequence

$$Q^{(j-1)}(S)_0 \xrightarrow{\bar{\partial}_S} Q^{(j)}(S)_0 \xrightarrow{\bar{\partial}_S} Q^{(j+1)}(S)_0$$

is an exact sequence.

We want to prove the following

THEOREM 3. *Let $S$ be a germ of generic submanifold of $\mathbf{C}^n$ near the origin* 0 *of real codimension $k < n$.*
*Assume that for some $\lambda \in \Sigma$*

$$e_0(\lambda) = (p, q) \quad \textit{with} \quad p + q = n - k \ (^1)$$

*and let $q > 0$. Then in complex* (10) *the Poincaré lemma fails in dimension $j = q$ at the origin.*

REMARK. Since Ker $\{Q_0^{(0)} \to Q_0^{(1)}\}$ contains the traces on $S$ near 0 of germs of holomorphic functions, that kernel is nonzero (and infinite dimensional). We may agree to say that the Poincaré lemma does not hold for $j = 0$. The previous statement can then be formulated by saying that, under the specified conditions, Poincaré lemma fails in dimensions $0, p$ and $q$.

The above theorem will be proved in the next two sections.

## 5. – Some a priori estimates.

*a*) Let $\Omega$ be open in $\mathbf{R}^n$ and let $\mathcal{E}(\Omega)$ denote the space of $C^\infty$ functions on $\Omega$. We set for $p > 0$, $\mathcal{E}^p(\Omega) = \mathcal{E}(\Omega) \times \ldots \times \mathcal{E}(\Omega)$, $p$ times. With $\mathcal{E}$ we will denote the sheaf of germs of $C^\infty$ functions on $\mathbf{R}^n$.

We assume that we have given a short complex of differential operators on $\Omega$:

$$\mathcal{E}^p(\Omega) \xrightarrow{A(x, D)} \mathcal{E}^q(\Omega) \xrightarrow{B(x, D)} \mathcal{E}^r(\Omega) \,. \tag{11}$$

LEMMA 3. *We assume that at a given point $x_0 \in \Omega$ the complex* (11) *admits the Poincaré lemma, i.e. we assume that the sequence*

$$\mathcal{E}^p_{x_0} \xrightarrow{A(x, D)} \mathcal{E}^q_{x_0} \xrightarrow{B(x, D)} \mathcal{E}^r_{x_0}$$

*is an exact sequence.*

(1) As $S$ is generic the analytic tangent space $H(S)_0$ has complex dimension $n - k$. The assumption means that one can find a $C^\infty$ hypersurface $\{\varrho(z) = 0\}$ containing $S$ near the origin and such that the hermitian form $(\sum \{\partial^2 \varrho / \partial z_\alpha \partial \bar{z}_\beta\}_0 u_\alpha \bar{u}_\beta) | H(S)_0$ is nondegenerate with signature $(p, q)$.

*Then for every open neighborhood $\omega$ of $x_0$ in $\Omega$ we can find an open neighborhood $\omega_1$ of $x_0$ in $\Omega$ ($x_0 \in \omega_1 \subset\subset \omega$) such that*

*for any $f \in \mathcal{E}^q(\omega)$ with $B(x, D)f = 0$ on $\omega$*

*there exists an $u \in \mathcal{E}^p(\omega_1)$ with*

$$A(x, D)u = f \quad \text{on } \omega_1 .$$

PROOF. For every open neighborhood $\omega$ of $x_0$ we set

$$Z(\omega) = \{f \in \mathcal{E}^p(\omega) | B(x, D)f = 0 \text{ on } \omega\}$$

the space $\mathcal{E}^p(\omega)$ with its natural Schwartz topology is a Fréchet space. Therefore $Z(\omega)$ as a closed subspace of $\mathcal{E}^p(\omega)$ is also a Fréchet space.

Now let $\{\omega^{(m)}\}_{m\in\mathbb{N}}$ denote a fundamental *sequence* of open (relatively compact) neighborhoods of $x_0$ in $\omega$. We define for each $m \in \mathbb{N}$

$$G_m = \{(u, f) \in \mathcal{E}^p(\omega^{(m)}) \times Z(\omega) | A(x, D)u = f \text{ on } \omega^{(m)}\} .$$

This, as a closed subspace of $\mathcal{E}^p(\omega^{(m)}) \times Z(\omega)$, is also a Fréchet space. Let $\pi_m\colon G_m \to Z(\omega)$ be the natural projection. It is linear and continuous. Also by the assumption of the validity of Poincaré lemma we must have

$$Z(\omega) = \bigcup_{m=1}^{\infty} \pi_m(G_m) .$$

By Baire's cathegory theorem one, say $\pi_{m_0}(G_{m_0})$, of the spaces $\pi_m(G_m)$ must be of second category. Then by the Banach open mapping theorem the linear continuous map

$$\pi_{m_0}\colon G_{m_0} \to Z(\omega)$$

must be surjective. This proves the lemma with $\omega_1 = \omega^{(m_0)}$.

COROLLARY. *With the same assumptions and notations of the previous lemma we have that*

*given a compact set $K_1 \subset \omega_1$ and an integer $m_1 \geqslant 0$ we can find a compact set $K = K(K_1, m_1) \subset \omega$, an integer $m = m(K_1, m_1) \geqslant 0$ and a constant $c > 0$ such that:*

*for any $f \in \mathcal{E}^q(\omega)$ with $B(x, D)f = 0$ on $\omega$*

*one can choose $u \in \mathcal{E}^p(\omega_1)$ with*

$$A(x, D)u = f \quad \text{on } \omega_1$$

*and with*

$$\sup_{|\alpha|\leqslant m_1}\sup_{K_1}|D^\alpha u|\leqslant c\sup_{|\alpha|\leqslant m}\sup_{K}|D^\alpha f|\ ^{(2)}\,.$$

PROOF. With the notations of the previous proof, the map

$$\pi_{m_0}\colon G_{m_0}\to Z(\omega)$$

is not only surjective but also open (Banach open mapping theorem). Now

$$U=\{(u,f)\in G_{m_0}|\sup_{|\alpha|\leqslant m_1}\sup_{K_1}|D^\alpha u|<1\}$$

is an open neighborhood of the origin in $G_{m_0}$. Therefore $\pi_{m_0}(U)$ contains an open neighborhood $W$ of the origin in $Z(\omega)$. Restricting $W$, if necessary, we may assume that

$$W=\Big\{f\in Z(\omega)|\sup_{|\alpha|\leqslant m}\sup_{K}|D^\alpha f|<\varepsilon\Big\}$$

for $\varepsilon>0$, $m$ integer $\geqslant 0$ and $K$ compact and conveniently chosen. We may as well assume that $\omega_1\subset K\subset\omega$, since we have chosen $(\omega_1\subset\subset\omega)$.

Let $f\in Z(\omega)$ be given, and assume first that

$$\sup_{|\alpha|\leqslant m}\sup_{K}|D^\alpha f|=\|f\|_{K,m}>0\,.$$

Then $(\varepsilon/2)(f/\|f\|_{K,m})\in W$ so that we can find $w\in\mathcal{E}^p(\omega_1)$ with $\big(w,(\varepsilon/2)(f/\|f\|_{K,m})\big)\in$ $\in U$. Setting $u=(2/\varepsilon)\|f\|_{K,m}w$ we must have

$$A(x,D)u=f\qquad\text{on }\omega_1$$

and

$$\sup_{|\alpha|\leqslant m_1}\sup_{K_1}|D^\alpha u|\leqslant\frac{2}{\varepsilon}\sup_{|\alpha|\leqslant m}\sup_{K}|D^\alpha f|\,.$$

We can therefore choose $c=2/\varepsilon$ and the corollary is proved in this case.

