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On the Barotropic Motion of Compressible Perfect Fluids.

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1. – Introduction.

In this paper we consider the equations of the barotropic motion of a compressible perfect fluid in an open, bounded and connected subset Ω of the euclidean space $\mathbb{R}^3$ (or $\mathbb{R}^2$). For an useful introduction to the proof given in this paper and for a clear description of the main ideas we refer the reader to [3].

We assume that Ω is locally situated on one side of its boundary Γ a differentiable manifold of class C^5. We denote by $n = n(x)$ the unit outward normal to the boundary Γ and we assume that the vector field $n(x)$ is defined and of class C^4 in a neighbourhood of Γ.

We denote by $H^k(\Omega)$, k non-negative integer, the Sobolev space of order k endowed with the usual norm $\| \ \|_k$, and by $(\ , \ )$ and $\| \ \|$ the scalar product and the norm in $H^0(\Omega) \equiv L^2(\Omega)$. We denote also by $H^k(\Omega)$ the space $\big(H^k(\Omega)\big)^3$ of the vector fields $v = (v_1, v_2, v_3)$ such that $v_i \in H^k(\Omega)$, $i = 1, 2, 3$ and by $\|v\|_k$ the norm of v in $\big(H^k(\Omega)\big)^3$. A similar convention applies to the other functional spaces and norms used in this paper.

We denote by $L^\infty(0, T; H^k)$ the Banach space of the (measurable) essentially bounded functions defined on $[0, T]$ whith values in $H^k(\Omega)$. The norm in this space is denoted by $\| \ \|_{k,T}$. The subspace of the continuous [resp. lipschitz continuous] functions on the closed interval $[0, T]$ is denoted by $C(0, T; H^k)$ [resp. Lip $(0, T; H^k)$]. We denote by $L^1(0, T; H^k)$ the Banach space of the integrable functions on $[0, T]$ with values in $H^k(\Omega)$, with the usual norm $[\ ]_{k,T}$.

We also use on Γ the Sobolev fractionary spaces $H^{k+\frac{1}{2}}(\Gamma)$ and the corresponding spaces $L^\infty\big(0, T; H^{k+\frac{1}{2}}(\Gamma)\big)$. The norms in these spaces will be denoted by $||| \ |||_{k+\frac{1}{2}}$ and $||| \ |||_{k+\frac{1}{2},T}$ respectively.

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For scalar or vector fields w we use the notations $w(0) = w(0, x)$, $\partial_t w = \partial w/\partial t$, $\partial_n w = \partial w/\partial n$, $\partial_i w = \partial w/\partial x_i$, $\partial^2_{ij} w = \partial_i \partial_j w$. We set also $(v \cdot \nabla) w = \sum_i v_i \partial_i w$, where v and w are vector fields.

In this paper we consider the equations of the barotropic motion of a compressible perfect (i.e. non-viscous) fluid in Ω. An interesting example of barotropic flow is given by a gas in isentropic motion (i.e. the entropy is constant). For the particular case of an ideal gas (i.e. the equation of state is Clapeyron's equation) with constant specific heats one has in particular $p = N\varrho^\gamma$ where $N > 0$ and $\gamma > 1$ are constants. Another interesting case is the isothermical motion of an ideal gas.

Let us denote by $v(t, x)$, $\varrho(t, x)$ and $p(t, x)$ the velocity, the density and the pressure of the fluid in the point x at the time t, and by $f(t, x)$ the density of the external mass forces. The equations of the motion are then (see for instance [16] sections C.I and E.I, II, [14] IV § 1, [10] § 1 and § 2)

$$
(1.1) \qquad \begin{cases} \varrho[\partial_t v + (v\cdot\nabla)v - f] = -\nabla p(\varrho) & \text{in } Q_{T_0} \equiv [0, T_0]\times\Omega\,, \\ \partial_t \varrho + \operatorname{div}(\varrho v) = 0 & \text{in } Q_{T_0}\,, \\ v|_{t=0} = a(x) & \text{in } \Omega\,, \\ \varrho|_{t=0} = \varrho_0(x) & \text{in } \Omega\,, \\ v\cdot n = 0 & \text{on } \Sigma_{T_0} \equiv [0, T_0]\times\Gamma\,. \end{cases}
$$

We assume that the initial velocity $a(x)$ and the initial density distribution $\varrho_0(x)$ are given. Moreover we assume that the fluid is barotropic that is the pression $p = p(\varrho)$ depends only on ϱ. The known function « $\xi \to p(\xi)$ » verifies the physical hypothesis $p'(\xi) > 0$ for $\xi > 0$. We assume that $p \in C^4(]0, +\infty[; \mathbb{R})$. We also assume that

$$
(1.2) \qquad \varrho_0(x) \geqslant m_0 \quad \text{in } \Omega\,,
$$

where m_0 is a positive constant, that

$$
(1.3) \qquad a(x)\cdot n = 0 \quad \text{on } \Gamma
$$

and that the following compatibility conditions are verified (which are independent from the extension of the vector field $n(x)$ to the neighborhood of Γ):

$$
(1.4) \qquad \frac{p'(\varrho_0)}{\varrho_0}\,\partial_n \varrho_0 = \sum_{i,j} (\partial_i n_j) a_i a_j + f|_{t=0}\cdot n \quad \text{on } \Gamma\,,
$$

$$
(1.5) \qquad \sum_{i,j} (\partial_i n_j)(\dot a_i a_j + a_i \dot a_j) + \partial_t f|_{t=0}\cdot n + \partial_n\left[\left(a\cdot\frac{\nabla\varrho_0}{\varrho_0} + \operatorname{div} a\right)p'(\varrho_0)\right] = 0 \quad \text{on } \Gamma\,,
$$

where we set

$$\dot{a} \equiv -(a\cdot\nabla)a + f|_{t=0} - p'(\varrho_0)\frac{\nabla\varrho_0}{\varrho_0}\,. \tag{1.6}$$

Assumptions (1.3), (1.4) and (1.5) are necessary to the solvability of (1.1), for by taking the scalar product of $(1.1)_1$ with the normal n one gets

$$\sum_{i,j}(\partial_i n_j)v_i v_j + f\cdot n = \frac{\nabla p}{\varrho}\cdot n \qquad \text{on } \Gamma, \tag{1.7}$$

which for $t=0$ gives (1.4). On the other hand from (1.1) it follows that $\partial_t v|_{t=0} = \dot{a}$. By differentiating (1.7) with respect to t for $t=0$ and using $(1.1)_2$ to determinate $\partial_t\varrho|_{t=0}$, we deduce (1.5).

In this paper we prove the following result, announced in [2] (see also [1]):

THEOREM 1.1. *Assume that the above conditions hold and that*

$$a,\ \varrho_0 \in H^3(\Omega)\,, \qquad \partial_t^j f \in L^\infty(0, T_0; H^{3-j})\,, \qquad j = 0, 1, 2\,.$$

Then there exists $T_1 \in\,]0, T_0]$, *depending only on* Ω, *on the particular function* « $\xi \to p(\xi)$ » *and on the norms* $\|a\|_3$, $\|\varrho_0\|_3$, $\|\partial_t^j f\|_{3-j,T_0}$, $j=0,1,2$ (T_1 *is a non decreasing function of these norms*) *such that* (1.1) *is uniquely solvable in* Q_{T_1}. *Moreover* $\partial_t^j v$, $\partial_t^j \varrho \in L^\infty(0, T_1; H^{3-j})$, $j = 0, 1, 2$ (1).

The uniqueness of the solution was proved by D. Graffi [6] and, in a more general context, by J. Serrin [15].

The assumption on the force field f can be weakened. Moreover results and proofs remain essentially the same if one use Sobolev spaces of greater order. Obviously one must then add supplementary compatibility conditions.

Finally put $Z \equiv \{(v, \varrho)\colon \partial_t^j(v, \varrho) \in L^\infty(0, T_1; H^{3-j}), j = 0, 1, 2\}$ and let $Z \hookrightarrow Y$ be a compact embedding. Our proof shows trivially (by a compactness argument) that v, ϱ depends Y-continuously on a, ϱ_0, f provided that these data remain bounded respect to the norms indicated in theorem 1.1 (under this condition the specific topology for the data is not substantial).

System (1.1) has been considered by T. Kato [8], [9] when Ω is the whole space. The boundary value problem in the case in which the initial velocity $a(x)$ is everywhere sub-sonic and the initial density $\varrho_0(x)$ is nearly constant (and $f \equiv 0$) has been studied by D. G. Ebin [4].

PLAN OF THE PROOF. By using the change of variables (4.1) and the identity (4.4) with $h(t, x) = p'[\exp g(t, x)]$, one sees that system (1.1) is

(1) It is not hard to prove a little more; actually

$$\partial_t^j v,\ \partial_t^j \varrho \in C(0, T_1; H^{3-j})\,, \quad j = 0, 1, 2\,.$$

equivalent to the system

$$
(1.8) \qquad \begin{cases} \partial_t v + (v\cdot\nabla)v - f = -h\nabla g & \text{in } Q_{T_0}, \\ \partial_t g + v\cdot\nabla g + \operatorname{div} v = 0 & \text{in } Q_{T_0}, \\ v|_{t=0} = a(x) & \text{in } \Omega, \\ g|_{t=0} = g_0(x) & \text{in } \Omega, \\ v\cdot n = 0 & \text{on } \Sigma_{T_0}, \end{cases}
$$

where $g_0(x) \equiv \log \varrho_0(x)$. On the other hand a vector field V vanishes identically if and only if (4.2) holds. Applying this result to equation $(1.8)_1$ and $(1.8)_3$ it is not difficult to show that system (1.8), hence system (1.1), is equivalent to the hyperbolic first order system (referred in [1])

$$
(1.9) \qquad \begin{cases} Qg + \delta = 0 & \text{in } Q_{T_0}, \\ Q\delta + \operatorname{div}(h\nabla g) = -F & \text{in } Q_{T_0}, \\ g|_{t=0} = g_0 & \text{in } \Omega, \\ \delta|_{t=0} = \gamma & \text{in } \Omega, \\ \partial_n g = G & \text{on } \Sigma_{T_0}, \end{cases}
$$

plus the system $(2.5)_\delta$, where $(2.5)_\delta$ denotes the system (2.5) (see the following section) with θ replaced by δ. To show this equivalence use also (4.3) and the definitions (2.1), (2.26). The operator Q is defined as

$$
(1.10) \qquad Q = \partial_t + v\cdot\nabla\,.
$$

Finally system (1.9) + $(2.5)_\delta$ is clearly equivalent to system $(2.5)_\delta$ + (2.23) + + (2.28).

We solve this last system as follows: for each $\theta \in \mathbb{K}_0$ (see the definition (2.4)) we solve (2.5) and we put for convenience $v = \Phi_1[\theta]$, and for each $q \in \mathbb{K}_1$ (another suitable convex set) we define $h = \Phi_2[q]$ by (2.15). Now we consider the solution $g = \Phi_3[v, h]$ of (2.23) and we define $\delta = \Phi_4[v, g]$ by (2.28). Finally we prove the existence of a fixed point for the map Φ: $(\theta, q) \to (\delta, g)$ in $\mathbb{K}_0 \times \mathbb{K}_1$. We remark that in general δ doesn't verify the condition $(2.2)_1$, hence a suitable device must be introduced.

The functions $v = \Phi_1[\theta]$ and g, corresponding to the fixed point of the map Φ, are obviously solutions of $(2.5)_\delta$ + (2.23) + (2.28), hence of (1.1).

2. – Some basic estimates.

Since Ω and the function « $\xi \to p(\xi)$ » are fixed, the quantities which depend only on these data are constants, and will be denoted by c, $\bar{c}$, $\bar{c}_0$, c_1, c_2, ... Different constants can be denoted by the same symbol c. We also denote by ψ, ψ_0, ψ_1, ..., non-decreasing functions in all their arguments. In general these functions depend on the arguments $\|a\|_3$, $\|\varrho_0\|_3$, $\|\partial_t^j f\|_{3-j,T_0}$, $j = 0, 1, 2$, and this dependence will not be explicitely indicated. Everytime a function ψ will depend also on other arguments this will be explicitely remarked. Notice that one can always replace a finite number of functions ψ by their maximum, which is again a ψ-type function on the same arguments. *This will be done many times, without any comment and without a change of the symbol* ψ.

In the following we will consider the tridimensional case since the proofs in the bidimensional one are similar. The following inequalities will be used without any comment:

$$\|uv_1 \dots v_m\|_k \leqslant c\|u\|_k \|v_1\|_2 \dots \|v_m\|_2 , \qquad k = 0, 1, 2,$$

$$\|uv\| \leqslant c\|u\|_1 \|v\|_1 ,$$

$$\|uvw\| \leqslant c\|u\|_1 \|v\|_1 \|w\|_1 ,$$

$$\|u\|_{L^6(\Omega)} \leqslant c\|u\|_1 ,$$

$$\|u\|_{L^\infty(\Omega)} \leqslant c\|u\|_2 ,$$

where $c = c(\Omega)$.

