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Monotonicity of the free boundary in the two-dimensional dam problem

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Monotonicity of the Free Boundary in the Two-Dimensional Dam Problem (*).

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Introduction.

Two reservoirs containing water with different levels are separated by a porous dam D: the water flowing from the first reservoir to the other wets a subset Ω of D.

The purpose of this work is to study monotonicity properties of the flow in the two-dimensional stationary case. Our conclusion, in its simplest form, is that the free boundary $D \cap \partial\Omega$ is a monotone graph provided ∂D itself satisfies suitable geometrical assumptions.

As part of our proof we obtain the result, which is interesting by itself, that the flow is tangential to the boundary of the dam at points of the seepage line.

At last we show how our technique allows us to control the number of monotone arcs of the free boundary in the case of several reservoirs.

It must be pointed out that our monotonicity assumptions on ∂D are always natural hypotheses since in the formulation of the problem no new reservoirs are allowed to be formed. It would be interesting to state the problem in which water can form such reservoirs and show monotonicity properties of the free boundary also in this case.

1. - Statement of the problem.

Let a, b, c, a_1, b_1, y_1, y_2 be real numbers and Y_0, Y_1 be two real functions satisfying the following assumptions:

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$$(1.1)\qquad \begin{cases} a_1 < a < c < b < b_1;\ y_1 > y_2 \\ Y_0,\ Y_1 \text{ are defined and } C^2 \text{ in } [a_1, b_1] \end{cases}$$

$$(1.2)\qquad \begin{cases} Y_0(a_1) = Y(a_1);\ Y_0(b_1) = Y(b_1);\ Y(a) = Y(c) = y_1;\ Y(b) = y_2 \\ Y_0(x) < Y(x) \text{ in }]a_1, b_1[;\ Y_0(x) \leqslant y_2 \text{ in } [a_1, b_1] \\ Y(x) < y_1 \text{ in } [a_1, a[;\ Y(x) < y_2 \text{ in }]b, b_1];\ Y(x) > y_1 \text{ in }]a, c[\end{cases}$$

$$(1.3)\qquad Y'(x) < 0 \quad \text{in } [c, b]\,.$$

We shall use the following notations:

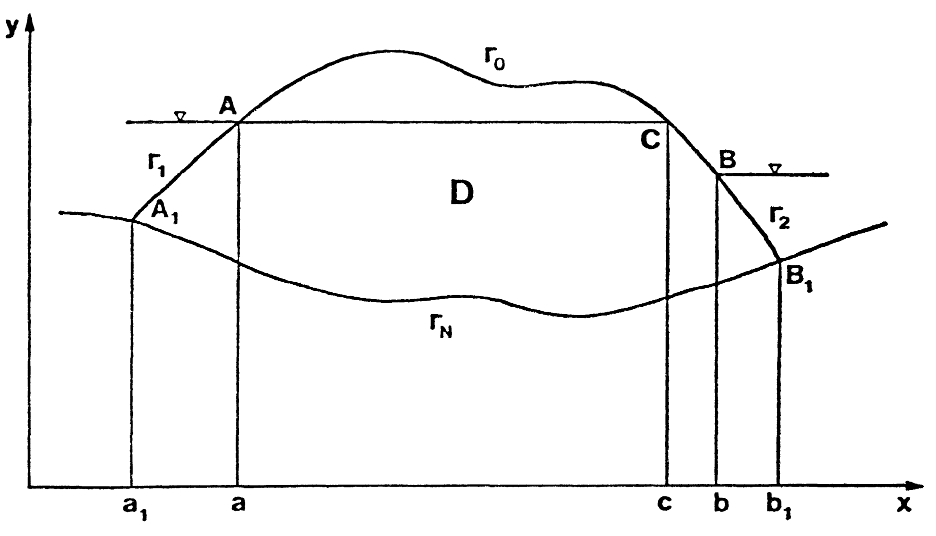
$$D = \{(x, y) \in \mathbb{R}^2\colon a_1 < x < b_1,\ Y_0(x) < y < Y(x)\}$$

$$\Gamma_N = \{(x, Y_0(x))\colon a_1 < x < b_1\};\ \Gamma_0 = \{(x, Y(x))\colon a < x < b\}$$

$$\Gamma_1 = \{(x, Y(x))\colon a_1 < x < a\};\ \Gamma_2 = \{(x, Y(x))\colon b < x < b_1\}$$

$$A_1 = (a_1, Y(a_1));\ A = (a, Y(a));\ C = (c, Y(c));\ B = (b, Y(b));\ B_1 = (b_1, Y(b_1))\,.$$

Finally ν will be the exterior normal unit vector to ∂D. ■



Consider the following problem:

