ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

HENRY B. LAUFER

Versal deformations for two-dimensional pseudoconvex manifolds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 3 (1980), p. 511-521

<http://www.numdam.org/item?id=ASNSP_1980_4_7_3_511_0>

Versal Deformations for Two-Dimensional Pseudoconvex Manifolds.

HENRY B. LAUFER (*)

Let M be a strictly pseudoconvex manifold with a one-dimensional exceptional set A. Let Θ be the holomorphic tangent sheaf to M. The general Kodaira-Spencer [11] theory shows that $H^1(M, \Theta)$ corresponds to first order infinitesimal deformations of M and that $H^2(M, \Theta)$ represents the obstructions to formally extending deformations to higher order. $H^1(M, \Theta)$ is finite dimensional since M is strictly pseudoconvex [1]. $H^2(M, \Theta) = 0$ essentially because A is one-dimensional. But it is known [6], [5] that there is no finite-dimensional deformation theory for M if one keeps track of the boundary. So in order to stay within the Kodaira-Spencer framework, given a deformation of M and a compact set K in M, we shall only worry about the deformation near K. Then M has a versal deformation $\omega: \mathcal{M} \to Q$ with Q a manifold of dimension $\dim H^1(M, \Theta)$ in case either (i) M is of arbitrary dimension and is a sufficiently small neighborhood of A (Definition 1, Theorem 2 and Theorem 5 below) or (ii) M is of dimension two (Theorem 8 below). The existence of ω was proved for arbitrary Stein M by Andreotti and Vesentini [2]. Openness of versality holds (Theorem 3 and Theorem 8 below).

Some applications of this paper are given in [16] and [17]. In [17], the dimension two analogue of [7] and [23, Theorem 2.1 and Proposition 2.3] is proved, i.e. if all of the fibers of a deformation are isomorphic, then the deformation is trivial.

Most of the results of this paper have been announced in [15].

Much of the research for this paper was done at Purdue University. The author thanks Purdue University for its generous hospitality.

(*) The author is an Alfred P. Sloan Fellow. This research was also partially supported by NSF Grant MCS 7604969A01.

Pervenuto alla Redazione il 17 Agosto 1979.

DEFINITION 1. Let M be a strictly pseudoconvex manifold. A *special* cover $\mathfrak{U} = \{U_i\}$, $0 \leqslant i \leqslant m$, is a finite cover of M such that each U_i is Stein and such that $\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k = \emptyset$ for $i \neq j \neq k$. ($^-$ denotes closure in M.)

THEOREM 2. *Let M^* be a strictly pseudoconvex manifold with a one-dimensional exceptional set A. Then there is a strictly pseudoconvex neighborhood M of A, a special cover $\mathfrak{U}$ of M, and a deformation $\omega\colon \mathcal{M} \to Q$ of $M = \omega^{-1}(0)$, with Q a manifold, such that the Kodaira-Spencer map $\varrho_0\colon {}_QT_0 \to$ $\to H^1(M, \Theta)$ is an isomorphism. ω may be chosen to be a 1-convex holomorphic map.*

PROOF. We first construct a larger cover $\mathfrak{V} = \{V_i\}$, $0 \leqslant i \leqslant m$. Let the V_i, $1 \leqslant i \leqslant m$, be small balls in local coordinates for M^* centered about the singular points $\{s_i\}$ of A. Choose $\overline{V}_i \cup \overline{V}_j = \emptyset$ for $i \neq j$; closure is in M^*. Should a connected component of A be non-singular, also choose such a V_i about some points s_i in the component. So we get points s_i, $1 \leqslant i \leqslant m$, lying in all irreducible components A_k of A. Let $S = \cup\, s_i$, $1 \leqslant i \leqslant m$. Let $T \subset \cup\, V_i$, $1 \leqslant i \leqslant m$, be a closed neighborhood of S in A. We choose V_0 to be a Stein neighborhood of $A - T$ as follows. Each $A_k - S$ is an open Riemann surface and thus Stein [9, Theorem IX. C. 10, p. 270]. Let f_k be a C^∞ strictly plurisubharmonic function on $A_k - S$ such that $f_k(z) \to \infty$ as $z \to S$, $z \in A_k$. By [18, Satz 3.3, p. 275], there is a neighborhood W_k of $A_k - S$ in M^* such that f_k has a C^∞ plurisubharmonic extension, also denoted by f_k, to W_k. Let g be a C^∞ function defined in a neighborhood W of the connected component A' of A containing A_k such that $g = 0$ on A', $g > 0$ off A', g is plurisubharmonic on W, and g is strictly plurisubharmonic on $W - A'$. Let N be sufficiently large so that $f_k(z) < N - 1$ for $z \in A_k - T$. Then for r a sufficiently large real number, $V_{0,k} = \{z \in W \cap W_k | f_k(z) +$ $+ rg(z) < N\}$ is a Stein neighborhood of $A_k - T$. Moreover, for large r the various $\overline{V}_{0,k}$ will be disjoint. Let $V_0 = \cup\, V_{0,k}$, all k. Then $\overline{V}_i \cap \overline{V}_j \cap$ $\cap\, \overline{V}_k = \emptyset$ for $i \neq j \neq k$.