If $\|f\|_{K,m}=0$ since $\omega_1\subset K$ we can take $u=0$. The corollary is therefore also verified in this case and therefore in general.

*b*) We consider the space $\mathcal{D}(\Omega)$ of $C^\infty$ compactly supported functions on $\Omega$ and we denote by $dx$ the Lebesgue measure. For $u\in\mathcal{E}^S(\Omega)$ and

(2) For $\alpha\in\mathbb{N}^n$ we have set $D^\alpha=(\partial^{\alpha_1+\dots+\alpha_n})/(\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n})$, $\alpha=(\alpha_1,\dots,\alpha_n)$. Also $|\cdot|$ denotes a norm on the spaces $\mathbb{C}^p$ or $\mathbb{C}^q$, for instance the euclidean norm.

$\nu \in \mathcal{D}^S(\Omega)$ we consider the scalar product

$$(u, \nu)_\Omega = \int_\Omega {}^t u \bar{\nu}\, dx .$$

Given the differential operator $A(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)$ the formal adjoint of it is defined as the differential operator

$$A^*(x, D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^p(\Omega)$$

characterized by the property

$$\big(A(x, D)u, \nu\big)_\Omega = \big(u, A^*(x, D)\nu\big)_\Omega$$

for every $u \in \mathcal{E}^p(\Omega)$ and every $\nu \in \mathcal{D}^q(\Omega)$.

Explicitly if $A(x, D) = \sum C_\alpha(x) D^\alpha$ with $C_\alpha(x)$ matrices of type $(q \times p)$ of $C^\infty$ functions on $\Omega$, then we have

$$A^*(x, D)\nu = \sum (-1)^{|\alpha|} D^\alpha\big({}^t\overline{C_\alpha(x)}\nu\big) .$$

PROPOSITION 5. *We assume that the complex* (11) *admits the Poincaré lemma at a point* $x_0 \in \Omega$.

*Then for any given open neighborhood* $\omega$ *of* $x_0$ *in* $\Omega$ *we can find*

*an open neighborhood* $\omega'$ *of* $x_0$ *in* $\omega$ *with* $x_0 \in \omega' \subset\subset \omega$;

*a compact subset* $K$ *in* $\omega$;

*an integer* $m \geqslant 0$;

*a constant* $c > 0$;

*such that*

*for every* $f \in \mathcal{E}^q(\omega)$ *with* $B(x, D)f = 0$ *in* $\omega$;

*for every* $\nu \in \mathcal{D}^q(\omega')$;

*we have*

$$\left| \int_{\omega'} {}^t f \bar{\nu}\, dx \right| \leqslant c \sup_{\omega'} |A^*(x, D)\nu| \sup_{|\alpha| \leqslant m} \sup_K |D^\alpha f| .$$

PROOF. Given $\omega$ we select $\omega_1$, with $x_0 \in \omega_1 \subset\subset \omega$ as in lemma 3. We then choose $\omega'$ open with $x_0 \in \omega' \subset\subset \omega_1$ and set $K_1 = \bar{\omega}'$ and $m_1 = 0$. By the corollary to lemma 3, we can then choose a compact set $K \subset \omega$ an integer

$m \geqslant 0$ and a constant $c' > 0$ such that

for every $f \in \mathcal{E}^q(\omega)$ with $B(x, D)f = 0$ in $\omega$

one can find $u \in \mathcal{E}^p(\omega_1)$ with

$$A(x, D)u = f \quad \text{on } \omega_1$$

and

$$\sup_{\omega'} |u| \leqslant c' \sup_{|\alpha| \leqslant m} \sup_{K} |D^\alpha f| .$$

Now given any $v \in \mathcal{D}^q(\omega')$ we do have

$$\begin{aligned}\left|\int_{\omega'} {}^t f \bar{v}\, dx\right| &= \left|\int_{\omega'} {}^t(A(x, D)u)\bar{v}\, dx\right. \\ &= \left|\int {}^t u \overline{A^*(x, D)v}\, dx\right| \\ &\leqslant \operatorname{vol}(\omega') \sup_{\omega'} |u| \sup_{\omega'} |A^*(x, D)v| \\ &\leqslant c \sup_{\omega'} |A^*(x, D)v| \sup_{|\alpha| \leqslant m} \sup_{K} |D^\alpha f|\end{aligned}$$

where $c = c' \operatorname{vol}(\omega')$. Here $\operatorname{vol}(\omega')$ denotes the volume .of $\omega'$.

## 6. – Proof of Theorem 3.

*a*) We first prepare the equations of the submanifold $S$ near the origin.

By a linear change of holomorphic coordinates at the origin we may assume that, setting $l = n - k$, we have

$$\varrho_j \equiv \operatorname{Im}(z_{l+j}) + O(2) , \quad 1 \leqslant j \leqslant k .$$

This is because $\bar{\partial}\varrho_1 \wedge \ldots \wedge \bar{\partial}\varrho_k \neq 0$ at the origin. Then $\operatorname{Im}(z_{l+j}) = 0$, $1 \leqslant j \leqslant k$ are the equations of the real tangent space to $S$ at $0$ and $\mathbf{C}^l = \{z_{l+1} = \ldots = z_n = 0\}$ is the holomorphic tangent space to $S$ at $0$.

Using the implicit function theorem we may therefore assume that in a neighborhood $U$ of the origin the equations of $S$ are in the form

$$(*) \qquad \begin{cases} s_\alpha = g_\alpha(z_1, \ldots, z_l, t_1, \ldots, t_k) \\ 1 \leqslant \alpha \leqslant k \end{cases}$$

where we have set $z_{l+\alpha} = t_\alpha + is_\alpha$, $1 \leqslant \alpha \leqslant k$ and where the $g_\alpha$'s vanish at the origin of second order and are in a small neighborhood of the origin $C^\infty$ functions.

Since $s_\alpha = (1/2i)(z_{l+\alpha} - \bar{z}_{l+\alpha})$ we derive from equations (*)

$$\left(I + i\left(\frac{\partial g_\alpha}{\partial t_j}\right)\right)\begin{pmatrix} d\bar{z}_{l+1} \\ \vdots \\ d\bar{z}_n \end{pmatrix} = -2i\left(\frac{\partial g_\alpha}{\partial \bar{z}_s}\right)\begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_l \end{pmatrix} \bmod \boldsymbol{I}^{(1)}(S)$$

where $\boldsymbol{I}^{(1)}(S)$ is the space of one-forms of type $\sum_{\alpha=1}^{k} A_\alpha \varrho_\alpha + \sum_{\alpha=1}^{k} B_\alpha \bar{\partial}\varrho_\alpha$, $\varrho_\alpha \equiv$ $\equiv s_\alpha - g_\alpha$, $1 \leqslant \alpha \leqslant k$.

Let $C = \left(C_{sj}(z_1, \ldots, z_l, t_1, \ldots, t_k)\right)_{\substack{1 \leqslant j \leqslant l \\ 1 \leqslant s \leqslant k}}$ be the matrix

$$C = -2i\left(I + i\left(\frac{\partial g_\alpha}{\partial t_j}\right)\right)^{-1}\left(\frac{\partial g_\alpha}{\partial \bar{z}_s}\right)$$

and define for $f\colon S \to \mathbf{C}$, $C^\infty$,

$$\bar{L}_j f = \frac{\partial f}{\partial \bar{z}_j} + \sum_{s=1}^{k} C_{sj}\frac{\partial f}{\partial t_s}, \qquad \leqslant j < l$$

where $z_1, \ldots, z_l, t_1, \ldots, t_k$ denote the $C^\infty$ coordinates on $S$ induced from the tangent space of $S$ at 0. Note that $C_{sj} = 0$ at the origin.