PROBLEM 1. *Find a pair* $\{\varphi, p\}$ *satisfying the following conditions* (1.4)-(1.7):

$$(1.4)\qquad \begin{cases} \varphi \text{ is defined and continuous in } [a_1, b_1]; \\ \varphi(x) = Y(x) \text{ in } [a_1, a] \cup [b, b_1] \\ Y_0(x) < \varphi(x) \leqslant Y(x) \text{ and } y_2 \leqslant \varphi(x) \leqslant y_1 \text{ in } [a, b]; \end{cases}$$

setting $\Omega = \{(x, y) \in D\colon y < \varphi(x)\}$, $\Lambda = \bar{D} - \Omega$ *and* $\chi =$ *characteristic function of* Ω,

$$(1.5)\qquad \begin{cases} p \in C^0(\bar{D}) \cap H^1(D);\; p > 0 \text{ in } \Omega,\; p = 0 \text{ in } \Lambda, \\ \Delta p + \chi_y = 0 \text{ in } \mathcal{D}'(D) \end{cases}$$

$$(1.6)\qquad p = y_i - y \text{ on } \bar{\Gamma}_i \; (i = 1, 2);\; p = 0 \text{ on } \Gamma_0;$$

finally, $p_\nu = -\cos \nu y$ *on* Γ_N *and* $p_\nu \leqslant -\cos \nu y$ *on* $\Gamma_0 \cap \partial\Omega$ *in the following weak sense* (1):

$$(1.7)\qquad \begin{cases} \int_D (\nabla p \cdot \nabla v + \chi v_y) \leqslant 0 \text{ for every smooth function } v \text{ which vanishes} \\ \text{on } \Gamma_1 \cup \Gamma_2 \text{ and is non negative on } \Gamma_0. \end{cases} \quad \blacksquare$$

The graph of φ will be called free boundary.

It is well known that this problem has one solution, that φ is analytic where $\varphi(x) < Y(x)$ and $p \in C^{0,\alpha}(\bar{D})$ $\forall \alpha \in]0, 1[$ (see [1], [2]).

In this paper we prove that φ is strictly decreasing in $[a, b]$ and that $\varphi'(x)$ exists and $\varphi'(x) = Y'(x)$ at point $x \in [c, b[$ satisfying $\varphi(x) = Y(x)$ (*i.e.* $\partial\Omega$ and ∂D are tangential).

Results of this kind were known only in some particular cases for the shape of D (see *e.g.* [3], [5], [6]. For complete references about problem 1 see [4]).

We recall some known results that will be useful later:

LEMMA 1.1. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *If* $\bar{x} \in \bar{\Omega}$, $\varrho_0 > 0$, $u \in C^0(\bar{\Omega} \cap \bar{B}_{2\varrho_0}(\bar{x}))$ (2) *and* $\Delta u \geqslant 1$ *in* $\Omega \cap B_{2\varrho_0}(\bar{x})$, $u \geqslant 0$ *in* $\Omega \cap B_{2\varrho_0}(\bar{x})$, $u = 0$ *on* $\partial\Omega \cap B_{2\varrho_0}(\bar{x})$ *then*

$$(1.8)\qquad \sup_{\bar{\Omega} \cap \bar{B}_\varrho(x_0)} \{u(x) - u(x_0)\} \geqslant \frac{1}{2n}\varrho^2, \qquad \forall x_0 \in \overline{\Omega \cap B_{\varrho_0}(\bar{x})}, \qquad \forall \varrho \leqslant \varrho_0 .$$

(1) Clearly (1.7) also contains the equation $\Delta p + \chi_y = 0$.

(2) $B_\varrho(y)$ and $\bar{B}_\varrho(y)$ are respectively the open and closed ball of center y and radius ϱ.

PROOF. It is sufficient to prove (1.8) with $x_0 \in \Omega \cap B_{\varrho_0}(\bar{x})$ since the constant $1/2n$ in (1.8) does not depend on x_0. Consider $x_0 \in \Omega \cap B_{\varrho_0}(\bar{x})$ and $\varrho \leqslant \varrho_0$ and take $v(x) = u(x) - u(x_0) - (1/2n)|x - x_0|^2$. If $v \equiv 0$ (1.8) is obvious. Let us suppose $v \not\equiv 0$. We have $v \in C^0(\bar{\Omega} \cap \bar{B}_{2\varrho_0}(\bar{x}))$, $\Delta v \geqslant 0$ in $\Omega \cap B_\varrho(x_0)$; thus if y_0 is a maximum point for v in $\bar{\Omega} \cap \bar{B}_\varrho(x_0)$ we have $y_0 \in \partial(\Omega \cap B_\varrho(x_0))$. In particular $v(y_0) > 0$ since $v(x_0) = 0$. If follows that $y_0 \in \partial B_\varrho(x_0)$ since $v \leqslant 0$ on $\partial\Omega \cap B_\varrho(x_0)$. Thus $v(y_0) = u(y_0) - u(x_0) - (1/2n)\varrho^2$, and from $v(y_0) > 0$ (1.8) clearly follows. ■

LEMMA 1.2. *With the notations of problem 1, let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that*

$$[\alpha, \beta] \times \{\gamma\} \subset D\,; \qquad \{(x, Y(x)) \colon \alpha \leqslant x \leqslant \beta\} \subset \Gamma_0\,. \tag{1.9}$$

Introduce the set $\tilde{D} =]\alpha, \beta[\times]\gamma, \infty[$ and the function (3)

$$w(x, y) = \int_y^\infty p(x, t)\,dt\,, \qquad (x, y) \in \tilde{D}\,. \tag{1.10}$$

Then

$$\Delta p + \chi_y \geqslant 0 \quad \text{and} \quad \Delta w \geqslant \chi \quad \text{in } \tilde{D}\,. \tag{1.11}$$