Let M_1 be a strictly pseudoconvex neighborhood of A contained in $\bigcup V_i$, $0 \leqslant i \leqslant m$. Replace $\{V_i\}$ by $\{V_i \cap M_1\}$, which we will also denote by $\{V_i\} = \mathfrak{V}$. Since $\mathfrak{V}$ is a Leray cover of M_1, $H^1(M_1, \Theta) \approx H^1(N(\mathfrak{V}), \Theta)$. Let $\theta_1, \ldots, \theta_n$ be vector fields on $\{V_i \cap V_j\}$ which represent a basis of $H^1(M_1, \Theta)$. If $M_1 \supset M$, also a strictly pseudoconvex neighborhood of A, then the restriction map $H^1(M_1, \Theta) \to H^1(M, \Theta)$ is an isomorphism [13, Lemma 3.1, p. 599]. So $\{\theta_k\}$ will also be a basis for $H^1(M, \Theta)$ for M smaller than M_1 and for refinements $\mathfrak{U}$ of $\mathfrak{V}$.

Using just the specialness of the cover $\mathfrak{V}$, we shall construct $\mathcal{M}$ via coordinate patches. These patches will be modified in the course of the construction. Let $\mathfrak{V}' = \{V'_i\}$, $0 \leqslant i \leqslant m$ with $V'_i \Subset V_i$ be a refinement of $\mathfrak{V}$. Given any compact set K in M_1, we may choose $\mathfrak{V}'$ to be a cover of K. Now let $^-$ denote closure in M_1. Let $K = \overline{M}$. Take an initial Q to be a polydisc of dimension $n = \dim H^1(M_1, \Theta)$. Start with patches $V_i' \times Q$, $0 \leqslant i \leqslant m$. We must give the g_{ij}, the transition functions for $\mathcal{M}$. For each small $t = (t_1, \ldots, t_n)$ in Q, integration along $t_1\theta_1 + \ldots + t_n\theta_n$ for time 1 gives a map $h_{ij}(t)\colon \overline{V}_i' \cap \overline{V}_j' \to V_i \cap V_j$. Restrict Q to these small values of t and define an initial $g_{ij}\colon (V_i' \cap V_j') \times Q \to (V_i \cap V_j) \times Q$ by $g_{ij} = \big(h_{ij}(t), t\big)$. There will be no compatibility conditions to verify for these changes of coordinates since no three coordinate patches intersect. However, for these changes of coordinates to define a manifold and in particular to insure that the space is Hausdorff, we still must modify the domains and ranges of the g_{ij}. Let B be the set of non-interior points of $V_j' - V_i'$. Then the points of $V_j' \times Q$ which might not be separated from points in $V_i \times Q$ (which are not identified by g_{ij}) lie in $B \times Q$. $\overline{B}$ is disjoint from the compact set $C = \overline{(V_i' - V_j') \cap V_j} \cap K$. Let D be a neighborhood of C such that $\overline{D}$ is compact and $\overline{D} \cap \overline{B} = \emptyset$. Then for small Q, $h_{ij}(B \times Q) \cap \overline{D} = \emptyset$. So far, g_{ij} maps $(V_i' \cap V_j') \times Q \subset V_j' \times Q$ biholomorphically to an open subset R_{ij} of $V_i \times Q$. R_{ij} lies near to $(V_i' \cap V_j') \times Q$, as a subset of $V_i \times Q$. In the cover for $\mathcal{M}$, replace $V_i \times Q$ by the subset $[\{(V_i' - \overline{V}_j) \cup D\} \times Q] \cup R_{ij} = T_i$. This modifies $V_i' \times Q$ only near V_j and makes Hausdorff the space $(V_j' \times Q) \cup T_i$ with points identified under g_{ij}.

Since $\overline{V}_i \cap \overline{V}_j \cap \overline{V}_k = \emptyset$ for $i \neq j \neq k$, the construction of the above paragraph leaves $V_i \cap V_k$ and $V_j \cap V_k$ unchanged. Thus to complete the construction of coordinate patches for $\mathcal{M}$, we look at an unordered pair (i, j), $i \neq j$. We favor one element of the unordered pair, say i, and form T_i as in the previous paragraph. This changes the range of g_{ij} and the domain of $g_{ji} = [g_{ij}]^{-1}$ to R_{ij}. We then consider a different unordered pair and repeat the construction of the previous paragraph. After considering all unordered pairs, we have a Hausdorff space $\mathcal{M}'$ and a projection map $\omega'\colon \mathcal{M}' \to Q$ which shows that $\mathcal{M}'$ is a family of deformations of $M' = (\omega')^{-1}(0)$. $K \subset M'$.

