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Some Estimates of Solutions to Monge-Ampère Type Equations in Dimension Two.

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1. – Introduction.

One of the purposes of the present paper is to point out the efficiency of an approach to elliptic second order p.d.e.'s, in which special aspects of the geometry of solutions are stressed. The point of view we try to explain here is that, for an elliptic second order p.d.e., the development of solutions from boundary data or from another input and under suitable boundary constraints is, among other things, a process geometric in nature and crucial information (such as content or perimeter) on level sets of such solutions, hence basic estimates for these solutions, is conveniently and cheaply inferred from steps of this process. This opinion is suggested by earlier results [64] [65] on Dirichlet problems for linear and quasilinear equations in divergence form. Indeed it has been shown that a priori estimates of Luxemburg-Zaanen norms of solutions to such problems—roughly speaking, a Luxemburg-Zaanen norm is any norm which depends on the content of level sets only—are simply another way of interpreting a special differential inequality for the distribution function of the solutions in question. On the other hand, such a differential inequality can be derived as a straightforward consequence of the equations themselves and of a general geometric principle—the isoperimetric inequality. Remarkably, this approach to second order linear and quasilinear Dirichlet problems also shows in a natural manner what are the worst problems in the relevant framework, i.e. those which have the largest solutions. Consequently it allows one to offer a sharp form of a priori estimates. For further references on this matter see [41] [42] [45] [57].

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In the present paper an attempt is made to test the above statements on a class of genuinely nonlinear equations. Specifically, we consider Monge-Ampère equations of the following type

$$a\left(\sqrt{u_x^2+u_y^2}\right)(u_{xx}u_{yy}-u_{xy}^2)=H(x,y,u), \tag{1.1}$$

where x and y are independent real variables, u is the unknown, $a(r)$ is a (measurable) nonnegative function of a single variable r (having a suitable, not too singular, behaviour as $r \to 0$), $H(x, y, u)$ depends monotonically on the last argument and has suitable integrability properties. Special instances might be the standard Monge-Ampère equation

$$u_{xx}u_{yy}-u_{xy}^2=H(x,y,u) \tag{1.2}$$

and

$$(1+u_x^2+u_y^2)^{-2}(u_{xx}u_{yy}-u_{xy}^2)=H(x,y,u) \tag{1.3}$$

the equation of surfaces with prescribed Gauss curvature. ■

For the sake of completeness, let us mention some essential references on Monge-Ampère equations. The interest for second order p.d.e.'s, where the hessian determinant of the unknown function is linearly combined with second order derivatives, goes back to Monge [27] and Ampère [2]. An old fashioned theory of these equations is outlined in Goursat treatises [16, vol. 1, chap. 2] [17, vol. 3, chap. 24]. A modern approach to Monge-Ampère equations began with works [24] by H. Lewy, who proved a priori bounds for solutions to such equations and with their help solved questions on Minkowski and Weyl problems in differential geometry [25] [26]. Indeed Monge-Ampère equations have been studied in recent years in close connection with the latter problems; see Nirenberg [28] for a penetrating treatment of these. Subsequent contributions were made by Heinz [18] [19] [20], who greatly extended Lewy's technique. Bakel'man [3] and Aleksandrov [1] introduced an appropriate notion of generalized solution for a class of Monge-Ampère equations and proved existence theorems. Further massive investigations on the subject have been made by Bakel'man and Pogorelov; see, for instance, [4] [5] [6] [7] [8] [9] [30] [31] [32] [33] [34]. Very comprehensive results on the Dirichlet problem and on a priori bounds for solutions to Monge-Ampère equations—perhaps the most comprehensive results on this matter, up to now—are in papers [11] [12] by Cheng and Yau. We do not mention here complex Monge-Ampère equations, which are nowadays receiving a great deal of attention but which are out of the scope of the present paper. Additional references are listed at the end of the paper. ■

Both results, and proofs we present later, should bring into evidence a geometric structure of equation (1.1). To prepare the way, it is perhaps worth considering first: (i) a special representation of the hessian matrix, that brings out level lines and lines of steepest descent; (ii) a special case of equation (1.1), namely the equation: $u_{xx}u_{yy} - u_{xy}^2 = 0$.

(i) The formula we have in mind is a close relative to the following one:

$$\Delta u = \frac{\partial^2 u}{\partial(Du)^2} + \operatorname{div}\left(\frac{Du}{|Du|}\right)\frac{\partial u}{\partial(Du)}, \tag{1.4}$$

a representation of the laplacian Δu of a smooth function u of n variables $x_1, x_2, \ldots, x_n$. Here

$$\frac{\partial}{\partial(Du)} = \frac{Du}{|Du|}\times D = \frac{1}{|Du|}\sum_{k=1}^{n} u_{x_k}\frac{\partial}{\partial x_k}$$

is the derivative (with respect to the arclength) along the trajectories of Du —i.e. the *lines of steepest descent*—and $\partial^2/\partial(Du)^2$ is the square of $\partial/\partial(Du)$. Obviously $\partial u/\partial(Du) = |Du|$; and

$$\frac{\partial^2 u}{\partial(Du)^2} = \left(D^2u\cdot\frac{Du}{|Du|}\right)\times\frac{Du}{|Du|},$$

since

$$\frac{\partial^2}{\partial(Du)^2} = |Du|^{-2}\sum_{i,k=1}^{n} u_{x_i}\,u_{x_k}\frac{\partial^2}{\partial x_i\partial x_k} +$$
$$+ \text{(a derivative along a tangent field to the level surfaces of } u).$$

We use the notations:

$$D = \text{gradient}, \qquad D^2u = \text{hessian matrix of } u. \tag{1.5}$$

Formula (1.4) follows from a simple inspection of the right-hand side. The coefficient

$$\operatorname{div}\left(\frac{Du}{|Du|}\right),$$

appearing in (1.4), has a remarkable geometric meaning: it is the *mean curvature of the level surfaces of u*. Formula (1.4) shows that Δ acts as an ordinary differential operator along the lines of steepest descent; more precisely, the value of Δu at a point only involves derivatives of u along the line of steepest descent passing through that point and the mean curvature of the level surface through the point. Of course, (1.4) is a generalization of the formula

$$\Delta u(\sqrt{x_1^2 + \ldots + x_n^2}) = u''(r) + \frac{n-1}{r}\, u'(r)$$

for the laplacian of spherically symmetric functions.

We claim

$$u_{xx}u_{yy} - u_{xy}^2 = -\,k|Du|\,\frac{\partial^2 u}{\partial(Du)^2} - h^2|Du|^2, \tag{1.6}$$

where h is the *curvature of the lines of steepest descent* and k is the *curvature of the level lines* of u—the sign of k here is so chosen that the principal normal to the line $u = \text{const}$ (i.e. a normal pointing towards the center of curvature) is exactly $k\,Du/|Du|$. Note that (1.6) yields in particular

$$\text{hessian of } u(\sqrt{x^2 + y^2}) = \frac{1}{r}\, u'(r)\, u''(r)\,, \tag{1.7}$$

a formula for the hessian determinant of circularly symmetric functions.

An expression of h and k can be exhaustively obtained with the following device. Let X be a smooth vector field in the n-dimensional euclidean space and let $\partial/\partial X$ be the derivative

$$\frac{1}{|X|}\sum_{k=1}^{n} X_k\,\frac{\partial}{\partial x_k}$$

along the trajectories of X. The principal normal to these trajectories appears in the expression of the second order derivative $\partial^2/\partial X^2 = (\partial/\partial X)^2$ according to the rule

$$\frac{\partial^2}{\partial X^2} = |X|^{-2}\sum_{i,k=1}^{n} X_i X_k\,\frac{\partial^2}{\partial x_i\,\partial x_k} + (\text{principal normal})\times D\,.$$

By expanding $\partial^2/\partial X^2$ one then sees that the principal normal has the following components: $(\partial/\partial X)(X_k/|X|)$ $(k = 1, \ldots, n)$. In two dimensions the result can be put into the form

$$\text{principal normal} = \frac{iX}{|X|}\,\operatorname{div}\left(-\,i\,\frac{X}{|X|}\right),$$

where i stands for the rotation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; hence the curvature of the trajectories of X is exactly

$$-\operatorname{div}\left(i\frac{X}{|X|}\right).$$

Thus we conclude this way:

$$h = \text{curvature of the lines of steepest descent} \tag{1.8}$$
$$= -\operatorname{div}\left(i\frac{Du}{|Du|}\right) = |Du|^{-3}\left[(u_{xx}-u_{yy})\,u_x u_y - u_{xy}(u_x^2-u_y^2)\right],$$

$$k = \text{curvature of the level lines} \tag{1.9}$$
$$= -\operatorname{div}\frac{Du}{|Du|} = -|Du|^{-3}(u_{xx}u_y^2 - 2u_{xy}u_y u_x + u_{yy}u_x^2),$$

since the level lines of u are just the trajectories of iDu. It should be remarked that (1.9) is a special case of the well-known Bonnet's formula for the geodesic curvature, see Bianchi [43, chap. 5, section 99].

Equation (1.6) is a straightforward consequence of (1.8) and (1.9). A more comprehensive formula for the hessian matrix, which yields both (1.6) and (the two-dimensional case of) (1.4), is:

$$\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} = \tag{1.10}$$
$$= \begin{pmatrix}\dfrac{\partial^2 u}{\partial(Du)^2} & \dfrac{\partial}{\partial(iDu)}\dfrac{\partial u}{\partial(Du)}\\ & -k|Du|\end{pmatrix},$$

since

$$\frac{\partial}{\partial(iDu)}\frac{\partial u}{\partial(Du)} = \frac{1}{|Du|}\left(-u_y\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial y}\right)|Du| = -h|Du|$$

according to (1.8). Here φ is the angle between Du and the x-axis, an angle which can be defined by

$$Du = |Du|\exp(i\varphi) \tag{1.11}$$

(provided 2-vectors are identified with complex numbers) and which has the properties:

$$\frac{\partial\varphi}{\partial(Du)} = -h\,, \qquad \frac{\partial\varphi}{\partial(iDu)} = -k\,.$$

(ii) A key role may be played by the level lines of the first order derivatives, if the following equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0 \tag{1.12}$$

is to be discussed. Equation (1.12) means that the Gauss curvature of the graph of u vanishes at every point (equivalently, all points on the graph of u are parabolic). Hence (1.12) characterizes *developable* (nonparametric) surfaces, i.e. surfaces which are locally isometric to an euclidean plane. The following theorem is proved in textbooks of differential geometry (see [54], section 6.3; see also [58]): a developable surface is either trivial (i.e. a plane, or a cone, or a cylinder) or the ruled surface spanned by the tangent straight lines to some saddle curve. Below a short proof of this theorem is sketched, which comes from a simple inspection of equation (1.12).