Extending $f$ to a neighborhood of the origin in $\mathbf{C}^n$ by taking it independent of the coordinates $s_\alpha$ we have

$$\bar{\partial} f = \sum_{j=1}^{l} (\bar{L}_j f)\, d\bar{z}_j \bmod \boldsymbol{I}^{(1)}(S)\,.$$

With these notations one realizes that, near the origin on $S$,

$$Q^{(j)}(S) = \{\varphi = \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant l} a_{i_1 \ldots i_j}\, d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_j}\}$$

where the a's are $C^\infty$ on $S$ and that

$$\bar{\partial}_S \varphi = \sum_{1 \leqslant i_0 < \ldots < i_j \leqslant l} \left(\sum (-1)^h \bar{L}_{i_h} a_{i_0 \ldots \hat{i}_h \ldots i_l}\right) d\bar{z}_{i_0} \wedge \ldots \wedge d\bar{z}_{i_j}\,.$$

We note explicitly that, since $\bar{\partial}_S \bar{\partial}_S f = 0$ for any $f$ $C^\infty$ on $S$ we must have the commutation rules

$$[\bar{L}_j, \bar{L}_h] = \bar{L}_j \bar{L}_h - \bar{L}_h \bar{L}_j = 0\,, \quad 1 \leqslant j,\ h \leqslant l\,.$$

*b*) Let us now describe the assumption. There exists a $C^\infty$ function $\varrho$, vanishing on $S$, such that

$$\left(\sum\left\{\frac{\partial^2\varrho}{\partial z_\alpha\,\partial\bar z_\beta}\right\}_0 u_\alpha\bar u_\beta\right)\Big|H(S)_0$$

is nondegenerate with $p$ positive and $q$ negative eigenvalues.

Now $H(S)_0 = \mathbf{C}^l = \{u_{l+1} = \ldots = u_n = 0\}$ and $\varrho$ must have the form $\sum_{\alpha=1}^{k} A_\alpha(s_\alpha - g_\alpha)$ with $A_\alpha$, $C^\infty$ near the origin. Since the $g_\alpha$ vanish at the origin of second order, the assumption means that there are $k$ real numbers $\lambda_1, \ldots, \lambda_k$ not all zero such that the hermitian form on $\mathbf{C}^l$

$$\sum_{\mu,\nu=1}^{l}\left\{\frac{\partial^2\left(\sum_1^k \lambda_\alpha g_\alpha\right)}{\partial z_\mu\,\partial\bar z_\nu}\right\}_0 u_\mu\bar u_\nu$$

is nondegenerate with $p$ positive and $q$ negative eigenvalues ($p+q=l$).

We set $g = \sum_1^k \lambda_\alpha g_\alpha$. By a linear change of coordinates inside the analytic tangent space $\mathbf{C}^l$ to $S$ at 0 we may assume the above hermitian form to be in diagonal form i.e.

$$\left(\left\{\frac{\partial^2 g}{\partial z_\mu\,\partial\bar z_\nu}\right\}_0\right)_{\substack{1\le\mu\le l\\1\le\nu\le l}} = \begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}, \qquad p+q=l.$$

*c*) Let $M>0$ be a positive constant. We define on $S$ near the origin the following functions:

$$\psi(z,t) = \sum_1^k \lambda_\alpha t_\alpha + i\left\{g(z,t) - 2\sum\frac{\partial^2 g}{\partial z_\mu\,\partial z_\nu}(0)z_\mu z_\nu - 2\sum\frac{\partial^2 g}{\partial t_\alpha\,\partial z_\nu}(0)(t_\alpha+ig_\alpha)z_\nu + M\sum(t_\alpha+ig_\alpha)^2 + M\sum_{p+1}^{l}|z_j|^2\right\}$$

$$\chi(z,t) = \sum_1^k \lambda_\alpha t_\alpha + i\left\{-g(z,t) + 2\sum\frac{\partial^2 g}{\partial\bar z_\mu\,\partial\bar z_\nu}(0)\bar z_\mu\bar z_\nu + 2\sum\frac{\partial^2 g}{\partial t_\alpha\,\partial\bar z_\nu}(0)(t_\alpha-ig_\alpha)\bar z_\nu + M\sum(t_\alpha-ig_\alpha)^2 + M\sum_1^{p}|z_j|^2\right\}.$$

We claim that the hessian of the imaginary part of $\psi$ and the hessian of the imaginary part of $\chi$, evaluated at the origin, are positive definite quadratic forms, provided $M$ is chosen sufficiently large.

Indeed, with obvious notations,

$$\begin{aligned}\mathrm{Hess}_0\{\mathrm{Im}\,(\psi(z,t))\} &= \sum \frac{\partial^2 g}{\partial z_\mu\,\partial \bar z_\nu}(0) z_\mu \bar z_\nu + \sum \frac{\partial^2 g}{\partial t_\alpha\,\partial t_\beta}(0) t_\alpha t_\beta \\ &\quad + M \sum t_\alpha^2 + M \sum_{p+1}^{l} |z_j|^2 = \\ &= \left\{ M \sum t_\alpha^2 + \sum \frac{\partial^2 g}{\partial t_\alpha\,\partial t_\beta}(0) t_\alpha t_\beta \right\} + \sum_1^p |z_j|^2 + (M-1)\sum_{p+1}^{l} |z_j|^2\end{aligned}$$

and, similarly

$$\mathrm{Hess}_0\{\mathrm{Im}\,(\chi(z,t))\} = \left\{ M \sum t_\alpha^2 - \sum \frac{\partial^2 g}{\partial t_\alpha\,\partial t_\beta}(0) t_\alpha t_\beta \right\} + (M-1)\sum_1^p |z_j|^2 + \sum_{p+1}^{l} |z_j|^2 .$$

These expressions establish our claim.

*d*) Now we remark that $\mathrm{Im}\,\psi(z,t)$ and $\mathrm{Im}\,\chi(z,t)$ vanish at the origin of second order. Therefore, if $\omega$ is a sufficiently small neighborhood of the origin in the real tangent space to $S$ at 0 (where $z_1, \ldots, z_l, t_1, \ldots, t_k$ are taken as coordinates), we will have, with $\varepsilon > 0$, in $\omega$

$$\mathrm{Im}\,\psi(z,t) \geqslant \varepsilon\left( \sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \right)$$

$$\mathrm{Im}\,\chi(z,t) \geqslant \varepsilon\left( \sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \right).$$

Moreover

$$\begin{aligned}\psi(z,t) - \overline{\chi(z,t)} &= iM\left\{ \sum_1^l |z_j|^2 + 2\sum_1^k (t_\alpha + i g_\alpha)^2 \right\} = \\ &= iM\left( 2\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \right) - 4M\sum_1^k t_\alpha g_\alpha - 2iM \sum_1^k g_\alpha^2 .\end{aligned}$$

Therefore, if $\omega$ is sufficiently small, we will have in $\omega$

$$\mathrm{Im}\,\left(\psi(z,t) - \overline{\chi(z,t)}\right) \geqslant \frac{M}{2}\left( \sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \right)$$

and

$$\left| 4M \sum_1^k t_\alpha g_\alpha - 2iM \sum_1^k g_\alpha^2 \right| \leqslant \frac{M}{4}\left( \sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \right).$$

This last inequality holds because the left hand side vanishes at the origin at least of third order.