M, the interior of K, is the desired strictly pseudoconvex manifold. Let $U_i = M \cap V_i$. $\mathfrak{U} = \{U_i\}$ is then a special cover. Let $\mathcal{M}$ be a neighborhood of M in $\mathcal{M}'$ such that $\mathcal{M} \cap (\omega')^{-1}(0) = M$. Then, after possibly shrinking Q, $\omega = \omega' | \mathcal{M}$ is the desired deformation. By construction, $\varrho_0\colon {}_QT_0 \to H^1(M, \Theta)$ is an isomorphism. Recall [19, Satz 1, p. 547]:

Let $\pi\colon Z \to S$ be a holomorphic mapping of complex spaces with strictly pseudoconvex special fiber $X = \pi^{-1}(s_0)$, $s_0 \in S$ fixed. Then for every compact set $K \subset X$, there exist open sets $U \subset Z$ and $V \subset S$, with $K \subset U$, $s_0 \in V$, $\pi(U) \subset V$, such that $\pi | U\colon U \to V$ is a 1-convex map.

We shall use this result several times in this paper. In particular, ω can be chosen to be 1-convex. This completes the proof of the Theorem.

THEOREM 3. *Let $\omega: \mathcal{M} \to Q$ be a deformation of a strictly pseudoconvex manifold $M_0 = \omega^{-1}(0)$ which has a special cover $\mathfrak{U}$. Let Θ_q be the tangent sheaf on $M_q = \omega^{-1}(q)$. Suppose that ω is 1-convex, Q is a manifold and $\varrho_0: {}_QT_0 \to H^1(M_0, \Theta_0)$ is surjective. Then $\varrho_q: {}_QT_q \to H^1(M_q, \Theta_q)$ is surjective for all small q.*

PROOF. Let Θ be the sheaf of germs of vector fields on $\mathcal{M}$ which lie in the direction of the fibers. Let $\omega_*^1(\Theta)$ be the first direct image sheaf of Θ under the map ω. Then $\omega_*^1(\Theta)$ is a coherent analytic sheaf on Q [21, Main Theorem (i), p. 213].

Using ω, we may shrink $\mathcal{M}$ along the fibers and not change any map ϱ_q for small q. Then, as in the proof of Theorem 2, we may use [18] to extend the special cover $\mathfrak{U}$ on M to a special cover on the shrunken $\mathcal{M}$. Without loss of generality, we may thus assume that $\mathcal{M}$ has a special cover. Then $\omega_*^r(\mathcal{F}) = 0$ for $r>1$ and $\mathcal{F}$ any coherent sheaf on $\mathcal{M}$. In particular, $\omega_*^r(\mathcal{F})$ is $\mathcal{O}$-flat. Θ is locally free and so is ω-flat. Let $\mathfrak{m}_q$ be the ideal sheaf of $q \in Q$. Then [22, Proposition 2.2, p. 208] $H^1(M_q, \Theta_q) \approx \omega_*^1(\Theta)/\mathfrak{m}_q\omega_*^1(\Theta)$. Let $\mathcal{T}$ be the tangent sheaf on Q. Then the Kodaira-Spencer map [11] $\varrho: \mathcal{T} \to \omega_*^1(\Theta)$ is a map of coherent analytic sheaves. Since ${}_QT_0 \approx \mathcal{T}/\mathfrak{m}_0\mathcal{T}$, the given hypothesis that ϱ_0 is surjective says that $\varrho_0: \mathcal{T}/\mathfrak{m}_0\mathcal{T} \to \omega_*^1(\Theta)/\mathfrak{m}_0\omega_*^1(\Theta)$ is surjective. By Nakayama's Lemma, ϱ is surjective at 0. Then ϱ is surjective near 0 by coherence. Then ϱ_q is surjective for q near 0.

To deal with non-reduced parameter spaces, we need the following easy strengthening of [2].

THEOREM 4. *Let M be a Stein manifold and $\omega: \mathcal{M} \to S$ a deformation of $M = M_0 = \omega^{-1}(0)$ with S a possibly non-reduced analytic space. Then given any compact set $K \subset M$, there is a neighborhood $\mathcal{M}_1$ or K in $\mathcal{M}$ such that $\omega|\mathcal{M}_1$ is a trivial deformation.*

PROOF. ω is given to be locally trivial. As in [9, p. 266-269], we may use a C^∞ strictly plurisubharmonic exhaustion function on M to write $M = \cup M^{(i)}$ $1 \leqslant i < \infty$, with $M^{(i)} \subset M^{(i+1)}$, $M^{(i)}$ a strictly pseudoconvex Stein manifold, and $M^{(i+1)} = M^{(i)} \cup N^{(i)}$ with $N^{(i)}$ a Stein manifold near which ω is a trivial deformation. We may assume that ω is a trivial deformation near $M^{(1)}$.

