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Analytic Convexity.

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De Giorgi [6] and Piccinini [12] were the first to make the following observation. Consider the Laplace operator in two variables x, y:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

as operating on functions of three variables x, y, t, and consider the equation

$$\Delta u = f$$

for u and f functions of x, y, t. Then one has the following facts

(α) for $f \in C^\infty(\boldsymbol{R}^3)$ there exists $u \in C^\infty(\boldsymbol{R}^3)$ such that $\Delta u = f$;

(β) there exist some f real analytic in $\boldsymbol{R}^3$ such that the equation $\Delta u = f$ has no real analytic solution u defined on $\boldsymbol{R}^3$.

Let $\mathcal{E}$ denote the sheaf of germs of C^∞ functions on $\boldsymbol{R}^3$ and let $\mathcal{A}$ denote the sheaf of germs of real analytic functions on $\boldsymbol{R}^3$. We consider the two exact sequences of sheaves

$$0 \longrightarrow \mathcal{E}_\Delta \longrightarrow \mathcal{E} \xrightarrow{\Delta} \mathcal{E} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{A}_\Delta \longrightarrow \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \longrightarrow 0$$

where $\mathcal{E}_\Delta$ and $\mathcal{A}_\Delta$ represent respectively the kernels of the sheaves homomorphisms defined by the operator Δ on $\mathcal{E}$ or $\mathcal{A}$. Then we deduce exact

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cohomology sequences

$$H^0(\boldsymbol{R}^3, \mathcal{E}) \xrightarrow{\Delta} H^0(\boldsymbol{R}^3, \mathcal{E}) \longrightarrow H^1(\boldsymbol{R}^3, \mathcal{E}_\Delta) \longrightarrow 0$$

$$H^0(\boldsymbol{R}^3, \mathcal{A}) \xrightarrow{\Delta} H^0(\boldsymbol{R}^3, \mathcal{A}) \longrightarrow H^1(\boldsymbol{R}^3, \mathcal{A}_\Delta) \longrightarrow 0$$

as $H^1(\boldsymbol{R}^3, \mathcal{E}) = 0 = H^1(\boldsymbol{R}^3, \mathcal{A})$. From the remark of de Giorgi and Piccinini we derive that one has

$$H^1(\boldsymbol{R}^3, \mathcal{E}_\Delta) = 0 \qquad \text{and} \qquad H^1(\boldsymbol{R}^3, \mathcal{A}_\Delta) \neq 0$$

indeed this statement is equivalent to their remark. This suggests the following generalization (section 1). We consider a Hilbert complex of sheaves

$$0 \longrightarrow \mathcal{E}_{A_0} \longrightarrow \mathcal{E}^{p_0} \xrightarrow{A_0(D)} \mathcal{E}^{p_1} \xrightarrow{A_1(D)} \mathcal{E}^{p_2} \longrightarrow \dots$$

or

$$0 \longrightarrow \mathcal{A}_{A_0} \longrightarrow \mathcal{A}^{p_0} \xrightarrow{A_0(D)} \mathcal{A}^{p_1} \xrightarrow{A_1(D)} \mathcal{A}^{p_2} \longrightarrow \dots$$

i.e., a complex of differential operators with constant coefficients on $\boldsymbol{R}^n$ obtained by Fourier transform from a Hilbert resolution of a module over the ring of polynomials in n variables and we let the complex of operators act on C^∞ or real analytic functions to obtain the two exact sequences of sheaves given above. Then given an open set Ω in $\boldsymbol{R}^n$ we will say that it is C^∞ or analytically convex if

$$H^j(\Omega, \mathcal{E}_{A_0}) = 0 \quad \forall j > 0 \qquad \text{or respectively} \qquad H^j(\Omega, \mathcal{A}_{A_0}) = 0 \quad \forall j > 0 .$$

The example of De Giorgi and Piccinini shows that these two notions of convexity may be different.

After some general remarks on elliptic operators (section 2) that we need later on, we begin the study of analytic convexity by the following procedure. We consider the given complex as the complex of Cauchy data on a linear subspace $\boldsymbol{R}^n$ of a Hilbert complex in several more variables in some $\boldsymbol{R}^N$ (sections 3 and 4); this we call a suspension to $\boldsymbol{R}^N$ of the given complex. If the suspension complex has the first operator elliptic and if $\boldsymbol{R}^n$ in $\boldsymbol{R}^N$ is in a Cauchy-Kowalewska position (i.e., $\boldsymbol{R}^n$ is non characteristic for the suspension complex) then we are able to reduce the study of the analytic convexity of the given complex to the study of the C^∞ convexity of its suspension.