Let u be a smooth nontrivial solution to (1.12) and look at the level lines $p(x, y) = \text{const}$ and $q(x, y) = \text{const}$ of the first order derivatives $p = u_x$ and $q = u_y$. First claim: *the level lines of p coincide with those of q.* In fact equation (1.12) can be rewritten as

$$\left(-p_y \frac{\partial}{\partial x} + p_x \frac{\partial}{\partial y}\right) q = 0\,,$$

hence reads as follows: q is annihilated by the derivative along a vector field which is orthogonal to the gradient of p. Thus q must be constant along the lines where p is constant, in other words the level lines of p are level lines of q too. The converse being also true, the first claim is proved (clearly, we have here rephrased nothing but the vanishing of the jacobian of the hodograph map $(x, y) \to Du(x, y) = \big(p(x, y), q(x, y)\big)$ associated with a solution of (1.12)). Second claim: *the level lines of p are straight lines.* In fact, as we know from the preceeding paragraph, the curvature of a line $p(x, y) = \text{const}$ is $c = -\operatorname{div}\big(Dp/|Dp|\big)$. Hence

$$c = |Dp|^{-3}\left[-u_{xx}\frac{\partial}{\partial x}\,(u_{xx}u_{yy} - u_{xy}^2) + u_{xxx}(u_{xx}u_{yy} - u_{xy}^2)\right],$$

so that $c = 0$ because of (1.12). Third claim: *the graph of u is a ruled surface.* More precisely, u is linear along the level lines of p. This fact follows trivially from claims 1 and 2 and from the chain rule. Thus we have proved that the graph of any solution to equation (1.12) is a ruled surface. The most significant part of our proof ends here. To conclude one should prove that the rulings of a ruled (non-flat, non-conical, non-cylindrical) surface, whose Gauss curvature vanishes, are all tangent to the line of striction (the terminology here is as in [46]). This is a routine exercise and will be omitted. ∎

We are interested in global a priori bounds for solutions to *Dirichlet* problems for equation (1.1). Thus a basic ingredient is

(1.13a) $\quad G =$ *an open bounded subset of the euclidean plane*

and the solutions we are concerned with are real-valued functions (from suitable function classes, to be specified later) verifying equation (1.1) in G and having prescribed values on the boundary ∂G of G. Although our method works when the boundary datum is any bounded function, for the sake of simplicity we shall restrict ourselves to the following boundary condition:

$$u = 0 \quad \text{on } \partial G . \tag{1.14}$$

We assume ellipticity and conditions on the right-hand side that ensure uniqueness. As is well-known and easy to see, the elliptic solutions to equation (1.1) are precisely those which are convex or concave, and uniqueness of smooth convex (or concave) solutions—constrained by Dirichlet boundary conditions—holds if $H(x, y, u)$ is an increasing (or decreasing) function of the last argument. Accordingly, we look for solutions u which fulfil the following requirement:

(1.15) $\quad u$ *is concave*

and we assume the decrease of $u \to H(x, y, u)$. The less demanding condition:

$$\big(H(x, y, u) - H(x, y, 0)\big)\, u \leqslant 0 \tag{1.16a}$$

will suffice. Clearly, (1.15) and boundary condition (1.14) guarantee that *our solutions are positive* and oblige us to assume:

(1.13b) $\quad G$ *is convex*

$$H(x, y, u) \geqslant 0 . \tag{1.16b}$$

It is of some interest to notice that most of our results continue to hold even if the ellipticity condition (1.15) is replaced by a weaker one, namely

(1.17a) $\quad u$ *is quasi-concave.*

Quasi-concave and quasi-convex functions have been estensively studied in connection with optimization problems and mathematical programming.

A real-valued function u is said quasi-concave if its domain is a convex subset of an euclidean space and if the inequality $u\big(\lambda z_1 + (1-\lambda) z_2\big) \geqslant \min\big(u(z_1), u(z_2)\big)$ holds for all λ in $[0,1]$ and for all z_1, z_2 in the domain of u. Equivalently, *a function u is quasi-concave if its level sets $\{z\colon u(z) > t\}$ all are convex.* It can be shown that a twice continuously differentiable function u of two real variables x and y (defined in an open convex region of the euclidean plane) is quasi-concave if and only if

$$\text{(1.17}b\text{)}\qquad \begin{vmatrix} 0 & u_x & u_y \\ u_x & u_{xx} & u_{xy} \\ u_y & u_{xy} & u_{yy} \end{vmatrix} \equiv 2u_{xy}u_xu_y - u_{xx}u_y^2 - u_{yy}u_x^2 \geqslant 0\,,$$

i.e. the curvature (1.9) of the level lines is positive. See [51] [55] for presentations of quasi-convex functions. Formula (1.6) may confirm that (1.17) is an appropriate ellipticity condition for equation (1.1). It will be clear from the sequel, where the divergence structure of equation (1.1) is deeply exploited, that (1.17) is just an analogue of that weak ellipticity condition —an ad hoc ellipticity condition for quasilinear equations of the form: $\operatorname{div} a(x_1, \ldots, x_n, u, Du) + H(x_1, \ldots, x_n, u, Du) = 0$—which only demands the positivity of $a(x_1, \ldots, x_n, u, Du) \times Du$ on the solutions u under examination.

Concerning the gradient-dependent term in equation (1.1), we make the following hypotheses:

$$\text{(1.18}a\text{)}\qquad a(r) \geqslant 0\,,$$

as we have already said; and

$$\text{(1.18}b\text{)}\qquad \int_0 r a(r)\, dr \qquad \text{converges}\,,$$

a restriction on the behaviour of $a(r)$ as $r \to 0$; furthermore we assume either

$$\text{(1.19)}\qquad r^3 a(r) \qquad \text{increases}$$

or

$$\text{(1.20)}\qquad r\int_0^r t a(t)\, dt \geqslant \quad \text{a convex nonnegative function of } r\,.$$

Clearly, (1.19) is more stringent than (1.20) since (1.19) is equivalent to the convexity of the left-hand side of (1.20). For example, equation (1.2) satisfies (1.19), while equation (1.3) satisfies (1.20) since

$$r\int_0^r ta(t)\,dt = r^3/2(1+r^2) \geqslant r^3/2(1+r)^2 \qquad \text{convex}\,,$$

if $a(r) = (1+r^2)^{-2}$. ■

Our main results are summarized below. For the sake of simplicity, only results on the standard Monge-Ampère equation (1.2) are mentioned here. Results on the more general equation (1.1) are discussed in subsequent sections.

Let G be a domain as in (1.13), let H be as in (1.16) and suppose further

$$\text{(1.21}a\text{)} \qquad \int_G H(x, y, 0)\,dx\,dy \qquad \text{converges}\,.$$

Without loss of generality, we assume

$$\text{(1.21}b\text{)} \qquad \int_G H(x, y, 0)\,dx\,dy = 1\,.$$

Let u verify equation (1.2) in G, as well as the boundary condition (1.14) and either condition (1.15) or condition (1.17). As a precaution, one may assume that u is twice continuously differentiable; if the full concavity condition (1.15) is taken for granted, then generalized solutions in the sense of Bakel'man-Aleksandrov are allowed, see section **3** for details.

Our argument is based on the consideration of

$$\text{(1.22)} \qquad \{(x, y) \in G\colon u(x, y) > t\}\,,$$

the level sets of the solution u. More specifically, a key role is played by

$$\text{(1.23)} \qquad \lambda(t) = \text{perimeter of the level set (1.22)}.$$

Note that the level sets (1.22) are *convex* open subsets of G and have positive distance from the boundary of G if $t > 0$, by virtue of our hypotheses. Moreover $\lambda(t)$ is a *decreasing* function of t, varying from $\lambda(0) =$ perimeter of G to 0 as t runs from 0 to $\max u$; indeed, if A and B are regions such that A is convex and $A \subseteq B$, then perimeter of $A \leqslant$ perimeter of B, see [48] for example.

The following differential inequality for the function $\lambda(t)$ is proved:

$$(1.24) \qquad 1 \leqslant (-\lambda'(t))\left[\frac{1}{4\pi^3}\int_0^{\lambda(t)^2/4\pi} H(\ ,\ ,0)^*(s)\,ds\right]^{1/2},$$

and most of our results are derived from it. In formula (1.24) $H(\ ,\ ,0)^*$ is the decreasing rearrangement in the sense of Hardy-Littlewood of the function $H(\ ,\ ,0)$; namely the unique nonnegative nonincreasing function from $[0,\infty]$ into $[0,\infty]$, which is equidistributed with $H(\ ,\ ,0)$. We refer to [52] [59] [60] concerning rearrangements of functions.

Incidentally, (1.24) and (1.21) yield at once

$$(1.25) \qquad 2\pi^{3/2}(\max u - t) \leqslant \lambda(t) \leqslant (\text{perimeter of } G) - 2\pi^{3/2}t\,,$$

an *estimate of the length of level lines* of solutions to Monge-Ampère equations. The simplicity of this result is perhaps due, among other things, to the convexity of the relevant solutions. Available estimates of the length of level lines of solutions look much more complicated for Laplace's equation [30] or other linear elliptic equations [53]; for example, Gerver proved that, if u is harmonic and bounded in a disc, then the length-function $t \to$ length of $\{(x, y) \in \text{a smaller disc}: u(x, y) = t\}$ is equidistributed with a function of t which grows not faster than the square of a logarithm.

Inequality (1.24) is derived directly from equation (1.2) via simple arguments of differential geometry. The classical isoperimetric inequality —which allows one to relate the perimeter and area of level sets (1.22), hence $\lambda(t)$ (1.23) with the standard distribution function of u—helps to infer from (1.24) the following conclusions. Let $\|\ \|$ be any Luxemburg-Zaanen norm [56], and consider $\|u\|$ as a functional of domain G and right-hand side H. Here u is the solution under estimation. Let G vary in the class of all bounded convex domains having a fixed perimeter, and let H vary in the class of all measurable functions such that (1.6) (1.21) hold and the distribution function of $H(\ ,\ ,0)$ is fixed. *Claim*: *$\|u\|$ attains its maximum value when G is a disc and H is a circularly symmetric function independent on u, say $H(x, y, u) = f(\sqrt{x^2 + y^2})$; the same statement holds if $\|u\|$ is replaced by $\int_G |Du|\,dx\,dy$, the total variation of u.*

A more precise form of this theorem is the following pair of inequalities

$$(1.26a) \qquad u^*(s) \leqslant \frac{1}{2\pi}\int_s^{L^2/4\pi}\left(\frac{1}{r}\int_0^r H(\ ,\ ,0)^*(t)\,dt\right)^{1/2} dr\,,$$

$$\int_G |Du|\,dx\,dy \leqslant \int_0^{L^2/4\pi} \left(\frac{1}{\pi}\int_0^r H(\ ,\ ,0)^*(t)\,dt\right)^{1/2} dr\,, \tag{1.26b}$$

together with the assertion that equality holds under the circumstances described above. In (1.26) stars stand for decreasing rearrangements and

$$L = \text{perimeter of } G\,. \tag{1.27}$$

As an application, let us quote the estimates

$$\int_G |Du|\,dx\,dy \leqslant \frac{1}{4}\pi^{-3/2}L^2$$

$$\|u\|_{L^p(G)} \leqslant \frac{L^{1+2/p}}{2\pi^{3/2}[2\pi(p+1)(p+2)]^{1/p}} \qquad (1 \leqslant p \leqslant \infty)$$

which easily come from (1.26) under the normalization (1.21*b*). We point out that *these estimates are sharp.* In particular

$$\max u \leqslant \tfrac{1}{2}\pi^{-3/2}L\,. \tag{1.28}$$

The geometry of the ground domain (1.13) is involved in the theorem above, and in all consequences of it, through the length (1.27) of the boundary. The question arises: do sharp estimates of the solution to the problem in question hold, which involve that geometry through quantities other than perimeter? For example, Bakel'man [9] proved a sharp estimate of the maximum of solutions to Monge-Ampère equations—even slightly more general than (1.1)—in terms of diameter of the domain. The following inequality

$$(\max u)^2 \leqslant \frac{4}{27}\,(\text{area } G) \tag{1.29}$$

is a partial answer to the proposed question. In (1.29), u is any solution to problem (1.2) (1.13) (1.14) (1.16) (1.21); note that we are able to prove (1.29) only if u is fully concave, as in (1.15). Inequalities (1.28) and (1.29) can be summarized with the following theorem.

Let G be any bounded open convex domain, and let

$$j(G) = \inf\Big\{(\max u)^{-2}\int_G (\text{hessian of } u)\,dx\,dy: \quad u \text{ is concave in } G,\ u = 0 \text{ on } \partial G\Big\}; \tag{1.30}$$

then: (i) $j(G)\times$(perimeter of $G)\geqslant 4\pi^3$, and equality holds if G is a disc; (ii) $27/4\leqslant j(G)\times$(area of $G)\leqslant \pi^2$, equality holds on the right-hand side if G is any ellipse, equality holds at the left if G is any triangle.

2. – On the divergence structure of equation (1.1).

The following lemma is the starting point of our method.