We can now prove the theorem by induction on i. The case $i = 1$ is given. $M^{(i+1)} = M^{(i)} \cup N^{(i)}$. After shrinking a little, we may assume by induction that ω is trivial near $M^{(i)}$ and $N^{(i)}$. Then near $M^{(i+1)}$, ω may be defined by giving just one transition map $g_{12}: U_1 \cap U_2 \to U_1 \cap U_2$ with $U_1 \approx M^{(i)} \times S$ and $U_2 \approx N^{(i)} \times S$. Shrinking $M^{(i+1)}$ a little more, we shall extend ω to a (non-singular) ambient neighborhood Δ of $0 \in S$. The theorem will then follow from the original formulation in [2].

To extend ω, let $M'' \Subset M' \Subset M^{(i)}$ and $N'' \Subset N' \Subset N^{(i)}$ with M'', M', N'' and N' Stein. Then for T a sufficiently small neighborhood of 0 in S, g_{12} restricts to give a map $(h_{12}(s), s): (M'' \cap N'') \times T \to (M' \cap N') \times T$. Here, in the domain of h_{12}, we are using the product structure on U_2. In the range of h_{12}, we are using the product structure on U_1. $h_{12}(0)$ is the inclusion map. So that $h_{12}(s)$ may be given by a set of functions, embed the Stein manifold $M' \cap N'$ in C^n for some n. By [9, Theorem VIII, C. 8, p. 257], there is a neighborhood V of $M' \cap N'$ in C^n and a holomorphic retraction map $\varrho: V \to M' \cap N'$. Let the initial ambient neighborhood Δ' of 0 in S be Stein with $\Delta' \cap T$ a subvariety of Δ'. Then the functions defining $h_{12}(s)$ extend to functions on $(M'' \cap N'') \times \Delta'$. By restricting to a smaller neighborhood Δ'', we may assume that the image of the extended $h_{12}(s)$ lies in V. Composing with ϱ gives $(f_{12}(s), s): (M'' \cap N'') \times \Delta'' \to (M' \times N') \times \Delta''$. Since $f_{12}(0) = h_{12}(0)$ is the identity map onto its image, $f_{12}(s)$ is a biholomorphic map onto its image for all sufficiently small $s \in \Delta''$. Proceeding as in the proof of Theorem 2, we may shrink $M^{(i+1)}$ a little more and form the desired deformation which extends ω. This completes the proof of Theorem 4.

THEOREM 5. *Let M be a strictly pseudoconvex manifold with a special cover $\mathfrak{U}$. Let Θ_0 be the tangent sheaf to M. Let $\omega: \mathcal{M} \to Q$ be a deformation of $M = M_0 = \omega^{-1}(0)$ such that Q is a manifold and $\varrho_0: {}_QT_0 \to H^1(M_0, \Theta_0)$ is surjective. Let $\lambda: \mathfrak{R} \to S$ be any deformation of $M = M_0 = \lambda^{-1}(0)$ with S a possibly non-reduced analytic space. Then, given any compact set K in M, there are neighborhoods $\mathcal{M}_1$ and $\mathfrak{R}_1$ of K in $\mathcal{M}$ and $\mathfrak{R}$ respectively, neighborhoods Q_1 and S_1 of 0 in Q and S respectively, and a holomorphic map $f: S_1 \to Q_1$ such that $\omega|\mathcal{M}_1 = \omega_1: \mathcal{M}_1 \to Q_1$ and $\lambda|\mathfrak{R}_1 = \lambda_1: \mathfrak{R}_1 \to S_1$ are deformations with λ_1 induced by f. If ϱ_0 is also injective, then the tangent map of f at the origin is uniquely determined.*

PROOF. Shrinking M and $\mathfrak{U}$ a little, we may assume by Theorem 4 that λ is trivial near 0 on each U_i. As in the proof of Theorem 4, we may shrink M further and extend λ to a non-singular ambient neighborhood Δ of 0 in S.