We consider then sufficient conditions for analytic convexity in terms of the suspension complex (section 5), we give some examples and we investigate the case of a convex open set. We show that in this case all analytic cohomology groups in dimension ≥ 2 vanish.

To turn the sufficient conditions into necessary conditions, one needs an approximation theorem of Runge type (sections 6 and 7). We are able to establish an approximation theorem of this sort for convex open sets (section 8) or for starshaped open sets for operators represented by homogeneous matrices of polynomials (section 9). We end up our investigation with the study of the tangential Cauchy-Riemann complex for a real space $\mathbf{R}^{n+h}$ in some complex space $\mathbf{C}^n$. Setting $z = x + iy$, since $\Delta = 4(\partial^2/\partial z \partial \bar{z})$, we recover in a more precise cohomological form the example of de Giorgi and Piccinini. Also we show with an example that for non convex sets we may again have analytic cohomology in dimension ≥ 2 without having C^∞ cohomology.

We hope to come back to this subject with an extension of the principle of Phragmén-Lindelöf of Hörmander [8] to the situation we have considered.

1. – C^∞ and analytic convexity.

a) Let Ω be an open set in $\mathbf{R}^n$ and let $\mathcal{E}(\Omega)$ denote the space of complex valued C^∞ functions defined on Ω. Set $\mathcal{E}^p(\Omega) = \mathcal{E}(\Omega) \times \ldots \times \mathcal{E}(\Omega)$ p times. Let

$$(1) \qquad (\mathcal{E}^*(\Omega), A_*) \equiv \left\{\mathcal{E}^{p_0}(\Omega) \xrightarrow{A_0(D)} \mathcal{E}^{p_1}(\Omega) \xrightarrow{A_1(D)} \mathcal{E}^{p_2}(\Omega) \xrightarrow{A_2(D)} \ldots\right\}$$

be a complex of differential operators with constant coefficients. Here $A_j(D)$ is a matrix of type $p_{j+1} \times p_j$ with entries differential operators with constant coefficients. The assumption that (1) is a complex means that

$$A_{j+1}(D) A_j(D) = 0, \qquad \forall j.$$

Complexes of this kind can be obtained as follows. Let $\mathcal{T}_n = \mathbf{C}[\xi_1, \ldots, \xi_n]$ be the ring of polynomials in the n indeterminates $\xi_1, \ldots, \xi_n$. Let $A_0(\xi) = (a_{0ij}(\xi))$ be a $p_1 \times p_0$ matrix with polynomial entries, let ${}^tA_0(\xi)\colon \mathcal{T}_n^{p_1} \to \mathcal{T}_n^{p_0}$ be considered as a $\mathcal{T}_n$-homomorphism, and set $N = \operatorname{coker} \{{}^tA_0(\xi)\colon \mathcal{T}_n^{p_1} \to \mathcal{T}_n^{p_0}\}$. By a theorem of Hilbert (cf. [3]) we can continue this homomorphism by a finite sequence of $\mathcal{T}_n$-homomorphisms

$$(2) \qquad 0 \longleftarrow N \longleftarrow \mathcal{T}_n^{p_0} \xleftarrow{{}^tA_0(\xi)} \mathcal{T}_n^{p_1} \xleftarrow{{}^tA_1(\xi)} \mathcal{T}_n^{p_2} \longleftarrow \ldots$$
$$\ldots \longleftarrow \mathcal{T}_n^{p_{d-1}} \xleftarrow{{}^tA_{d-1}(\xi)} \mathcal{T}_n^{d} \longleftarrow 0$$

to obtain an exact sequence, i.e., a free resolution of N. We also can assume that $d \leq \max(2, n)$.

Replacing the matrices ${}^tA_j(\xi)$ by their transposed and the indeterminates ξ_j by $\partial/\partial x_j$ we obtain a complex (1) of differential operators with constant coefficients which moreover has the property of being exact on open convex sets Ω.