We start by remarking that (see for istance [7], [5]) there exist N vector fields $u^{(l)}(x)$, $l = 1, \dots, N$, defined in Ω, which are a basis for the linear space of the solutions of the system $\operatorname{div} w = 0$, $\operatorname{rot} w = 0$ in Ω, $w \cdot n = 0$ on Γ (N is the number of cuts needed to make Ω simply connected). We assume that $(u^{(l)}, u^{(j)}) = \delta_{lj}$.

Let us put

$$(2.1) \qquad \left\{ \begin{array}{ll} \alpha = \operatorname{rot} a & \text{in } \Omega, \\ \gamma = \operatorname{div} a & \text{in } \Omega, \\ g_0 = \log \varrho_0 & \text{in } \Omega, \\ g_1 = -(a \cdot \nabla g_0 + \gamma) & \text{in } \Omega, \\ f_1 = \operatorname{div} f & \text{in } Q_{T_0}, \\ f_2 = \operatorname{rot} f & \text{in } Q_{T_0}. \end{array} \right.$$

We assume for convenience that $T_0 \leqslant 1$. For the moment, T is any real number such that $T \in]0, T_0]$. Later on we will impose on T the condition (2.6) and in section 3 we will put $T = T_1$ (see definition (3.3)).

Consider now a scalar field $\theta(t, x)$, defined in Q_T, and verifying the qualitative conditions

$$(2.2)\quad \begin{cases} \int_\Omega \theta(t, x)\, dx = 0 & \forall t \in [0, T], \\ \theta|_{t=0} = \gamma(x) & \text{in } \Omega, \\ \partial_t \theta|_{t=0} = -\, a \cdot \nabla\gamma - \operatorname{div}[p'(e^{g_0})\nabla g_0] - \sum_{i,j}(\partial_i a_j)(\partial_j a_i) + f_1|_{t=0} & \text{in } \Omega, \end{cases}$$

and the bounds

$$(2.3)\quad \begin{cases} \|\theta\|_{1,T} \leqslant A\,, \quad \|\partial_t\theta\|_{0,T} \leqslant A_1\,, \\ \|\theta\|_{2,T} \leqslant B\,, \quad \|\partial_t\theta\|_{1,T} \leqslant B_1\,, \quad \|\partial_t^2\theta\|_{0,T} \leqslant B_2\,. \end{cases}$$

For the sake of clearness the values of the positive constants A, E, A_1, L, B, B_1, B_2 and T_1, which appear in this number, will be specified only later in equations (3.1) and (3.3). It turns out from the definitions that these constants *depend only* on $\|a\|_3$, $\|\varrho_0\|_3$, $\|\partial_t^j f\|_{3-j,T_0}$, $j = 0, 1, 2$.

We remark that it would be easy to obtain explicit (but not significant!) expressions for the functions ψ_0, ψ_1, ψ_2, ψ_3, ψ_4, ψ appearing in (3.1), (3.3).

Define the convex set

$$(2.4)\quad \mathbf{K}_0 = \{\theta : \theta \text{ verifies (2.2) and (2.3)}\}\,.$$

This set is bounded in $\operatorname{Lip}(0, T; H^1)$ and $\|\theta(t)\|_2 \leqslant B$ for each $t \in [0, T]$ and each $\theta \in \mathbf{K}_0$. Hence, by Ascoli-Arzelà's theorem, $\mathbf{K}_0$ is a relatively compact set in $C(0, T; H^1)$. Moreover, by using a weak* topology argument in the spaces $L^\infty(0, T; H^k)$, one easily sees that $\mathbf{K}_0$ is closed. Hence $\mathbf{K}_0$ is a compact set in $C(0, T; H^1)$.

We consider now the following problem where θ is a fixed scalar field:

$$(2.5)\quad \begin{cases} \operatorname{div} v = \theta & \text{in } Q_{T_0}, \\ \operatorname{rot} v = \zeta & \text{in } Q_{T_0}, \\ v \cdot n = 0 & \text{on } \Sigma_{T_0}, \\ (v(0) - a, u^{(l)}) = 0 & l = 1, \dots, N, \\ (\partial_t v + (v \cdot \nabla) v - f, u^{(l)}) = 0 & \forall t \in [0, T_0],\ l = 1, \dots, N, \\ \partial_t \zeta + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla) v + \theta\zeta = f_2 & \text{in } Q_{T_0}, \\ \zeta|_{t=0} = \alpha & \text{in } \Omega. \end{cases}$$

We recall that this system of equations was solved in [1] (see (1.1′), (1.1″) in [1]) and that it is equivalent to system (1.1) of [1] [2].

From the theorems 1.1, 1.2 and 1.3 of [1] it follows in particular (see section 4 in [1]) that there exists a constant $\bar{c}_0 = \bar{c}_0(\Omega)$ such that if

$$T \leqslant \frac{\bar{c}_0}{\|a\|_3 + B + [f]_{3,T_0}} \tag{2.6}$$

then for each $\theta \in \mathrm{K}_0$ there exists a unique solution v of system (2.5) in $[0, T]$, which we denote by $V = \Phi_1[\theta]$. Moreover $(1.3)_3$ holds. From the equations $(1.9)^*$, $(1.10)^*$ and $(1.12)^*$ of [1] used for $k = 1$ it follows that

$$\begin{cases} \|v\|_{3,T} \leqslant c(\|a\|_3 + B + [f]_{3,T}), \\ \|\partial_t v\|_{2,T} \leqslant cB_1 + c\|v\|_{3,T}^2 + c\|f\|_{2,T}, \\ \|\partial_t^2 v\|_{1,T} \leqslant cB_2 + c\|v\|_{3,T}\|\partial_t v\|_{2,T} + c\|\partial_t f\|_{1,T}, \end{cases} \tag{2.7}$$

where $c = c(\Omega)$. Furthermore ([1] lemma 4.2 and 4.3 with $k = 1$)

(2.8) If $\theta_n \in \mathrm{K}_0$, $\theta_n \to \theta$ in $C(0, T; H^1)$ then $v_n \to v$ in $C(0, T; H^2)$ and $\partial_t v_n \to \partial_t v$ in $C(0, T; H^1)$ where $v_n = \Phi_1[\theta_n]$, $v = \Phi_1[\theta]$.

We need also the following estimates of v and $\partial_t v$ in terms of A and A_1:

Lemma 2.1. *Let $v = \Phi_1[\theta]$. Then*

$$\begin{cases} \|v\|_{2,T} \leqslant c(\|a\|_2 + A + [f]_{2,T}), \\ \|\partial_t v\|_{1,T} \leqslant c(\|v\|_{2,T}^2 + A_1 + \|f\|_{1,T}), \end{cases} \tag{2.9}$$

where $c = c(\Omega)$.

Proof. One easily verifies that the proof of the estimate (2.8) of [1] holds if $k = 0$ and gives, when $f \not\equiv 0$,

$$\|v\|_{2,T} \leqslant c(\|\theta\|_{1,T} + \|\zeta\|_{1,T} + \|a\| + [f]_{0,T_0}) \exp[c(\|\theta\|_{1,T} + \|\zeta\|_{1,T})\,T].$$

Analogously, for $k = 0$ and $f \not\equiv 0$, the estimate $(2.26)_1$ of [1] gives

$$\|\zeta\|_{1,T} \leqslant (\|\alpha\|_1 + [f_2]_{1,T}) \exp[c\|v\|_{3,T}\,T]$$

[2] See section 5 of [1] for the case $f \not\equiv 0$.

(this also follows from $(2.5)_6$, $(2.5)_7$). The above estimates, together with $(2.7)_1$ and (2.6) yield $(2.9)_1$.

On the other hand, from equation $(2.5)_6$ it follows that

$$\|\partial_t \zeta\|_{0,T} \leqslant \|v\|_{2,T}^2 + \|f\|_{1,T} . \tag{2.10}$$

Moreover, as in the proof of theorem 1.1 in section 3 of [1], one gets

$$\|\partial_t v^0\|_{1,T} \leqslant c\|\partial_t \theta\|_{0,T} + \|\partial_t \zeta\|_{0,T} \tag{2.11}$$

and

$$\left\| \frac{d\theta_l}{dt} \right\|_{L^\infty(0,T)} \leqslant c\|v\|_{0,T}\|v\|_{1,T} + c\|f\|_{0,T} .$$

From this last estimate, using equation (2.9) of [1] and estimates (2.10) and (2.11) one gets $(2.9)_2$. □

From now on we assume that T verifies (2.6).

Consider now functions $q(t, x)$ such that

$$\begin{cases} q|_{t=0} = g_0(x) & \text{in } \Omega, \\ \partial_t q|_{t=0} = g_1(x) & \text{in } \Omega, \end{cases} \tag{2.12}$$

and that

$$\begin{cases} \|q\|_{2,T} \leqslant E , \\ \|q\|_{3,T} \leqslant L , \quad \|\partial_t q\|_{2,T} \leqslant L , \quad \|\partial_t^2 q\|_{1,T} \leqslant L \end{cases} \tag{2.13}$$

and define the convex and compact subset $\mathbb{K}_1$ of $C(0, T; H^2)$ as

$$\mathbb{K}_1 = \{q: q \text{ verify (2.12) and (2.13)}\} . \tag{2.14}$$

Let us define $h = \Phi_2[q]$ by the equation

$$h(t, x) \equiv p'\big(\exp[q(t, x)]\big) \quad \text{in } Q_T , \tag{2.15}$$

for each $q \in \mathbb{K}_1$. Let $\bar{c}$ be a positive constant such that $\|w\|_{L^\infty(\Omega)} \leqslant \bar{c}\|w\|_2$, $\forall w \in H^2$, and put

$$m = \min_{\xi} p'(\xi) , \qquad \xi \in \big[\exp[-\bar{c}E], \exp[\bar{c}E]\big]. \tag{2.16}$$

Notice that $m = m(E)$ is a non-increasing function of E. For each q such that $\|q\|_{2,T} \leqslant E$, hence for each $q \in \mathbb{K}_1$, one has

$$h(t, x) \geqslant m \quad \text{in } Q_T, \tag{2.17}$$

since $|q(t, x)| \leqslant \bar{c}E$. We will now get a bound for $\partial_t^j h$, $j = 0, 1, 2$, in terms of E and L. Put

$$K = \max_{\xi} \{|p'(\xi)|, |p''(\xi)|, |p'''(\xi)|, |p^{(4)}(\xi)|\} \exp[3\bar{c}E], \quad \text{for } \xi \in [\exp[-\bar{c}E], \exp[\bar{c}E]]; \tag{2.18}$$

$K = K(E)$ is a non decreasing function of E. With some calculations one sees that

$$\begin{cases} \|h\|_{2,T} \leqslant cK(1 + E^2), \\ \|\partial_t^j h\|_{3-j,T} \leqslant cK(1 + L^3), \quad j = 0, 1, 2. \end{cases} \tag{2.19}$$

Moreover from (2.15) it follows that

(2.20) If $q_n \in \mathbb{K}_1$, $q_n \to q$ in $C(0, T; H^2)$ then $h_n \to h$ in $C(0, T; H^2)$, where $h_n = \Phi_2[q_n]$, $h = \Phi_2[q]$.

We introduce now a sufficient condition in order that the sets $\mathbb{K}_0$ and $\mathbb{K}_1$ are non empty. We remark that $\mathbb{K}_0$ is non empty if and only if $\mathbb{K}$ (see definition (3.2)) is non empty, as follows from (3.8).

Let now $\gamma_1(x)$ be the right hand side of $(2.2)_3$. The quantities $\|\gamma\|_{k+1}$, $\|\gamma_1\|_k$, $\|g_0\|_{k+2}$, $k = 0, 1$ and $\|g_1\|_2$ are bounded by a known fixed function $\psi_0 = \psi_0(\|a\|_3, \|f_0\|_3, \|f|_{t=0}\|_2)$. We shall see now that there exists $c_1 = c_1(\Omega)$ such that if

$$\min\{A, E, A_1, L, B, B_1, B_2\} \geqslant c_1 \psi_0 \tag{2.21}$$

then $\mathbb{K}$ and $\mathbb{K}_1$ are non empty. In fact let Ω_1 be a ball such that $\bar{\Omega} \subset \Omega_1$ and let $\Lambda\colon H^k(\Omega) \to H^k(\Omega_1)$, $k = 0, 1, 2, 3$, be a linear continuous operator such that for each $u \in L^2(\Omega)$ the function Λu has compact support in Ω_1 and $(\Lambda u)|_\Omega = u$. Consider now the solution θ of the Cauchy-Dirichlet problem for the wave operator

$$\begin{aligned} &\partial_t^2 \theta - \Delta\theta = 0 && \text{in } \Omega_1 \times]0, 1], \\ &\theta|_{t=0} = \Lambda\gamma && \text{in } \Omega_1, \\ &\partial_t \theta|_{t=0} = \Lambda\gamma_1 && \text{in } \Omega_1, \\ &\theta = 0 && \text{on } \Gamma_1 \times]0, 1], \end{aligned}$$

where Γ_1 is the boundary of Ω_1. From well known estimates for the wave equation it follows that $\sum_{j=0}^{2} \|\partial_t^j \theta\|_{2-j,T} \leqslant c_1 \psi_0$. Hence $\theta(t)|_\Omega$ belongs to $\mathbb{K}$ if $\min\{A, A_1, B, B_1, B_2\} \geqslant c_1 \psi_0$.