PROOF. If $\psi \in \mathcal{D}(\tilde{D})$, $\psi \geqslant 0$, we deduce by (1.5), (1.7):

$$ {}_{\mathcal{D}'(\tilde{D})}\langle \Delta p + \chi_y, \psi \rangle_{\mathcal{D}(\tilde{D})} = -\int_{\tilde{D}} (\nabla p \cdot \nabla \psi + \chi \psi_y) = -\int_{D \cap \tilde{D}} (\nabla p \cdot \nabla \psi + \chi \psi_y) = $$
$$ = -\int_D (\nabla p \cdot \nabla \psi + \chi \psi_y) \geqslant 0 $$

that is $\Delta_p + \chi_y \geqslant 0$ in $\tilde{D}$. Thus $D_y(\Delta w - \chi) \leqslant 0$ in $\tilde{D}$. But $\Delta w - \chi = 0$ in $\tilde{D} \setminus D$. Hence (1.11) follows. ■

We deduce now the following

COROLLARY 1.1. *Let (x_0, y_0) be such that $c \leqslant x_0 < b$, $y_0 = \varphi(x_0) = Y(x_0)$ (i.e. (x_0, y_0) belongs to the seepage line).*

Then positive constants ϱ_0, c_0 exist such that

$$\sup\{p(x, y) \colon (x, y) \in B_\varrho(x_0, y_0)\} \geqslant c_0 \varrho \qquad \forall \varrho \leqslant \varrho_0\,. \tag{1.12}$$

(3) We extend p and χ to the whole of $\mathbf{R}^2$ by: $p = 0$ and $\chi = 0$ in $\mathbf{R}^2 \setminus D$. The function (1.10) has been introduced first in [3] to solve the dam problem in its simplest form and has been used later in many other cases (see e.g. [5]).

PROOF. By the previous lemmas, if ϱ_0 is small enough, we have (setting B_ϱ instead of $B_\varrho(x_0, y_0)$):

$$\sup_{B_\varrho} w \geqslant \tfrac{1}{4}\varrho^2 \qquad \forall \varrho \leqslant \varrho_0 . \tag{1.13}$$

But for some $\delta \in]0, 1[$ (4) we have

$$w(x, y) \leqslant 2\delta\varrho \cdot \sup_{B_\varrho} p \qquad \forall (x, y) \in B_{\delta\varrho}, \qquad \forall \varrho \leqslant \varrho_0 . \tag{1.14}$$

Hence, from (1.13), (1.14):

$$\tfrac{1}{4}\delta^2\varrho^2 \leqslant \sup_{B_{\delta\varrho}} w \leqslant 2\delta\varrho \cdot \sup_{B_\varrho} p \qquad \forall \varrho \leqslant \varrho_0$$

from which (1.12) follows with $c_0 = \delta/8$. ∎

2. – Behavior of the free boundary at the seepage line.

The aim of this section is to prove that $\partial\Omega$ and ∂D are tangential at the seepage line; i.e. at points $P_0 = (x, Y(x))$ such that $c \leqslant x < b$, $Y(x) = \varphi(x)$.

The basic idea is the following: the linear growth of p (given by corollary 1.1) in a neighborhood of such a point P_0 is possible only if $\partial\Omega$ and ∂D are tangential at P_0.

Our proof uses proposition 2.1, which we consider to be of independent interest and state in n-dimensions. In order not to distinguish between $n = 2$ and $n > 2$, we will use the definition of capacity of a compact subset K of an open set $R \subset \mathbb{R}^n$ as given in [7]. Here and later R will be the half ball B_1^+ (5) and the capacity of K with respect to B_1^+ will be simply denoted by cap K.

We first need some lemmas.

LEMMA 2.1. *For any $n \geqslant 2$ and $\varepsilon \in]0, \frac{1}{2}[$ there exists a constant $c_0(\varepsilon)$ such that for any function g, which is continuous and non negative in $\bar{B}_1^+$ and superharmonic in B_1^+, the following estimate holds*:

$$g(y) \geqslant c_0(\varepsilon) y_n \cdot \operatorname{cap}\{x \in B_{\frac{1}{2}}^+ : x_n \geqslant \varepsilon,\ g(x) \geqslant 1\}, \qquad \forall y \in B_{\frac{1}{2}}^+ . \tag{2.1}$$

PROOF. Let $G(x, y)$ be the Green function of B_1^+ (which could be computed explicitly from the one of B_1 by reflecting the pole). It is easy to see

(4) ϱ_0, δ depend on (x_0, y_0) and the geometry.
(5) With the notations: $B_\varrho = B_\varrho(0)$ and $B_\varrho^+ = B_\varrho \cap \boldsymbol{R}_+^n$.

that for some constant $c_1(\varepsilon)$ we have

$$G(x, y) \geqslant c_1(\varepsilon) y_n \quad \text{if } x, y \in B_{\frac{1}{2}}^+ , \ x_n \geqslant \varepsilon . \tag{2.2}$$

Let K be the set $\{x \in B_{\frac{1}{2}}^+ : x_n \geqslant \varepsilon, \ g(x) \geqslant 1\}$ with g satisfying the assumptions of the present lemma, and consider the capacitary distribution μ_K and the capacitary potential V_K of K. We have

$$\operatorname{cap} K = \mu_K(K) \tag{2.3}$$

$$V_K(y) = \int_K G(x, y) \, d\mu_K(x) \quad \forall y \in B_1^+ . \tag{2.4}$$

From (2.2)-(2.4) and the definition of K we deduce immediately

$$V_K(y) \geqslant \int_K c_1(\varepsilon) y_n \, d\mu_K(x) = c_1(\varepsilon) y_n \operatorname{cap} K \quad \forall y \in B_{\frac{1}{2}}^+ .$$

But $g(y) \geqslant V_K(y)$ in B_1^+ since g is non negative and superharmonic in B_1^+ and $g \geqslant 1$ on K.