The condition for the complex (1) to be exact on open convex sets characterizes the complexes obtained from Hilbert resolutions (2) (cf. [3]). We shall therefore call those complexes in the sequel *Hilbert complexes.*

The Hilbert complex (1) is obtained from complex (2) by the following procedure:

α) we consider $\mathcal{E}(\Omega)$ as a $\mathcal{T}_n$-module by letting a polynomial $p(\xi) \in \mathcal{T}_n$ operate on $f \in \mathcal{E}(\Omega)$ by

$$p(\xi)\cdot f = p(D)f$$

where $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$.

β) We apply the functor $\operatorname{Hom}_{\mathcal{T}_n}(\cdot, \mathcal{E}(\Omega))$ to the sequence (2).

Let $\mathfrak{b} = \mathcal{T}_n(\varphi_1(\xi), \ldots, \varphi_l(\xi))$ be an ideal of $\mathcal{T}_n$. Any $\mathcal{T}_n$-homomorphism $\sigma\colon \mathfrak{b} \to \mathcal{E}(\Omega)$ is an assignment

$$\sigma(\varphi_i(\xi)) = f_i(x) \in \mathcal{E}(\Omega) \qquad 1 \leq i \leq l$$

with the condition that whenever $\sum_{i=1}^{l} a_i(\xi)\varphi_i(\xi) = 0$ with $a_i(\xi) \in \mathcal{T}_n$ we have $\sum a_i(D)f_i(x) = 0$. And conversely any such assignment defines a $\mathcal{T}_n$-homomorphism $\sigma\colon \mathfrak{b} \to \mathcal{E}(\Omega)$. Let us now recall the following criterion (cf. [7] p. 6).

A left $\mathcal{T}_n$-module F is injective if for every ideal $\mathfrak{b}$ in $\mathcal{T}_n$ and every $\mathcal{T}_n$-homomorphism $\sigma\colon \mathfrak{b} \to F$ we can find $f \in F$ with $\sigma(p) = p\cdot f$, $\forall p \in \mathfrak{b}$.

From the above remark on the Hilbert complex it follows then that for Ω *open and convex the module* $\mathcal{E}(\Omega)$, *as* $\mathcal{T}_n$-*module, is injective.*

In particular, denoting by $\mathcal{E}$ the sheaf of germs of C^∞ functions on $\boldsymbol{R}^n$ and by $\mathcal{E}_{A_0}$ the subsheaf of $\mathcal{E}^{p_0}$ of germs of solutions of $A_0(D)u = 0$ we have an exact sequence of sheaves

$$(3) \qquad 0 \longrightarrow \mathcal{E}_{A_0} \xrightarrow{\ \ |\ \ } \mathcal{E}^{p_0} \xrightarrow{A_0(D)} \mathcal{E}^{p_1} \xrightarrow{A_1(D)} \mathcal{E}^{p_2} \xrightarrow{A_2(D)} \ldots$$

which is a «resolution» of the sheaf $\mathcal{E}_{A_0}$ by fine sheaves. This because a Hilbert complex (1) admits the so called Poincaré lemma or equivalently because for every $x \in \boldsymbol{R}^n$ the stalk $\mathcal{E}_x = \varinjlim_{\Omega \ni x} \mathcal{E}(\Omega)$ is an injective $\mathcal{T}_n$-module.

From de Rham theorem we then deduce that for any open set $\Omega \subset \boldsymbol{R}^n$ we have

$$H^j(\Omega, \mathcal{E}_{A_0}) \cong H^j(\mathcal{E}^*(\Omega), A_*) \cong \operatorname{Ext}^j(N, \mathcal{E}(\Omega)), \quad j \geq 0 .$$

Note that these groups depend only on the $\mathcal{P}_n$-module N and not on the particular resolution (2).