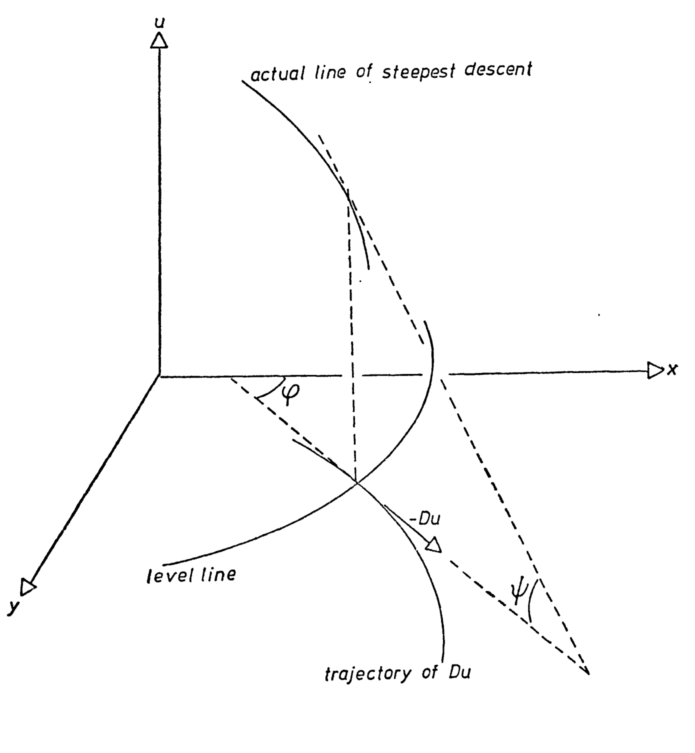
LEMMA 2.1. *Let $0\leqslant r\to a(r)$ be a real-valued measurable function such that $\int\limits_0 ra(r)\,dr$ converges, and set:*

$$b(r)=\int\limits_0^r ta(t)\,dt\,. \tag{2.1}$$

The following equality holds:

$$a(|Du|)\begin{vmatrix} u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{vmatrix}=\operatorname{div}\left\{\frac{b(|Du|)}{|Du|^2}\begin{pmatrix} u_{yy} & -u_{xy}\\ -u_{xy} & u_{xx}\end{pmatrix}Du\right\}, \tag{2.2}$$

where u is any smooth real-valued function of two real variables.



For example, lemma 2.1 enables us to write the standard Monge-Ampère equation (1.2) in the following form:

$$\frac{1}{2}\operatorname{div}\left\{\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} Du\right\} = H(x, y, u);$$

while the equation (1.3) for surfaces with prescribed Gauss curvature can be written this way:

$$\frac{1}{2}\operatorname{div}\left\{\frac{1}{1+|Du|^2}\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} Du\right\} = H(x, y, u) .$$

Obviously, a proof of lemma 2.1 may consist of an inspection of the right-hand side of formula (2.2). A more interesting proof, which perhaps shows how formula (2.2) could be discovered, goes as follows. Consider the Gauss map from the graph of u into the (upper half of the) unit sphere and consider the geographical coordinates φ ($=$ longitude, $0 \leqslant \varphi \leqslant 2\pi$) and ψ ($=$ colatitude, $0 \leqslant \psi < \pi/2$) of the image points. In other words, set:

$$-\frac{p}{\sqrt{1+p^2+q^2}} = \cos\varphi\sin\psi , \qquad -\frac{q}{\sqrt{1+p^2+q^2}} = \sin\varphi\sin\psi ,$$
$$\frac{1}{\sqrt{1+p^2+q^2}} = \cos\psi ,$$

where $p = u_x$ and $q = u_y$. Equivalently:

$$\text{(2.3}a) \qquad -p = \tan\psi\cos\varphi , \qquad -q = \tan\psi\sin\varphi ,$$

or:

$$\text{(2.3}b) \qquad \sqrt{p^2+q^2} = \tan\psi , \qquad -Du/|Du| = e^{i\varphi}.$$

Thus φ stands for the direction of the gradient and ψ is the slope, with respect to a horizontal plane, of the actual lines of steepest descent, i.e. the (saddle) lines which lie on the graph of u and whose vertical projections on the x, y plane are the trajectories of Du.

From (2.3) one infers

$$\text{(2.4)} \qquad d\varphi = \frac{q\,dp - p\,dq}{p^2+q^2} , \qquad d\psi = \frac{p\,dp + q\,dq}{(1+p^2+q^2)\sqrt{p^2+q^2}} ,$$

hence

$$\text{(2.5)} \qquad d\varphi \wedge d\psi = \frac{\cos^2\psi}{\tan\psi}\, dp \wedge dq ,$$

where $\wedge$ indicates the exterior product of differential forms. Since

$$a(\tan\psi)\tan\psi\,\frac{1}{\cos^2\psi}\,d\varphi\wedge d\psi = d\varphi\wedge db(\tan\psi)$$

because of (2.1) and since $d\varphi\wedge db(\tan\psi) = -\,d\big(b(\tan\psi)\,d\varphi\big)$, we have from (2.5):

$$a(|Du|)\,dp\wedge dq = -\,d\big(b(\tan\psi)\,d\varphi\big)\,. \tag{2.6}$$

As

$$dp\wedge dq = \begin{vmatrix} p_x & q_x \\ p_y & q_y \end{vmatrix} dx\wedge dy$$

and since $d\varphi$ has the expression given in (2.4), (2.6) yields

$$a(|Du|)\begin{vmatrix} p_x & q_x \\ p_y & q_y \end{vmatrix}\,\mathrm{d}x\wedge dy = d\left\{\frac{b(|Du|)}{|Du|^2}\begin{vmatrix} p & q \\ dp & dq \end{vmatrix}\right\}, \tag{2.7}$$

an equivalent form of (2.2).

We refer to [63] concerning further identities, which represent the invariants (i.e. the elementary symmetric functions of the eigenvalues) of the hessian matrix (of a function of n variables) in the form of a divergence.

3. – Main results.

In this section we develop a method for obtaining a priori bounds of solutions to equation (1.1). We first present our method and then state a theorem.

Let u satisfy equation (1.1) in a region G, and boundary condition (1.14). Suppose that hypotheses (1.13) on the region, and hypotheses (1.16) (1.18) (1.20) (1.21) on the equation, hold. Suppose moreover that an ellipticity condition holds, namely u is concave (1.15). It will be clear from the sequel that our method works even if (1.15) is replaced by the less demanding quasi-concavity condition (1.17).

We emphasize that in this paper only a priori bounds are considered and no existence theorem is discussed. Thus we assume the existence of a solution with the desired properties and we limit ourselves to prove estimates of this solution.

The solution in question is automatically continuous, positive and bounded, because of (1.15) (in imposing the boundary condition (1.14) we have tacitly assumed that u is continuous up to the boundary). For simplicity, we suppose that u is twice continuously differentiable; we explain later how this restriction could be relaxed. Although it is not really necessary, we assume occasionally that u is strictly concave, i.e. the smallest eigenvalue of $-D^2u$ (= the negative of the hessian matrix) is bounded from below by a positive constant E. Our final results do not depend on the constant E: consequently, the last hypothesis is made for convenience only and can be eventually dropped by using an approximation argument. ■

The basic ingredients and tools of our proof are:

(i) The distribution function of u, i.e.

$$\mu(t) = \text{area of } \{(x, y) \in G\colon u(x, y) > t\}\,, \tag{3.1}$$

and the rearrangements u^*, $u^\star$ of u. Formal definitions of these rearrangements are

$$u^*(s) = \inf\{t \geqslant 0\colon \mu(t) < s\} \tag{3.2a}$$

$$u^\star(x, y) = u^*\big(\pi(x^2 + y^2)\big); \tag{3.3a}$$

equivalently:

$$u^* = \int_0^{+\infty} \mathbf{1}_{[0,\mu(t)]}\, dt \tag{3.2b}$$

$$u^\star = \int_0^{+\infty} \mathbf{1}_{\{(x,y):x^2+y^2<\mu(t)/\pi\}}\, dt\,, \tag{3.3b}$$

where $\mathbf{1}$ stands for characteristic function. In other words, $u^\star$ is the circularly symmetric function equidistributed with u, namely the function whose level sets are concentric disks having the same area as the level sets of u. An analogous characterization of u^* holds. Recall that any (positive integrable) function is the superimposition of the characteristic functions of its level sets, as the following formula

$$u = \int_0^{+\infty} \mathbf{1}_{\{(x,y)\in G:u(x,y)>t\}}\, dt \tag{3.4}$$

(where the integral is Bochner's) shows.

(ii) The perimeter $\lambda(t)$ of the level sets of u, or

$$\lambda(t) = \text{length of } \{(x, y) \in \bar{G}\colon u(x, y) = t\}\,. \tag{3.5}$$

Note that, thanks to our hypotheses, the set at the right of (3.5) is a smooth convex curve and actually is the boundary of the level set $\{(x, y) \in G\colon u(x, y) > t\}$, provided $0 \leqslant t < \max u$. In fact u has exactly one maximum point in G and Du cannot vanish away from this point. The last remark will be frequently used in the sequel; a quantitive form of it is the following inequality:

$$|Du(x, y)|^2 \geqslant 2E(\max u - u(x, y)) \tag{3.6a}$$

where E is any constant such that

$$-\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \geqslant E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6b}$$

(iii) The curvature k of the level lines of u. An expression of k is given in (1.9); for convenience, we rewrite it as follows:

$$k = -\,|Du|^{-3}\left(\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} Du,\, Du\right). \tag{3.7}$$

Obviously

$$k \geqslant 0\,,$$

for u is concave (or quasi-concave).

(iv) The isoperimetric inequality

$$\mu(t) \leqslant \frac{1}{4\pi}\,\lambda(t)^2, \tag{3.8}$$

connecting area and perimeter of the level sets of u. We refer to [61] for an exhaustive survey of isoperimetric theorems.

(v) The following special (and elementary) case of Umlaufsatz of H. Hopf (or Gauss-Bonnet theorem):

$$\int\limits_{u(x,y)=t} k\sqrt{dx^2 + dy^2} = 2\pi\,. \tag{3.9}$$

Formula (3.9), where k is given by (3.7), expresses the geometrically obvious fact that the winding number of a closed convex curve is exactly 1; see [46], for example.

(vi) The following formula for the derivative of length-function (3.5):

$$-\lambda'(t) = \int_{u(x,y)=t} \frac{k}{|Du|} \sqrt{dx^2 + dy^2} \qquad (0 < t < \max u)\,. \tag{3.10}$$

(vii) Lemma 2.1.

PROOF OF (vi). Let φ be any test function from $C_0^1(]0, \max u[)$. Federer's coarea formula [49] tells us

$$\int_0^{\max u} dt \int_{u(x,y)=t} \varphi'(u) \sqrt{dx^2 + dy^2} = \int_G \varphi'(u)|Du|\, dx\, dy\,,$$

hence we have

$$\int_0^{\max u} \varphi'(t)\, \lambda(t)\, dt = \int_G \frac{Du}{|Du|} \times D\varphi(u)\, dx\, dy\,.$$

A partial integration transforms the right-hand side into

$$-\int_G \varphi(u)\, \operatorname{div}\left(\frac{Du}{|Du|}\right) dx\, dy\,,$$

hence applying again Federer's formula yields

$$\int_0^{\max u} \varphi'(t)\, \lambda(t)\, dt = \int_0^{\max u} \varphi(t)\, dt \int_{u(x,y)=t} \left[-\operatorname{div}\left(\frac{Du}{|Du|}\right)\right] \frac{\sqrt{dx^2 + dy^2}}{|Du|}\,,$$

an equivalent form of (3.10) (compare with (1.9)).