*e*) We claim that the functions $\psi$ and $\chi$ satisfy the following differential equations

$$\bar{L}_j\psi = 0 \quad \text{for } 1 \leqslant j \leqslant p$$

and

$$L_j\chi = 0 \quad \text{for } p+1 \leqslant j \leqslant l .$$

Indeed $\psi = \Theta + iM \sum_{p+1}^{l} |z_j|^2$ where

$$\Theta = \sum \lambda_\alpha(t_\alpha + ig_\alpha) - 2i \sum \frac{\partial^2 g}{\partial z_\mu \partial z_\nu}(0) z_\mu z_\nu - 2i \sum \frac{\partial^2 g}{\partial t_\alpha \partial z_\nu}(0)(t_\alpha + ig_\alpha) z_\nu + iM \sum (t_\alpha + ig_\alpha)^2 .$$

Now $t_\alpha + ig_\alpha$ is the restriction to $S$ of the holomorphic function $z_{l+\alpha} = t_\alpha + is_\alpha$, $1 \leqslant \alpha \leqslant k$. It follows that $\Theta$ is the restriction to $S$ of a holomorphic function defined in a neighborhood $U$ of the origin in $\mathbf{C}^n$. We have therefore $\bar{L}_j\Theta = 0$ for $1 \leqslant j \leqslant l$. On the other hand we have $\bar{L}_j \bar{z}_s = 0$ if $j \neq s$, $1 \leqslant j$, $s \leqslant l$; this by the explicit form of the operators $\bar{L}_j$ given in *a*). Therefore $\bar{L}_j \sum_{p+1}^{l} |z_j|^2 = 0$ if $1 \leqslant j \leqslant p$. We obtain therefore the first set of assertions.

Similarly $\chi = \eta + iM \sum_{1}^{p} |z_j|^2$ where

$$\eta = \sum \lambda_\alpha(t_\alpha - ig_\alpha) + 2i \sum \frac{\partial^2 g}{\partial \bar{z}_\mu \partial \bar{z}_\nu}(0) \bar{z}_\mu \bar{z}_\nu + 2i \sum \frac{\partial^2 g}{\partial_\alpha t \partial \bar{z}_\nu}(0)(t_\alpha - ig_\alpha) \bar{z}_\nu + iM \sum (t_\alpha - ig_\alpha)^2$$

is the restriction to $S$ of a antiholomorphic function defined in $U$.

With the same argument as before we obtain the second set of equations. However this second set of equations will not be *explicitly* needed.

*f*) Let us now assume that the Poincaré lemma holds for $\bar{\partial}_S$ in dimension $q$ at the origin.

Let $\omega$ be that small neighborhood of the origin in $S$ in which the inequalities established in *d*) hold. According to proposition 5 we can then find another open neighborhood $\omega'$ of the origin in $S$ with $0 \subset \omega' \subset\subset \omega$ so that the conclusion of that proposition holds for every $f$ defined in $\omega$ with $f \in Q^{(q)}(\omega)$ and $\bar{\partial}_S f = 0$ on $\omega$, and for every $\nu \in Q^{(q)}(\omega')$ with $\nu$ compactly supported in $\omega'$.

We now choose $f$ and $\nu$. Let $\tau > 0$ be a real parameter. For every $\tau > 0$ we define

$$f_\tau = \exp\left[\frac{i}{\tau}\psi(z, t)\right] d\bar{z}_{p+1} \wedge \ldots \wedge d\bar{z}_l \in Q^{(q)}(\omega) .$$

We have

$$\bar{\partial}_S f_\tau = 0\,.$$

Indeed

$$\bar{\partial}_S f_\tau = \sum_{j=1}^{p} \bar{L}_j \exp\left[\frac{i}{\tau}\psi(z,t)\right] d\bar{z}_j \wedge d\bar{z}_{p+1} \wedge \ldots \wedge d\bar{z}_l = 0$$

because $\bar{L}_j \psi = 0$ for $1 \leqslant j \leqslant p$ as established just above.

*g*) We now proceed to define a convenient element $\nu_\tau \in Q^{(q)}(\omega')$ for any $\tau > 0$, compactly supported in $\omega'$.

First we remark that if we set on $\mathbf{C}^n$

$$\gamma(z) = \sum_1^k \lambda_\alpha \bar{z}_{l+\alpha} + 2i \sum_{\mu,\nu=1}^{l} \frac{\partial^2 g}{\partial \bar{z}_\mu \partial \bar{z}_\nu}(0)\bar{z}_\mu \bar{z}_\nu + 2i \sum_{\substack{1 \leqslant \alpha \leqslant k \\ 1 \leqslant \nu \leqslant l}} \frac{\partial^2 g}{\partial t_\alpha \partial \bar{z}_\nu}(0)\bar{z}_{l+\alpha}\bar{z}_\nu + iM \sum_1^k \bar{z}_{l+\alpha}^2 + iM \sum_1^p z_j \bar{z}_j$$

we have that $\chi = \gamma|\omega$. Let us define for $\tau > 0$ the form of type $(n, p)$ in $\mathbf{C}^n$

$$\eta_\tau = \exp\left[-\frac{i}{\tau}\overline{\gamma(z)}\right] dz_1 \wedge d\bar{z}_1 \ldots dz_p \wedge d\bar{z}_p \wedge dz_{p+1} \wedge \ldots \wedge dz_n\,.$$

Since $\overline{\gamma(z)}$ is the sum of a holomorphic function and the function $-iM \sum_1^p |z_j|^2$ which does not depend from $z_{p+1}, \ldots, z_n$, we do have that

$$\bar{\partial}\eta_\tau = 0 \quad \text{ on } \mathbf{C}^n\,.$$

We fix on $S$ the eucledian volume element

$$dx = (-2i)^{-l} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_l \wedge d\bar{z}_l \wedge dt_1 \wedge \ldots \wedge dt_k$$

and let $\alpha, \beta \in Q^{(r)}(\omega')$ be given explicitly in the form

$$\alpha = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant l} \alpha_{i_1 \ldots i_r} d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_r}$$

$$\beta = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant l} \beta_{i_1 \ldots i_r} d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_r}\,.$$

We define the sesquilinear form on $Q^{(r)}(\omega')$:

$$\langle \alpha, \beta \rangle = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant l} \alpha_{i_1 \ldots i_r} \bar{\beta}_{i_1 \ldots i_r}\,.$$

If one of the forms $\alpha$ or $\beta$ has compact support in $\omega'$ we define then their scalar product

$$(\alpha, \beta)_{\omega'} = \int_{\omega'} \langle \alpha, \beta \rangle \, dx .$$

Now given the operator $\bar{\partial}_S \colon Q^{(r)}(\omega') \to Q^{(r+1)}(\omega')$ its formal adjoint

$$(\bar{\partial}_S)^* \colon Q^{(r+1)}(\omega') \to Q^{(r)}(\omega')$$

is uniquely defined by the property

$$(\bar{\partial}_S \alpha, \beta)_{\omega'} = (\alpha, (\bar{\partial}_S)^* \beta)_{\omega'}$$

$\forall \alpha \in Q^{(r)}(\omega')$ compactly supported and $\forall \beta \in Q^{(r+1)}(\omega')$.