We shall say that *an open set Ω is C^∞-convex* for the Hilbert complex (1) *if* $H^j(\Omega, \mathcal{E}_{A_0}) = 0$ for $j > 0$.

b) We replace in the previous consideration the space $\mathcal{E}(\Omega)$ by the space $\mathcal{A}(\Omega)$ of complex valued real analytic functions on Ω. Then the complex (1) is replaced by the complex

$$(4) \qquad (\mathcal{A}^*(\Omega), A_*) \equiv \left\{\mathcal{A}^{p_0}(\Omega) \xrightarrow{A_0(D)} \mathcal{A}^{p_1}(\Omega) \xrightarrow{A_1(D)} \mathcal{A}^{p_2}(\Omega) \xrightarrow{A_2(D)} \dots\right\}.$$

Let us consider $\mathcal{A}(\Omega)$ as a $\mathcal{P}_n$-module by letting $p(\xi) \in \mathcal{P}_n$ operate on $f \in \mathcal{A}(\Omega)$ by

$$(5) \qquad p(\xi)\cdot f = p(D) f .$$

Then the complex (4) is obtained from the resolution (2) by application of the functor $\operatorname{Hom}_{\mathcal{P}_n}(\cdot, \mathcal{A}(\Omega))$. Now it is no more true in general (if $n \geq 3$) that $\mathcal{A}(\Omega)$ is injective if Ω is open and convex. However, if we denote by $\mathcal{A}$ the sheaf of germs of complex valued real analytic functions and by $\mathcal{A}_x$ the stalk of $\mathcal{A}$ at $x \in \boldsymbol{R}^n$ we can consider on $\mathcal{A}_x$ the structure of $\mathcal{P}_n$-module induced by (5) and we have the following

PROPOSITION 1. *The $\mathcal{P}_n$-module $\mathcal{A}_x$ is an injective module.*

PROOF. We use the criterion for injectivity mentioned above. Let $\mathfrak{b} = \mathcal{P}_n(\varphi_1(\xi), \dots, \varphi_\nu(\xi))$ be an ideal of $\mathcal{P}_n$. Without loss of generality we may assume x at the origin $0 \in \boldsymbol{R}^n$. Let

$$\sigma\colon \mathfrak{b} \to \mathcal{A}_0$$

be a $\mathcal{P}_n$-homomorphism. This is an assignment

$$\sigma(\varphi_i) = f_i(x) \in \mathcal{A}_0$$

with the property that whenever $\sum a_i(\xi)\varphi_i(\xi) = 0$ with $a_i(\xi) \in \mathcal{P}_n$ we have

$$\sum a_i(D) f_i(x) = 0 .$$

Let us now consider $\boldsymbol{R}^n \subset \boldsymbol{C}^n$ where $z = x + iy$ are holomorphic coordinates. Denoting by $\mathcal{O}_0$ the germs of holomorphic functions at the origin in $\boldsymbol{C}^n$ and by $\mathcal{E}_0$ the germs of C^∞ functions at the origin in $\boldsymbol{R}^{2n} = \boldsymbol{C}^n$ we have natural inclusions of rings

$$\mathcal{A}_0 \subset \mathcal{O}_0 \subset \mathcal{E}_0 .$$

Let $\mathcal{P}_{2n}$ denote the ring of polynomials in the $2n$-variables $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ and consider the ideal

$$\hat{\mathfrak{b}} = \mathcal{P}_{2n}(\varphi_1(\xi), \ldots, \varphi_l(\xi), \xi_1 + i\eta_1, \ldots, \xi_n + i\eta_n) .$$

We consider $\mathcal{E}_0$ as a $\mathcal{P}_{2n}$-module by letting $p(\xi, \eta) \in \mathcal{P}_{2n}$ operate on $f(x, y) \in \mathcal{E}_0$ by

$$p(\xi, \eta)\cdot f(x, y) = p(D_x, D_y) f(x, y)$$

where $D_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $D_y = (\partial/\partial y_1, \ldots, \partial/\partial y_n)$. We define a $\mathcal{P}_{2n}$-homomorphism

$$\hat{\sigma}\colon \hat{\mathfrak{b}} \to \mathcal{E}_0$$

by setting

$$\begin{cases} \hat{\sigma}(\varphi_i) = f_i(x + iy) & 1 \le i \le l \\ \hat{\sigma}(\xi_j + i\eta_j) = 0 & 1 \le j \le n . \end{cases}$$

Indeed suppose that

$$\sum_{j=1}^{l} p_j(\xi, \eta)\varphi_j(\xi) + \sum_{j=1}^{n} q_j(\xi, \eta)(\xi_j + i\eta_j) = 0 .$$

We have to show that

$$\sum p_j(D_x, D_y) f_j(x + iy) = 0$$

in order that $\hat{\sigma}$ be well defined.