We point out that $\lambda(t)$ is *Lipschitz continuous*, as the same proof above shows. Incidentally, from (3.6) and (3.10) we have:

$$0 \leqslant -\lambda'(t) \leqslant \sqrt{2}\pi\big(E(\max u - t)\big)^{-1/2}.$$

For the sake of completeness, let us quote the following proposition: distribution function (3.1) is Lipschitz continuous and the estimate:

$0 \leqslant \mu(t) - \mu(t+h) \leqslant (2\pi/E)\,h$ holds. A proof comes immediately from the inequality $k|Du| \geqslant E$, and Federer's formula together with equation (3.9), which yield

$$\int\limits_{t<u(x,y)\leqslant t+h} k|Du|\,dx\,dy = \int\limits_t^{t+h} ds \int\limits_{u(x,y)=s} k\sqrt{dx^2+dy^2} = \int\limits_t^{t+h} 2\pi\,ds\,. \qquad \blacksquare$$

Having declared the rules of our game, we look now at the equation (1.1) and we integrate both sides over the level set

$$\{(x, y) \in G\colon u(x, y) > t\}\,.$$

Here

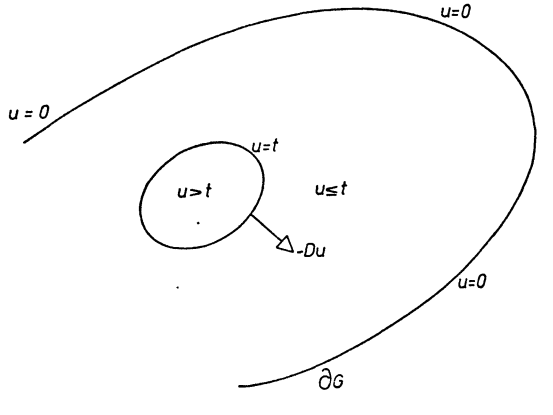
$$0 \leqslant t < \max u\,.$$

As we have already said, the boundary of this set is the smooth convex curve

$$\{(x, y) \in \bar{G}\colon u(x, y) = t\}$$

and the inner normal to the same boundary is exactly

$$\frac{Du}{|Du|}\,.$$



Then lemma 2.1 and equation (3.7) give at once

$$(3.11) \qquad \int\limits_{u(x,y)=t} b(|Du|)\,k\sqrt{dx^2+dy^2} = \int\limits_{u(x,y)>t} H(x, y, u)\,dx\,dy\,,$$

since the critical points give no contribution. Here

$$b(r) = \int_0^r ta(t)\,dt\,. \tag{3.12a}$$ ■

Now we estimate both sides of equation (3.11).

(viii) The monotonicity (1.16*a*) of H and the positivity of u tell us that the right-hand side of (3.11) does not exceed

$$\int_{u(x,y)>t} H(x, y, 0)\,dx\,dy\,.$$

A well-known theorem by Hardy and Littlewood on rearrangements of functions [52] [59] ensures that the latter integral is estimated by

$$\int_0^{\mu(t)} H(\ ,\ , 0)^*(s)\,ds\,.$$

Finally we obtain

$$\int_{u(x,y)>t} H(x, y, u)\,dx\,dy \leqslant \int_0^{\lambda(t)^2/4\pi} H(\ ,\ , 0)^*(s)\,ds \tag{3.13}$$

via the isoperimetric inequality (3.8).

(ix) In order to estimate from below the left-hand side of (3.11) we need the coerciveness assumption (1.20), namely

$$r\int_0^r ta(t)\,dt \equiv rb(r) \geqslant C(r)\,, \tag{3.12b}$$

where

(3.12*c*) $C(r)$ is some *convex nonnegative* function.

Note that one can take $C(r) = rb(r)$ if (1.19) holds; for

$$rd^2\big(rb(r)\big)/dr^2 = d\big(r^3 a(r)\big)/dr\,.$$

In any case, $C(r)$ can be defined as the greatest convex nonnegative minorant of $rb(r)$. Obviously (3.12*b*, *c*) imply $C(0+) = C'(0+) = 0$ and $C'(r) > 0$;

thus

$$C(r) = r\int_0^r tA(t)\,dt\,, \tag{3.12d}$$

where $A(r)$ is a nonnegative function such that

$$\frac{d}{dr}\,(r^3A(r))\geqslant 0\,. \tag{3.12e}$$

Jensen's inequality for convex functions gives

$$\frac{\int\limits_{u(x,y)=t} C(|Du|)\,k|Du|^{-1}\sqrt{dx^2+dy^2}}{\int\limits_{u(x,y)=t} k|Du|^{-1}\sqrt{dx^2+dy^2}} \geqslant C\left(\frac{\int\limits_{u(x,y)=t} k\sqrt{dx^2+dy^2}}{\int\limits_{u(x,y)=t} k|Du|^{-1}\sqrt{dx^2+dy^2}}\right).$$

hence

$$\int\limits_{u(x,y)=t} b(|Du|)\sqrt{dx^2+dy^2}\geqslant 2\pi B\left(\frac{2\pi}{-\lambda'(t)}\right) \tag{3.14}$$

because of (3.12b), (3.10) and (3.9). Here

$$B(r)=\frac{1}{r}\,C(r)\equiv\int_0^r tA(t)\,dt \tag{3.15}$$

is a *monotone* function which *increases* from $B(0+)=0$ to $B(+\infty)=$ $=\int_0^{+\infty} tA(t)\,dt$, as is easily seen from (3.12). ■

Coupling (3.13) with (3.14) gives

$$B\left(\frac{2\pi}{-\lambda'(t)}\right)\leqslant\frac{1}{2\pi}\int_0^{\lambda(t)^2/4\pi} H(\;,\;,0)^*(s)\,ds \tag{3.16}$$

and this inequality can be written as

$$1\leqslant\frac{-\lambda'(t)}{2\pi}\,B^{-1}\left(\frac{1}{2\pi}\int_0^{\lambda(t)^2/4\pi} H(\;,\;,0)^*(s)\,ds\right), \tag{3.17}$$

provided the following *compatibility condition* holds:

$$\int_G H(x,y,0)\,dx\,dy<2\pi B(+\infty)\,. \tag{3.18a}$$

Condition (3.18a) is merely a repetition of (1.21a) if the right-hand side is $+\infty$; otherwise, it ensures that the right-hand side of (3.16) is in the range of B, for

$$\int_G H(x, y, 0)\, dx\, dy = \int_0^{+\infty} H(\ ,\ , 0)^*(s)\, ds\ .$$

Condition (3.18a) allows us to go ahead; it will be clear from the sequel that (3.18a) is *necessary* for the consistency of the results we have in mind. Note that (3.18a) implies

$$\text{(3.18}b\text{)} \qquad \int_G H(x, y, 0)\, dx\, dy < 2\pi \int_0^{+\infty} r a(r)\, dr\ .$$

Integrating both sides of (3.17) between 0 and t gives

$$\text{(3.19)} \qquad t \leqslant \int_{\lambda(t)^2/4\pi}^{L^2/4\pi} B^{-1}\left(\frac{1}{2\pi}\int_0^s H(\ ,\ , 0)^*(r)\, dr\right)\frac{ds}{\sqrt{4\pi s}}$$

after straightforward changes of variables on the right. These changes of variables are legitimate since $\lambda(t)$ is absolutely continuous. Here $L = \lambda(0)$, that is:

$$\text{(3.20)} \qquad L = \text{perimeter of } G.$$

Inequality (3.19), isoperimetric inequality (3.8) and the definitions (3.2) (3.3) of rearrangements imply

$$\text{(3.21}a\text{)} \qquad u^\star(x, y) \leqslant v(x, y)\ ,$$

where v is defined by

$$\text{(3.21}b\text{)} \qquad v(x, y) = \int_{\pi(x^2+y^2)}^{L^2/4\pi} B^{-1}\left(\frac{1}{2\pi}\int_0^s H(\ ,\ , 0)^*(r)\, dr\right)\frac{ds}{\sqrt{4\pi s}}\ . \qquad \blacksquare$$

Inequality (3.21) is a basic result. It can be summarized and interpreted as in the following theorem.

THEOREM 3.1. *Let u be a sufficiently smooth solution to the problem*

$$\text{(3.22)} \qquad \begin{cases} a(|Du|)(u_{xx}u_{yy} - u_{xy}^2) = H(x, y, u) & \text{in } G \\ u = 0 & \text{on the boundary of } G \\ u & \text{is concave in } G. \end{cases}$$

Assume the following hypotheses:

G is (open, bounded and) convex;

$H(x, y, u) \geqslant 0\,, \qquad \big(H(x, y, u) - H(x, y, 0)\big)\, u \leqslant 0\,;$

$a(r) \geqslant 0\,, \qquad \int\limits_0 ra(r)\, dr < \infty\,;$

a nonnegative function $A(r)$ exists such that

$$\text{(3.23)} \qquad \begin{cases} 0 < r \to r^3 A(r) & \text{increases}\,, \\ \int\limits_0^r ta(t)\, dt \geqslant \int\limits_0^r tA(t)\, dt & \text{for every } r > 0\,, \\ \int\limits_G H(x, y, 0)\, dx\, dy < 2\pi \int\limits_0^{+\infty} rA(r)\, dr\,. \end{cases}$$

Moreover, set:

(3.24a) $\qquad G^{\star} =$ the disk having the same perimeter L as G,

and call v the solution to the following problem:

$$\text{(3.24}b\text{)} \qquad \begin{cases} A(|Dv|)(v_{xx} v_{yy} - v_{xy}^2) = H(\;,\;,0)^{\bigstar}(x, y) & \text{in } G^{\star} \\ v = 0 & \text{on the boundary of } G^{\star} \\ v & \text{is concave and continuous in the closure of } G^{\star}. \end{cases}$$

Conclusions:

$$\text{(3.25)} \qquad u^{\bigstar} \leqslant v\,;$$

$$\text{(3.26)} \qquad \int\limits_G |Du|\, dx\, dy \leqslant \int\limits_{G^{\star}} |Dv|\, dx\, dy\,.$$

PROOF OF (3.25). This statement is exactly the inequality (3.21).

In fact problem (3.24) has a circularly symmetric solution; for the domain is a disk, the differential operator commutes with rotations and the right-hand side

$$H(\;,\;,0)^{\bigstar}(x, y) = H(\;,\;,0)^*(\pi r^2)$$

is a function of $r = \sqrt{x^2 + y^2}$ only.

Let v be such a solution. Then the equation for v becomes (compare with (1.7)):

$$A\left(-\frac{\partial v}{\partial r}\right)\frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial^2 v}{\partial r^2}=H(\ ,\ ,0)^*(\pi r^2) \tag{3.27a}$$

provided the concavity of v is taken into account. Thus we have an ordinary differential equation for determining v, namely:

$$\frac{\partial}{\partial r}B\left(-\frac{\partial v}{\partial r}\right)=rH(\ ,\ ,0)^*(\pi r^2) \tag{3.27b}$$

where B is given by

$$B(r)=\int_0^r tA(t)\,dt\,. \tag{3.28}$$

Note that conditions (3.12) are all satisfied. In fact the function B, defined by (3.28), satisfies (3.15), i.e.

$$C(r)=rB(r) \qquad \text{is convex,}$$

thanks to our hypothesis (3.23*a*).

On the other hand, problem (3.24) cannot have more than one solution. Then the solution v, we are looking for, is obtained just by solving equation (3.27); and has exactly the expression (3.21*b*).