So, without loss of generality, we shall now assume that S is non-singular. Let the transition maps for λ be given by $g_{ij}(s)$, $s \in S$. Let the transition maps for ω be given by $h_{ij}(q)$, $q \in Q$. Let $U''_i \Subset U'_i \Subset U_i$ be two refinements of $\mathfrak{U}$. Choose Q_1 and S_1 small so that $h_{ij}(q) \circ g_{ij}(s) = k_{ij}(q, s): U''_i \cap U''_j \to \to U'_i \cap U'_j$ is well defined for $(q, s) \in Q_1 \times S_1$. Then, as in the proof of Theorem 2, the k_{ij} may be used to construct a deformation $\tau\colon \mathcal{Y} \to B$ of a slightly shrunk M. B is a Cartesian product $Q_1 \times S_1$ of neighborhoods Q_1 and S_1 of 0 in Q and S respectively. Above $0 \times S_1$, τ coincides with λ. Above $Q_1 \times 0$, τ coincides with ω. Let $\mathcal{T}$ be the tangent sheaf of B. Let ${}_Q\mathcal{T}$ be the subsheaf of $\mathcal{T}$ of germs of vector fields on B in the Q_1 directions. Choose [19, Satz 1, p. 547] τ to be a 1-convex map. Then, by the proof of Theorem 2, $\varrho_Q\colon {}_Q\mathcal{T} \to \tau_*^1(\Theta)$ is surjective near $0 \times 0 = 0$. Let $\nu_1, \ldots, \nu_n$ be vector fields on B such that $\nu_1(0), \ldots, \nu_n(0)$ project onto a basis of ${}_ST_0$. Since ϱ_Q is surjective near 0, we may modify $\nu_1, \ldots, \nu_n$ by sections of ${}_Q\mathcal{T}$ and assume that $\varrho(\nu_i) = 0$ in $\tau_*^1(\Theta)$ for all i and small B. Then, for sufficiently small B, $\varrho(\nu_i) = 0$ in $H^1(\tau^{-1}(B), \Theta)$. Then, by the nature of ϱ, for each i there exists a vector field θ_i on $\tau^{-1}(B)$ such that at each point b of $\tau^{-1}(B)$, τ_* maps $\theta_i(b)$ to $\nu_i(\tau(b))$. Let $(t_1, \ldots, t_n)$ be near $(0, \ldots, 0)$. Then, integrating along $t_1\theta_1 + \ldots + t_n\theta_n$ and $t_1\nu_1 + \ldots + t_n\nu_n$ for time 1 and for small $(t_1, \ldots, t_n)$ gives a Cartesian product structure $\mathcal{Y} \approx \mathcal{M} \times S_1$ with a projection map $\omega \times id\colon \mathcal{M} \times S_1 \to Q_1 \times S_1$ which shows that $\mathcal{Y}$ is a deformation of a slightly smaller M. There is also an automorphism of $B = Q_1 \times S_1$ near 0×0 which shows that τ and $\omega \times id$ are equivalent deformations. $\lambda\colon \mathfrak{K} \to S_1$ is a subspace of $\tau\colon \mathcal{Y} \to B$. Projecting $\mathcal{Y}$ onto $\mathcal{M}$ via the Cartesian product structure gives the desired map $f\colon S_1 \to Q_1$.

This concludes the proof of Theorem 5 except for the last sentence. But the tangent map of f at the origin just agrees with the infinitesimal Kodaira-Spencer map in this case.

Let M be as in Theorem 5. Then $H^2(M, \mathcal{O}) = 0$. [19, Satz 5, p. 562] says that under such circumstances we can form its simultaneous-blow-down subspace T of Q, as in Definition 9 below. The versality result of Theorem 5 implies versality for deformations of germs of M near A. Blow down M to V. Let p be the singular point of V. Then [19, Satz 7, p. 562] says that the simultaneous blow-down over T is versal for deformations which can be simultaneously resolved.

The following corollary about the rigidity of exceptional curves of the first kind is known. For example, use [10, Theorem 3, p. 85], which says that A lists above S, and [19, Satz 2, p. 547], which says that one can simultaneously blow down near the lifting. We shall use it to strengthen our results in the two-dimensional case.

COROLLARY 6. *Let M be a two-dimensional manifold. Let A be a submanifold of M which is a compact Riemann surface of genus 0 with $A \cdot A = -1$. Let $\lambda\colon \mathcal{M} \to S$ be a deformation of $M = \lambda^{-1}(0)$. Then in a neighborhood of A in $\mathcal{M}$, λ is the trivial deformation.*

PROOF. It suffices to see that for any small strictly pseudoconvex neighborhood N of A in M, $H^1(N, \Theta) = 0$.

Since A is in fact an exceptional curve of the first kind, $H^1(N, \Theta)$ can be directly computed via a Leray cover to give 0. Or, one may use [8, Satz 1, p. 355] and [14, (3.9), p. 85].