Now we remark that $\dim_{\mathbf{R}} S = 2l + k = l + n$. Therefore the exterior form in $\mathbf{C}^n$ of total degree $2l + k = l + n$

$$(d\bar{z}_{p+1} \wedge \ldots \wedge d\bar{z}_l) \wedge (dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_p \wedge d\bar{z}_p) \wedge (dz_{p+1} \wedge \ldots \wedge dz_l) \wedge dz_{l+1} \wedge \ldots \wedge dz_n$$

when restricted to $S$ can be written as

$$\sigma(z, t)\, dx , \qquad (z, t) = (z_1, \ldots, z_l, t_1, \ldots, t_k) ,$$

with $\sigma(z, t)$, $C^\infty$ on $S$. We note that

$$\sigma(0, 0) = (-2i)^l (-1)^{\frac{1}{2}q(q+1)} \neq 0 , \qquad (q + p = l) .$$

We set, for $\tau > 0$,

$$w_\tau = \bar{\sigma}(z, t) \exp\left[\frac{i}{\tau} \chi(z, t)\right] d\bar{z}_{p+1} \wedge \ldots \wedge d\bar{z}_l \in Q^{(q)}(\omega') .$$

For any $\alpha \in Q^{(q)}(\omega')$ and with compact support we have the following formula of integration:

$$(\alpha, w_\tau)_{\omega'} = \int_\omega \alpha_{p+1 \ldots l} \exp\left[-\frac{i}{\tau} \overline{\chi(z, t)}\right] \sigma(z, t)\, dx = \int_S \alpha \wedge \eta_\tau .$$

Let $\beta \in Q^{(q-1)}(\omega')$ with compact support. We can find a form $\tilde{\beta}$ in $\mathbf{C}^n$ with compact support and of type $(0, q-1)$ whose image in $Q^{(q-1)}(\omega')$ is $\beta$ (under the natural map given by the definition of the space $Q^{(q-1)}$ as a quotient of the space of $(0, q-1)$ forms in the surrounding space $\mathbf{C}^n$). We have therefore by definition

$$\bar\partial_S \beta = \bar\partial \tilde\beta \bmod (\varrho_1, \dots, \varrho_k, \bar\partial\varrho_1, \dots, \bar\partial\varrho_k)\,.$$

Now we have for any such $\beta$ the following formula:

$$\begin{aligned}(\bar\partial_S\beta, w_\tau)_{\omega'} &= \int_S \bar\partial\tilde\beta \wedge \eta_\tau \\ &= \int_S \bar\partial(\tilde\beta\wedge\eta_\tau) \\ &= \int_S d(\tilde\beta\wedge\eta_\tau) \\ &= 0\,.\end{aligned}$$

The first of these equalities is due to the formula established just above and to the fact that for a compactly supported form $\pi^{(n,l-1)}$ of type $(n, l-1)$ (defined in the neighborhood $U$ of the origin where $S$ is given) we have

$$\begin{aligned}\int_S \bar\partial\varrho_j \wedge \pi^{(n,l-1)} &= \int_S \bar\partial(\varrho_j\wedge\pi^{(n,l-1)}) \quad \text{as } \varrho_j = 0 \quad \text{on } S \\ &= \int_S d(\varrho_j \wedge \pi^{(n,l-1)}) \ \text{ by reasons of bidegree} \\ &= 0 \ \text{ by Stokes theorem}\,.\end{aligned}$$

The second equality follows by the fact that $\bar\partial\eta_\tau = 0$. The third equality is valid because of reasons of bidegree and the last by Stokes' theorem.

But this shows that, for any $\tau > 0$,

$$(\bar\partial_S)^* w_\tau = 0 \qquad \text{on } \omega'\,.$$

Let $R > 0$ be so chosen that

$$\left\{\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R\right\} \subset\subset \omega'.$$

We select a $C^\infty$ function $\nu(z, t)$ compactly supported in $\omega'$ with $0 \leqslant \nu \leqslant 1$ in $\omega'$ and $\nu = 1$ on $\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R$.

We then define for any $\tau > 0$,

$$v_\tau = \tau^{-l-k/2}\nu(z,t)\,w_\tau = \tau^{-l-k/2}\mu(z,t)\exp\left[\frac{i}{\tau}\chi(z,t)\right]d\bar z_{p+1}\wedge\ldots\wedge d\bar z_l$$

where $\mu(z,t) = \nu(z,t)\overline{\sigma(z,t)}$ has compact support in $\omega'$ and equals $\overline{\sigma(z,t)}$ on the ball $\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R$.

We have $v_\tau \in Q^{(q)}(\omega')$ and compactly supported. Moreover on the ball $\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R$, $v_\tau = \tau^{-l-k/2} w_\tau$. Therefore

$$(\bar\partial_S)^* v_\tau = \beta(z,t)\exp\left[\frac{i}{\tau}\chi(z,t)\right]\tau^{-l-k/2}$$

with $\beta(z,t) \in Q^{(q-1)}(\omega')$ with compact support, independent from $\tau$, and vanishing on the ball $\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R$.

If we now apply to this choice of $f_\tau$ and $v_\tau$ the statement of proposition 5 we realize that we can find a constant $c > 0$, a compact $K \subset \omega$ and an integer $m \geqslant 0$, all independent from $\tau$, such that

$$(*) \qquad \left|\tau^{-l-k/2}\int_{\omega'}\mu(z,t)\exp\left[\frac{i}{\tau}\left(\psi(z,t) - \overline{\chi(z,t)}\right)\right]dx\right| \leqslant$$
$$\leqslant c\sup_{\omega'}\left|\tau^{-l-k/2}\beta(z,t)\exp\left[\frac{i}{\tau}\chi(z,t)\right]\right|\sup_{|\alpha|\leqslant m}\sup_K\left|D^\alpha\exp\left[\frac{i}{\tau}\psi(z,t)\right]\right|$$

for any $\tau > 0$.

*h*) We evaluate the left hand side of $(*)$. Replacing $z$ with $\sqrt{\tau}\,z$ and $t$ with $\sqrt{\tau}\,t$ we get if $0 < \tau < 1$

$$\tau^{-l-k/2}\int_{\omega'}\mu(z,t)\exp\left[\frac{i}{\tau}\left(\psi(z,t) - \overline{\chi(z,t)}\right)\right]dx =$$
$$= \int_{\mathbf{C}^l\times\mathbf{R}^k}\mu(\sqrt{\tau}\,z, \sqrt{\tau}\,t)\exp\left[\frac{i}{\tau}\left(\psi(\sqrt{\tau}\,z, \sqrt{\tau}\,t) - \overline{\chi(\sqrt{\tau}\,z, \sqrt{\tau}\,t)}\right)\right]dx\,.$$

We remark that, taking into account the expression and inequalities given in *d*),

- the integrand on the right hand side is bounded in absolute value by $\exp\left[-\frac{3}{4}M\left(\sum_1^l |z_j|^2 + \sum_1^k t_\alpha^2\right)\right]$ and this function is summable over the whole space $\mathbf{C}^l\times\mathbf{R}^k$;

– the same integrand for $\tau \to 0$ converges pointwise to the function

$$\mu(0,0)\exp\left[-M\left(\sum_1^l |z_j|^2 + 2\sum_1^k t_\alpha^2\right)\right].$$

By Lebesgue's theorem on dominated convergence we deduce that

$$\lim_{\tau\to 0^+}\{\text{left hand side of } (*)\} = \frac{1}{2M}\left(\frac{1}{\sqrt{2}}\right)^k \sigma(k+2l)|\mu(0,0)|$$

where $\sigma(s)$ denotes the eucledian volume of the unit sphere in $\mathbf{R}^s$.