The above argument also shows: *compatibility condition* (3.23*c*) (compare with (3.18)) *is necessary and sufficient for the* (*maximizing*) *problem* (3.24) *to have a solution.*

PROOF OF (3.26). Formula (3.21*b*) yields

$$|Dv(x,y)|=B^{-1}\left(\frac{1}{2\pi}\int_0^{\pi(x^2+y^2)}H(\ ,\ ,0)^*(s)\,ds\right). \tag{3.21c}$$

On the other hand, from Federer's coarea formula and inequality (3.17) we have

$$\int_G |Du|\,dx\,dy=\int_0^{\max u}\lambda(t)\,dt\leqslant\int_0^{\max u}B^{-1}\left(\frac{1}{2\pi}\int_0^{\lambda(t)^2/4\pi}H(\ ,\ ,0)^*(s)\,ds\right)\frac{\lambda(t)}{2\pi}\,(-d\lambda(t))=$$
$$=2\pi\int^{L/2\pi}B^{-1}\left(\frac{1}{2\pi}\int_0^{\pi r^2}H(\ ,\ ,0)^*(s)\,ds\right)r\,dr\,.$$

The last quantity equals

$$\int_{G^{\star}} |Dv|\, dx\, dy ,$$

because of (3.21c) and the definition (3.24a) of $G^{\star}$. ∎

REMARK. Estimates of the maximum of a function u, which is concave in a (convex) domain G and vanishes on ∂G, are equivalent to estimates of the total variation $\int_G |Du|\, dx\, dy$ of u. In fact, *the following inequality holds*:

$$1 \leqslant \frac{\max u}{\int_G |Du|\, dx\, dy} \times (\text{perimeter of } G) \leqslant 2 , \tag{3.29}$$

provided u and G are as above. A proof of (3.29) comes from Federer's formula

$$\int_G |Du|\, dx\, dy = \int_0^{\max u} \lambda(t)\, dt ,$$

where $\lambda(t)$ has the meaning (3.5). Indeed, the convexity of the level sets of u implies:

$$\lambda(t) \leqslant \text{perimeter of } G ,$$

namely the left part of (3.29). Note that this part of (3.29) remains valid even if u is only quasi-concave. On the other hand, $u \geqslant U$ since u is concave; here U indicates the Minkowski function defined this way: the graph of U is the cone which projects ∂G from the highest point of the graph of u. An elementary argument shows that $(1 - t/\max u) \times (\text{perimeter of } G)$ is the length of the level line $U(x, y) = t$; hence

$$\lambda(t) \geqslant (1 - t/\max u) \times (\text{perimeter of } G) ,$$

so that the right part of (3.29) also follows.

Incidentally, let us insert in (3.29) the following function: $u(x, y) =$ $=$ distance of (x, y) from the complementary set of G. It is easy to see that u is concave if G is convex. On the other hand, $|Du(x, y)| = 1$ almost everywhere in G; moreover the maximum of u is the so-called inradius of G, that is the radius of the largest disk contained in G. Thus we have derived from (3.24) a new proof of the following well-known [44] proposition: *the inequality*

$$1 \leqslant \frac{\text{inradius} \times \text{perimeter}}{\text{area}} \leqslant 2 \tag{3.30}$$

holds for any bounded convex two dimensional region. ∎

THEOREM 3.2. *Let the situation be as in theorem* 3.1. *Then the estimate*

$$(3.31)\qquad \int_G P(|Du|)\, k\, dx\, dy \leqslant \int_{G^{\star}} P(|Dv|)(x^2+y^2)^{-1/2}\, dx\, dy$$

holds. Here k is the curvature (1.9) (3.7) *of the level lines of u and $P(r)$ is any function such that*

$$(3.32)\qquad P(0+) = P'(0+) = 0\,, \quad P'(r) > 0\,, \quad r\frac{P''(r)}{P'(r)} \leqslant \frac{(r^3 A(r))'}{\left(r\int_0^r tA(t)\, dt\right)'}\,.$$

Equality holds in (3.31) *if G is a disk, u is circularly symmetric and $a(r) = A(r)$.*

For example, theorem 3.2 (together with formula (3.21c)) enables one to obtain the following estimate:

$$(3.33)\qquad \int_G |Du|^m k\, dx\, dy \leqslant \int_0^{L^2/4\pi} \left[\frac{1}{\pi}\int_0^r H(\ ,\ , 0)^*(s)\, ds\right]^{m/2} \sqrt{\frac{\pi}{r}}\, dr\,,$$

provided the equation for u is (1.2) and $1 < m \leqslant 3$. In particular, under the same circumstances we have:

$$(3.34)\qquad \int_G |Du|^m k\, dx\, dy \leqslant (\text{perim. of } G)\left[\frac{1}{\pi}\int_G H(x, y, 0)\, dx\, dy\right]^{m/2},$$

a sharp estimate, as are all in this section.

PROOF OF THEOREM 3.2. The conditions on $P(r)$ at $r = 0$ guarantee that $P(|Du|)\, k/|Du|$ is integrable on the whole of G. Then we can use Federer's coarea formula and we get

$$(3.35)\qquad \int_G P(|Du|)\, k\, dx\, dy = \int_0^{\max u} dt \int_{u(x,y)=t} P(|Du|)\frac{k}{|Du|}\sqrt{dx^2+dy^2}.$$

Define a function Φ with the following rule:

$$(3.36)\qquad \Phi(s) = rB(r) \quad \text{if} \quad s = P(r) \quad \text{and} \quad 0 < r < +\infty\,,$$

where $B(r)$ is as in (3.28). As is easy to see, $\Phi(s)$ is an *increasing convex* function of s, precisely because of our assumptions (3.32). Thus we have the following chain of inequalities:

$$\Phi\left(\frac{\int_{u(x,y)=t} P(|Du|)\,k|Du|^{-1}\sqrt{dx^2+dy^2}}{\int_{u(x,y)=t} k|Du|^{-1}\sqrt{dx^2+dy^2}}\right)$$

$$\leqslant \frac{\int_{u(x,y)=t} \Phi\big(P(|Du|)\big)\,k|Du|^{-1}\sqrt{dx^2+dy^2}}{\text{same denominator}}$$

(Jensen's inequality)

$$=\frac{-1}{\lambda'(t)}\int_{u(x,y)=t} B(|Du|)\,k\sqrt{dx^2+dy^2}$$

(formulas (3.10) and (3.36))

$$\leqslant\frac{-1}{\lambda'(t)}\int_{u(x,y)=t} b(|Du|)\,k\sqrt{dx^2+dy^2}$$

(hypothesis (3.23b) and formulas (3.12a) (3.28))

$$\leqslant \square B^{-1}(\square)\,, \qquad \square=\frac{1}{2\pi}\int_0^{\lambda(t)^2/4\pi} H(\ ,\ ,0)^*(s)\,ds$$

(inequalities (3.11) (3.13) (3.17)).

Thanks to the definition (3.36) (and the monotonicity) of Φ, the obtained result can be rewritten in the form:

$$\int_{u(x,y)=t} P(|Du|)\,\frac{k}{|Du|}\sqrt{dx^2+dy^2} \leqslant (-\lambda'(t))\,P\left(B^{-1}\left(\frac{1}{2\pi}\int_0^{\lambda(t)^2/4\pi} H(\ ,\ ,0)^*(s)\,ds\right)\right), \tag{3.37}$$

provided (3.10) is used once more. From (3.35) and (3.37) we get

$$\int_G P(|Du|)\,k\,dx\,dy \leqslant 2\pi\int_0^{L/2\pi} P\left(B^{-1}\left(\frac{1}{2\pi}\int_0^{\pi r^2} H(\ ,\ ,0)^*(s)\,ds\right)\right)dr\,, \tag{3.38}$$

namely the desired inequality (3.31), via formula (3.21c). ∎

REMARK. Consider the Dirichlet problem (1.2) (1.13) (1.14) (1.15) (1.16) for the standard Monge-Ampère equation. Theorem 3.1 enables us to obtain sharp estimates of the following form:

(3.39) a Luxemburg-Zaanen norm of $u \leqslant$ a function of these arguments only: perimeter of G, $\int_G H(x, y, 0)\, dx\, dy$;

provided u is a smooth solution. We claim that these estimates continue to hold even if any smoothness assumption on u is dropped and only the concavity and the continuity of u on $\bar{G}$ are retained: in other words, estimates of the type (3.39) are valid for *generalized solutions* of the Monge-Ampère equation (1.2). This claim is basically a consequence of a smoothing procedure and the following monotonicity property of the generalized hessian: if u, v are concave in G, continuous in $\bar{G}$ and $u = v$ on ∂G, then $u \geqslant v$ implies

$$\text{total variation of } \operatorname{hess} u \geqslant \text{total variation of } \operatorname{hess} v .$$

This property comes easily from the definition of generalized hessian. Recall that $\operatorname{hess} u$, *the generalized hessian* of a concave (or convex) function u on an open set G, is the measure (actually a countably additive measure, as one could prove) whose value on any Borel subset E of G is

$$(\operatorname{hess} u)(E) = \text{Lebesgue area of } \bigcup_{(x,y)\in E} (Du)(x, y) . \tag{3.40}$$

Here Du is a generalized (set-valued) gradient, namely $(Du)(x, y)$ is the set of all 2-vectors (p, q) such that the plane through the point $\big(x, y, u(x, y)\big)$, and normal to the direction $(-p, -q, 1)$, is supporting (i.e. above) the graph of u. We refer to [1] [3] [12] concerning properties of the generalized hessian and existence theorems of generalized solutions to Monge-Ampère equations.

4. – Surfaces with prescribed Gauss curvature.

Applications of theorem 3.1 to the standard Monge-Ampère equation (1.2) are sketched in the introduction. In this section we apply theorems 3.1 and 3.2 to equation (1.3), the equation for surfaces with prescribed Gauss curvature. Our results on this equation can be summarized as follows.

THEOREM 4.1. *Let u be a* (smooth) *solution to the problem*

$$(4.1)\qquad \begin{cases} (1+u_x^2+u_y^2)^{-2}(u_{xx}u_{yy}-u_{xy}^2)=H(x,y,u) & \text{in } G\\ u=0 & \text{on the boundary of } G\\ u & \text{is concave in } G\,, \end{cases}$$

and assume the following hypotheses:

$$(4.2)\qquad \begin{cases} G & \text{is open, bounded and convex;}\\ H(x,y,u)\geqslant 0\,, & \big(H(x,y,u)-H(x,y,0)\big)\,u\leqslant 0;\\ \int\limits_G H(x,y,0)\,dx\,dy<\pi\,. \end{cases}$$

The following estimates hold:

$$(4.3)\qquad u^\star\leqslant v$$

$$(4.4)\qquad \int\limits_G |Du|\,dx\,dy\leqslant\int\limits_{G^{\star}}|Dv|\,dx\,dy$$

$$(4.5)\qquad \int\limits_G |Du|^3(1+|Du|)^{-2}k\,dx\,dy\leqslant\int\limits_{G^{\star}}|Dv|^3(1+|Dv|)^{-2}\frac{dx\,dy}{\sqrt{x^2+y^2}}\,.$$

Here:

$$(4.6a)\qquad G^{\star}=\text{the disk having the same perimeter } (L,\text{ say}) \text{ as } G$$

and v is the solution to the following problem:

$$(4.6b)\qquad \begin{cases} (1+|Dv|)^{-3}(v_{xx}v_{yy}-v_{xy}^2)=H(\;,\;,0)^\star(x,y) & \text{in } G^{\star}\\ v=0 & \text{on the boundary of } G^{\star}\\ v & \text{is concave and continuous in the closure of } G^{\star}. \end{cases}$$

Concerning a proof of theorem 4.1, it is enough to observe that problem (4.1) satisfies all the hypotheses of theorem 3.1 if (4.2) holds. In particular, the crucial condition (3.23) is fulfilled with the following choice:

$$(4.7)\qquad A(r)=(1+r)^{-3}.$$

Note that an *explicit representation* of v is:

$$(4.8)\qquad v(x,y)=\int_{\pi(x^2+y^2)}^{L^2/4\pi}\frac{\sqrt{\frac{1}{\pi}\int_0^r H(\ ,\ ,0)^*(s)\,ds}}{1-\sqrt{\frac{1}{\pi}\int_0^r H(\ ,\ ,0)^*(s)\,ds}}\frac{dr}{\sqrt{4\pi r}},$$

hence *explicit a priori estimates* can be derived from (4.3) (4.4) (4.5) via formula (4.8). For example:

$$(4.9)\qquad \max u\leqslant Lc/2\pi(1-c)$$

$$(4.10)\qquad \int_G u\,dx\,dy\leqslant L^3c/24\pi^2(1-c)$$

$$(4.11)\qquad \int_G |Du|\,dx\,dy\leqslant L^2c/4\pi(1-c)$$

$$(4.12)\qquad \int_G |Du|^3(1+|Du|)^{-2}k\,dx\,dy\leqslant Lc^3/(1-c)\,,$$

where:

$$(4.13)\qquad c^2=(1/\pi)\int_G H(x,y,0)\,dx\,dy$$

and $L=$ perimeter of G. We recall that the object k, appearing in formulas (4.5) and (4.12), is the curvature (1.9) of the level lines of u. ■

THEOREM 4.2. *Let u be a smooth concave function on a convex domain G, and let u vanish on the boundary of G. Set*

$$(4.14)\qquad K=(1+u_x^2+u_y^2)^{-2}(u_{xx}u_{yy}-u_{xy}^2)\,,$$

the Gauss curvature of the graph of u. The following inequalities hold:

$$(4.15)\qquad \int_G u\,dx\,dy\leqslant(L^3/24\pi^2)\sqrt{c(2-c)}/(1-c)$$

$$(4.16)\qquad \int_G |Du|\,dx\,dy\leqslant(L^2/4\pi)\sqrt{c(2-c)}/(1-c)\,,$$

where $L=$ length of ∂G (i.e. the length of the boundary of the graph of u) and

$$(4.17)\qquad 2\pi c=\int_G K\sqrt{1+|Du|^2}\,dx\,dy$$

is the total curvature of the graph of u.