PROPOSITION 7. *Let M be a strictly pseudoconvex two-dimensional manifold. Let A be the exceptional set. Then there are a finite number of points $p_i \in M - A$ such that the manifold M' obtained from M by quadratic transformations at the p_i can be written $M' = U_1 \cup U_2$ with U_1 and U_2 open Stein subsets of M'.*

PROOF. Let M^* be a strictly pseudoconvex manifold with $M \Subset M^*$ and also with the same exceptional set A. Let $\mathfrak{J}$ be the ideal sheaf of A. By [12, Lemma 4.10, p. 61], we can find a divisor D on A with $A_i \cdot D$ arbitrarily negative for all irreducible components A_i of A. Let $\mathfrak{I}$ be the ideal sheaf corresponding to D. Then, by [12, Lemma 6.19, p. 117] (and its proof in case A lacks normal crossings), for the $A_i \cdot D$ sufficiently negative, $H^1(M^*, \mathfrak{J}\mathfrak{I}) = H^1(M^*, \mathfrak{J}^2\mathfrak{I}) = 0$. Then $\Gamma(M^*, \mathfrak{I}) \to \Gamma(M^*, \mathfrak{I}/\mathfrak{J}\mathfrak{I})$ and $\Gamma(M^*, \mathfrak{J}\mathfrak{I}) \to \Gamma(M^*, \mathfrak{J}\mathfrak{I}/\mathfrak{J}^2\mathfrak{I})$ are surjective. Then we can find $f_1, f_2 \in \Gamma(M^*, \mathfrak{I})$ such that $(f_1) - D$ and $(f_2) - D$ contain no A_i and also if $p \in \operatorname{supp}\big((f_1) - D\big) \cap \operatorname{supp}\big((f_2) - D\big) \cap M$, then $p \notin A$ and p is a point of normal crossing for $(f_1) - D$ and $(f_2) - D$. There are only a finite number of such p_i. Let M' be obtained from M by quadratic transformations at the p_i. Let D_1 and D_2 be the proper transforms on M' of $(f_1) - D$ and $(f_2) - D$ respectively. Let $U_i = M' - \operatorname{supp} D_i$, $i = 1, 2$. Then U_1 and U_2 are the desired Stein subsets of M'. One may construct the needed holomorphic functions on the U_i by considering f_3/f_i, with $f_3 \in \Gamma(M^*, \mathfrak{I})$ or $f_3 \in \Gamma(M^*, \mathfrak{J}\mathfrak{I})$. Then U_i is holomorphically convex and the f_3/f_i will give local coordinates. This concludes the proof of Proposition 7.

THEOREM 8. *Let M be a strictly pseudoconvex two-dimensional manifold. Then there exists a deformation $\omega: \mathcal{M} \to Q$ of $M = \omega^{-1}(0)$ such that ω is 1-convex, Q is a manifold and the Kodaira-Spencer map $\varrho_0: {}_QT_0 \to H^1(M, \Theta_0)$ is an isomorphism. Let $M_q = \omega^{-1}(q)$. $\varrho_q: {}_QT_q \to H^1(M_q, \Theta_q)$ is surjective for all small $q \in Q$. Let $\lambda: \mathfrak{R} \to S$ be any deformation of $M = M_0 = \lambda^{-1}(0)$ with S a possibly non-reduced analytic space. Then, given any compact set K in M, there are neighborhoods $\mathcal{M}_1$ and $\mathfrak{R}_1$ of K in $\mathcal{M}$ and $\mathfrak{R}$ respectively, neighborhoods Q_1 and S_1 of 0 in Q and S respectively, and a holomorphic map $f: S_1 \to Q_1$ such that $\omega|\mathcal{M}_1 = \omega_1: \mathcal{M}_1 \to Q_1$ and $\lambda|\mathfrak{R}_1 = \lambda_1: \mathfrak{R}_1 \to S_1$ are deformations with λ_1 induced from ω_1 by f. The tangent map of f at 0 is uniquely determined.*

PROOF. For any coherent sheaf $\mathcal{F}$ on M, $H^1(M, \mathcal{F})$ is determined by small neighborhoods of the exceptional set. If N is a small holomorphically convex neighborhood of an exceptional curve of the first kind, then $H^1(N, \Theta) = 0$. Hence quadratic transformations off the exceptional set have no effect on $H^1(M, \Theta)$.

To construct ω, let M^* be a strictly pseudoconvex manifold with $M \Subset M^*$. Let $M^{*\prime}$ be obtained from M^* by a finite number of quadratic transformations and have a special cover (Proposition 7). $\pi: M^{*\prime} \to M^*$. By the proof of Theorem 2, there is a deformation $\omega': \mathcal{M}' \to Q$ of $M' = \pi^{-1}(M)$ with $\varrho_0': {}_QT_0 \to H^1(M', \Theta)$ an isomorphism and ω' a 1-convex map. By Corollary 6, the exceptional curves of the first kind in M' which are the result of quadratic transformations in M have neighborhoods on which ω' is a trivial deformation. Simultaneously blow down the exceptional curves of the first kind in these neighborhoods. This gives a deformation $\omega: \mathcal{M} \to Q$ of M. ω is 1-convex. ϱ_0 is an isomorphism by the observation of the previous paragraph. ϱ_q' is surjective for small q by Theorem 3. $\cup A_q'$, with A_q' the exceptional set in M_q', is the subvariety of $\mathcal{M}'$ where the Remmert reduction is not an isomorphism [19, p. 553]. Hence for small q, M_q' is obtained from M_q by quadratic transformations off the exceptional set. Then also ϱ_q is surjective for small q.