In particular for $\tau \to 0$ the left hand side of $(*)$ tends to a finite limit different from zero $\left(\text{since } \mu(0,0) = \overline{\sigma(0,0)} \neq 0\right)$.

We now evaluate the right hand side of $(*)$. We first remark that for $0 < \tau < 1$ we have for any $\alpha \in \mathbf{N}^{2l+k}$ estimates of the form

$$\sup_K \left| D^\alpha \exp\left[\frac{i}{\tau}\psi(z,t)\right]\right| \leqslant c(\alpha)\,\tau^{-|\alpha|}$$

where $c(\alpha)$ is a constant independent of $\tau$, (here we made use of the first inequality established in $d$)).

Also with a constant $c_1 > 0$ independent of $\tau$ we have

$$\sup_{\omega'} |(\bar\partial_S)^* v_\tau| = \sup_{\omega'}\left|\tau^{-l-k/2}\beta\exp\left[\frac{i}{\tau}\chi(z,t)\right]\right| \leqslant c_1\tau^{-l-k/2}\exp\left[-\frac{\varepsilon R^2}{\tau}\right]$$

because of the second inequality established in point $d$) and the fact that $\beta = 0$ on the ball $\sum_1^k t_\alpha^2 + \sum_1^l |z_j|^2 \leqslant R$.

But then, with a constant $c_2 > 0$ independent of $\tau$ we have

$$|\text{right hand side of } (*)| \leqslant c_2\tau^{-m-l-k/2}\exp\left[-\frac{\varepsilon R^2}{\tau}\right].$$

For $\tau \to 0$ the right hand side of this inequality tends to zero.

This says that the inequality $(*)$ cannot hold for $\tau > 0$ small. This achieves the proof.

*i*) At the end of this proof the following remark is in order

REMARK. *Let $S$ be generic in $\mathbf{C}^n$, $0 \in S$, and for some $\lambda \in \Sigma$, $e(\lambda) = (p, q)$ with $p + q = n - k$.*

*Set*

$$\mathcal{H}^q_{\bar\partial_S,0} = \frac{\operatorname{Ker}\left\{Q^{(q)}(S)_0 \xrightarrow{\bar\partial_S} Q_0^{(q+1)}(S)_0\right\}}{\operatorname{Im}\left\{Q^{(q-1)}(S)_0 \xrightarrow{\bar\partial_S} Q^{(q)}(S)_0\right\}}.$$

*We claim that*

$$\dim_{\mathbf{C}} \mathcal{H}^q_{\bar\partial_S,0} = \infty .$$

PROOF. If $q = 0$ the remark is obvious as $\mathcal{H}^0_{\bar\partial_S}$ contains the traces of germs of holomorphic functions on $S$. We can thus assume $q > 0$.

Let $\omega$ be a small relatively compact neighborhood of 0 in $S$ such that at any point $y \in \bar\omega$, $e_y(\lambda) = (p, q)$ for some $\lambda \in \Sigma_y$, the unit sphere in the normal bundle to $S$ at $y$.

For any point $y \in \omega$ we can find according to the proof of the previous theorem an element

$$f_y \in Z(\omega) = \{f \in Q^{(q)}(\omega) |\ \bar\partial_S f = 0 \text{ in } \omega\}$$

such that in no neighborhood $\omega(y)$ of $y$ we can write

$$f_y = \bar\partial_S g$$

with $g \in Q^{(q-1)}(\omega(y))$. Indeed if this is not the case the argument of lemma 3 applies and therefore we can find a fixed neighborhood $\omega_1(y)$ of $y$ in $\omega$ such that for any $f \in Z(\omega)$ we can find $g \in Q^{(q-1)}(\omega_1(y))$ with $\bar\partial_S g = f$ on $\omega_1(y)$. Then also the conclusion of the corollary to lemma 3 holds, and then also the estimate of proposition 5. This is what is contradicted in the proof of theorem 3 with the choice of elements $f_\tau \in Z(\omega)$ for all $\tau > 0$.

Let us select a countable dense set $\{y_h\}_{h\in\mathbf{N}}$ in $\omega$ and for each $y_h$ a fundamental sequence of open neighborhoods $\{\omega_k(y_h)\}_{h\in\mathbf{N}}$ of $y_h$ in $\omega$. We set

$$G(k, h) = \{(u, f) \in Q^{(q-1)}(\omega_k(y_h)) \times Z(\omega) |\ \bar\partial_S u = f \text{ on } \omega_k(y_h)\}$$

and we denote by $\pi_{kh}\colon G(k, h) \to Z(\omega)$ the natural projection.

We have that $\pi_{kh}(G(k, h)) \subsetneq Z(\omega)$. Therefore $\bigcup_{hk} \pi_{kh}(G(k, h))$ is of first category in $Z(\omega)$ and hence

$$\mathcal{B} = \bigcup_{hk} \pi_{kh}(G(k, h)) \subsetneq Z(\omega) .$$

It will be enough to prove that

$$\dim_{\mathbf{C}} Z(\omega)/\mathcal{B} = \infty .$$

Let $f \in Z(\omega)$ with $f \notin \mathcal{B}$ which is possible by the above considerations. Assume that $\dim_{\mathbf{C}} Z(\omega)/\mathcal{B} < \infty$.

We consider the natural map

$$\lambda\colon Z(\omega) \to Z(\omega)/\mathfrak{B}$$

and for $t \in \mathbb{R}$ the image $\lambda(\exp[tz_1]f)$. Let $V$ be the space generated by these elements over $\mathbb{C}$. Let $k = \dim_{\mathbb{C}} V$. By the assumption $k < \infty$. Let $\lambda(\exp[t_i z_1]f)$ be generators of $V$ for $1 \leqslant i \leqslant k$. Let $t_0 \neq t_1, t_2, \ldots, t_k$. There must exist constants $c_1, \ldots, c_k$ not all zero such that

$$\big(\exp[t_0 z_1] - c_1 \exp[t_1 z_1] - \ldots - c_k \exp[t_k z_1]\big) f \in \mathfrak{B}\,.$$

This means that for some $h$ and $k$ we can find $g \in Q^{(q-1)}(\omega_k(y_h))$ with $\bar\partial_S g = \big(\exp[t_0 z_1] - c_1 \exp[t_1 z_1] - \ldots - c_k \exp[t_k z_1]\big) f$. Let

$$\Theta = \exp[t_0 z_1] - c_1 \exp[t_1 z_1] - \ldots - c_k \exp[t_k z_1]\,.$$

Since the $t_j$'s are two by two distinct $\Theta|S$ is not identically zero. Therefore we can find $y_r \in \omega_k(y_h)$ and an $s \geqslant 0$ such that $\omega_s(y_r) \subset \omega_k(y_h)$ and such that $\Theta|\omega_s(y_s)$ is always different from zero. We have therefore

$$f = \bar\partial_S \frac{g}{\Theta} \qquad \text{on} \quad \omega_s(y_s)\,.$$

This contradicts the assumption $f \notin \mathfrak{B}$. We must therefore have

$$\dim_{\mathbb{C}} Z(\omega)/\mathfrak{B} = \infty.$$

## 7. – Local cohomology for $\bar\partial_S$.

*a*) Let $S$ be a smooth real submanifold of the complex manifold $X$.