Now $\sum p_j(\xi, i\xi)\varphi_j(\xi) = 0$; therefore as σ is a $\mathcal{P}_n$-homomorphism we get $\sum p_j(D_x, iD_x) f_j(x) = 0$. But for $f_j(x + iy) \in \mathcal{O}_0$ we have $D_x f_j(x + iy) = - iD_y f_j(x + iy)$, i.e., $iD_x f_j(x + iy) = D_y f_j(x + iy)$. Therefore we obtain from the last identity $\sum p_j(D_x, D_y) f_j(x + iy) = 0$ as we wanted.

Now $\mathcal{E}_0$ is an injective $\mathcal{P}_{2n}$ module, therefore there exists $g(x, y) \in \mathcal{E}_0$ with the property

$$\hat{\sigma}(\lambda(\xi, \eta)) = \lambda(D_x, D_y) g(x, y)$$

for every $\lambda \in \hat{\mathfrak{b}}$. Taking $\lambda = \xi_j + i\eta_j$ $1 \leq j \leq n$ we obtain that g is holomorphic, i.e., $g \in \mathcal{O}_0$ and $g = g(x + iy)$. Thus $g(x) \in \mathcal{A}_0$ and therefore $\sigma(\lambda(\xi)) = \lambda(D_x) g(x)$ $\forall \lambda \in \mathfrak{b}$. This shows that $\mathcal{A}_0$ is an injective $\mathcal{P}_n$-module.

As a consequence from (2) applying the functor $\mathrm{Hom}_{\mathcal{P}_n}(\cdot, \mathcal{A}_x)$ we obtain an exact sequence of sheaves

$$(6) \qquad 0 \longrightarrow \mathcal{A}_{A_0} \xrightarrow{\ \ \mid\ \ } \mathcal{A}^{p_0} \xrightarrow{A_0(D)} \mathcal{A}^{p_1} \xrightarrow{A_1(D)} \mathcal{A}^{p_2} \xrightarrow{A_2(D)} \cdots$$

where $\mathcal{A}_{A_0}$ denotes the subsheaf of $\mathcal{A}^{p_0}$ of germs u satisfying the equation $A_0(D)u = 0$. In other words the resolution (6) admits the Poincaré lemma.

We now remark that for any open set $\Omega \subset \mathbf{R}^n$ we have

$$H^j(\Omega, \mathcal{A}) = 0 \qquad \forall j > 0 .$$

This is a consequence of a theorem of Grauert [1] that states that « *every open subset of $\mathbf{R}^n$ admits in the complexification $\mathbf{C}^n$ of $\mathbf{R}^n$ a fundamental system of open neighborhoods which are open sets of holomorphy* ». We can therefore apply again the de Rham theorem and we obtain for Ω open in $\mathbf{R}^n$,

$$H^j(\Omega, \mathcal{A}_{A_0}) \cong H^j(\mathcal{A}^*(\Omega), A_*) \cong \mathrm{Ext}^j\,(N, \mathcal{A}(\Omega))\,, \qquad j \geq 0.$$

We will say that *the open set $\Omega \subset \mathbf{R}^n$ is analytically convex* for the « Hilbert complex » (4) *if* $H^j(\Omega, \mathcal{A}_{A_0}) = 0$ for $j > 0$.

2. – Elliptic operators.

a) Let $A_0(\xi) = (a_{0_{ij}}(\xi))$ be a $p_1 \times p_0$ matrix with polynomial entries. We consider ${}^tA_0(\xi)$ as a $\mathcal{P}_n$-homomorphism and denote by N its cokernel, so that we have the exact sequence of $\mathcal{P}_n$-modules

$$(1) \qquad 0 \longleftarrow N \longleftarrow \mathcal{P}_n^{p_0} \xleftarrow{{}^tA_0(\xi)} \mathcal{P}_n^{p_1} .$$

Let us introduce the following ideals of $\mathcal{P}_n$,

$\mathfrak{b} = \mathfrak{b}(N) =$ ideal generated by the p_0-rowed minor determinants of the matrix $A_0(\xi)$ (the 0-ideal if $p_0 > p_1$);

$\mathfrak{b}' = \mathfrak{b}'(N) = \{p \in \mathcal{P}_n | pN = 0\}$ the annihilator-ideal of the module N.