Note that the inequality

$$0 < c < 1 \tag{4.18}$$

is automatically true. For $2\pi c$ is the area of the image of the graph of u under the Gauss map, and this image is a part of the upper half of the unit sphere.

Inequalities (4.15) and (4.16) are isoperimetric-type theorems, connecting the volume of the ipograph, or the total variation of u, with the perimeter of the graph. Note that (4.16) yields

$$\int_G \sqrt{1+|Du|^2}\, dx\, dy \equiv \text{area of the graph} \leqslant (L^2/4\pi)[1+\sqrt{c(2-c)}/(1-c)]\,, \tag{4.19}$$

an isoperimetric inequality for a two-dimensional manifold with positive Gauss curvature.

Proof of theorem 4.2. Rewrite formula (4.14) as

$$(1+u_x^2+u_y^2)^{-3/2}\,(u_{xx}u_{yy}-u_{xy}^2) = f(x,y)\,, \tag{4.20a}$$

$$f = K(1+u_x^2+u_y^2)^{1/2}, \tag{4.20b}$$

and apply theorem 3.1 to equation (4.20). All the hypotheses of theorem 3.1 are satisfied, for

$$a(r) = (1+r^2)^{-3/2}$$

is such that:

$$r^3\,a(r) \qquad \text{increases}$$

and

$$\int_0^{+\infty} ra(r)\, dr = 1 > (1/2\pi)\int_G f(x,y)\, dx\, dy$$

because of (4.18). Hence (4.15) and (4.16) are straightforward consequences of (3.25) and (3.26), via the same arguments we have used for the proof of theorem 4.1. ■

Remark. The a priori bounds we have listed in theorem 4.1, and all corollaries of these bounds, are obtained by solving problem (4.6). The differential operator involved in the latter problem is different from (and in a sense worse than) that which is involved in the original problem. As the differential operator appearing in (4.1) is invariant under rotations, the question arises: can the solution u to problem (4.1) be estimated in terms of circularly symmetric solutions to the following equation

$$(1+v_x^2+v_y^2)^{-2}(v_{xx}v_{yy}-v_{xy}^2)=H(\ ,\ ,0)^{\star}, \tag{4.21}$$

where the differential operator is retained unchanged?

In this connection, let us mention the following curious result.

THEOREM 4.3. *Let u be a smooth solution to problem* (4.1). *Assume hypotheses* (4.2), *assume further the following a priori bound*:

$$\max|Du|\leqslant\sqrt{3}\,. \tag{4.22}$$

Then inequalities (4.3) (4.4) *hold, if v is the circularly symmetric concave solution to equation* (4.21) *which vanishes on the boundary of the disk* (4.6*a*).

The proof of this theorem is exactly the same as that of theorem 3.1. The key fact is the following: the coefficient $a(r)=(1+r^2)^{-2}$ is such that $r^3a(r)$ increases for $0\leqslant r\leqslant\sqrt{3}$ (and decreases for $r\geqslant\sqrt{3}$); in other words

$$r\int_0^r ta(t)\,dt=r^3/2(1+r^2) \qquad \text{is convex for } 0\leqslant r\leqslant\sqrt{3}\,. \tag{4.23}$$

As is easy to see, property (4.23) and assumption (4.22) guarantee that all the arguments of section **3** still work.

Incidentally, similar phenomena occur when problems, which involve the *mean* curvature, are considered. The following theorem can be proved with the technique of [65].

THEOREM 4.4. *Let u be a* (sufficiently nice) *solution to the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{u_{x_i}}{\sqrt{1+|Du|^2}}=H(x,u) & \text{in } G\\ u=0 & \text{on the boundary of } G, \end{cases} \tag{4.24}$$

where G is any domain (*with finite measure*) *in n-dimensional euclidean space R^n and the right-hand side H satisfies the conditions*

$$\begin{cases} \big(H(x,u)-H(x,0)\big)\,u\leqslant 0\,,\\ \displaystyle\int_G|H(x,0)|^n\,dx<(n\sqrt{\pi})^n/\Gamma(1+n/2)\,. \end{cases} \tag{4.25}$$

Assume the following a priori bound:

$$\max |Du| \leqslant \sqrt{2}\,. \tag{4.26}$$

Then

$$u^\star \leqslant v\,, \tag{4.27}$$

where v is the (*spherically symmetric*) *solution to the problem*

$$\begin{cases} -\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i} \dfrac{v_{x_i}}{\sqrt{1+|Dv|^2}} = H(\ ,0)^\star & \text{in } G^\star \\ v = 0 & \text{on the boundary of } G^\star, \end{cases} \tag{4.28}$$

and $G^\star$ is the ball having the same measure as G. If condition (4.26) *is dropped, then* (4.27) *is still true provided* (4.28) *is replaced by*

$$\begin{cases} -\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i} \dfrac{v_{x_i}}{1+|Dv|} = H(\ ,0)^\star & \text{in } G^\star \\ v = 0 & \text{on the boundary of } G^\star. \end{cases} \tag{4.29}$$

5. – On a theorem by N. V. Efimov.

In [47, section 77] Efimov observed that, if a convex body is given in 3-space, then the diameter of the body and the Gauss curvature of its boundary cannot be simultaneously large. More precisely, Efimov proved the following: Suppose that u is a concave function on a convex region G and u vanishes on ∂G, call K the Gauss curvature of the graph of u; then $(\min K)(\max u)(\operatorname{diam} G) \leqslant 2\pi$. Below we present a theorem, which generalizes (and improves) this inequality. Such a theorem shows that, if the coefficient $a(r)$ decays fast enough as $r \to +\infty$, then both the maximum of concave (or convex) solutions to equation (1.1) and the size of the domain can be estimated in terms of a *lower bound* of the right-hand side.

THEOREM 5.1. *Suppose that*

$$\alpha = \int_0^{+\infty} r a(r)\, dr\,, \qquad \beta = \int_0^{+\infty} r^2 a(r)\, dr \tag{5.1}$$

are finite, and suppose

$$H(x, y, u) \geqslant \lambda = \text{a positive constant}\,. \tag{5.2}$$

If u is a (smooth) solution to problem (1.1) (1.13) (1.14) (1.15), *then both*

$$\tfrac{1}{2}(\text{perimeter of } G)(\max u) \tag{5.3}$$

and

$$[(\text{area } G)((\text{area } G) + \pi(\max u)^2)]^{1/2} \tag{5.4}$$

do not exceed

$$(2\pi/\lambda)[\alpha^{2/3} + \beta^{2/3}]^{3/2}. \tag{5.5}$$

PROOF. By lemma 1.1, we can rewrite equation (1.1) in the form

$$H(x, y, u)\sqrt{1 + |Du|^2} = \operatorname{div}\left\{\frac{c(|Du|)}{|Du|^2}\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx}\end{pmatrix} Du\right\}, \tag{5.6}$$

where:

$$c(r) = \int_0^r t\sqrt{1 + t^2}\, a(t)\, dt\,. \tag{5.7}$$

Integrating both sides of (5.6) over G gives

$$\int_G H(x, y, u)\sqrt{1 + |Du|^2}\, dx\, dy = \int_{\partial G} c(|Du|)\, k\sqrt{dx^2 + dy^2}, \tag{5.8}$$

since ∂G is exactly the level line $u = 0$ and the critical points of u give no contribution. Thus k is the curvature of the boundary, and

$$\int_{\partial G} k\sqrt{dx^2 + dy^2} = 2\pi \tag{5.9}$$

because of Umlaufsatz and the convexity of G. From (5.8) (5.9) and (5.7) (5.2) one infers

$$\int_G \sqrt{1 + |Du|^2}\, dx\, dy \leqslant (2\pi/\lambda)\int_0^{+\infty} r\sqrt{1 + r^2}\; a(r)\, dr\,. \tag{5.10}$$

The conclusion follows from lemma 5.2 below and the inequality

$$\int_0^\infty r\sqrt{1 + r^2}\, a(r)\, dr \leqslant \left[\left(\int_0^\infty r a(r)\, dr\right)^{2/3} + \left(\int_0^\infty r^2\, a(r)\, dr\right)^{2/3}\right]^{3/2}. \tag{5.11}$$

LEMMA 5.2. *If u is any concave function on a convex region G and u vanishes on ∂G, then* (5.3) *and* (5.4) *do not exceed the area of the graph of u.*

PROOF. Let Γ be the cone which projects ∂G from the highest point of the graph of u. By the concavity of u, Γ bounds a *convex subset* of the ipograph of u. Hence

$$\text{area graph}\, u \geqslant \text{area } \Gamma\,. \tag{5.12}$$

On the other hand, it is easily seen that

$$\text{area } \Gamma \geqslant \tfrac{1}{2} hL\,, \tag{5.13}$$

where $h = \max u$ is the height of Γ and L is the perimeter of G.
Furthermore, a theorem on Steiner symmetrization tells us

$$\text{area } \Gamma \geqslant [A(A + \pi h^2)]^{1/2}, \tag{5.14}$$

where A is the area of G. In fact the right side of (5.14) is the area of the right circular cone which has the same height as Γ and is erected on a disk with area A; this cone is an image of Γ under a Steiner symmetrization.
The lemma is proved.

6. – Perimeter or area?

In this section we prove

$$(\max u)^2 \leqslant \frac{4}{27}\,(\text{area } G) \int_G H(x, y, 0)\, dx\, dy\,, \tag{6.1}$$

a sharp bound for concave solutions to the standard Monge-Ampère equation (1.2). Inequality (6.1) is a corollary of theorem 6.1 below and holds for smooth or generalized solutions to problem (1.2) (1.13) (1.14) (1.16). The difference between (6.1) and the estimates, which can be derived from theorem 3.1, is that (6.1) involves the geometry of the ground domain through the content only and ignores whims of the boundary.

THEOREM 6.1. *Let G be an open bounded convex* (2-dimensional) *domain and let $j(G)$ be defined as in* (1.30). *Then*

$$\frac{27}{4} \leqslant j(G) \times (\text{area } G) \leqslant \pi^2. \tag{6.2}$$

Equality holds at the right if G is an ellipse, equality holds at the left if G is a triangle.

Note that $27/4 = 6.75$ and $\pi^2 = 9.8696...$, so that $j(G)$ does not differ very much from a numerical multiple of (area $G)^{-1}$. Note also that $j(G) \times \times$(area G) is invariant under affine transformations (i.e. linear changes of coordinate = translations + rotations + dilatations), a property which shows little sensitivity of $j(G) \times$(area G) to the shape of G. Incidentally, theorem 6.1 settles the two-dimensional case of a problem by Aleksandrov [40, section 10].

A proof of theorem 6.1 runs as follows.