A similar argument works for $\mathbb{K}_1$.

To prove theorem 1.1 we need the following result on linear hyperbolic equations:

THEOREM 2.1. *Let $g_i \in H^{3-i}$, $i = 0, 1$, $\partial_t^j F \in L^\infty(0, T; H^{2-j})$ $\partial_t^j G \in L^\infty(0, T; H^{\frac{5}{2}-j}(\Gamma))$, $j = 0, 1, 2$, and assume that the following compatibility conditions hold for $i = 0, 1$*

$$\partial_n g_i = \partial_t^i G|_{t=0} \quad \text{on } \Gamma \text{ for } t = 0. \tag{2.22}$$

Moreover let $\partial_t^j v$, $\partial_t^j h \in L^\infty(0, T; H^{3-j})$, $j = 0, 1, 2$, $v \cdot n = 0$ on Σ_T, $h(t, x) \geqslant m > 0$ in Q_T. Then there exists polynomials

$$P_0 = P_0\left(\frac{1}{m}, \|v\|_{2,T}, \|h\|_{2,T}, \|\partial_t v\|_{1,T}\right)$$

$$P_2 = P_2\left(\frac{1}{m}, \|\partial_t^j v\|_{3-j,T}, \|\partial^j h\|_{3-j,T}\right), \qquad j = 0, 1, 2,$$

whose coefficients depend only on Ω such that if

$$TP_0 \leqslant 1$$

the linear hyperbolic mixed problem

$$\begin{cases} Q^2 g - \operatorname{div}(h \nabla g) = F & \text{in } Q_T, \\ g|_{t=0} = g_0 & \text{in } \Omega, \\ \partial_t g|_{t=0} = g_1 & \text{in } \Omega, \\ \partial_n g = G & \text{on } \Sigma_T, \end{cases} \tag{2.23}$$

has a unique solution $g(t, x)$. Moreover

$$\sum_{j=0}^{2} \|\partial_t^j g\|_{3-j,T}^2 \leqslant P_0 (\exp[P_2 T]) \left(\sum_{i=0}^{1} \|g_i\|_{3-i}^2 + \|F|_{t=0}\|_1^2\right) + \\ + P_2 T (\exp[P_2 T]) \sum_{j=0}^{2} \left(\|\partial_t^j F\|_{2-j,T}^2 + |||\partial_t^j G|||_{\frac{5}{2}-j,T}^2\right). \tag{2.24}$$

Condition $TP_0 \leqslant 1$ can be dropped and the hypothesis on the data F and G can be weakened by minor changes in (2.24).

Existence, uniqueness and regularity for the solution of (2.23) with regular coefficients is due to S. Miyatake [11], [12], [13]. Estimate (2.24) will be proved in the appendix. We take the opportunity to thank S. Miyatake for some useful conversations.

From theorem 2.1 one gets the following result:

COROLLARY 2.2. *There exists* $\psi_1 = \psi_1(A, E, A_1)$ *and* $\psi = \psi(A, E, A_1, L, B, B_1, B_2)$ *such that if*

$$T\psi \leqslant 1 \tag{2.25}$$

the following result holds: let $(\theta, q) \in \mathbb{K}_0 \times \mathbb{K}_1$, $v = \Phi_1[\theta]$, $h = \Phi_2[q]$, *and consider the problem* (2.23) *with* F *and* G *given by*

$$\begin{cases} F \equiv \sum_{i,j} (\partial_i v_j)(\partial_j v_i) - f_1 , \\ G \equiv \dfrac{1}{h}\Big(\sum_{i,j} (\partial_i n_j) v_i v_j + f \cdot n\Big) , \end{cases} \tag{2.26}$$

and with g_0 *and* g_1 *given by* (2.1). *Then the problem* (2.23) *has a unique solution* $g(t, x)$ *in* Q_T. *Moreover*

$$\sum_{j=0}^{2} \|\partial_t^j g\|_{3-j,T} \leqslant \psi_1 + T^{\frac{1}{2}} \psi . \tag{2.27}$$

The solution g *of this problem will be indicated by* $g = \Phi_3[v, h]$.

PROOF. Condition $(2.22)_1$ follows from (1.4). On the other hand if one puts

$$V_0 = \left(\partial_t v + (v \cdot \nabla) v - f + \frac{\nabla p(\varrho_0)}{\varrho_0}\right)\bigg|_{t=0}$$

it follows from (4.3), $(2.5)_6$, $(2.2)_3$, $(2.5)_3$, (1.4), $(1.1)_3$ and $(2.5)_5$ that V_0 verifies the system (4.2), consequently $V_0 = 0$. Hence $\partial_t v|_{t=0} = \dot{a}$ for each $\theta \in \mathbb{K}_0$ and the compatibility condition $(2.22)_2$ follows easily from (1.4) and (1.5). For the determination of $h|_{t=0}$ and $\partial_t h|_{t=0}$ use equation (2.12).

On the other hand from the definition of P_2, from (2.7) and from $(2.19)_2$ it follows that $P_2 \leqslant \psi(A, E, A_1, L, B, B_1, B_2)$. Recall that m^{-1} and K are non-decreasing functions of E. Analogously from the definition of P_0, (2.9) and $(2.19)_1$ one gets $P_0 \leqslant \psi_1(A, E, A_1)$. In particular condition (2.25) implies $TP_2 \leqslant 1$.

Using now the expressions of F and G and the bounds (2.7), $(2.19)_2$ and (2.17) one easily gets

$$\sum_{j=0}^{2} \left(\| \partial_t^j F \|^2_{2-j,T} + ||| \partial_t^j G |||^2_{\frac{5}{2}-j,T} \right) \leqslant \psi \,.$$

Using the above results one show that (2.24) yields (2.27). □

Let us define $\delta = \Phi_4[v, g]$ by

$$\delta = - Qg \quad \text{ in } Q_T, \tag{2.28}$$

where g is the solution described in corollary 2.2. The function δ solves the problem

$$\begin{cases} Q\delta + \operatorname{div}(h \nabla g) = - \sum_{i,j} (\partial_i v_j)(\partial_j v_i) + f_1 & \text{in } Q_T, \\ \delta|_{t=0} = \gamma & \text{in } \Omega, \end{cases} \tag{2.29}$$

as one easily verifies by a direct computation. Define now

$$X \equiv C(0, T; H^1) \times C(0, T; H^2)$$

and consider the map

$$\Phi : \mathbb{K}_0 \times \mathbb{K}_1 \to X$$

defined as $\Phi[(\theta, q)] = (\delta, g)$ where $g = \Phi_3[v, h]$, $\delta = \Phi_4[v, g]$ (recall that v and h are given by $v = \Phi_1[\theta]$, $h = \Phi_2[q]$). In other words, given a pair (θ, q) we solve (2.5) in order to get v and we use (2.15) to define h. Once v and h are known, we solve (2.23) to obtain g and we use (2.28) to define δ. The pair (δ, g) is the image under Φ of (θ, q).

The following result holds:

LEMMA 2.3. *The map Φ is continuous with respect to the X topology.*

PROOF. Let $(\theta_n, q_n) \in \mathbb{K}_0 \times \mathbb{K}_1$, $(\theta_n, q_n) \to (\theta, q)$ in X and put $v_n = \Phi_1[\theta_n]$, $h_n = \Phi_2[q_n]$, $g_n = \Phi_3[v_n, h_n]$, $\delta_n = \Phi_4[v_n, g_n]$. From (2.8), (2.20) and (2.26) it follows that $v_n \to v$ in $C(0, T; H^2)$, $h_n \to h$ in $C(0, T; H^2)$, $\partial_t v_n \to \partial_t v$ in $C(0, T; H^1)$, $F_n \to F$ in $C(0, T; H^1)$ and $G_n \to G$ in $C(0, T; H^{\frac{3}{2}}(\Gamma))$; recall that $\|v_n\|_{3,T}$ and $\|h_n\|_{3,T}$ are bounded sequences. Write now the equation $(2.23)_1$ for g, the corresponding equation $(2.23)_1^n$ for g_n and subtract the two equations obtained. By adding and subtracting suitable terms we easily verify (use also the bounds (2.27)) that $g - g_n$ verifies the equation $(2.23)_1$ with F replaced by a function $\overline{F}_n$ such that $\|\overline{F}_n\|_{C(0,T;H^1)} \to 0$ when $n \to +\infty$. Moreover $(g - g_n)|_{t=0} = 0$, $(\partial_t g - \partial_t g_n)|_{t=0} = 0$ and $(\partial/\partial n)(g - g_n) = G - G_n$ on Σ_T.

In particular $\|\overline{F}_n\|_{0,T} \to 0$ and $|||G - G_n|||_{\frac{1}{2},T} \to 0$, hence $g_n \to g$ in $L^\infty(0, T; H^1)$ by well known estimates for linear hyperbolic second order mixed problems, as for instance estimate (6.19) [3] used for $g - g_n$. From (2.27), by a compactness argument, it follows then that $g_n \to g$ in $C(0, T; H^2)$ and $\partial_t g_n \to \partial_t g$ in $C(0, T; H^1)$ (this can be derived also from estimate (8.2)). Hence $\delta_n \to \delta$ in $C(0, T; H^1)$. $\square$

LEMMA 2.4. *Let* $(\theta, q) \in \mathbb{K}_0 \times \mathbb{K}_1$, $v = \Phi_1[\theta]$, $h = \Phi_2[q]$, $g = \Phi_3[v, h]$, $\delta = \Phi_4[v, g]$. *Then*

$$\|\delta\|_{1,T} \leqslant c\{\|a\|_3 + T[K(1 + E^2)\|g\|_{3,T} + \|f\|_{2,T_0} + 1]\}, \tag{2.30}$$

$$\|\partial_t \delta\|_{0,T} \leqslant c[K(1 + E^2)\|g_0\|_2 + \|f\|_{1,T}] + \\ + c(\|v\|_{2,T}\|\delta\|_{1,T} + \|v\|^2_{2,T}) + cTK(1 + E^2)\|\delta\|_{2,T}. \tag{2.31}$$

PROOF. Set for convenience

$$M \equiv -\operatorname{div}(h \nabla g) - \sum_{i,j} (\partial_i v_j)(\partial_j v_i) + f_1 .$$

Using the equation (2.29) and recalling that $v \cdot n = 0$ on Σ_T one easily sees (see for instance [1]) that

$$\frac{d}{dt}\|\delta(t)\|_1 \leqslant c\|v\|_{3,T}\|\delta(t)\|_1 + \|M(t)\|_1, \quad \forall t \in [0, T];$$

hence by comparison theorems for ordinary equations and by $(2.29)_2$

$$\|\delta\|_{1,T} \leqslant (\|\gamma\|_1 + T\|M\|_{1,T}) \exp[c\|v\|_{3,T} T].$$

Using now $(2.7)_1$, $(2.19)_1$ and (2.6) one gets (2.30). On the other hand from $(2.29)_1$ it follows that

$$\|\partial_t \delta\|_{0,T} \leqslant c(\|v\|_{2,T}\|\delta\|_{1,T} + \|h\|_{2,T}\|g\|_{2,T} + \|v\|^2_{2,T} + \|f\|_{1,T}). \tag{2.32}$$

[3] Adapted to Ω, as done in section 9.