(1) H. Grauert, *On Levi's problem and the imbedding of real analytic manifolds*, Ann. Math., **58** (1958), pp. 460-472.

One can show ([13] p. 5) that the first of these ideals depends only on the module N and not on the presentation (1) we have considered. For the second we remark that

$$\mathfrak{b}'(N) = \{p \in \mathcal{P}_n | p\mathcal{P}_n^{p_0} \subset \operatorname{Im} {}^tA_0(\xi)\} .$$

PROPOSITION 2. *We have*

$$\sqrt{\mathfrak{b}(N)} = \sqrt{\mathfrak{b}'(N)} \,.$$

PROOF. Let

$$\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \varepsilon_{p_1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

be the canonical basis of $\mathcal{P}_n^{p_1}$, so that $C_1(\xi) = {}^tA_0(\xi)\varepsilon_1, \ldots, C_{p_1}(\xi) = {}^tA_0(\xi)\varepsilon_{p_1}$ are the column vectors of the matrix ${}^tA_0(\xi)$.

Set $L = \big(C_{i_1}(\xi), \ldots, C_{i_{p_0}}(\xi)\big)$, a minor of ${}^tA_0(\xi)$ of order p_0. From Cramer's rule we deduce that for any

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_{p_0} \end{pmatrix} \in \mathcal{P}_n^{p_0}$$

we have

$$\det L \cdot X = \sum C_{i_h}(\xi) \det \big(C_{i_1}(\xi), \ldots, \underset{(h)}{X}, \ldots, C_{i_{p_0}}(\xi)\big) .$$

Therefore, by the last remark, we deduce that

$$\mathfrak{b}(N) \subset \mathfrak{b}'(N) .$$

Now if $\mu \in \mathfrak{b}'(N)$ we must have

$$\mu \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha_{11}(\xi)\, {}^tA_0(\xi)\varepsilon_1 + \ldots + \alpha_{1p_1}(\xi)\, {}^tA_0(\xi)\varepsilon_{p_1}$$

. . . .

$$\mu \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \alpha_{p_01}(\xi)\, {}^tA_0(\xi)\varepsilon_1 + \ldots + \alpha_{p_0p_1}(\xi)\, {}^tA_0(\xi)\varepsilon_{p_1} .$$

This gives the identity

$$\mu \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\alpha_{ij}(\xi))\ {}^tA_0(\xi)$$

and therefore taking determinants we get

$$\mu^{p_0} \in \mathfrak{b}(N) .$$

This proves our contention.

Let $\mathfrak{a} = \mathfrak{a}(N)$ denote the *asymptotic ideal* of the ideal $\mathfrak{b}$, i.e., the homogeneous ideal of the principal parts of polynomials of $\mathfrak{b}$. We denote by

$V = V(\mathfrak{b})$, *the characteristic variety of* N (*or* A_0), the variety of common zeros in $\boldsymbol{C}^n$ of the elements of $\mathfrak{b}$. With self-consistent notations we have $V(\mathfrak{b}) = V(\mathfrak{b}')$ because of proposition 2;

$V_0 = V(\mathfrak{a})$, *the asymptotic variety of* N (*or* A_0), the variety of common zeros in $\boldsymbol{C}^n$ of the elements of $\mathfrak{a}$. Note that V_0 is a cone, if $\xi \in V_0$ then $\forall \lambda \in \boldsymbol{C}$, $\lambda\xi \in V_0$.

PROPOSITION 3. *The following conditions are equivalent*:

i) *For some constants* $c_1, c_2 > 0$ *we have*

$$|\xi| \leq c_1|\mathrm{Re}\,\xi| + c_2 , \qquad \forall \xi \in V .$$

ii) *For some constant* $c > 0$ *we have*

$$|\xi| \leq c|\mathrm{Re}\,\xi| , \qquad \forall \xi \in V_0 .$$

iii) *If* $\xi \in V_0$, $\xi \in \boldsymbol{R}^n$ *then* $\xi = 0$, i.e. $V_0 \cap \boldsymbol{R}^n \subset \{0\}$.

iv) *There exists a homogeneous polynomial* $p \in \mathfrak{a}$ *such that*

$$p(\xi) \neq 0 \qquad \forall \xi \in \boldsymbol{R}^n - \{0\} .$$

v) *There exists a polynomial* $q \in \mathfrak{b}$, *and some constants* $c_3, c_4 > 0$ *such that*

$$|q(i\xi)| \geq c_3|\xi|^{\deg q} - c_4 \qquad \forall \xi \in \boldsymbol{R}^n ,$$

where deg q *denotes the degree of the polynomial* q.