Consider $\lambda: \mathfrak{R} \to S$, a deformation of M. λ is locally trivial. So we may perform quadratic transformations simultaneously on all M_s, s small, to get a deformation $\lambda': \mathfrak{R}' \to S$ of M'. Then by Theorem 5, with $K' = \pi^{-1}(K)$, we get $\mathcal{M}_1'$, $\mathfrak{R}_1'$, Q_1, S_1 and $f: S_1 \to Q_1$ for λ'. Simultaneously blowing down the exceptional curves of the first kind on $\mathcal{M}_1'$ and $\mathfrak{R}_1'$ yields the desired $\mathcal{M}_1$ and $\mathfrak{R}_1$. This completes the proof of Theorem 8.

We now wish to blow down a deformation $\omega: \mathcal{M} \to Q$ of M. $M = M_0 = \omega^{-1}(0)$. We essentially follow ideas and work of Riemenschneider [19] and of Artin and Schlessinger [4], [3, especially Theorem 4, p. 341]. Choose ω to be 1-convex. $M_q = \omega^{-1}(q)$ is an open manifold of dimension two, so $H^2(M_q, \mathcal{O}) = 0$ [22]. Then [19, Satz 5, p. 558] says that there is a maximal reduced subspace T of Q near 0 such that, letting $\mathcal{A} = \omega^{-1}(T)$, the family $\omega_a = \omega|\mathcal{A}: \mathcal{A} \to T$ simultaneously blows down to a flat deformation $\pi_a: \mathfrak{X} \to T$ of the blow down $V = X_0 = \pi_a^{-1}(0)$ of M. $T = \{q \in Q | \dim H^1(M_q, \mathcal{O}) = \dim H^1(M_0, \mathcal{O})\}$.

DEFINITION 9. Let $\omega: \mathcal{M} \to Q$ be a 1-convex deformation of $M = M_0 = \omega^{-1}(0)$. Let the reduced space T be given by $T = \{q \in Q | \dim H^1(M_q, \mathcal{O}) = \dim H^1(M_0, \mathcal{O})\}$. Then T is the *simultaneous-blow-down* subspace of Q.

THEOREM 10. *Let M be a strictly pseudoconvex two-dimensional manifold with exceptional set A. Let $\omega: \mathcal{M} \to Q$ be as in Theorem 8. Suppose that M is the minimal resolution of the normal two-dimensional analytic space V. Let T be the simultaneous-blow-down subspace of Q. Then the blow-down $\pi_a: \mathfrak{X} \to T$ of ω over T is the unique deformation of V which is versal for deformations with reduced parameter spaces that can be simultaneously resolved, i.e. given any deformation $\pi: \mathfrak{Y} \to S$ of $V = X_0 = \pi^{-1}(0)$ with S reduced such that π may be simultaneously resolved and any compact set $K \subset V$, then there exist neighborhoods $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ of K in $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, neighborhoods T_1 and S_1 of 0 in T and S respectively, and a holomorphic map $f: S_1 \to T_1$ such that $\pi_a|\mathfrak{X}_1: \mathfrak{X}_1 \to T_1$ and $\pi_1 = \pi|\mathfrak{Y}_1: \mathfrak{Y}_1 \to S_1$ are deformations with π_1 induced by f. The induced map f_* on the Zariski tangent space of S at 0 to the Zariski tangent space of T at 0 is unique.*

For all points $t \in T$ sufficiently near to 0, π_a is versal near t except for the uniqueness of the map f_*.

If $\mathfrak{X}'$, open in $\mathfrak{X}$, has $\pi_a' = \pi_a|\mathfrak{X}': \mathfrak{X}' \to T$ a deformation with $V' = (\pi_a')^{-1}(0)$ being a strictly pseudoconvex neighborhood of the singular points of V, then π_b' is the unique deformation of V' which is versal for deformations with reduced parameter spaces which can be simultaneously resolved.

PROOF. Let $\lambda: \mathcal{R} \to S$ be a simultaneous resolution of π. Then $R = \lambda^{-1}(0)$ is a resolution of $V = \pi^{-1}(0)$. Suppose that $A_i \subset R$ is an exceptional curve of the first kind. Then by Corollary 6, we can simultaneously blow down A_i and nearby exceptional curves of the first kind and still have a deformation of the blown down R. Thus, without loss of generality, we may assume that R is the minimal resolution of V. Since minimal resolutions are unique [20], [12, pp. 87-88], $R \approx M$. Let $\tau_0: M \to V$ be the resolving map. Apply Theorem 8, using the compact set $\tau_0^{-1}(K)$. We need that $f(S_1) \subset T$. But since λ may be simultaneously blown down, for $s \in S$, $\dim H^1(R_s, \mathcal{O}) = \dim H^1(M, \mathcal{O})$. Hence $f(s) \in T$. The first paragraph of the Theorem now follows by letting $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ be the blow downs of $\mathcal{M}_1 \cap \omega^{-1}(T_1)$ and R respectively. (The uniqueness of π_a is proved in the usual way from the uniqueness of f_*.)