Moreover from (2.28) one gets by standard techniques

$$\frac{d}{dt}\|g(t)\|_2 \leqslant c(\|v\|_{3,T}\|g(t)\|_2 + \|\delta\|_{2,T}) .$$

Again by comparison theorems, and by (2.6) and $(2.7)_1$, it follows that

$$\|g\|_{2,T} \leqslant c\|g_0\|_2 + cT\|\delta\|_{2,T} . \tag{2.33}$$

By (2.32), (2.33) and $(2.19)_1$ one gets (2.31). □

LEMMA 2.5. *Under the hypothesis of lemma* 2.4 *the following estimates hold*:

$$\|\delta\|_{2,T} \leqslant \|\partial_t g\|_{2,T} + c\|v\|_{2,T}\|g\|_{3,T} , \tag{2.34}$$

$$\|\partial_t \delta\|_{1,T} \leqslant \|\partial_t^2 g\|_{1,T} + c(\|v\|_{2,T}\|\partial_t g\|_{2,T} + \|\partial_t v\|_{1,T}\|g\|_{3,T}) , \tag{2.35}$$

$$\begin{aligned}\|\partial_t^2 \delta\|_{0,T} \leqslant c[&\|\partial_t f\|_{1,T} + \|\partial_t v\|_{1,T}\|v\|_{3,T} + \|v\|_{2,T}\|\partial_t \delta\|_{1,T} + \\ &+ \|\partial_t v\|_{2,T}\|\delta\|_{1,T} + K(1+E^2)\|\partial_t g\|_{2,T} + K(E+E^2)L\|g\|_{3,T}] .\end{aligned} \tag{2.36}$$

PROOF. Estimates (2.34) and (2.35) follows from (2.28). Estimate (2.36) follows easily by differentiating $(2.29)_1$ with respect to t and by using (2.19) □

THEOREM 2.6. *There exist* c_2, c_3, $\psi_2 = \psi_2(A, E)$, $\psi_3 = \psi_3(A, E, A_1)$, $\psi_4 = \psi_4(A, E, A_1, L, B, B_1)$, $\psi = \psi(A, E, A_1, L, B, B_1, B_2)$ *such that if*

$$T\psi \leqslant 1 \tag{2.37}$$

then

$$\left\{\begin{aligned}&\|\delta\|_{1,T} \leqslant c_2\|a\|_3 + T\psi ,\\ &\|\partial_t\delta\|_{0,T} \leqslant \psi_2 + T\psi ,\\ &\|g\|_{2,T} \leqslant c_3\|g_0\|_2 + T\psi ,\\ &\|\delta\|_{2,T} \leqslant \psi_3 + T^{\frac{1}{2}}\psi ,\\ &\|\partial_t\delta\|_{1,T} \leqslant \psi_3 + T^{\frac{1}{2}}\psi ,\\ &\|\partial_t^2\delta\|_{0,T} \leqslant \psi_4 + T^{\frac{1}{2}}\psi ,\\ &\|g\|_{3,T} \leqslant \psi_1 + T^{\frac{1}{2}}\psi ,\\ &\|\partial_t g\|_{2,T} \leqslant \psi_1 + T^{\frac{1}{2}}\psi ,\\ &\|\partial_t^2 g\|_{1,T} \leqslant \psi_1 + T^{\frac{1}{2}}\psi .\end{aligned}\right. \tag{2.38}$$

PROOF. The proof follows directly from (2.30), (2.31), (2.27), (2.9), (2.33), (2.34), (2.35), (2.36). □

3. – Construction of the fixed point.

Let A, E, A_1, L, B, B_1, B_2 be fixed as follows

$$(3.1)\qquad \begin{cases} A = c_2\|a\|_3 + 1 + c_1\psi_0 , \\ E = c_3\|g_0\|_2 + 1 + c_1\psi_0 , \\ A_1 = \psi_2(A, E) + 1 + c_1\psi_0 , \\ L = \psi_1(A, E, A_1) + 1 + c_1\psi_0 , \\ B = \psi_3(A, E, A_1) + 1 + c_1\psi_0 , \\ B_1 = \psi_3(A, E, A_1) + 1 + c_1\psi_0 , \\ B_2 = \psi_4(A, E, A_1, L, B, B_1) + 1 + c_1\psi_0 . \end{cases}$$

Put also

$$(3.2)\qquad \mathbf{K} = \{\theta : \theta \text{ verifies } (2.2)_2,\ (2.2)_3 \text{ and } (2.3)\} .$$

From (2.21) it follows in particular that $\mathbf{K}\times\mathbf{K}_1$ is non-empty. Finally define T_1 by

$$(3.3)\qquad T_1 = \frac{1}{\psi^2}$$

where $\psi = \psi(A, E, A_1, L, B, B_1, B_2)$ is the function appearing in theorem 2.6. Possibly taking a larger ψ we may assume that T_1 verifies also condition (2.6). It follows from the above definitions that

$$(3.4)\qquad T_1 = T_1\big(\|a\|_3,\ \|\varrho_0\|_3,\ \|\partial_t^j f\|_{3-j,T_0}\big) , \qquad j = 0, 1, 2,$$

is a positive function, non-increasing in all the five variables.

The following result holds:

THEOREM 3.1. *Let A, E, A_1, L, B, B_1, B_2, T_1 be given by* (3.1), (3.3). *Then for each $T \in\,]0, T_1]$ the set $\mathbf{K}_0\times\mathbf{K}_1$ is non empty, the map*

$$(3.5)\qquad (\theta, q) \xrightarrow{\Phi} (\delta, g) ,$$

is continuous in the X topology for $(\theta, q) \in \mathbf{K}_0\times\mathbf{K}_1$, and

$$(3.6)\qquad \Phi(\mathbf{K}_0\times\mathbf{K}_1) \subset \mathbf{K}\times\mathbf{K}_1 .$$

PROOF. The continuity follows from lemma 2.3 and the inclusion (3.6) from theorem 2.6 and from definitions (3.1). $\square$

We define now the linear operator

$$\boldsymbol{\pi} u = u - \frac{1}{|\Omega|} \int_{\Omega} u(y)\, dy\,, \tag{3.7}$$

which acts on functions u defined in Ω or in Q_T. One has $\|\boldsymbol{\pi}\| = 1$ in H^k, $k \geqslant 0$, hence in $L^\infty(0, T; H^k)$. Furthermore

$$\boldsymbol{\pi} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \boldsymbol{\pi}\,,$$

$\boldsymbol{\pi}\theta|_{t=0} = \theta|_{t=0}$ and $\boldsymbol{\pi}\partial_t\theta|_{t=0} = \partial_t\theta|_{t=0}$ as one verifies by straightforward calculations (see (2.2)). From these properties it follows that

$$\boldsymbol{\pi}(\mathbb{K}) = \mathbb{K}_0\,. \tag{3.8}$$

Using theorem 3.1, the properties just stated for $\boldsymbol{\pi}$ and the Schauder fixed point theorem one gets the following result, where I denotes the identity map on $\mathbb{K}_1$:

COROLLARY 3.2. *Let the assumptions of theorem* 3.1 *hold, then the map* $(\boldsymbol{\pi} \times I) \circ \Phi$, *i.e. the map*

$$(\theta, q) \to (\boldsymbol{\pi}\delta, g)\,,$$

has a fixed point in $\mathbb{K}_0 \times \mathbb{K}_1$.

We will now establish the existence of a fixed point for the map Φ:

THEOREM 3.3. *Under the assumptions of theorem* 3.1 *the map* Φ: $(\theta, q) \to$ $\to (\delta, g)$ *has a fixed point in* $\mathbb{K}_0 \times \mathbb{K}_1$.

The following lemma will be useful to prove theorem 3.3:

LEMMA 3.4. *Let* $(\theta, q) \in \mathbb{K}_0 \times \mathbb{K}_1$, $v = \Phi_1[\theta]$, $h = \Phi_2[q]$, $g = \Phi_3[v, h]$, $\delta = \Phi_4[v, g]$. *Then*

$$\left\{\begin{aligned} &\frac{d}{dt} \int_{\Omega} \delta\, dx = \int_{\Omega} \theta(\delta - \theta)\, dx\,, \qquad \forall t \in [0, T]\,, \\ &\int_{\Omega} \delta(0, x)\, dx = 0\,. \end{aligned}\right. \tag{3.9}$$

PROOF. Using the divergence theorem one gets easily

$$\int_\Omega \operatorname{div}(h\nabla g)\,dx = \int_\Gamma \Big[\sum_{i,j} (\partial_i n_j) v_i v_j + f\cdot n\Big] d\Gamma . \tag{3.10}$$

On the other hand an easy computation yields the well known formulaes [4]

$$\begin{cases} \operatorname{div}[(v\cdot\nabla)v - f] = \sum\limits_{i,j} (\partial_i v_j)(\partial_j v_i) + v\cdot\nabla\theta - f_1 & \text{in } \Omega, \\ [(v\cdot\nabla)v - f]\cdot n = -\sum\limits_{i,j} (\partial_i n_j) v_i v_j - f\cdot n & \text{on } \Gamma, \end{cases} \tag{3.11}$$

hence by the divergence theorem

$$-\int_\Omega \sum_{i,j} (\partial_i v_j)(\partial_j v_i)\,dx + \int_\Omega f_1\,dx - \int_\Omega v\cdot\nabla\theta\,dx = \int_\Gamma \Big[\sum_{i,j} (\partial_i n_j) v_i v_j + f\cdot n\Big] d\Gamma . \tag{3.12}$$

Integrating now both sides of $(2.29)_1$ in Ω, using (3.10), (3.12) and $(2.5)_1$ one easily gets $(3.9)_1$. □

PROOF OF THEOREM 3.3. Let $(\theta, q) = (\boldsymbol{\pi}\delta, g)$ be the fixed point in the statement of corollary 3.2. To prove theorem 3.3 it suffices to show that $\boldsymbol{\pi}\delta = \delta$, since $\boldsymbol{\pi}\delta = \theta$. Put for convenience

$$y(t) = \int_\Omega \delta(t, x)\,dx .$$

One has

$$\|(\delta - \theta)(t)\| = |\Omega|^{-\frac{1}{2}}|y(t)| ,$$

since $\theta = \boldsymbol{\pi}\delta$. On the other hand from (3.9) one gets

$$\begin{cases} \left|\dfrac{dy(t)}{dt}\right| \leqslant \|\theta\|_{0,T}|\Omega|^{-\frac{1}{2}}|y(t)| , \\ y(0) = 0 . \end{cases}$$

By comparison theorems for ordinary differential equations it follows then that $y(t) \equiv 0$ on $[0, T]$. □

[4] The vectors v and $\nabla(v\cdot n)$ are orthogonal since $v\cdot n = 0$, hence

$$0 = \sum_{i,j} v_i\,\partial_i(v_j n_j) = [(v\cdot\nabla)v]\cdot n + \sum_{i,j} (\partial_i n_j) v_i v_j .$$

4. – Equivalence of the two formulations.

The functions v and g corresponding to the fixed point of theorem 3.3 solve the system (2.5) + (2.23) with $h = p'(e^g)$ and with $\theta = \delta$ given by (2.28). We shall verify in this section that (v, ϱ) is a solution of system (1.1) in $[0, T_1]$, where by definition

$$\varrho(t, x) \equiv \exp[g(t, x)] . \tag{4.1}$$

Let us put for convenience

$$V = \partial_t v + (v\cdot\nabla)v - f + \frac{\nabla p(\varrho)}{\varrho} .$$

It is true that

$$\begin{cases} \operatorname{rot} V = 0 & \text{in } Q_{T_1} , \\ \operatorname{div} V = 0 & \text{in } Q_{T_1} , \\ V\cdot n = 0 & \text{on } \Sigma_{T_1} , \\ (V, u^{(l)}) = 0 & l = 1, 2, \ldots, N. \end{cases} \tag{4.2}$$

The first equation follows from $(2.5)_2$, $(2.5)_6$ since

$$\frac{\nabla p(\varrho)}{\varrho} = \nabla\Psi(\varrho) , \qquad \text{where } \Psi(\xi) = \int_1^\xi p'(\eta)\eta^{-1}\, d\eta . \tag{4.3}$$

The second equation follows from $(2.5)_1$ and $(2.29)_1$ since

$$\frac{\nabla p(\varrho)}{\varrho} = h\nabla g . \tag{4.4}$$

The third equation follows from $(2.5)_3$, $(3.11)_2$, $(2.23)_4$ and (4.4). Finally $(4.2)_4$ follows from $(2.5)_5$ (recall (4.3) and the properties of $u^{(l)}$). Now from (4.2) one gets $V \equiv 0$, i.e. equation $(1.1)_1$.

On the other hand (2.28) and definition (4.1) yield equation $(1.1)_2$. The initial condition $(1.1)_3$ holds since $v(0) - a$ verifies the system (4.2) (see [1]). The two remaining equations $(1.1)_4$ and $(1.1)_5$ are trivially verified. □

APPENDIX

In this appendix we prove theorem 2.1.

In paper [13] (see also [11], [12]) S. Miyatake introduces the algebrical condition (H) which connects the coefficients of a second order regularly hyperbolic operator with the coefficients of a first order boundary operator. Condition (H) is necessary and sufficient in order that the solution of the initial—boundary value problem exists and verifies estimates like (2.24). However in this last estimate we need a particular dependence for the « constants » P_0, P_2 in terms of the coefficients v and h of the hyperbolic operator P, dependence suggested heuristically by our non-linear problem (1.1). This particular dependence is crucial to get the basic estimates (2.38).

In spite of the fact that some of the essential tools to prove estimate (2.24) are those utilized in Miyatake's proofs (as for instance the use of $(Pg, (Q + \varepsilon\partial_x)g)_{0,\gamma(0,t)}$ and the method of section 6) the exact form of (2.24) can not be claimed directly from his papers.

Our essential aim is to prove the a priori estimates. The proof of the existence of a solution is then based on these a priori estimates (the exact dependence on the coefficients being now superfluous). To shorten this appendix we show the existence by combining our estimates with the existence results of [13] instead of repeating known arguments.

5. – Notations and some basic estimates.

Let us put $y = (y_1, y_2)$ and consider real functions $w_k(t, y, x)$, $l_k(t, y, x)$, $k = 1, 2, 3$, defined for $(t, y, x) \in \mathbb{R}_t^+ \times \mathbb{R}_y^2 \times \mathbb{R}_x^+$ and constants ($w_k = 0$ and $l_k = 1$) outside a fixed compact set in $\overline{\mathbb{R}_t^+ \times \mathbb{R}_y^2 \times \mathbb{R}_x^+}$. We denote by w the vector field (w_1, w_2, w_3) and by l the triplet (l_1, l_2, l_3), which play here the part of (v_1, v_2, v_3) and (h, h, h) in the preceding sections.