We have on $X$ the Dolbeault complex of sheaves

$$\mathcal{E}^{(0)} \xrightarrow{\bar\partial} \mathcal{E}^{(1)} \xrightarrow{\bar\partial} \mathcal{E}^{(2)} \xrightarrow{\bar\partial} \ldots$$

and the subcomplex of sheaves

$$\boldsymbol{I}^{(0)} \xrightarrow{\bar\partial} \boldsymbol{I}^{(1)} \xrightarrow{\bar\partial} \boldsymbol{I}^{(2)} \xrightarrow{\bar\partial} \ldots$$

where $\boldsymbol{I}^{(j)}$ denotes the sheaf $U \to \boldsymbol{I}^{(j)}(S \cap U, U)$ defined in section 3. If $x \notin S$ then, for every $j \geqslant 0$, $\boldsymbol{I}^{(j)}_x = \mathcal{E}^{(j)}_x$.

Therefore the quotient complex of sheaves

$$Q^{(0)} \xrightarrow{\bar{\partial}_S} Q^{(1)} \xrightarrow{\bar{\partial}_S} Q^{(2)} \xrightarrow{\bar{\partial}_S} \ldots$$

is concentrated on $S$.

We set

$$\mathcal{O}_S = \operatorname{Ker} \{Q^{(0)} \xrightarrow{\bar{\partial}_S} Q^{(1)}\}$$

and

$$\mathcal{H}^j_{\bar{\partial}_S} = \frac{\operatorname{Ker}\{Q^{(j)} \xrightarrow{\bar{\partial}_S} Q^{(j+1)}\}}{\operatorname{Im}\{Q^{(j-1)} \xrightarrow{\bar{\partial}_S} Q^{(j)}\}} \quad \text{for } j \geqslant 1 .$$

We have $Q^{(0)} \simeq \mathcal{E}_S$ the sheaf of germs of $C^\infty$ functions on $S$. If $S$ is *generic* the sheaves $Q^{(j)}$ for $j \geqslant 1$ are also locally free sheaves of modules over $\mathcal{E}_S$ and the maps $\bar{\partial}_S$ are given by differential operators.

Let $\phi$ be a paracompactifying family of supports on $S$ and let us denote by $H^j_\phi(S, \bar{\partial}_S)$ the $j$-th cohomology group of the complex

$$\Gamma_\phi(S, Q^{(0)}) \xrightarrow{\bar{\partial}_S} \Gamma_\phi(S, Q^{(1)}) \xrightarrow{\bar{\partial}_S} \Gamma_\phi(S, Q^{(2)}) \xrightarrow{\bar{\partial}_S} \ldots$$

for $j \geqslant 0$.

From a general theorem of Godement ([6], theorem 4.6.1, p. 178) we derive the following

PROPOSITION 6. *For any paracompactifying family of supports $\phi$ on $S$ we have a spectral sequence*

$$E_2^{pq} = H^p_\phi(S, \mathcal{H}^q_{\bar{\partial}_S}) \Rightarrow H^s_\phi(S, \bar{\partial}_S)$$

$s = p + q$.

Indeed we have only to remark that the sheaves $\mathcal{E}^{(j)}$ and $I^{(j)}$ are soft (indeed fine) sheaves thus also the sheaves $Q^{(j)}$ are soft sheaves ([6], theorem 3.5.3, p. 154). Hence $H^q_\phi(S, Q^{(j)}) = 0$ for any $q > 0$ and any $j \geqslant 0$. This enables us to apply the quoted theorem of Godement.

*b*) As an illustration we consider the following three cases:

CASE 1. $\dim_{\mathbf{R}} S = n$, $S$ is generic, i.e. $S$ is « *totally real* ». Then the boundary complex reduces to the trivial complex

$$Q^{(0)} \to 0$$

and thus $\mathcal{O}_S \simeq \mathcal{E}_S \simeq Q^{(0)}$.

For instance consider on the Riemann sphere $\mathbf{P}_1(\mathbf{C})$ an affine holomorphic coordinate $z$ (we allow $z$ to take the value $\infty$) and set $D_1 = \{z \in \mathbf{P}_1(\mathbf{C}) |\ |z| \leqslant 1\}$, $D_2 = \{z \in \mathbf{P}_1(\mathbf{C}) |\ |z| \geqslant 1\}$.

Let $X = \mathbf{P}_1(\mathbf{C}) \times \ldots \times \mathbf{P}_1(\mathbf{C})$ ($n$ times) and set

$$\Omega^+ = D_1 \times \ldots \times D_1$$

$$\Omega^- = D_2 \times \ldots \times D_2$$

$$S = \Omega^+ \cap \Omega^- \quad \text{and} \quad \Omega = \Omega^+ \cup \Omega^- .$$

Then $S$ is totally real. Moreover $\Omega^+$ and $\Omega^-$ are isomorphic and also isomorphic to a polycylinder in $\mathbf{C}^n$. For a closed polycylinder $P$ in $\mathbf{C}^n$ we have

$$H^j(P, \bar\partial) = 0 \quad \text{for } j \geqslant 1 .$$

Indeed this follows from the expression of the homotopy operator for the Dolbeault complex as given by H. K. Nickerson [14] and from the fact that for a function $\varphi(\xi)$, $C^\infty$ on the closed disc $\{\xi \in \mathbf{C} |\ |\xi| \leqslant 1\}$, the function

$$g(z) = \int_{|\xi| \leqslant 1} \frac{\varphi(\xi)}{\xi - z}\, d\xi\, d\bar\xi$$

is also $C^\infty$ in the closed disc $\{z \in \mathbf{C} |\ |z| \leqslant 1\}$.

From the Mayer-Vietoris sequence we deduce then, that

$$0 \to H^0(\Omega, \bar\partial) \to H^0(\Omega^+, \bar\partial) \oplus H^0(\Omega^-, \bar\partial) \to \mathcal{E}(S) \to H^1(\Omega, \bar\partial) \to 0$$

is an exact sequence. Moreover one has $H^j(\Omega, \bar\partial) = 0$ for $j \geqslant 2$.

CASE 2. $S$ is a hypersurface with nondegenerate definite Levi form. In this case we have

$$\mathcal{H}^0_{\bar\partial_S} \simeq \mathcal{O}_S , \quad \mathcal{H}^{n-1}_{\bar\partial_S} \neq 0 ,$$

moreover $\mathcal{H}^j_{\bar\partial_S} = 0$ if $j \neq 0, n-1$ as it follows from the Mayer-Vietoris sequence and a result of ([3], part II, p. 802).

We derive from this that

(i) $H^j(S, \bar\partial_S) \simeq H^j(S, \mathcal{O}_S)$ for $0 \leqslant j \leqslant n-2$;

(ii) the sheaf $\mathcal{H}^{n-1}_{\bar\partial_S}$ is a soft sheaf since

$$Q^{(n-2)} \xrightarrow{\bar\partial_S} Q^{(n-1)} \longrightarrow \mathcal{H}^{n-1}_{\bar\partial_S} \longrightarrow 0$$

is an exact sequence and the sheaves $Q^{(j)}$ are soft sheaves;

(iii) from the spectral sequence of proposition 6 we deduce then an exact sequence

$$0 \to H^0(S, \mathcal{H}^{n-1}_{\bar\partial_S}) \to H^{n-1}(S, \bar\partial_S) \to H^{n-1}(S, \mathcal{O}_S) \to 0$$

and also that $H^j(S, \mathcal{O}_S) = 0$ if $j \geqslant n$.