Therefore (2.1) follows with $c_0(\varepsilon) = c_1(\varepsilon)$. ∎

REMARK 2.1. If $n = 2$ there are some relations between capacity and length of arcs. We will be interested only in circular arcs and confine ourselves to the proof of the following:

LEMMA 2.2. *There exist constants $c_1, c_2, c_3 > 0$ such that for any compact circular arc L contained in $B_{\frac{1}{2}}^+$ and having radius $\geqslant c_3$ the following inequality holds*

$$\operatorname{cap} L \geqslant c_2 |\ln (c_1 \cdot \text{length } L)|^{-1} . \tag{2.5}$$

PROOF. Notice first that the Green function G of B_1^+ satisfies

$$G(x, y) \leqslant -\ln \left(\tfrac{1}{2}|x - y|\right) \quad \forall x, y \in B_1^+ \tag{2.6}$$

since, for fixed $x \in B_1^+$, the function $g(y) = G(x, y) + \ln \left(\frac{1}{2}|x - y|\right)$ is harmonic in B_1^+ and $\leqslant 0$ on ∂B_1^+.

Consider now a compact circular arc L contained in B_1^+ and denote by a its length and by C the circle containing L. If z and R are the center and the radius of C, suppose $R \geqslant 4$ and notice that the closed ball $\bar{B}_{R/2}(z)$ does not intersect B_1^+. It will be convenient to use the following notations: for $x \in B_1^+$ define $\bar{x} \in L$ according to $|x - \bar{x}| = \min \{|x - y| : y \in L\}$; moreover for $x, y \in L$ denote by $\alpha(x, y)$ the length of the arc contained in L and having x, y as endpoints. We have for $x, y \in B_1^+$

$$\alpha(\bar{x}, \bar{y}) = 2R \arcsin \frac{|\bar{x}-\bar{y}|}{2R}.$$

Thus, choosing $\delta > 0$ such that $t^{-1} \arcsin t \leqslant 2$ for $0 < t \leqslant \delta$, we get

$$\alpha(\bar{x}, \bar{y}) \leqslant 2|\bar{x}-\bar{y}| \leqslant 4|x-y| \quad \text{if} \quad x, y \in B_1^+,\ R \geqslant 2/\delta.$$

Fix now $y \in B_1^+$ and denote by L' the arc contained in C and having length $2a$ and $\bar{y}$ as midpoint. Choosing $c_3 = \max\{4; 2/\delta\}$ and denoting by ds the differential of arc along C, we have if $R \geqslant c_3$:

$$\int_L G(x, y)\,ds(x) \leqslant -\int_L \ln\left(\tfrac{1}{2}|x-y|\right) ds(x) \leqslant -\int_L \ln\left(\tfrac{1}{8}\alpha(x, \bar{y})\right) ds(x) \leqslant$$

$$\leqslant -\int_{L'} \ln\left(\tfrac{1}{8}\alpha(x, \bar{y})\right) ds(x) = -2\int_0^a \ln(r/8)\,dr = 2a\left(1-\ln(a/8)\right) \leqslant c_2^{-1} a|\ln(a/8)|$$

for some constant $c_2 > 0$ (since clearly $a \leqslant 4$, for instance). Therefore, if we consider the uniform distribution μ on L of total mass $c_2|\ln(a/8)|^{-1}$, we get

$$\int_L G(x, y)\,d\mu(x) = c_2|\ln(a/8)|^{-1} \cdot a^{-1} \int_L G(x, y)\,ds(x) \leqslant 1 \qquad \text{for any } y \in B_1^+.$$

Hence we conclude

$$\operatorname{cap} L \geqslant \mu(L) = c_2|\ln(a/8)|^{-1} \quad \text{if } R \geqslant c_3,$$

i.e. (2.5) with $c_1 = \frac{1}{8}$. ■

From lemma 2.1 we deduce immediately

LEMMA 2.3. *For any $n \geqslant 2$ and $\varepsilon \in]0, \frac{1}{2}[$ there exists a constant $c(\varepsilon) > 0$ such that for any function v which is continuous and subharmonic in B_1^+ and satisfies $v(x) \leqslant x_n$, the following estimate holds*

$$v(y) \leqslant y_n\left(1 - c(\varepsilon) \operatorname{cap}\{x \in B_{\frac{1}{2}}^+ : x_n \geqslant \varepsilon,\ v(x) \leqslant 0\}\right) \quad \forall y \in B_{\frac{1}{2}}^+. \tag{2.7}$$

PROOF. Choosing $g(y) = \varepsilon^{-1}\left(y_n - v(y)\right)$ and applying lemma 2.1 we get (2.7) with $c(\varepsilon) = \varepsilon c_0(\varepsilon)$. ■

Now we are able to prove the following basic proposition.