PROOF. i) $\Rightarrow$ ii). For any $\xi \in V_0$ we can find a sequence $\{\xi_\nu\}_{\nu=1,2,\ldots} \subset V$ such that $\xi_\nu/\nu \to \xi$, as ν goes to infinity. From the inequality i) we get

$$\left|\frac{\xi_\nu}{\nu}\right| \leq c_1 \left|\frac{\operatorname{Re} \xi_\nu}{\nu}\right| + \frac{c_2}{\nu}.$$

For $\nu \to \infty$ we get $|\xi| \leq c_1 |\operatorname{Re} \xi|$ which is inequality ii) with $c = c_1$.

ii) $\Rightarrow$ iii). If inequality ii) holds and $\xi \in V_0$ then $i\xi \in V_0$ so that for every $\xi \in V_0$ we also have the inequality

$$|\xi| \leq c |\operatorname{Im} \xi| .$$

Thus if $\xi \in V_0 \cap \boldsymbol{R}^n$ then $\operatorname{Im} \xi = 0$ and therefore $\xi = 0$.

iii) $\Rightarrow$ i). By contradiction, assume that i) does not hold, so that for every $\nu = 1, 2, \ldots$ we can find $\xi_\nu \in V$ with

$$|\xi_\nu| > \nu |\operatorname{Re} \xi_\nu| + \nu .$$

We necessarily have $\xi_\nu \neq 0$ so that we can consider the sequence $\{\xi_\nu/|\xi_\nu|\}$, and since this is bounded we can extract a convergent subsequence

$$\frac{\xi_{k_\nu}}{|\xi_{k_\nu}|} \to \xi .$$

We must have $|\xi| = 1$ so that $\xi \neq 0$. Moreover, since $|\xi_\nu| > \nu$ we must have $\xi \in V_0$. Because $\left|\operatorname{Re} \xi_\nu/|\xi_\nu|\right| < 1/\nu$ we must have $\operatorname{Re} \xi = 0$. But then $i\xi \subset V_0 \cap \boldsymbol{R}^n$ and $i\xi \neq 0$. This contradicts iii). Thus the statement is proved.

iii) $\Leftrightarrow$ iv). Let $\varphi_1(\xi), \ldots, \varphi_l(\xi)$ be a homogeneous basis of the asymptotic ideal $\mathfrak{a}$ and let m_j be the degree of φ_j $(1 \leq j \leq l)$.

Set with $m = \sup m_j$,

$$p(\xi) = \sum (\xi_1^2 + \ldots + \xi_n^2)^{m-m_j} \bar{\varphi}_j(\xi) \varphi_j(\xi)$$

where $\bar{\varphi}_j(\xi)$ denotes the polynomial obtained from $\varphi_j(\xi)$ by complex conjugation of its coefficients. Clearly $p \in \mathfrak{a}$. If iii) holds and $\xi \in \boldsymbol{R}^n - \{0\}$ then one of the $\varphi_j(\xi)$ is different from zero, thus

$$p(\xi) = \sum (\xi_1^2 + \ldots + \xi_n^2)^{m-m_j} |\varphi_j(\xi)|^2 > 0 .$$

Hence iii) $\Rightarrow$ iv). Conversely, if $p \in \mathfrak{a}$ verifying iv) exists, as p vanishes on V_0, we must have $V_0 \cap \boldsymbol{R}^n \subset \{0\}$ i.e., iv) $\Rightarrow$ iii).

iv) $\Rightarrow$ v). Let us select a polynomial $q \in \mathfrak{b}$ with principal part a polynomial p satisfying condition iv), and let m denote its degree. Note that for $\xi \in \boldsymbol{R}^n - \{0\}$, $p(i\xi) = i^m p(\xi) \neq 0$. Thus

$$\mu = \inf_{\substack{|\xi|=1 \\ \xi \in \boldsymbol{R}^n}} |p(i\xi)| > 0 .$$

We do have therefore

$$|p(i\xi)| \geq \mu |\xi|^m .$$

Set $q = p + p_1$ with p_1 a polynomial of degree $\leq m-1$, so that for some positive constant $c > 0$ we have

$$|p_1(i\xi)| \leq c(1 + |\xi|^{m-1}) \qquad \forall \xi \in \boldsymbol{R}^n .$$