CASE 3. $S$ is a hypersurface with nondegenerate Levi form with $p = q = (n-1)/2$ (hence $n$ is odd) eigenvalues of each sign.

In this case we have

$$\mathcal{H}^0_{\bar\partial_S} \simeq \mathcal{O}_S\,, \qquad \mathcal{H}^p_{\bar\partial_S} \neq 0$$

while for the already quoted result we have $\mathcal{H}^j_{\bar\partial_S} = 0$ if $j \neq 0$, $j \neq p$.

From this and the spectral sequence of proposition 6 we deduce that

(i) $H^j(S, \bar\partial_S) \simeq H^j(S, \mathcal{O}_S)$ for $0 \leqslant j \leqslant p-1$.

(ii) We have an exact sequence

$$\begin{aligned}
0 &\to H^0(S, \mathcal{H}^p_{\bar\partial_S}) \to H^p(S, \bar\partial_S) \to H^p(S, \mathcal{O}_S) \to \\
&\to H^1(S, \mathcal{H}^p_{\bar\partial_S}) \to H^{p+1}(S, \bar\partial_S) \to H^{p+1}(S, \mathcal{O}_S) \to \\
&\ldots \to H^{n-1}(S, \bar\partial_S) \to H^{n-1}(S, \mathcal{O}_S) \to \\
&\to H^{n-p}(S, \mathcal{H}^p_{\bar\partial_S}) \to 0\,.
\end{aligned}$$

REMARK. We do also have in this last case $H^{n+j}(S, \mathcal{O}_S) \simeq H^{n-p+1+j}(S, \mathcal{H}^p_{\bar\partial_S})$ for any $j \geqslant 0$. But one can expect that these groups vanish.

*c*) The case of $S$ totally real or the case of an hypersurface are particularly simple.

In general we may expect the failure of the Poincaré lemma in any dimension or in some privileged dimensions.

Here are two examples of this situation.

EXAMPLE 1. Let $k \geqslant 2$, $S$ generic, $0 \in S$ with equations

$$\begin{cases} s_\alpha = g_\alpha(z_1, \ldots, z_l, t_1, \ldots, t_k) \\ 1 \leqslant \alpha \leqslant k \end{cases}$$

$z_{l+\alpha} = t_\alpha + i s_\alpha$, $H(S)_0 = \{z_{l+1} = \ldots = z_n = 0\}$.

We assume that

$$A = \sum_1^l \frac{\partial^2 g_1}{\partial z_\nu \, \partial \bar{z}_\mu}(0) u_\nu \bar{u}_\mu > 0 \qquad \text{(positive definite)}$$

$$B = \sum_1^l \frac{\partial^2 g_2}{\partial z_\nu \, \partial \bar{z}_\mu}(0) u_\nu \bar{u}_\mu \qquad \text{is such that}$$

$$\det\left(\frac{\partial^2 g_2}{\partial z_\nu \, \partial \bar{z}_\mu}(0) - \lambda \frac{\partial^2 g_1}{\partial z_\nu \, \partial \bar{z}_\mu}(0)\right) = 0 \qquad \text{has } l \text{ distinct roots}\,.$$

Under these conditions the hermitian form $B - \lambda A$ assumes all possible signatures $(0, l), \ldots, (l, 0)$ as one realizes by a convenient $\mathbf{C}$-linear change of coordinates on $H(S)_0$ which reduces $A$ to the identity matrix and $B$ to diagonal form.

In this case the *Poincaré lemma fails in all dimensions* $0, 1, \ldots, l$.

EXAMPLE 2. Let $n = 2r + 2$ and set $\mathbf{C}^n = \mathbf{C}^r \times \mathbf{C}^r \times \mathbf{C}^2$ where $z \in \mathbf{C}^r$, $u \in \mathbf{C}^r$, $w = (t_1 + is_1, t_2 + is_2) \in \mathbf{C}^2$ are holomorphic coordinates. Let $k = 2$ and $S$ be defined by the equations

$$\begin{cases} s_1 = {}^t z\bar{z} - {}^t u\bar{u} \\[2mm] s_2 = ({}^t z, {}^t u)\begin{pmatrix} 0 & M \\ {}^t\bar{M} & 0 \end{pmatrix}\begin{pmatrix} \bar{z} \\ \bar{u} \end{pmatrix} \end{cases}$$

where $M$ is an $r \times r$ matrix with complex elements and $\det M \neq 0$.

The Levi form is given for $(\lambda_1, \lambda_2) \in \mathbf{R}^2$ at the origin by the hermitian matrix

$$\lambda_1 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & M \\ {}^t\bar{M} & 0 \end{pmatrix} = H(\lambda_1, \lambda_2)\,.$$

One easily recognizes that $\det H(\lambda_1, \lambda_2) \neq 0$ if $(\lambda_1, \lambda_2) \neq (0, 0)$.

Therefore for all $(\lambda_1, \lambda_2) \in \mathbf{R}^2 - \{0\}$ the signature of the hermitian matrix $H(\lambda_1, \lambda_2)$ is constant and equals $(r, r)$.

In this case we are sure that the *Poincaré lemma fails in dimension* $r$ on $S$ at the origin.

One may presume that it holds in all dimensions different from $0$ and $r$. From a result of Naruki [13] one can deduce that the Poincaré lemma holds in dimension $r + 1, \ldots, 2r$.

*d*) Let us assume that $X = \mathbf{C}^n$ and $0 \in S$ while $S$ is generic at the origin.

We can assume $S$ is given by equations as in point $a$) of the proof of theorem 3. From the explicit expression of the operator $\bar{\partial}_S$ given in $a$) we deduce that:

The symbolic complex at the origin $0 \in S$ for the complex of $\bar{\partial}_S$ on $S$ is a Hilbert complex (actually a Koszul complex).

We can then apply the results of [4] and we conclude that

PROPOSITION 7. *If $S$ is generic at a point $x_0 \in S$ then the tangential complex of $\bar{\partial}_S$ admits the formal Poincaré lemma in any dimension $j \geqslant 1$.*

*If moreover $S$ is real analytic then the tangential complex of $\bar{\partial}_S$ admits the analytic Poincaré lemma in any dimension $j \geqslant 1$.*

$e$) We end up with an easy application of the Mayer-Vietoris sequence.

Let $\Omega$, $\Omega^+$, $\Omega^-$, $S$ be defined in the complex manifold $X$ as usual. Assume

$$S = \{x \in X | \varrho_1(x) = \dots = \varrho_k(x) = 0\}$$

is generic at each point. Then $H^j(S, \bar{\partial}_S) = 0$ if $j > l = n - k$.

From the exact sequence of proposition 4 we deduce therefore that

$$H^j(\Omega, \bar{\partial}) \simeq H^j(\Omega^+, \bar{\partial}) \oplus H^j(\Omega^-, \bar{\partial})$$

for all $j \geqslant l + 2$.

For $l = 1$ we have moreover on $S$ the exact sequence (derived from the spectral sequence of proposition 6)

$$0 \to H^0(S, \bar{\partial}_S) \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{H}^1_{\bar{\partial}_S}) \to$$
$$\to H^1(S, \bar{\partial}_S) \to H^1(S, \mathcal{O}_S) \to 0 .$$

The last zero is obtained by the fact that $\mathcal{H}^1_{\bar{\partial}_S}$ is a soft sheaf and thus its first cohomology group is zero. Note that we have $H^j(S, \mathcal{O}_S) = 0$ if $j \geqslant 2$.

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