STEP 1. Set

$$J(z_0, G) = \inf\{u(z_0)^{-2}(\operatorname{hess} u)(G) \colon u \text{ is concave in } G, \; u \text{ is continuous in } \bar{G} \text{ and } u = 0 \text{ on } \partial G\} \tag{6.3}$$

for any convex domain G and any point z_0 in G. Here $(\operatorname{hess} u)(G)$ is $\int_G (u_{xx}u_{yy} - u_{xy}^2)\, dx\, dy$ if u is twice continuously differentiable; it is the total variation of the generalized hessian of u (see the concluding remark, section 3) if u is merely concave in G. We have

$$j(G) = \inf\{J(z_0, G) \colon z_0 \in G\}\,, \tag{6.4}$$

because of the definition (1.30) of $j(G)$ (the function class and the notion of hessian, relevant to this definition, are understood to be as in (6.3), of course). ■

Observe that the infimum in (6.3) actually is a minimum. In fact, set:

$$U(z) = 1 - \operatorname{Inf}\{\lambda > 0 \colon (z/\lambda) + (1 - 1/\lambda)z_0 \in G\}\,. \tag{6.5a}$$

In convex analysis, $1 - U$ is called a Minkowski function: the graph of U is just the cone which projects ∂G from the point $(x_0, y_0, 1)$ (here x_0 and y_0 are the coordinates of z_0). Clearly U is concave, U vanishes on ∂G and has the value 1 at z_0. Warning: *U is the smallest function having these properties.* Hence, if u is any member from the function class indicated in (6.3), we must have

$$u(z_0)^{-2}(\operatorname{hess} u)(G) \geqslant (\operatorname{hess} U)(G)$$

thanks to the monotonicity property of the generalized hessian (see section3). We conclude that

$$J(z_0, G) = \text{total variation of hess } U\,. \quad ■ \tag{6.5b}$$

By the way, a more careful analysis could lead to the equation

$$\text{hess}\, U = J(z_0, G)\,\delta(\cdot - z_0)\,, \tag{6.5c}$$

where $\delta(\cdot - z_0)$ denotes the Dirac mass concentrated at z_0. This equation is perhaps the most comprehensive way of saying that the graph of U is a developable surface with a crucial singularity at a point.

STEP 2. *Suppose that G has a smooth boundary.* Then:

$$J(z_0, G) = \frac{1}{2}\int_{\partial G} \frac{k}{(n(z), z_0 - z)^2}\,|dz| \tag{6.6a}$$

where:

$$k = \text{curvature of } \partial G \ \ (\geqslant 0) \tag{6.6b}$$

$$n(z) = \text{the inner normal to } \partial G \text{ at } z\,. \tag{6.6c}$$

Here $|dz|$ is a shorthand for $\sqrt{dx^2 + dy^2}$, and we use brackets to denote the scalar product between vectors.

Formula (6.6) is a consequence of (6.5). For, if the boundary is smooth, the function U is continuously differentiable out of z_0 and the value of the (conventional) gradient of U at a point $z \neq z_0$ is

$$DU(z) = n(w)/\big(n(w), z_0 - w\big)\,,$$

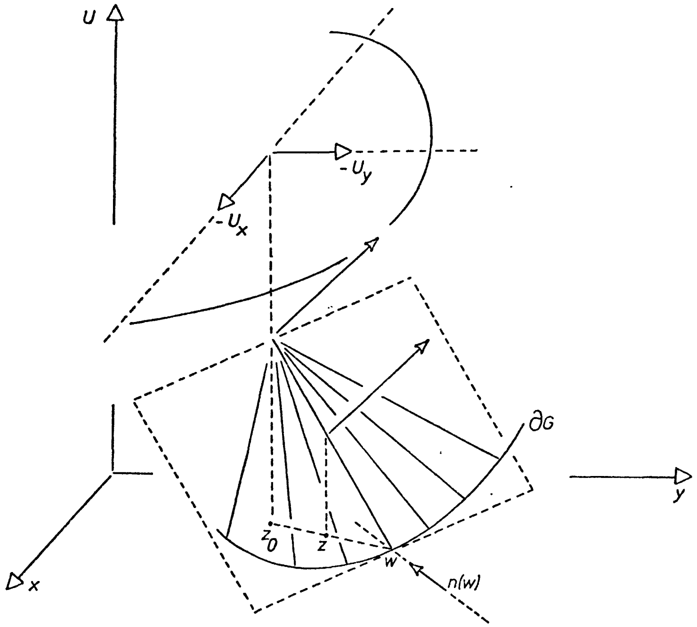
w being the point where the half-axis from z_0 towards z meets the boundary. Notice that the denominator is the distance between z_0 and the tangent straight-line to ∂G at w. On the other hand

$$DU(z_0)\,,$$

the value at z_0 of the generalized gradient of U, is the region (a convex region, as one may see) bounded by the curve

$$\partial G \ni w \to n(w)/\big(n(w), z_0 - w\big)\,.$$

The area of this region is precisely the total mass of the generalized hessian of U. Computing this area with a contour integral gives (6.6). ∎



An alternative argument, which would lead to (6.6), is the following. Let u be a smooth concave function, which vanishes on the boundary of G. From lemma 2.1 and formula (1.9) we have

$$\int_G (u_{xx} u_{yy} - u_{xy}^2)\, dx\, dy = \frac{1}{2} \int_{\partial G} |Du|^2 k |dz| ,$$

since ∂G is exactly the level line $u = 0$. For the same reason

$$Du(z) = |Du(z)|\, n(z) \qquad \text{at every } z \in \partial G ,$$

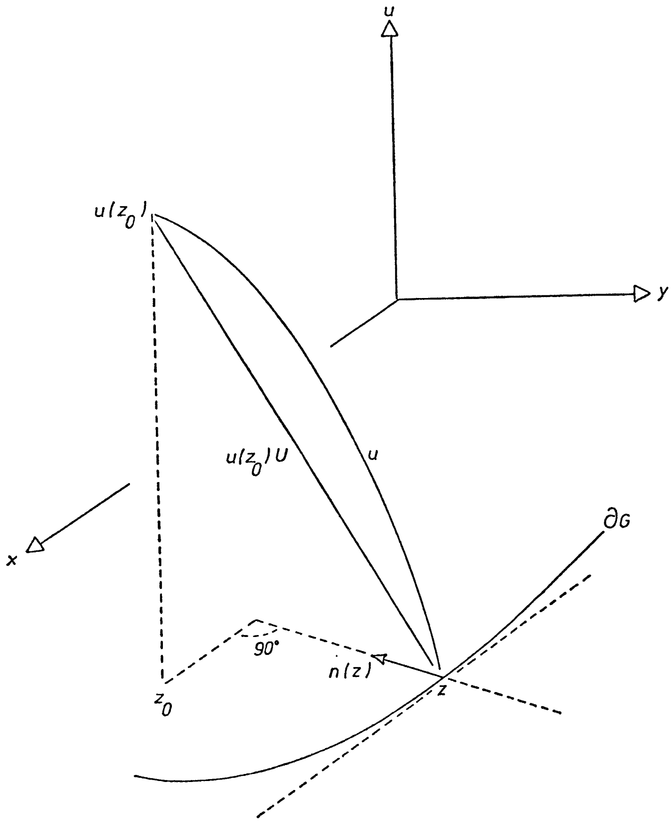
while the concavity of u implies

$$u(z_0) \leqslant \big(Du(z), z_0 - z\big) \qquad \text{provided } z \in \partial G .$$

Putting together the three formulas above we obtain

$$\int_G (u_{xx} u_{yy} - u_{xy}^2)\, dx\, dy \geqslant u(z_0)^2 \frac{1}{2} \int_{\partial G} \frac{k}{\big(n(z), z_0 - z\big)^2} |dz| ,$$

namely (6.6a) with the equality sign replaced by $\geqslant$. ■



For example, formula (6.6) gives

$$J(x_0, y_0; G) = \frac{\pi}{ab}\left[1 - \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right]^{-3/2}, \tag{6.7a}$$

if G is the ellipse $(x/a)^2 + (y/b)^2 < 1$. Hence we obtain via (6.4):

$$j(\text{ellipse}) = \pi^2/\text{area}\,. \tag{6.7b}$$

STEP 3. *Let G be a (convex) polygon with n sides.* We number counterclockwise the vertices of G and we denote them by $z_1, z_2, \dots, z_n$ ($z_{n+1} = z_1$, $z_{n+2} = z_2$). Then

$$J(z_0, G) = \frac{1}{4}\sum_{k=1}^{n} \frac{P_k}{A_k A_{k+1}}, \tag{6.8a}$$

where

$$P_k = \frac{1}{2}\begin{vmatrix} x_k & y_k & 1 \\ x_{k+1} & y_{k+1} & 1 \\ x_{k+2} & y_{k+2} & 1 \end{vmatrix} \tag{6.8b}$$

$= \text{area of the triangle with vertices at } z_k,\ z_{k+1},\ z_{k+2}$,

$$A_k = \frac{1}{2}\begin{vmatrix} x_0 & y_0 & 1 \\ x_k & y_k & 1 \\ x_{k+1} & y_{k+1} & 1 \end{vmatrix} \tag{6.8c}$$

$= \text{area of the triangle with vertices at } z_0,\ z_k,\ z_{k+1}$,

x_k and y_k being the coordinates of z_k.

Formula (6.8) should be considered a discretization of (6.6). It follows easily from equation (6.5). For, in the present situation, the function U, defined by (6.5*a*), is piecewise linear and has the values

$$U(x, y) = \frac{1}{2A_k}\begin{vmatrix} x & y & 1 \\ x_k & y_k & 1 \\ x_{k+1} & y_{k+1} & 1 \end{vmatrix}$$

inside the triangle with vertices at z_0, z_k, z_{k+1}. Hence the image of our polygon G under the generalized gradient of U fills the polygon whose vertices have coordinates

$$p_k = \frac{1}{2A_k}(y_k - y_{k+1}), \qquad q_k = \frac{1}{2A_k}(x_{k+1} - x_k).$$

The area of this new polygon is what we need to compute, namely the total mass of the generalized hessian of U. This area is

$$\sum_{k=1}^{n} \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ p_k & q_k & 1 \\ p_{k+1} & q_{k+1} & 1 \end{vmatrix} = \frac{1}{8}\sum_{k=1}^{n} \frac{1}{A_k A_{k+1}}\begin{vmatrix} x_{k+1} - x_k & y_{k+1} - y_k \\ x_{k+2} - x_{k+1} & y_{k+2} - y_{k+1} \end{vmatrix},$$

the right-hand side of (6.8*a*). ■

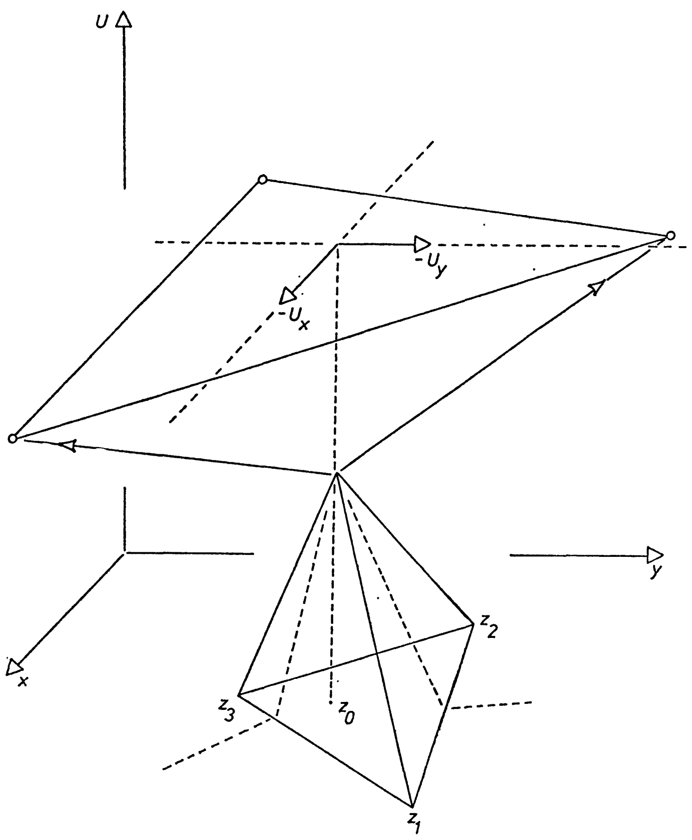
Two cases of formula (6.8) have a special interest: the first one is that of a *triangle*, the second one is that of a *regular polygon*.