The second paragraph of the Theorem follows from Theorem 8 and the above argument, which proved the first paragraph.

Let $M' = \tau_0^{-1}(V')$. Let K' be a compact set in M' with $A \subset K'$. By [19, Satz 1, p. 547], there is a neighborhood $\mathcal{M}'$ of K' in $\mathcal{M}$ and a neighborhood Q' of $0 \in Q$ such that $\omega' = \omega|\mathcal{M}': \mathcal{M}' \to Q'$ is a 1-convex map. Since in $\mathcal{M}$ the union of the exceptional sets of M_q is the subvariety of $\mathcal{M}$ where the Remmert reduction is not an isomorphism [19, p. 553], $M_q = \omega^{-1}(q)$ and $M_q' = (\omega')^{-1}(q)$ have the same exceptional set for all small q. Then

[13, Lemma 3.1, p. 599] the restriction map $H^1(M_q, \mathcal{O}) \to H^1(M'_q, \mathcal{O})$ is an isomorphism for all small q. Thus ω and ω' have the same simultaneous-blow-down subspace T of Q for small q. This concludes the proof of Theorem 10.

REFERENCES

[1] A. Andreotti - H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, **90** (1962), pp. 193-259.

[2] A. Andreotti - E. Vesentini, *On the pseudo-rigidity of Stein manifolds*, Ann. Scuola Norm. Sup. Pisa, **16** (1962), pp. 213-223.

[3] M. Artin, *Algebraic construction of Brieskorn's resolutions*, J. Algebra, **29** (1974), 330-348.

[4] M. Artin - M. Schlessinger, *Algebraic construction of Brieskorn's resolutions*, Preprint.

[5] D. Burns - S. Schnider - R. Wells, *Deformations of strongly pseudoconvex domains*, Preprint 1976.

[6] S. Chern - J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math., **133** (1974), pp. 219-271.

[7] W. Fischer - H. Grauert, *Locale triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1965), pp. 89-94.

[8] H. Grauert, *Über Modifikationcn and exzeptionelle analytische Mengen*, Math. Ann., **146** (1962), pp. 331-368.

[9] R. Gunning - H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1965.

[10] K. Kodaira, *On stability of compact submanifolds of complex manifolds*, Amer. J. Math., **85** (1963), pp. 79-94.

[11] K. Kodaira - D. Spencer, *On deformations of complex analytic structures I, II*, Ann. of Math., **67** (1958), pp. 328-466.

[12] H. Laufer, *Normal Two-Dimensional Singularities*, Ann. of Math. Studies no. 71, Princeton University Press, 1971.

[13] H. Laufer, *On rational singularities*, Amer. J. Math., **94** (1972), pp. 597-608.

[14] H. Laufer, *Deformations of resolutions of two-dimensional singularities*, Rice University Studies, **59**, vol. 1 (1973), pp. 53-96.

[15] H. Laufer, *Deformations of two-dimensional pseudoconvex manifolds*, Proc. of International Conf. on Several Complex Variables, Cortona, Italy, 1976-77, Scuola Normale Superiore, Pisa 1978, pp. 217-221.

[16] H. Laufer, *Ambient deformations for exceptional sets in two-manifolds*, Invent. Math., **55** (1979), pp. 1-36.

[17] H. Laufer, *Lifting cycles to deformations of two-dimensional pseudoconvex manifolds*, in preparation.

[18] R. Richberg, *Stetige streng pseudokonvexe Functionen*, Math. Ann., **175** (1968), pp. 257-286.

[19] O. RIEMENSCHNEIDER, *Familien komplexer R–ume mit streng pseudokonvexer spezieller Faser*, Comment. Math. Helv., **51** (1976), pp. 547-565.
[20] I. SHAFAVEVICH, *Algebraic Surfaces*, American Mathematical Society, Providence, R.I., 1967.
[21] Y.-T. SIU, *The 1-convex generalization of Grauert's direct image theorem*, Math. Ann., **190** (1971), pp. 203-214.
[22] Y.-T. SIU, *Dimension of sheaf cohomology groups under holomorphic deformation*, Math. Ann., **192** (1971), pp. 203-215.
[23] B. TEISSIER, *Deformations à type topologique constant - I*, Astérisque, **16** (1974), pp. 215-227.

State University of New York
Department of Mathematics
Stony Brook, New York 11794