PROPOSITION 2.1. *Let $\varepsilon \in]0, \frac{1}{2}[$ and $c_0 > 0$ be given constants and u a continuous function in $\bar{B}_1^+$, subharmonic in B_1^+, satisfying*

$$0 \leqslant u(x) \leqslant c_0 x_n \qquad \forall x \in B_1^+ . \tag{2.8}$$

Define

$$\gamma_k = \operatorname{cap}\{x \in B_{\frac{1}{2}}^+ : x_n \geqslant \varepsilon,\ u(2^{-k}x) = 0\}, \qquad k = 0, 1, \ldots \tag{2.9}$$

and suppose that

$$\sum \gamma_k = \infty . \tag{2.10}$$

Then

$$\lim_{x \to 0} \frac{u(x)}{|x|} = 0 . \tag{2.11}$$

PROOF. We will prove (2.11) by a recurrence argument.

By hypotesis (2.8) holds. Suppose, inductively, that for fixed $k \geqslant 0$

$$u(x) \leqslant c_k x_n \qquad \text{in } B_{2^{-k}}^+ . \tag{2.12}$$

Define $v(x) = c_k^{-1} \cdot 2^k \cdot u(x \cdot 2^{-k})$ in B_1^+ and apply lemma 2.2. Then (2.7) holds with $c(\varepsilon)$ independent of k and we have

$$c_k^{-1}\, 2^k u(x \cdot 2^{-k}) \leqslant x_n\big(1 - c(\varepsilon)\gamma_k\big) \qquad \text{in } B_{\frac{1}{2}}^+$$

i.e.

$$u(x) \leqslant c_k\big(1 - c(\varepsilon)\gamma_k\big) x_n \qquad\qquad \text{in } B_{2^{-k-1}}^+ . \tag{2.13}$$

That is we can choose $c_{k+1} = c_k\big(1 - c(\varepsilon)\gamma_k\big)$. Hence we get

$$u(x) \leqslant x_n \cdot c_0 \prod_{i=1}^{k} \big(1 - c(\varepsilon)\gamma_i\big) \qquad \text{in } B_{2^{-k}}^+,\ k = 0, 1, \ldots . \tag{2.14}$$

But, since $\ln(1 - t) \leqslant -t$, we have

$$\ln \prod_{i=1}^{k} \big(1 - c(\varepsilon)\gamma_i\big) = \sum_{i=1}^{k} \ln\big(1 - c(\varepsilon)\gamma_i\big) \leqslant -c(\varepsilon) \sum_{i=1}^{k} \gamma_i .$$

Hence if (2.10) holds the product in the right side of (2.14) tends to 0 as $k \to \infty$.

This completes the proof. ∎

In order to apply proposition 2.1 to our situation we need the following

LEMMA 2.4. *Let u be a real function which is continuous in the half ball $\bar{B}_1^+$ of $\mathbb{R}^n$, subharmonic in B_1^+, and satisfies*

$$\partial B_1 \cap \operatorname{supp} u \subset \mathbb{R}_+^n\,; \qquad u = 0 \ \text{on}\ B_1 \cap \partial \mathbb{R}_+^n. \tag{2.15}$$

Then for some $c_0 > 0$ we have

$$u(x) \leqslant c_0 x_n \quad \text{in}\ B_1^+. \tag{2.16}$$

PROOF. Define $\delta = \operatorname{dist}(\partial\mathbb{R}_n^+; \partial B_1 \cap \operatorname{supp} u)$ and $M = \sup\{u(x); x \in \bar{B}_1^+\}$. By (2.15) $\delta > 0$. Choose $c_0 = M/\delta$.

On $\partial B_1 \cap \operatorname{supp} u$ we have $u \leqslant M \leqslant c_0 x_n$; moreover $u = 0 \leqslant c_0 x_n$ elsewhere on ∂B_1^+. Therefore (2.17) follows from subharmonicity. ■

We now apply the previous results to our situation and prove the following

THEOREM 2.1. *Let $P_0 \equiv (x_0, y_0)$ be a point of the seepage line, i.e. $c \leqslant x_0 < b$, $y_0 = \varphi(x_0) = Y(x_0)$. Then*

$$\varphi'(x_0) \ \text{exists and} \quad \varphi'(x_0) = Y'(x_0). \tag{2.18}$$

PROOF. By contradiction, suppose that (2.18) does not hold. Then there exist $\mu > 0$ and a sequence $\{x_m\}$ satisfying

$$\lim x_m = x_0 \quad \text{and} \quad \varphi(x_m) \leqslant Y(x_m) - \mu|x_m - x_0|. \tag{2.19}$$

Notice that the segments $S_m = \{x_m\} \times [\varphi(x_m), Y(x_m)]$ do not intersect Ω.

By lemma 1.2, p is subharmonic in some neighbourhood of P_0. Since Y is C^2, there exist $R > 0$ and $Q_0 \in \mathbb{R}^2$ such that the ball $B = B_R(Q_0)$ does not intersect D and is tangential to ∂D at P_0. Define the inversion with respect to ∂B by means of

$$\hat{Q} = F(Q) = Q_0 + R^2|Q - Q_0|^{-2}(Q - Q_0), \qquad Q \neq Q_0 \tag{2.20}$$

and set

$$\hat{D} = F(D) \quad \text{and} \quad \hat{\Omega} = F(\Omega). \tag{2.21}$$

Define now the Kelvin transform of p:

$$\hat{p}(\hat{Q}) = p(Q). \tag{2.22}$$

It is well known that subharmonicity is preserved under this transformation.