Norms of vector functions are defined as the maximum of the norms of the components. We write $\partial_j = \partial_{y_j}$, $j = 1, 2$, $\partial_3 = \partial_x$, and we denote by ∂_y one or both the derivatives ∂_1, ∂_2. For instance $\|\partial_y w\| = \max \|\partial_j w_k\|$, $j = 1, 2$, $k = 1, 2, 3$. Analogously ∂ denotes spatial first derivatives. For instance $\|\partial w\| = \max \|\partial_j w_k\|$, $j, k = 1, 2, 3$. The meaning of symbols like $\|\partial_y^2 w\|$, $\|\partial^2 w\|$ and so on is now clear.

Unless otherwise stated the indices of summation take values from 1 to 3.

We assume that w and l verify the following conditions ($k = 1, 2, 3$):

$$l_k(t, y, x) \geqslant m > 0\,, \qquad m \text{ constant}, \tag{5.1}$$

$$w_3(t, y, 0) = 0\,, \tag{5.2}$$

$$\partial_t^j w\,, \qquad \partial_t^j(l_k - 1) \in L^\infty(\mathbb{R}_t^+; H^{3-j})\,, \qquad j = 0, 1, 2, \tag{5.3}$$

where $H^s = H^s(\mathbb{R}_y^2 \times \mathbb{R}_x^+)$. In the following we shall use the differential operators

$$Qg = \partial_t g + \Sigma w_k \partial_k g\,, \qquad Pg = Q^2 g - \Sigma \partial_k(l_k \partial_k g)\,.$$

Let γ be a positive parameter. We put $g_\gamma(t, y, x) = \exp[-\gamma t]\, g(t, y, x)$ and

$$\begin{aligned}
((g, u))_{0,\gamma} &= \iint \exp[-2\gamma t]\, g(t, y, x)\, \bar{u}(t, y, x)\, dy\, dx\,,\\
[g]_{0,\gamma}^2 &= ((g, g))_{0,\gamma}\,,\\
\|g\|_{l,\gamma}^2 &= \sum_{|\alpha| \leqslant l} [\partial^\alpha g]_{0,\gamma}^2\,, \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3)\,,\\
[g]_{l,\gamma}^2 &= \sum_{j+|\alpha|=l} [\partial_t^j \partial^\alpha g]_{0,\gamma}^2\,,\\
|g|_{l,\gamma}^2 &= \sum_{j+|\alpha|\leqslant l} [\partial_t^j \partial^\alpha g]_{0,\gamma}^2\,, \qquad l = 0, 1, 2, \ldots
\end{aligned}$$

where the integrals are over $\mathbb{R}_y^2 \times \mathbb{R}_x^+$. Moreover

$$\begin{aligned}
(g, u)_{0,\gamma(0,t)} &= \int_0^t ((g, u))_{0,\gamma}\, ds = \int_0^t\!\!\iint \exp[-2\gamma s]\, g(s)\, \bar{u}(s)\, dx\, dy\, ds\,,\\
[g]_{0,\gamma(0,t)}^2 &= (g, g)_{0,\gamma(0,t)}\,,\\
[g]_{l,\gamma(0,t)}^2 &= \int_0^t [g]_{l,\gamma}^2\, ds\,,\\
|g|_{l,\gamma(0,t)}^2 &= \int_0^t |g|_{l,\gamma}^2\, ds\,.
\end{aligned}$$

A star [resp. an apostrophe] added to a norm means that the derivative of greater order in the x variable [resp. t variable] does not appear in the norm. For instance $(\beta = (\beta_1, \beta_2))$

$$|g|^{*2}_{2,\gamma} = \sum_{\substack{j+|\beta|+i\leqslant 2\\ i\neq 2}} [\partial_t^j \partial_y^\beta \partial_x^i g]^2_{0,\gamma} .$$

We consider also the following norms and semi-norms on the boundary $x = 0$:

$$\begin{aligned}
\langle\langle g, u\rangle\rangle_{0,\gamma} &= \int \exp[-2\gamma t]\, g(t, y, 0)\, \overline{u}(t, y, 0)\, dy ,\\
\langle\langle g\rangle\rangle^2_{0,\gamma} &= \langle\langle g, g\rangle\rangle_{0,\gamma} ,\\
\langle\langle g\rangle\rangle^2_{l,\gamma} &= \sum_{j+|\beta|=l} \langle\langle \partial_t^j \partial_y^\beta g\rangle\rangle^2_{0,\gamma} ,\\
\langle g, u\rangle^2_{0,\gamma(0,t)} &= \int_0^t \langle\langle g, u\rangle\rangle_{0,\gamma}\, ds ,\\
\langle g\rangle^2_{l,\gamma(0,t)} &= \int_0^t \langle\langle g\rangle\rangle^2_{l,\gamma}\, ds , \qquad l = 0, 1, 2, \ldots
\end{aligned}$$

These definitions will be used also for the time intervals $\mathbb{R}_t^+$ and $\mathbb{R}_t$.

Let now $g(t, y, x)$ denote a complex valued C^∞ function with compact support in $\overline{\mathbb{R}_t^+ \times \mathbb{R}_y^2 \times \mathbb{R}_x^+}$. Starting from the bilinear form $2\,\mathrm{Re}\,(Pg, (Q + \varepsilon\partial_x) g)_{0,\gamma(0,t)}$ and performing suitable integrations by parts it is easy to check that

$$\begin{aligned}
(5.4)\quad & \Big\{[Qg(s)]^2_{0,\gamma} + \sum_k [(l_k^{\frac12} \partial_k g)(s)]^2_{0,\gamma}\Big\}\Big|_{s=0}^t - 2\,\mathrm{Re}\,\big(Pg, (Q + \varepsilon\partial_x) g\big)_{0,\gamma(0,t)} + \\
& + 2\gamma \int_0^t\!\!\iiint \exp[-2\gamma s]\Big\{|Qg|^2 + \sum_k l_k |\partial_k g|^2\Big\} dx\, dy\, ds + \\
& + \varepsilon \int_0^t\!\!\iint \exp[-2\gamma s]\{|Qg|^2 + l_3 |\partial_x g|^2\} dx\, ds - \varepsilon \sum_{j=1}^2 \int_0^t\!\!\iint \exp[-2\gamma s]\, l_j |\partial_j g|^2 dy\, ds + \\
& + 2\,\mathrm{Re} \int_0^t\!\!\iint \exp[-2\gamma s](l_3 \partial_x g)(Q\bar{g})\, dy\, ds = A ,
\end{aligned}$$

where A (a sum of terms which we don't write explicitely) verifies

$$\begin{aligned}
(5.5)\quad |A| \leqslant \{&|\mathrm{div}\, w|_\infty (1 + |l|_\infty) + |Ql|_\infty + 8|l|_\infty |\partial w|_\infty + 2\gamma\varepsilon + \varepsilon|\mathrm{div}\, w|_\infty + \\
&+ \varepsilon|\partial_x w|_\infty + \varepsilon|\partial_x l|_\infty\} \cdot \Big\{[Qg]^2_{0,\gamma(0,t)} + \sum_k [\partial_k g]^2_{0,\gamma(0,t)}\Big\} + \\
&+ \varepsilon\{[Qg(t)]^2_{0,\gamma} + [\partial_x g(t)]^2_{0,\gamma} + [Qg(0)]^2_{0,\gamma} + [\partial_x g(0)]^2_{0,\gamma}\} .
\end{aligned}$$

By definition $|u|_\infty \equiv \sup |u(t, y, x)|$. To get (5.4) the following identities are useful:

$$(Q+\gamma)g_\gamma = \exp[-\gamma t]Qg\,,$$
$$(Q+2\gamma)(gu) = [(Q+\gamma)g]u + g[(Q+\gamma)u]\,,$$
$$(Q+2\gamma)|g|^2 = 2\,\mathrm{Re}\,\{[(Q+\gamma)g]\bar{g}\}\,.$$

We denote by P_0, P_1 and P_2 polynomials (with real and non-negative coefficients) in the following variables:

$$(5.6)\qquad \begin{cases} P_0 = P_0\left(\dfrac{1}{m},\ \|w\|_{L^\infty(H^2)},\ \|\partial_t w\|_{L^\infty(H^1)},\ \|l-1\|_{L^\infty(H^2)}\right), \\ P_1 = P_1\left(\dfrac{1}{m},\ |w|_\infty,\ |\partial w|_\infty,\ |\partial_t w|_\infty,\ |l|_\infty,\ |\partial l|_\infty,\ |\partial_t l|_\infty\right), \\ P_2 = P_2\left(\dfrac{1}{m},\ \text{norms concerning (5.3)}\right). \end{cases}$$

In this section and in the next one we shall take for convenience

$$(5.6')\qquad P_0 = P_0\left(\frac{1}{m},\ |w|_\infty,\ |l|_\infty\right),$$

instead of $(5.6)_1$. We use the same symbol P_k, $k = 0, 1, 2$, to denote different polynomials (of the same type k) without any comment. However in the same equation a symbol P_k denotes the same polynomial. From well known Sobolev theorems it follows that $P_0 \leqslant P_1 \leqslant P_2$, when P_0 is defined by (5.6′). Moreover a polynomial P_0 of type (5.6′) is always of type $(5.6)_1$.

We return now to (5.4), (5.5). It follows from these equations that there exist $\bar{P}_0$ and $\bar{P}_1$ such that if

$$(5.7)\qquad \begin{cases} \varepsilon\bar{P}_0 \leqslant 1\,, \\ \gamma \geqslant \bar{P}_1\,, \end{cases}$$

then

$$(5.8)\qquad [g(t)]^2_{1,\gamma} + \gamma[g]^2_{1,\gamma(0,t)} + \varepsilon\langle\partial_t g\rangle^2_{0,\gamma(0,t)} + \varepsilon\langle\partial_x g\rangle^2_{0,\gamma(0,t)} \leqslant \varepsilon P_0 \sum_{j=1}^{2}\langle\partial_j g\rangle^2_{0,\gamma(0,t)} + \\ + P_0[g(0)]^2_{1,\gamma} + P_0|(Pg, (Q+\varepsilon\partial_x)g)_{0,\gamma(0,t)}| + P_0|\langle l_3\partial_x g, Qg\rangle_{0,\gamma(0,t)}|\,.$$

We denote by $\mathcal{F}$ a Fourier transform and by $\bar{\mathcal{F}}$ its inverse Fourier transform. For the sake of convenience the variables are sometimes indicated, as for instance

$$\mathcal{F}_{(y\to\eta)}u = \int \exp[-iy\cdot\eta]\, u(y)\, dy\, .$$

If

$$\Lambda^k_{y,\gamma}u \equiv \overline{\mathcal{F}}_{(\eta\to y)}[(|\eta|^2+\gamma^2)^{k/2}(\mathcal{F}_{(y\to\eta)}u)]\,, \tag{5.9}$$

where $k\in\mathbb{R}$, one easily sees that

$$\langle\!\langle \Lambda^k_{y,\gamma} g\rangle\!\rangle^2_{0,\gamma} = -2\,\mathrm{Re}\big((\partial_x g,\, \Lambda^{2k}_{y,\gamma} g)\big)_{0,\gamma}\,. \tag{5.10}$$

In particular one has, for $j=1,2$,

$$\langle\!\langle \Lambda^{-\frac12}_{,\gamma}(w_j\partial_j g)\rangle\!\rangle^2_{0,\gamma} = -2\,\mathrm{Re}\big(((\partial_x w_j)(\partial_j g) - (\partial_j w_j)(\partial_x g),\, \Lambda^{-1}_{y,\gamma}(w_j\,\partial_j g))\big)_{0,\gamma} - \\ -2\,\mathrm{Re}\big((\partial_j(w_j\,\partial_x g),\, \Lambda^{-1}_{y,\gamma}(w_j\,\partial_j g))\big)_{0,\gamma}\,,$$

hence, after some calculations

$$\langle\!\langle \Lambda^{-\frac12}_{y,\gamma}(w_j\,\partial_j g)\rangle\!\rangle^2_{0,\gamma} \leqslant P_1[g]^2_{1,\gamma}\,, \qquad j=1,2. \tag{5.11}$$

From (5.15) one easily gets

$$P_0|\langle l_3\,\partial_x g,\, Qg\rangle_{0,\gamma(0,t)}| \leqslant P_0|\langle l_3\,\partial_x g,\, \partial_t g\rangle_{0,\gamma(0,t)}| + \\ + \frac{P_1}{\gamma}\langle \Lambda^{\frac12}_{y,\gamma}(l_3\,\partial_x g)\rangle^2_{0,\gamma(0,t)} + \frac{\gamma}{4}[g]^2_{1,\gamma(0,t)}\,. \tag{5.12}$$

On the other hand

$$P_0|(Pg,\, (Q+\varepsilon\,\partial_x)g)_{0,\gamma(0,t)}| \leqslant \frac{\tilde{P}_0}{\gamma}|Pg|^2_{0,\gamma(0,t)} + \frac{\gamma}{4}[g]^2_{1,\gamma(0,t)}\,. \tag{5.13}$$