Let us look at the $n = 3$ case first. *If G is a triangle*, one infers at once from (6.8):

$$J(z_0, G) = \frac{(\text{area } G)^2}{4A_1A_2A_3}, \tag{6.9a}$$

since $P_1 = P_2 = P_3 = A_1 + A_2 + A_3 =$ area G. The numbers A_1, A_2, A_3 are known as the baricentric coordinates of z_0: they are three arbitrary (and positive, if z_0 is inside the triangle) numbers whose sum is the area of the triangle. Thus the geometric-arithmetic inequality tells us that the minimum of the right side of (6.9*a*) occurs when z_0 is the *center of mass* of the triangle, the point where $A_1 = A_2 = A_3 = \frac{1}{3}$ (area G). We conclude that:

$$j \text{ (triangle)} = 27/4 \text{ (area)} . \qquad \blacksquare \tag{6.9b}$$



Suppose now that G is a regular polygon with n sides. We can place the vertices of G in the following way:

$$z_k = r \exp(2\pi i k/n) ,$$

where r is some positive number. We then have from (6.8):

$$J(z_0, G) = (\text{area } G)\left(\sin\frac{\pi}{n}\right)^2 \frac{1}{n}\sum_{k=1}^{n}\frac{1}{A_k A_{k+1}}, \tag{6.10a}$$

for:

$$P_k = 2r^2 \sin\frac{2\pi}{n}\left(\sin\frac{\pi}{n}\right)^2$$

and:

$$\text{area } G = r^2\frac{n}{2}\sin\frac{2\pi}{n}\,.$$

The geometric-arithmetic inequality implies:

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{A_k A_{k+1}} \geqslant \left(\frac{1}{A_1A_2}\,\frac{1}{A_2A_3}\cdots\frac{1}{A_nA_1}\right)^{1/n}$$
$$= (A_1A_2\ldots A_n)^{-2/n} \geqslant \left(\frac{1}{n}\sum_{k=1}^{n}A_k\right)^{-2} = n^2(\text{area } G)^{-2};$$

where equality holds if $A_1 = A_2 = \ldots = A_n$, a circumstance occurring when z_0 is the center of the polygon. Thus

$$J(z_0, G) \geqslant \left(n\sin\frac{\pi}{n}\right)^2\Big/(\text{area } G)\,,$$

and the right-hand side is just the minimum with respect to z_0 of the left side. In other words:

$$j(\text{regular polygon}) \times \text{area} = \left(n\sin\frac{\pi}{n}\right)^2. \tag{6.10b}$$

A key fact for our argument is the following:

$$\left(n\sin\frac{\pi}{n}\right)^2 \quad \textit{increases as } n \textit{ increases.}$$

STEP 4. *Let G vary in the class of all convex polygons with n sides and fixed area; then $j(G)$ is maximum if G is affine to a regular polygon, $j(G)$ is minimum if G is a triangle* (we think here of a triangle as a n-gon with $n - 3$ superfluous vertices).

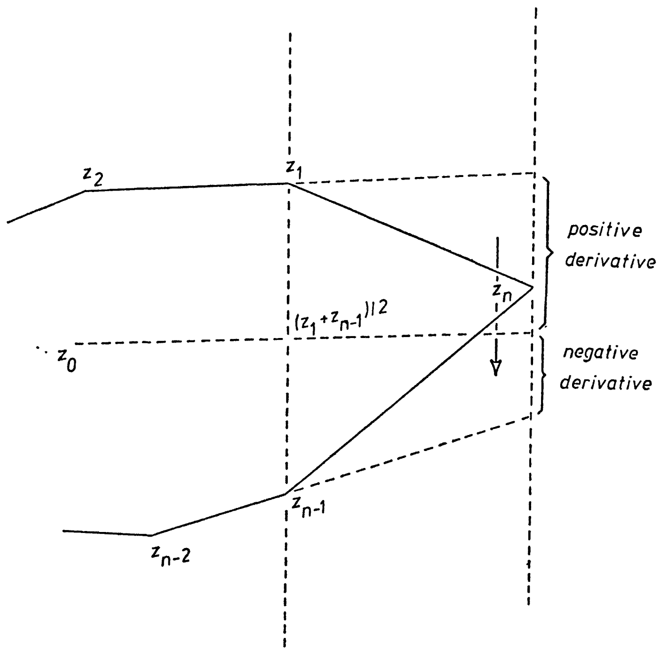
This assertion is basically a consequence of the following property. *Let $n \geqslant 4$ and let G be a convex n-gon with vertices at $z_1, z_2, \ldots, z_n$; keep $z_1, \ldots, z_{n-1}$ fixed and let z_n vary on a parallel straight-line to the chord from z_{n-1} to z_1; suppose that z_0 is in the convex hull of $z_1, \ldots, z_{n-1}$. Then $J(z_0, G)$ is minimum if z_n is on one of its extreme positions, namely on the axis joining either z_2 with z_1 or z_{n-2} with z_{n-1}. $J(z_0, G)$ is maximum if z_n is on the axis joining z_0 with the midpoint $(z_1 + z_{n-1})/2$* (provided z_n can meet this axis; otherwise the maximum occurs when z_n reaches its closest position to that axis).

For a proof, we denote $J(z_0, G)$ by $J(z_0, z_1, \ldots, z_{n-1}, z_n)$. Let $e^{i\varphi} = (z_{n-1} - z_1)/|z_{n-1} - z_1|$ be the unit vector parallel to the chord from z_{n-1} to z_1; form $J(z_0, z_1, \ldots, z_{n-1}, z_n + re^{i\varphi})$, where r is a real variable running near $r = 0$, and compute

$$\frac{d}{dr} J(z_0, z_1, \ldots, z_{n-1}, z_n + re^{i\varphi}) \qquad \text{at } r = 0\,. \tag{6.11a}$$

We omit the calculations, which are long and tedious, and we write down only the result. The derivative (6.11a) has the following value:

$$\frac{2P_{n-1}^2}{A_{n-1}^2 A_n^2 |z_{n-1} - z_1|}\, \frac{1}{2} \begin{vmatrix} x_0 & y_0 & 1 \\ \frac{1}{2}(x_1 + x_{n-1}) & \frac{1}{2}(y_1 + y_{n-1}) & 1 \\ x_n & y_n & 1 \end{vmatrix}. \tag{6.11b}$$



The first factor is positive; the second factor is the *signed* area of the triangle with vertices at z_0, $(z_1+z_{n-1})/2$, z_n. The announced proposition follows.

STEP 5. All conclusions of theorem 6.1 follow from step 3, step 4, (6.7) and (6.9). A continuity property of the functional j is involved, of course.

7. – On the Aleksandrov-Pucci maximum principle.

The following theorem is well known.

THEOREM (Aleksandrov-Pucci). *Let G be any open bounded subset of euclidean n-space R^n; then a constant C exists such that*

$$(\max|u|)^n \leqslant C\int_G |Eu|^n \frac{dx}{\det(a_{ik})}\,. \tag{7.1}$$

Here

$$E = \sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k} \tag{7.2a}$$

is a linear second order partial differential operator with measurable coefficients, satisfying the ellipticity condition:

$$\sum_{i,k=1}^{n} a_{ik}(x)\,\xi_i\xi_k > 0 \qquad \textit{for all } x \textit{ and } \xi \neq 0\,; \tag{7.2b}$$

u is any function having the properties:

$$u \in C^2(G) \cap C^0(\bar{G})\,, \qquad u(x) = 0 \textit{ for every } x \in \partial G\,. \tag{7.3}$$

Proofs of this theorem are presented in [40] and [62]. In this section we point out some estimates of the constant C appearing in (7.1). The estimates we present below are closely connected with estimates for solutions to Monge-Ampère equations and are derived in the two-dimensional case only.

Suppose $n = 2$; then the smallest constant C, for which (7.1) *holds, can be estimated as follows:*

$$C \leqslant \frac{1}{27}\ (\text{area of ch}\,G) \tag{7.4a}$$

$$C \leqslant \frac{1}{16\pi^3}\ (\text{perimeter of ch}\,G)^2. \tag{7.4b}$$

Here ch *stands for convex hull.*

The arguments of Aleksandrov and Pucci consist essentially of the following two steps.

(i) Let G as above and let a be any point in G; suppose that

$$u \in C^1(G) \cap C^0(\bar{G}) , \qquad u(x) \leqslant 0 \text{ for every } x \in \partial G,\ u(a) = 1 . \tag{7.5}$$

Then

$$(Du)(\{x \in G\colon u \text{ is concave at } x\}) \supseteq \bigcup_{x \in \mathrm{ch}\, G} (DU)(x); \tag{7.6}$$

where U is the Minkowski function defined by

$$U(x) = 1 - \inf\{\lambda > 0\colon (1/\lambda)\, x + (1 - 1/\lambda)\, a \in \mathrm{ch}\, G\} , \tag{7.7}$$

namely *U is the smallest function* which has the following properties: U is concave, U is positive in G, U has the value 1 at the point a (we say that a differentiable function u is concave at a point x if x is interior to the domain of u and the inequality $(Du(x), y - x) + u(x) \geqslant u(y)$ holds for every y from a neighbourhood of x).

The inclusion (7.6) is geometrically obvious and easy to prove. Note that the intersection of all sets at the left-hand side of (7.6) is exactly the right-hand side of (7.6).

(ii) If $u \in C^2(G)$ and (7.2) holds, then

$$\operatorname{meas}(Du)(\{x \in G\colon u \text{ is concave at } x\}) \leqslant n^{-n} \int_G |Eu|^n \frac{dx}{\det(a_{ik})} . \tag{7.8}$$

In fact

$$\operatorname{meas}(Du)(B) \leqslant \int_B |\det D^2 u|\, dx \tag{7.9}$$

for any Borel subset B of G, since $\det D^2 u$ is just the jacobian of the hodograph map $G \ni x \to (Du)(x) \in R^n$ and

$$\int_B |\det D^2 u|\, dx = \int_{(Du)(B)} \#\, (Du)^{-1}(y)\, dy .$$

On the other hand, the arithmetic-geometric inequality tells us that the inequality:

$$\det(a_{ik}(x))\, |\det D^2 u(x)| \leqslant n^{-n} |\mathrm{tr}\, (a_{ik}(x))\, D^2 u(x)|^n \tag{7.10}$$

holds at every point where the eigenvalues of the hessian matrix $D^2u(x)$ have all the same sign. As $D^2u(x)$ is negative-definite if u is concave at x, then (7.10) holds at every point where u is concave. Hence (7.9) and (7.10) yield (7.8).

Putting together (i) with (ii) gives the following result. Suppose that (7.2) and (7.3) hold; then

$$(\max |u|)^{-n}\int_G |Eu|^n \frac{dx}{\det(a_{ik})} \geqslant n^n j(\operatorname{ch} G)\,, \tag{7.11}$$

where

$$j(\operatorname{ch} G) = \inf\{J(a, \operatorname{ch} G) \colon a \in \operatorname{ch} G\}$$

and

$$J(a, \operatorname{ch} G) = \text{total mass of hess } U\,.$$

From section 6 we know that

$$\begin{aligned} j(\operatorname{ch} G) = \inf\{(\max\varphi)^{-n} \times \text{total mass of hess}\,\varphi\colon \\ \varphi \text{ is concave and continuous in } (\operatorname{ch} G)^-,\ \varphi = 0 \text{ on } \partial(\operatorname{ch} G)\}\,. \end{aligned} \tag{7.12}$$

Hence (7.4) follows via (7.11) (7.12) and theorems 3.1, 6.1.

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