Changing coordinates and units, we may assume that $P_0 = O$, $\hat{\Omega} \subset \hat{D} \subset \subset B_1^+ \subset \mathbb{R}_+^2$ and

(2.23) $\hat{p}$ is continuous in $\bar{B}_1^+$ and subharmonic in B_1^+.

From lemma 2.4 we get, for some $c_0 > 0$

$$\hat{p}(x, y) \leqslant c_0 y \quad \text{in } \bar{B}_1^+ . \tag{2.24}$$

In order to apply proposition 2.1 to the function $\hat{p}$, consider the circular arcs $\hat{S}_m = F(S_m)$: denoting their radii by R_m, notice that $R_m \geqslant \varrho_0$ for some $\varrho_0 > 0$. Clearly there exist sequences $\{m_j\}$ and $\{k_j\}$ and a constant $\nu_0 > 0$ such that

$$\hat{S}_{m_j} \subset B^+_{2^{-k_j-1}}; \qquad \text{length}\, \hat{S}_{m_j} \geqslant \nu \cdot 2^{-k_j} \qquad \forall j .$$

Moreover, if $\varepsilon \in]0, \frac{1}{2}[$ is small enough with respect to μ, the arcs

$$\hat{S}^{\varepsilon}_{m_j} = \{(x, y) \in \hat{S}_{m_j} : y \geqslant \varepsilon \cdot 2^{-k_j}\}$$

also satisfy

$$\text{length}\, \hat{S}^{\varepsilon}_{m_j} \geqslant \nu \cdot 2^{-k_j} \qquad \forall j$$

for some constant $\nu > 0$ (depending on μ, ε, ν_0). Therefore, if we define

$$L_j = 2^{k_j} \cdot \hat{S}^{\varepsilon}_{m_j}$$

we have: length $L_j \geqslant \nu$.

But, since the radius of L_j is $2^{k_j} R_{m_j} \geqslant \varrho_0 \cdot 2^{k_j}$, from lemma 2.2 we deduce the existence of $\lambda > 0$ such that

$$\text{cap}\, L_j \geqslant \lambda \quad \text{for } j \text{ large enough} . \tag{2.25}$$

Now we can apply proposition 2.1 to the function $\hat{p}$: recalling (2.23), (2.24) and that $\hat{p} \geqslant 0$, we only have to verify that (2.10) holds. Since clearly L_j is contained in the set $\{(x, y) \in B^+_{\frac{1}{2}};\, y \geqslant \varepsilon,\ \hat{p}(2^{-k_j} \cdot x, 2^{-k_j} \cdot y) = 0\}$ and capacity is an increasing set function, (2.10) is given by (2.25).

Therefore we obtain

$$\hat{p}(x, y) \cdot (x^2 + y^2)^{-\frac{1}{2}} \to 0 \qquad \text{as } (x, y) \to (0, 0)$$

that is

$$\lim_{Q\to P_0} \frac{p(Q)}{|Q-P_0|} = 0$$

which is impossible by corollary 1.1.

Thus we get a contradiction and the proof is complete. ∎

REMARK 2.2. It must be pointed out that the previous proof does not require the monotonicity assumption (1.3). Hence the same result is still valid in much more general cases.

It is sufficient to know that ∂D is a smooth graph, $D \cap \partial\Omega$ is also a graph and p satisfies (in some neighbourhood of P_0 on ∂D) a homogeneous Dirichlet boundary condition and the inequality $p_\nu \leqslant -\cos \nu y$ in the sense of (1.17).

In particular we will use the result given by proposition 2.1 also in the case of several reservoirs, that will be studied in sect. 4.

3. – Monotonicity of the free boundary.

In this section we study monotonicity properties of the free boundary.

We assume that hypotheses of sect. 1 are satisfied and prove that φ is itself monotone on $[a, b]$.

We begin with two lemmas.

LEMMA 3.1. *Let $x_0 \in [c, b[$ be such that $\varphi(x_0) = Y(x_0)$. Then φ cannot take at x_0 either a local maximum or a local minimum.*

PROOF. Clearly, from (1.3) and theorem 2.1. ∎

In the following u will denote the function $p + y$.

LEMMA 3.2. *Let y be a real number in $[y_2, y_1]$ and ω a non-empty connected component of the set $\{(x, y)\colon u(x, y) > \tilde{y}\}$ (resp. $< \tilde{y}$). Then $u|_{\bar{\omega}}$ can take a local maximum (resp. minimum) only on $\bar{\Gamma}_1$ (resp. $\bar{\Gamma}_2$).*

PROOF. Consider the first case. Let $Q_0 \in \omega$ be a local maximum for $u|_{\bar{\omega}}$.

For some ball B with center at Q_0, $\bar{\omega}$ contains $\bar{\Omega} \cap B$. Thus Q_0 is a local maximum for $u|_{\bar{\Omega}}$. It follows (by maximum and Hopf principles) that Q_0 belongs to $\partial\Omega$ but not to $D \cup \Gamma_N$. By Lemma 3.1 Q_0 cannot belong to Γ_0, so $Q_0 \in \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. But on $\bar{\Gamma}_2$ we have $u = y_2 \leqslant \tilde{y}$. Therefore $Q_0 \in \bar{\Gamma}_1$.