From (5.8), (5.12) and (5.13) it follows that

$$[g(t)]^2_{1,\gamma} + \gamma[g]^2_{1,\gamma(0,t)} + \varepsilon\langle\partial_t g\rangle^2_{0,\gamma(0,t)} + \varepsilon\langle\partial_x g\rangle^2_{0,\gamma(0,t)} \leqslant \\ \leqslant \varepsilon P_0\sum_{j=1}^{2}\langle\partial_j g\rangle^2_{0,\gamma(0,t)} + P_0[g(0)]^2_{1,\gamma} + \frac{P_0}{\gamma}|Pg|^2_{0,\gamma(0,t)} + \\ + \frac{P_1}{\gamma}\langle\Lambda^{\frac12}_{y,\gamma}\,\partial_x g\rangle^2_{0,\gamma(0,t)} + P_0|\langle l_3\,\partial_x g,\, \partial_t g\rangle_{0,\gamma(0,t)}| \tag{5.14}$$

for γ and ε verifying (6.11). We used also the estimate

$$\langle \Lambda^{\frac12}_{y,\gamma}(l_3\partial_x g)\rangle^2_{0,\gamma(0,t)} \leqslant P_1\langle\Lambda^{\frac12}_{y,\gamma}\,\partial_x g\rangle^2_{0,\gamma(0,t)}\,. \tag{5.15}$$

Estimate (5.14) will be useful to study the Neumann boundary condition. For the Dirichlet boundary condition we will use the estimate

$$(5.16)\qquad [g(t)]_{1,\gamma}^2 + \gamma[g]_{1,\gamma(0,t)}^2 + \varepsilon\langle\partial_t g\rangle_{0,\gamma(0,t)}^2 + \varepsilon\langle\partial_x g\rangle_{0,\gamma(0,t)}^2 \leqslant P_0[g(0)]_{1,\gamma}^2 + \frac{P_0}{\gamma}[Pg]_{0,\gamma(0,t)}^2 + \frac{P_0}{\varepsilon}\langle\partial_t g\rangle_{0,\gamma(0,t)}^2 + \frac{P_0}{\varepsilon}\sum_{j=1}^2\langle\partial_j g\rangle_{0,\gamma(0,t)}^2,$$

which holds for ε and γ verifying (5.7). Estimate (5.16) follows from (5.8), (5.13) and from

$$P_0|\langle l_3\,\partial_x g, Qg\rangle_{0,\gamma(0,t)}| \leqslant \frac{\varepsilon}{2}\langle\partial_x g\rangle_{0,\gamma(0,t)}^2 + \frac{\tilde P_0}{\varepsilon}\Big(\langle\partial_t g\rangle_{0,\gamma(0,t)}^2 + \sum_{j=1}^2\langle\partial_j g\rangle_{0,\gamma(0,t)}^2\Big).$$

Finally, starting from $(\partial_t g, g)_{0,\gamma(0,t)}$ one easily shows that

$$(5.17)\qquad [g(t)]_{0,\gamma}^2 + \gamma[g]_{0,\gamma(0,t)}^2 \leqslant [g(0)]_{0,\gamma}^2 + \frac{1}{\gamma}[\partial_t g]_{0,\gamma(0,t)}^2.$$

6. – The a priori bound of order zero for the Neumann condition.

In this section we use the notations $D_t = (1/i)\,\partial_t$, $D_k = (1/i)\,\partial_k$. Put $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, and define

$$(6.1)\quad \hat g(\tau,\eta) \equiv \mathcal{F}_{(y,t)\to(\eta,\sigma)}\{\exp[-\gamma t]\,g(y,t)\} = \iint_{\mathbb{R}_t\times\mathbb{R}_y}\exp[-i(\tau t + \eta\cdot y)]\,g(t,y)\,dy\,dt,$$

where x is a parameter. Notice that t runs in all of $(-\infty, +\infty)$. We assume that the known functions are extended to $\mathbb{R}_t\times\mathbb{R}_y^2\times\mathbb{R}_x^+$ with each norm bounded by a constant times the corresponding norm in $\mathbb{R}_t^+\times\mathbb{R}_y^2\times\mathbb{R}_x^+$: It is easy to show that

$$(6.2)\qquad \widehat{D_t g}(\tau,\eta) = \tau\hat g(\tau,\eta),$$

$$(6.3)\qquad \widehat{D_j g}(\tau,\eta) = \eta_j\hat g(\tau,\eta),\qquad j = 1, 2,$$

$$(6.4)\qquad \widehat{\Lambda_{y,\gamma}^k g}(\tau,\eta) = \big(|\eta|^2 + \gamma^2\big)^{k/2}\hat g(\tau,\eta),$$

$$(6.5)\qquad \langle\Lambda_{y,\gamma_0}^{\frac12} g\rangle_{0,\gamma(-\infty,+\infty)}^2 = -2\,\mathrm{Re}\,(\partial_x g, \Lambda_{y,\gamma_0} g)_{0,\gamma(-\infty,+\infty)}.$$

Consider now two real functions $\beta_i(\tau,\eta)$, $i=1,2$, defined in $\mathbb{C}\times\mathbb{R}^2/\{0\}$, homogeneous of degree zero i.e. $\beta_i(\delta\tau,\delta\eta)=\delta\beta_i(\tau,\eta)$, $\forall\delta>0$, and such that $0\leqslant\beta_i(\tau,\eta)\leqslant 1$, $i=1,2$, and $\beta_1+\beta_2\equiv 1$. These functions are determined by the values on the sphere

$$\Sigma\equiv\{(\tau,\eta)\colon |\tau|^2+|\eta|^2=1\}\,.$$

Finally, assume that

$$\text{(6.6)}\qquad \begin{cases} \operatorname{supp}\beta_1\subset\Sigma_1\equiv\{(\tau,\eta)\in\Sigma\colon |\tau|\leqslant c_\alpha|\eta|\}\,,\\ \operatorname{supp}\beta_2\subset\Sigma_2\equiv\{(\tau,\eta)\in\Sigma\colon |\eta|^2\leqslant\alpha|\tau|^2\}\,,\end{cases}$$

where the supports are taken in Σ and $\alpha>0$ is to be fixed later. The constant c_α depends only on α.

Consider now the pseudo-differential operators $\beta_i(D)=\beta_i(D_t,D_y)$, $i=1,2$, defined by

$$\text{(6.7)}\qquad \widehat{\beta_i(D)g}(\tau,\eta)=\beta_i(\tau,\eta)\hat{g}(\tau,\eta)\,.$$

Obviously

$$\text{(6.8)}\qquad g=\beta_1(D)\,g+\beta_2(D)\,g\equiv g_1+g_2\,.$$

It is clear that formulae (5.14) holds with $(0,t)$ replaced by any other time interval. In particular if g has compact support and ε, γ verify (5.7) one has

$$\text{(6.9)}\qquad \gamma[g]^2_{1,\gamma;I}+\varepsilon\langle\partial_t g\rangle^2_{0,\gamma;I}+\varepsilon\langle\partial_x g\rangle^2_{0,\gamma;I}\leqslant$$
$$\leqslant\varepsilon P_0\sum_{j=1}^{2}\langle\partial_j g\rangle^2_{0,\gamma;I}+\frac{P_0}{\gamma}\,|Pg|^2_{0,\gamma;I}+\frac{P_1}{\gamma}\,\langle\Lambda^{\frac12}_{y,\gamma}\,\partial_x g\rangle^2_{0,\gamma;I}+P_0|\langle l_3\,\partial_x g,\,\partial_t g\rangle_{0,\gamma;I}|\,,$$

where for convenience we put $I\equiv(-\infty,+\infty)$. Denoting now by H^2_γ the Hilbert space of the complex valued functions $g(t,y,x)$ such that $g_\gamma\in H^2(\mathbb{R}_t\times\mathbb{R}^2_y\times\mathbb{R}^+_x)$, it follows by approximation that (6.9) (and also (5.14)) holds for every $g\in H^2_\gamma$. In particular (6.9) holds for the functions g_i, $i=1,2$.

Now we want to prove estimates (6.12). From the identity $\langle u\rangle^2_{0,\gamma;I}=(2\pi)^{-3}\langle\hat{u}\rangle^2_{0,0;I}$ it follows that

$$\langle\Lambda^{-\frac12}_{y,\gamma}D_t g_1\rangle^2_{0,\gamma;I}=(2\pi)^{-3}\!\int\!\!\int(|\eta|^2+\gamma^2)^{-\frac12}|\tau|^2|\hat{g}_1(\tau,\eta)|^2\,d\sigma\,d\eta\,.$$

Since $|\tau|\leqslant c_\alpha|\eta|$ on the support of β_1, one gets

$$\langle\Lambda^{-\frac12}_{y,\gamma}D_t g_1\rangle^2_{0,\gamma;I}\leqslant(2\pi)^{-3}\,c^2_\alpha\langle\Lambda^{\frac12}_{y,0}g_1\rangle^2_{0,\gamma;I}\,.$$

By using (6.5) to estimate the right hand side, it follows that

$$\gamma\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t g\rangle_{0,\gamma;I}^2\leqslant 2(2\pi)^{-3}c_\alpha^2\gamma[g_1]_{1,\gamma;I}^2. \tag{6.10}$$

On the other hand from (6.9) with $\varepsilon=0$ one gets

$$\gamma[g_1]_{1,\gamma;I}^2\leqslant\frac{P_0}{\gamma}[Pg_1]_{0,\gamma;I}^2+\frac{P_1}{\gamma}\langle\Lambda_{y,\gamma}^{\frac{1}{2}}\partial_x g_1\rangle_{0,\gamma;I}^2+P_0|\langle l_3\partial_x g_1,\,\partial_t g_1\rangle_{0,\gamma;I}|. \tag{6.11}$$

Moreover from Schwartz inequality

$$P_0|\langle l_3\partial_x g_1,\,\partial_t g_1\rangle|_{0,\gamma;I}\leqslant\frac{(2\pi)^3}{4c_\alpha^2}\gamma\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t g_1\rangle_{0,\gamma;I}^2+\frac{c_\alpha^2}{(2\pi)^3\gamma}P_0^2\langle\Lambda_{y,\gamma}^{\frac{1}{2}}(l_3\partial_x g_1)\rangle_{0,\gamma;I}^2.$$

From this last estimate and from (6.10) and (6.11) (see also (5.15)) one easily shows that, for suitable P_0 and P_1, the following estimate holds (for $i=1$):

$$\gamma[g_i]_{1,\gamma;I}^2+\gamma\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}D_t g_i\rangle_{0,\gamma;I}^2\leqslant\frac{P_0}{\gamma}[Pg_i]_{0,\gamma;I}^2+\frac{P_1}{\gamma}\langle\Lambda_{y,\gamma}^{\frac{1}{2}}D_x g_i\rangle_{0,\gamma;I}^2, \tag{6.12}$$

for every $g\in H_\gamma^2$, and for γ verifying (5.7).

Now we prove (6.12) for $i=2$. By using estimate (6.9) for g_2 one easily gets

$$\begin{aligned}\gamma[g_2]_{1,\gamma;I}^2+\varepsilon\Big(\langle D_t g_2\rangle_{0,\gamma;I}^2-P_0\sum_{j=1}^{2}\langle D_j g_2\rangle_{0,\gamma;I}^2\Big)&\leqslant\\ \leqslant\frac{P_0}{\gamma}[Pg_2]_{0,\gamma;I}^2+\frac{P_1}{\gamma}\langle\Lambda_{y,\gamma}^{\frac{1}{2}}D_x g_2\rangle_{0,\gamma;I}^2&+\frac{\varepsilon}{4}\langle D_t g_2\rangle_{0,\gamma;I}^2+\frac{P_0}{\varepsilon}\langle D_x g_2\rangle_{0,\gamma;I}^2.\end{aligned} \tag{6.13}$$

Moreover, choosing $\alpha\leqslant(2P_0)^{-1}$ one has $|\tau|^2-P_0|\eta|^2\geqslant 2^{-1}|\tau|^2$ on Σ_2, hence

$$\langle D_t g_2\rangle_{0,\gamma;I}^2-P_0\sum_{j=1}^{2}\langle D_j g_2\rangle_{0,\gamma;I}^2\geqslant 2^{-1}\langle D_t g_2\rangle_{0,\gamma;I}^2. \tag{6.14}$$

Consequently from (6.13) one gets

$$\gamma[g_2]_{0,\gamma;I}^2+\frac{\varepsilon}{4}\langle D_t g_2\rangle_{0,\gamma;I}^2\leqslant\frac{P_0}{\gamma}[Pg_2]_{0,\gamma;I}^2+\frac{P_1}{\gamma}\langle\Lambda_{y,\gamma}^{\frac{1}{2}}D_x g_2\rangle_{0,\gamma;I}^2+\frac{P_0}{\varepsilon}\langle D_x g_2\rangle_{0,\gamma;I}^2.$$

Now from the general formula

$$\begin{cases}\gamma\langle\!\langle\Lambda_{y,\gamma}^{-\frac{1}{2}}u\rangle\!\rangle_{0,\gamma}^2\leqslant\langle\!\langle u\rangle\!\rangle_{0,\gamma}^2,\\ \gamma\langle\!\langle u\rangle\!\rangle_{0,\gamma}^2\leqslant\langle\!\langle\Lambda_{y,\gamma}^{\frac{1}{2}}u\rangle\!\rangle_{0,\gamma}^2,\end{cases} \tag{6.15}$$

it follows that

$$\gamma[g_2]^2_{1,\gamma;I}+\frac{\varepsilon}{4}\gamma\langle\Lambda^{-\frac{1}{2}}_{y,\gamma}D_tg_2\rangle^2_{0,\gamma;I}\leqslant\frac{P_0}{\gamma}[Pg_2]^2_{0,\gamma;I}+\left(P_1+\frac{P_0}{\varepsilon}\right)\frac{1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{y,\gamma}D_xg_2\rangle^2_{0,\gamma;I}.$$

Fixing now $\varepsilon=(\overline{P}_0+1)^{-1}$ (cf. (5.7)) and multiplying the above inequality by ε^{-1} it follows that there exist P_0 and P_1 such that, for $i=2$ and for γ verifying (5.7), estimate (6.12) holds for every $g\in H^2_\gamma$.