The proof of the other case in similar. ∎

Now we are able to prove the main theorem:

THEOREM 3.1. *The function φ is strictly decreasing in $[a, b]$.*

PROOF. First we observe that φ cannot be constant in any subinterval of $[a, b]$.

Indeed, by contradiction, suppose that $D \cap \partial\Omega$ contains some horizontal segment S. We have on S: u is constant and $u_\nu = 0$. Thus uniqueness results about the Cauchy problem for the Laplace equation imply that u is constant in Ω, which is impossible since $y_1 > y_2$. Therefore, if φ is monotone then it is strictly monotone.

Now we observe that if φ were not monotone, it would take a local maximum at some point of $]a, b[$, since φ is continuous and $\varphi(b) \leqslant \varphi(x) \leqslant \varphi(a)$.

Therefore it is sufficient to prove that φ cannot take in $]a, b[$ any local maximum and we shall do it by contradiction.

Let $x^* \in]a, b[$ be a local maximum for φ. Consider the point $P^* = (x^*, y^*) = \big(x^*, \varphi(x^*)\big)$: from Lemma 3.1 we get $P^* \in D$.

On the other hand, the continuous function φ on $[a, x^*]$ takes its minimum in some point $x_* \in [a, x^*]$. Clearly $x_* > a$ and $y_* = \varphi(x_*) \leqslant y^*$. Thus $P_* \equiv (x_*, y_*) \in D$.

But Hopf principle implies that P^* (resp. P_*) cannot be a local maximum (resp. minimum) for $u|_{\bar{\Omega}}$, because $\Delta u = 0$ in Ω and $u_\nu = 0$ at P^* (resp. P_*); so the open set $\{(x, y) \in \Omega\colon u(x, y) > y^*\}$ $\big($resp. $\{(x, y) \in \Omega\colon u(x, y) < y_*\}\big)$ cannot be empty and has P^* (resp. P_*) as a boundary point. We denote ω^* (resp. ω_*) its (or one of its) connected components whose boundary contains P^* (resp. P_*) and look for the point $\bar{P}$ (resp. $\underline{P}$) where $u|_{\bar{\omega}^*}$ (resp. $u|_{\bar{\omega}_*}$) takes its maximum (resp. minimum).

By Lemma 3.2 we have $\bar{P} \in \bar{\Gamma}_1$ and $\underline{P} \in \bar{\Gamma}_2$. Hence we conclude that $\bar{\omega}^*$ and $\bar{\omega}_*$ must have a common point Q.

Then we have $u(Q) < y_*$ and $u(Q) > y^*$, while $y_* \leqslant y^*$. Therefore we get a contradiction and the theorem is proved. ∎

4. – Generalizations.

We want to show how results about monotonicity properties could be generalized to more complicated situations.

It will be sufficient to deal with the case of three reservoirs R_1, R_2, R_3 with levels y_1, y_2, y_3. Without loss of generality we will suppose $y_1 \geqslant y_3$.

We shall use the following notations: Γ_N still is the impervious part of ∂D (which bounds D from below), $\Gamma_i = \partial D \cap \partial R_i$ and $\Gamma_0 = \partial D - (\bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 \cup \bar{\Gamma}_N)$. As in sect. 1, smoothenss properties of ∂D are assumed to be satisfied.

Monotonicity properties of ∂D (see (1.3)) are generalized as follows: defining $\underline{y} = \min y_i$ and $\bar{y} = \max y_i$, $D \cap (\mathbb{R} \times]\underline{y}, \bar{y}[)$ is supposed to have exactly two connected components, say D' and D'' (with D' between R_1 and R_2 and D'' between R_2 and R_3). Moreover, all connected components of $\partial D' \cap \Gamma_0$ and $\partial D'' \cap \Gamma_0$ are assumed to be monotone arcs (more precisely $Y' < 0$ or $Y' > 0$ on each of these arcs). Denoting the free boundary by Γ, Γ contains $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, and, clearly, we shall be only interested in monotonicity properties of the two connected components of $\Gamma - (\bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3)$ which are given by $\Gamma' = \Gamma \cap (\partial D' - \Gamma_1)$ and $\Gamma'' = \Gamma \cap (\partial D'' - \Gamma_3)$. φ_1 and φ_2 will denote the functions which represent Γ' and Γ'' respectively.

Under the previous assumptions we still have $\underline{y} \leqslant \varphi_j \leqslant \bar{y}$ $(j = 1, 2)$ and the result given by Theorem 2.1 holds in the present case. Therefore, if $P_0 \in \Gamma \cap \Gamma_0$ then Γ and ∂D are tangential at P_0; in particular Γ is a strictly monotone graph in a neighbourhood of P_0.

We can prove the following theorems:

THEOREM 4.1. *If $\bar{y} = y_1$ and $\underline{y} = y_2$, φ_1 is strictly decreasing. If $\bar{y} = y_2$ and $\underline{y} = y_3$, φ_2 is strictly decreasing.*

PROOF. We consider only the case $\bar{y} = y_1$ and $\underline{y} = y_2$, since the other case is quite similar.