Furthermore from the identity $Pg_i=\beta_iPg+(P\beta_i-\beta_iP)g$ one gets

$$[Pg_i]_{0,\gamma;I}\leqslant[Pg]_{0,\gamma;I}+P_2[g]_{1,\gamma;I} \tag{6.16}$$

since as in [4], section 4)

$$[(P\beta_i-\beta_iP)g]_{0,\gamma;I}\leqslant P_2[g]_{1,\gamma;I}.$$

Now from (6.8), (6.12) and (6.16) it follows that there exists a polynomial $\overline{P}_2$ such that if

$$\gamma\geqslant\overline{P}_2 \tag{6.17}$$

then

$$\gamma[g]^2_{1,\gamma;I}+\gamma\langle\Lambda^{-\frac{1}{2}}_{y,\gamma}D_tg\rangle^2_{0,\gamma;I}\leqslant\frac{P_0}{\gamma}[Pg]^2_{0,\gamma;I}+\frac{P_1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{y,\gamma}D_xg\rangle^2_{0,\gamma;I}. \tag{6.18}$$

From (5.14) with $\varepsilon=0$ and from (5.15), (6.18) and (5.17) one gets the following result:

THEOREM 6.1. *There exists a polynomial $\overline{P}_2$ such that if γ verifies* (6.17) *then*

$$|g(t)|^2_{1,\gamma}+\gamma|g|^2_{1,\gamma(0,t)}\leqslant P_0|g(0)|^2_{1,\gamma}+\frac{P_0}{\gamma}|Pg|^2_{0,\gamma;I}+\frac{P_1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{y,\gamma}\partial_xg\rangle^2_{0,\gamma;I} \tag{6.19}$$

for every $g\in H^2_\gamma$.

COROLLARY 6.2. *There exist polynomials P_0, P_1 and $\overline{P}_2$ such that if* (6.17) *holds then for each $t>0$ and for each $g\in H^2((0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x)$ we have*

$$\begin{aligned}|g(t)|^2_{1,\gamma}+\gamma|g|^2_{1,\gamma(0,t)}\leqslant P_0(\|g(0)\|^2_{1,\gamma}+\|\partial_tg(0)\|^2_{0,\gamma})&+\\ +\frac{P_0}{\gamma}|Pg|^2_{0,\gamma(0,t)}+\frac{P_1}{\gamma}\langle\Lambda^{\frac{1}{2}}_{y,\gamma}\partial_xg\rangle^2_{0,\gamma(0,t)}&.\end{aligned} \tag{6.20}$$

PROOF. To give a short proof we use also the existence and unicity result of [13]. First we assume that the coefficients of P are regular. Then

$$F \equiv Pg \in L^2\big((0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x\big),\ G \equiv \partial_x g|_{x=0} \in L^2(0,t;\, H^{\frac{1}{2}}(\mathbb{R}^2_y))\,,$$

$g_0 \equiv g|_{t=0} \in H^1(\mathbb{R}^2_y\times\mathbb{R}^+_x)$, $g_1 \equiv \partial_t g|_{t=0} \in L^2(\mathbb{R}^2_y\times\mathbb{R}^+_x)$ and there exist regular F_n, G_n and $g_0^{(n)}$ with $\partial_x g_0^{(n)}|_{x=0} = 0$, such that $F_n \to F$, $G_n \to G$ and $g_0^{(n)} \to g_0$ in the Banach spaces indicated above. Since the coefficients are regular, from theorem 2 of [13] for $k=1$ the problem $Pg^{(n)} = F_n$ in $(0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x$, $\partial_x g^{(n)}|_{x=0} = G_n$ in $(0,t)\times\mathbb{R}^2_y$, $g^{(n)}|_{t=0} = g_0^{(n)}$, $\partial_t g^{(n)}|_{t=0} = g_1$ has a unique solution $g^{(n)} \in H^2\big((0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x\big)$. By applying now (6.19) to $g^{(n)} - g^{(m)}$ it follows that the solution $g^{(n)}$ converges in the norm on the left hand side of (6.19) to a function $u \in C\big([0,t];\, H^1\big) \cap C^1\big([0,t];\, L^2\big)$. Now using the existence and unicity result stated in theorem 2 of [13] for $k=0$ it follows that $u = g$. Writing now estimate (6.19) for $g^{(n)}$ and passing to the limit one gets (6.20) for g.

When the coefficients of P are only as in (5.3) we prove (6.20) by approximating Pg with $P^{(n)}g$, where the coefficients $w^{(n)}$ and $l^{(n)}$ of $P^{(n)}$ are regular, uniformly bounded in all the norms utilized in the definitions (5.6) and converge to w and l in $C(0,t;\, H^2) \cap C^1(0,t;\, H^1)$.

7. – A priori bounds for the Dirichlet condition.

From (5.16) and (5.17) one gets the following result:

PROPOSITION 7.1. *There exists P_0 and $\overline{P}_1$ such that if*

$$\gamma \geqslant \overline{P}_1 \tag{7.1}$$

then for $t > 0$ and $g \in H^2\big((0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x\big)$ one has

$$|g(t)|^2_{1,\gamma} + \gamma|g|^2_{1,\gamma(0,t)} \leqslant P_0\big(\|g(0)\|^2_{1,\gamma} + \|\partial_t g(0)\|^2_{0,\gamma}\big) + \frac{P_0}{\gamma}\,|Pg|^2_{0,\gamma(0,t)} + P_0\langle g\rangle^2_{1,\gamma(0,t)}\,. \tag{7.2}$$

Now we want to prove corresponding estimates for the orders 1 and 2. Let $g \in H^3\big((0,t)\times\mathbb{R}^2_y\times\mathbb{R}^+_x\big)$. Using (7.2) for g and for the tangential derivatives $\partial_t g$ and $\partial_j g$, $j = 1, 2$, and adding one gets

$$|g(t)|^{*2}_{2,\gamma} + \gamma|g|^{*2}_{2,\gamma(0,t)} \leqslant P_0\big(\|g(0)\|^2_{2,\gamma} + \|\partial_t g(0)\|^2_{1,\gamma}\big) + \frac{\tilde{P}_0}{\gamma}\Big\{|Pg|^2_{1,\gamma(0,t)} + |P\partial_t g - \partial_t Pg|^2_{0,\gamma(0,t)} + \sum_{j=1}^{2}|P\partial_j g - \partial_j Pg|^2_{0,\gamma(0,t)}\Big\} + P_0\langle g\rangle^2_{2,\gamma(0,t)}\,, \tag{7.3}$$

where we used the estimate

$$\|\partial_t^2 g(0)\|_{0,\gamma}^2 \leqslant P_0([Pg(0)]_{0,\gamma}^2 + \|g(0)\|_{2,\gamma}^2 + \|\partial_t g(0)\|_{1,\gamma}^2)\,. \tag{7.4}$$

Now we shall estimate $\partial_x^2 g$. One has from the definition of P

$$(l_3 - w_3^2)\,\partial_x^2 g = \Phi \tag{7.5}$$

where

$$\Phi = \partial_t^2 g - \sum_{j=1}^{2} l_j \partial_j^2 g - \sum_k (\partial_k l_k)(\partial_k g) + \sum_k \sum_{j=1}^{2} w_k w_j (\partial_k \partial_j g) + \\ + 2\sum_k w_k \partial_t \partial_k g + \sum_k (\partial_t w_k)(\partial_k g) + \sum_{k,l} w_k (\partial_k w_l)(\partial_l g) - Pg\,. \tag{7.6}$$

From (7.6), using Sobolev's embedding theorems and interpolation theorems, it follows that

$$[\Phi(t)]_{0,\gamma}^2 \leqslant P_0 |g(t)|_{2,\gamma}^{*2} + |Pg(t)|_{0,\gamma}^2\,,$$

$$[\Phi]_{0,\gamma(0,t)}^2 \leqslant P_0 |g|_{2,\gamma(0,t)}^{*2} + [Pg]_{0,\gamma(0,t)}^2\,,$$

and using (5.17) to estimate $|Pg(t)|_{0,\gamma}^2$ one gets

$$[\Phi(t)]_{0,\gamma}^2 + \gamma[\Phi]_{0,\gamma(0,t)}^2 \leqslant P_0\big(|g(t)|_{2,\gamma}^{*2} + \gamma |g|_{2,\gamma(0,t)}^{*2}\big) + \\ + [Pg(0)]_{0,\gamma}^2 + \frac{1}{\gamma}[Pg]_{1,\gamma(0,t)}^2\,. \tag{7.7}$$

On the other hand

$$|w_3(t, y, x)| = |w_3(t, y, x) - w_3(t, y, 0)| \leqslant P_0 x^{\frac{1}{2}} \tag{7.8}$$

as follows from the Sobolev's embedding theorem $H^2 \hookrightarrow C^{0,\frac{1}{2}}$. Hence if $0 \leqslant x < r \equiv (2P_0)^{-1}$ one has

$$l_3 - w_3^2 \geqslant \frac{m}{2}\,. \tag{7.9}$$

Notice that with our conventions one has

$$r^{-1} = P_0 \tag{7.10}$$

since polynomials of type P_0 may depend on m^{-1}.

By using (7.5) and (7.7) it follows that for $0 < x < r$

$$(7.11)\qquad [\partial_x^2 g(t)]^2_{0,\gamma(x<r)} + \gamma[\partial_x^2 g]^2_{0,\gamma(0,t);(x<r)} \leqslant P_0(|g(t)|^{*2}_{2,\gamma} + \gamma |g|^{*2}_{2,\gamma(0,t)}) + \\ + P_0\left([Pg(0)]^2_{0,\gamma} + \frac{1}{\gamma}[Pg]^2_{1,\gamma(0,t)}\right)$$

where « $(x<r)$ » means that the norms concern only the strip $0<x<r$. Now we want to estimate $\partial_x^2 g$ in the interior. We take a real function $\varphi \in C^\infty(\mathbb{R}^+)$, $0 \leqslant \varphi(x) \leqslant 1$, such that $\varphi(x) = 0$ if $0 \leqslant x \leqslant \frac{1}{2}$ and $\varphi(x) = 1$ if $1 \leqslant x$. Putting $\varphi_r(x) = \varphi(x/r)$ one has (for each non negative integer n)

$$(7.12)\qquad \begin{cases} \left|\dfrac{d^n\varphi_r(x)}{dx^n}\right| \leqslant c_n r^{-n}, \\ \dfrac{d^n\varphi_r(x)}{dx^n} = 0 \qquad \text{if } 0 \leqslant x \leqslant \dfrac{r}{2} \text{ or if } r \leqslant x. \end{cases}$$

Using (7.2) for $\varphi_r \partial_x g$ and using also (7.12) and (7.10) one gets for $x > r$ (with obvious notations)

$$(7.13)\qquad |\partial_x g(t)|^2_{1,\gamma(r<x)} + \gamma|\partial_x g|^2_{1,\gamma(0,t):(r<x)} \leqslant P_0(\|g(0)\|^2_{2,\gamma} + \|\partial_t g(0)\|^2_{1,\gamma}) + \\ + \frac{P_0}{\gamma}\left(|Pg|^2_{1,\gamma(0,t)} + |P(\varphi_r \partial_x g) - \varphi_r(\partial_x Pg)|^2_{0,\gamma(0,t)}\right).$$

Now we shall estimate the norms of the commutators which appear in (8.3) and (8.13). Denoting for convenience by ∂ a derivative with respect to t, y_1, y_2 or x, we can write in a short form

$$(7.14)\qquad \partial Q^2 g - Q^2 \partial g \simeq (1+w)(\partial^2 w \cdot \partial g + \partial w \cdot \partial^2 g) + (\partial w)^2 \cdot \partial g,$$

where in the left hand side ∂ denotes a fixed derivative. On the contrary in the right hand side (where only the type of the terms obtained computing the left hand side is indicated) ∂ runs over all the derivatives, with the following exceptions: the derivative $\partial_t^2 g$ doesn't appear; the derivative $\partial_t^2 w$ appears only when we take $\partial = \partial_t$ in the left hand side.