As in the proof of Theorem 2.1, it is sufficient to show that φ_1 (6) cannot take any local maximum. By contradiction, let x^* be a local maximum for φ_1 and define x_* by: $\varphi_1(x_*) = \min\{\varphi_1(x) \colon x < x^*\}$. As above, the points $P^* = (x^*, \varphi_1(x^*))$ and $P_* = (x_*, \varphi_1(x_*))$ are in D and the open sets $\{(x, y) \in \Omega \colon u(x, y) > \varphi_1(x^*)\}$ and $\{(x, y) \in \Omega \colon u(x, y) < \varphi_1(x_*)\}$ have connected components ω^* and ω_* respectively such that $P^* \in \partial\omega^*$ and $P_* \in \partial\omega_*$.

Defining $\bar{P}$ and $\underline{P}$ by: $u(\bar{P}) = \max\{u(x, y) \colon (x, y) \in \bar{\omega}^*\}$ and $u(\underline{P}) = \min\{u(x, y) \colon (x, y) \in \bar{\omega}_*\}$ we get easily that $\bar{P} \in \bar{\Gamma}_1 \cup \bar{\Gamma}_3$ and $\underline{P} \in \bar{\Gamma}_2 \cup \bar{\Gamma}_3$. In any case it follows that $\omega^* \cap \omega_* \neq \emptyset$ which is impossible. ∎

THEOREM 4.2. *If $\bar{y} = y_1$ and $\underline{y} = y_3$, either φ_1 or φ_2 is strictly decreasing.*

PROOF. By contradiction suppose that φ_1 and φ_2 are both non-monotone.

We prove first that $\inf \varphi_1 < y_2$. By contradiction suppose $\inf \varphi_1 \geqslant y_2$. As above, considering a local maximum of φ_1 we get a contradiction. With similar arguments we obtain $\sup \varphi_2 > y_2$. Define now x^* and x_* by $\varphi_2(x^*) = \max \varphi_2$ and $\varphi_1(x_*) = \min \varphi_1$, and consider the sets $\{(x, y) \in \Omega \colon u(x, y) > \varphi_2(x^*)\}$ and $\{(x, y) \in \Omega \colon u(x, y) < \varphi_1(x_*)\}$. As in the previous proofs we conclude that their connected components ω^* and ω_* whose boundaries

(6) By definition, φ_1 is defined in an open interval.

contain $(x^*, \varphi_2(x^*))$ and $(x_*, \varphi_1(x_*))$ respectively must have a common point. Thus we get a contradiction. ■

REMARK 4.1. Theorems 4.1 and 4.2 allow us to conclude that in any case either φ_1 or φ_2 is strictly monotone.

The following theorem describes the shape of non-monotone arcs of the free boundary.

In all cases the theorem can be proved with the previous arguments.

THEOREM 4.3. *Suppose that either φ_1 or φ_2 is non-monotone. Then we have*

if $\bar{y} = y_1$ and $\underline{y} = y_2$, φ_2 has a maximum and no local minima;

if $\bar{y} = y_1$ and $\underline{y} = y_3$, either φ_1 has a minimum and no local maxima or φ_2 has a maximum and no local minima;

if $\bar{y} = y_2$ and $\underline{y} = y_3$, φ_1 has a minimum and no local maxima. ■

We conclude by showing that in some cases one of the two arcs of the free boundary cannot be monotone.

We have indeed:

THEOREM 4.4. *If either $|y_1 - y_2|$ or $|y_2 - y_3|$ is small enough with respect to $y_1 - y_3$ then φ_1 and φ_2 cannot be both monotone.*

PROOF. We consider for instance the case $\bar{y} = y_1$, since the case $\bar{y} = y_2$ is quite similar.

For fixed $y_0 < y_1$, take $y_2 = y_0 + \varepsilon$ and $y_3 = y_0 + \delta$ where ε and δ will be properly chosen in the interval $[0, y_1 - y_0[$. Denote by $\varphi_2(x; \varepsilon, \delta)$ the function which represents the corresponding second arc of the free boundary.

Consider now the case $\varepsilon = \delta = 0$. We have $y_0 \leqslant \varphi_2(x; 0, 0) \leqslant y_1$ and $\varphi_2(x; 0, 0)$ cannot be monotone without being constant since its limits, as x approaches the end-points of the interval where $\varphi_2(x; 0, 0)$ is defined, are equal to y_0. But $\varphi_2(x; 0, 0)$ cannot be constant by uniqueness results about the Cauchy problem for the Laplace equation unless $y_0 = y_1$, while $y_1 > y_0$.

Therefore $\varphi_2(x; 0, 0)$ is not monotone and, by Theorem 4.3 $\varphi_2(x; 0, 0)$ has a maximum point.

Denoting by y^* the corresponding maximum value, we have $y_0 < y^* \leqslant y_1$.

But it is easy to see that the pressure in D increases as levels increase, thus Ω also increases. Therefore, $\varphi_2(x; \varepsilon, \delta)$ is an increasing function with respect to each of the variables ε and δ. It follows that, for $\varepsilon, \delta \in [0, y^* - y_0[$, $\varphi_2(x; \varepsilon, \delta)$ cannot be monotone with respect to x. ■

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