The functions $(1+w)\partial w$ and $(\partial w)^2$ are bounded in norm L^∞ by P_1. On the other hand by Hölder's inequality in $\mathbb{R}^2_y \times \mathbb{R}^+_x$ one easily gets

$$[\partial^2 w \cdot \partial g]^2_{0,\gamma(0,t)} \leqslant \|\partial^2 w\|^2_{L^\infty(0,t;L^3)} \int_0^t \exp[-2\gamma s] \|\partial g(s)\|^2_{L^6} ds,$$

and from $H^1 \hookrightarrow L^p$, for $p = 3$ and $p = 6$, it follows that the left hand side is bounded by $P_2|g|^2_{2,\gamma(0,t)}$. Consequently

$$[\partial Q^2 g - Q^2 \partial g]^2_{0,\gamma(0,t)} \leqslant P_2|g|^2_{2,\gamma(0,t)}. \tag{7.15}$$

Moreover

$$\partial\Big[\sum_k \partial_k(l_k \partial_k g)\Big] - \sum_k \partial_k(l_k \partial_k \partial g) = \sum_k (\partial l_k)\, \partial_k^2 g - \sum_k (\partial \partial_k l_k)(\partial_k g)\,. \tag{7.16}$$

Computing the left hand side of (7.16) as above one gets finally

$$[\partial Pg - P\partial g]^2_{0,\gamma(0,t)} \leqslant P_2|g|^2_{2,\gamma(0,t)}. \tag{7.17}$$

On the other hand, using $\partial_t \varphi_r = \partial_y \varphi_r = 0$ one shows that

$$Q^2(\varphi_r \partial_x g) - \varphi_r Q^2 \partial_x g = (\partial_x \varphi_r)\, w_3(Q\partial_x g) + \\ + (\partial_x \varphi_r)\, Q[w_3(\partial_x g)] + (\partial_x^2 \varphi_r)\, w_3^2(\partial_x g)\,. \tag{7.18}$$

Using Sobolev's embedding theorems, (7.12) and (7.10) one shows that the norm $[\;]^2_{0,\gamma(0,t)}$ of the right hand side of (7.18) is bounded by $P_2|g|^2_{2,\gamma(0,t)}$. The same holds for the last term in the right hand side of the identity

$$Q^2(\varphi_r \partial_x g) - \varphi_r \partial_x Q^2 g = Q^2(\varphi_r \partial_x g) - \varphi_r Q^2 \partial_x g - \varphi_r(\partial_x Q^2 g - Q^2 \partial_x g) \tag{7.19}$$

as one shows using (7.15). Similarly one estimates the terms $\partial_k[l_k \partial_k(\varphi_r \partial_x g)] - - \varphi_r \partial_x \partial_k[l_k(\partial_k g)]$. Consequently

$$|P(\varphi_r \partial_x g) - \varphi_r(\partial_x Pg)|^2_{0,\gamma(0,t)} \leqslant P_2|g|^2_{2,\gamma(0,t)}. \tag{7.20}$$

From (7.3), (7.11), (7.13), (7.17) and (7.20) it follows

PROPOSITION 7.2. *There exist polynomials P_0 and $\overline{P}_2$ such that if*

$$\gamma \geqslant \overline{P}_2 \tag{7.21}$$

then for each $t > 0$ and each $g \in H^3$ one has

$$|g(t)|^2_{2,\gamma} + \gamma|g|^2_{2,\gamma(0,t)} \leqslant P_0\big(\|g_0\|^2_{2,\gamma} + \|\partial_t g(0)\|^2_{1,\gamma} + [Pg(0)]^2_{0,\gamma}\big) + \\ + \frac{P_0}{\gamma}\,|Pg|^2_{1,\gamma(0,t)} + P_0\,\langle g\rangle^2_{2,\gamma(0,t)}. \tag{7.22}$$

8. – A priori bounds of higher order and existence under the Neumann condition.

PROPOSITION 8.1. *There exist P_0, P_1 and $\overline{P}_2$ such that if*

$$\gamma \geqslant \overline{P}_2 \tag{8.1}$$

then for each $t > 0$ and each $g \in H^3$ one has

$$|g(t)|^2_{2,\gamma} + \gamma |g|^2_{2,\gamma(0,t)} \leqslant P_0\big(\|g(0)\|^2_{2,\gamma} + \|\partial_t g(0)\|^2_{1,\gamma} + |Pg(0)|^2_{0,\gamma}\big) + \\ + \frac{P_0}{\gamma} |Pg|^2_{1,\gamma(0,t)} + \frac{P_1}{\gamma} \langle \Lambda^{\frac{1}{2}}_{y,\gamma} \partial_x g \rangle^2_{1,\gamma(0,t)} . \tag{8.2}$$

To prove this result we apply (6.20) to the derivatives $\partial_t g$ and $\partial_y g$, and (7.2) to $\partial_x g$. We evaluate then the terms $[P\partial g - \partial P g]^2_{0,\gamma(0,t)}$ as in (7.17), and the term $\|\partial_t^2 g(0)\|^2_{0,\gamma}$ as in (7.4). Recall also $(6.15)_2$.

PROPOSITION 8.2. *There exist P_0, P_1 and $\overline{P}_2$ such that if γ verifies* (8.1) *then for each $t > 0$ and for each $g \in H^3$ one has the estimate*

$$|g(t)|'^2_{3,\gamma} + \gamma |g|'^2_{3,\gamma(0,t)} \leqslant P_0\big(\|g(0)\|^2_{3,\gamma} + \|\partial_t g(0)\|^2_{2,\gamma} + \|Pg(0)\|^2_{1,\gamma}\big) + \\ + \frac{P_0}{\gamma} |Pg|'^2_{2,\gamma(0,t)} + \frac{P_1}{\gamma} \langle \Lambda^{\frac{1}{2}}_{y,\gamma} \partial_x g \rangle^2_{2,\gamma(0,t)} . \tag{8.3}$$

To prove (8.3) we use (8.2) to estimate the derivatives $\partial_y g$ and (7.22) to estimate $\partial_x g$. We also use the estimate

$$|P\partial g - \partial P g|^2_{1,\gamma(0,t)} \leqslant P_2 |g|'^2_{3,\gamma(0,t)}, \qquad \text{if } \partial \neq \partial_t,$$

obtained by differentiating the right hand sides of (7.14) and (7.16) and by estimating then the $[\ ]_{0,\gamma(0,t)}$ norms. Notice that the terms $\partial_t^3 g$, $\partial_t^3 l$ and $\partial_t^3 w$ don't appear.

Existence of solutions.

Let now $\partial_t^j F \in L^2(0, +\infty; H^{2-j}(\mathbb{R}^2_y \times \mathbb{R}^+_x))$, $j = 0, 1$, $\partial_t^k G \in L^2(0, +\infty; H^{\frac{5}{2}-k}(\mathbb{R}^2_y)))$, $k = 0, 1, 2$ and $g_i \in H^{3-i}(\mathbb{R}^2_y \times \mathbb{R}^+_x)$, $i = 0, 1$, and assume that the compatibility conditions $\partial_x g_i|_{x=0} = \partial_t^i G|_{t=0}$, $i = 0, 1$ are verified. Assume also that the coefficients of P are regular. Then there exists a unique solution g of problem $Pg = F$ in $\mathbb{R}^+_t \times \mathbb{R}^2_y \times \mathbb{R}^+_x$, $\partial_x g|_{x=0} = G$ on $\mathbb{R}^+_t \times \mathbb{R}^2_y$, $\partial_t^i g|_{t=0} = g_i$, $i = 0, 1$ in $\mathbb{R}^2_y \times \mathbb{R}^+_x$. Moreover g verifies estimate (8.3), hence in particular there exist polynomials P_0, P_1 and $\bar{P}_2$ such that

$$(8.4)\qquad \sum_{k=0}^{2}\|\partial_t^k g\|^2_{L^\infty(0,t;H^{3-k})}\leqslant P_0(\exp[\bar{P}_2 t])\left(\sum_{i=0}^{1}\|g_i\|^2_{H^{3-i}}+\|F(0)\|^2_{H^1}\right)+$$
$$+P_1(\exp[\bar{P}_2 t])\int_0^t\left\{\sum_{j=0}^{1}\|\partial_t^j F(s)\|^2_{H^{2-j}}+\sum_{k=0}^{2}\langle\!\langle\partial_t^k G(s)\rangle\!\rangle^2_{H^{\frac{5}{2}-k}}\right\}ds\,,$$

where the Sobolev H^s are defined on $\mathbb{R}^2_y\times\mathbb{R}^+_x$ if s is integer and $\mathbb{R}^2_y$ otherwise. We get these results from the a priori bound (8.3) and from the existence theorem 2 of [13]. In fact the operator P and the boundary operator D_x verify all the conditions ([5]) required to use theorem 2 of [13], as the reader easily verifies; in particular the main condition (H) (II) of theorem 1 in [13] is verified since α and β vanish identically. When the coefficients of P verify only assumptions (5.1), (5.2) and (5.3), one easily shows by approximating the coefficients by regular ones and by using the a priori bound (8.3) (which holds uniformly) that the result holds again.

PROOF OF THEOREM 2.1. We denote here by (x_1, x_2, x_3) the cartesian coordinates of a point $x\in\Omega$ and by P_x the operator in the left hand side of $(2.23)_1$. Let $\{\Omega_i\}$, $i=0,1,\dots,m$, be an open «regular» covering of Ω such that $\bar{\Omega}_0\subset\Omega$ and $\bar{\Omega}_i\cap\Gamma\equiv\Gamma_i\neq\emptyset$, $i\neq 0$. By multiplying the data g_0, g_1, F and G by a suitable partition of the unity we get for $i=0$ a Cauchy problem and for $i=1,2,\dots,m$, m mixed problems whose data have compact support in $\Omega_i\cup\Gamma_i$. We shall take in account the case $i\neq 0$ since the case $i=0$ is easier. We assume without loss of generality that for each point $P\in\Omega_i$ there exists a unique $Q\in\Gamma_i$ such that $\overline{PQ}$ is orthogonal to Γ_i.

The coordinate y_3 of P is defined as the length of $\overline{PQ}$. The lines of curvature on Γ_i are taken as parametric curves and the coordinates (y_1, y_2) of P are defined as the coordinates of Q on Γ_i. The system of coordinates (y_1, y_2, y_3) is orthogonal on Ω_i since on parallel surfaces curves corresponding to lines of curvature are lines of curvature. The boundary Γ_i is characterized by $y_3=0$, moreover $y_3>0$ when $x\in\Omega_i$. The operator P_x is trasformed into the operator

$$P_y=\left(\partial_t+\sum_{k=1}^{3}w_k\,\partial_{y_k}\right)^2-\sum_{k=1}^{3}\partial_{y_k}(l_k\,\partial_{y_k})+h\sum_{k=1}^{3}a_k\,\partial_{y_k}$$

where $w_k=\sum_i(\partial_{x_i}y_k)v_i$ and $l_k=|\nabla_x y_k|^2 h$. The (known) coefficients a_k are

([5]) The definition of $\tilde{\tau}$ in [13] has to be replaced by: $\tilde{\tau}$ is the root of $P(s,t,qv+\eta,\tau)=\tilde{q}^2-\tilde{\tau}^2+d(\eta)^2$ such that $\partial\tilde{\tau}/\partial\tau>0$ (communicated by the author).

of class C^2 (6). One easily shows that conditions (5.1), (5.2) and (5.3) hold again. Hence each problem in Ω_i is equivalent to a problem in the half space $y_3 > 0$ as long as the support of the solution remains in the image of Ω_i under the change of coordinates $x \to y$. Hence by solving m mixed problems in the half space (and a Cauchy problem in all of the space) we get a (local in time) solution in Ω, which verifies (2.24) since $x \leftrightarrow y$ is of class C^4. Now we give a lower bound T_1 for the time interval in which the supports of the solutions verify the above property. T_1 is clearly bounded by a constant c_0 (7) times an upper bound for the propagation speed of the solutions. The propagation speed is bounded above by the maximum of the absolute value of the roots $\tau_\pm$ of the characteristic polynomial

$$\tau_\pm = -\sum_k w_k(y)\eta_k \pm \sqrt{\sum_k l_k(y)\eta_k}\,, \qquad |\eta| = 1\,,$$

hence is bounded above by a polynomial of type P_0. Consequently problem (2.23) is solvable in Q_T for T verifying $TP_0 \leqslant 1$. Moreover from (8.4) applied to each i-solution it easily follows (2.24).

REMARK. After doing the first step in $[0, T_1]$ we can consider T_1 as an initial time and do a second step in $[T_1, 2T_1]$. By using repeatedly this argument, and recalling that $T_1 = P_0^{-1}$, one easily gets estimates as (2.24) without condition $TP_0 \leqslant 1$.

(6) In the preceeding sections we didn't take in account the lower order terms $a_k h \partial_k g$, since they are trivially estimated.

(7) Depending only on the minimal distance to be covered, hence on Ω.

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