Since for every $\varepsilon > 0$ we can find a constant $C(\varepsilon)$ sufficiently large such that

$$1 + |\xi|^{m-1} \leq \varepsilon |\xi|^m + C(\varepsilon) \qquad \forall \xi \in \boldsymbol{R}^n ,$$

we obtain

$$\begin{aligned} |q(i\xi)| &\geq |p(i\xi)| - |p_1(i\xi)| \\ &\geq (\mu - c\varepsilon)|\xi|^m - cC(\varepsilon) . \end{aligned}$$

If ε is sufficiently small we obtain the desired conclusion.

v) $\Rightarrow$ iv). Let $q \in \mathfrak{b}$ be a polynomial of degree m satisfying

$$|q(i\xi)| > c_3 |\xi|^m - c_4 \qquad \forall \xi \in \boldsymbol{R}^n .$$

If p is the principal part of q, $p = q - p_1$ with p_1 a polynomial of degree $\leq m-1$. Thus for some $c' > 0$ we have

$$|p(\xi)| = |p(i\xi)| > c_3 |\xi|^m - c'(1 + |\xi|^{m-1}) \qquad \forall \xi \in \boldsymbol{R}^n .$$

Replacing ξ by $t\xi$ we get

$$|p(\xi)| \geq c_3 |\xi|^m - c' \left(\frac{1}{|t|^m} + \frac{|\xi|^{m-1}}{|t|} \right).$$

For $t \to \infty$ we thus obtain

$$|p(\xi)| \geq c_3 |\xi|^m \qquad \forall \xi \in \boldsymbol{R}^n .$$

But this implies iv).

DEFINITION. A differential operator with constant coefficients

$$A_0(D)\colon \mathcal{E}^{p_0}(\Omega) \to \mathcal{E}^{p_1}(\Omega) \qquad (\Omega \text{ open in } \boldsymbol{R}^n)$$

is said to be *elliptic*, if for the matrix of polynomials $A_0(\xi)$ the equivalent conditions of proposition 3 are satisfied.

b) We set for Ω open in $\boldsymbol{R}^n$

$$\mathcal{E}_{A_0}(\Omega) = \{u \in \mathcal{E}^{p_0}(\Omega) | A_0(D)u = 0\}$$
$$\mathcal{A}_{A_0}(\Omega) = \{u \in \mathcal{A}^{p_0}(\Omega) | A_0(D)u = 0\}\,.$$

We have the following theorem (of Petrowski) (cf. [8] Corollary 4.4.1. p. 114).

THEOREM 1. *α*) *If* $\mathcal{E}_{A_0}(\boldsymbol{R}^n) = \mathcal{A}_{A_0}(\boldsymbol{R}^n)$ *then the operator* $A_0(D)$ *is an elliptic operator.*

β) *If* $A_0(D)$ *is an elliptic operator then for any open set* $\Omega \subset \boldsymbol{R}^n$

$$\mathcal{E}_{A_0}(\Omega) = \mathcal{A}_{A_0}(\Omega)\,.$$

PROOF. *α*) The space $\mathcal{E}_{A_0}(\boldsymbol{R}^n)$ as a closed subspace of the Fréchet space $\mathcal{E}^{p_0}(\boldsymbol{R}^n)$ (with the Schwartz topology) is a Fréchet space.

Let us imbed in the natural way $\boldsymbol{R}^n$ into $\boldsymbol{C}^n$ and let $z = x + iy$ denote holomorphic coordinates on $\boldsymbol{C}^n$, and set $|z| = \left(\sum_{\alpha=1}^{n} |z_\alpha|^2\right)^{\frac{1}{2}}$. Let $U(\varepsilon) = \{z \in \boldsymbol{C}^n \big| |z| < \varepsilon\}$ be the ε-ball in $\boldsymbol{C}^n$ and let $\mathcal{O}(U(\varepsilon))$ denote the space of holomorphic functions on $U(\varepsilon)$ with the Fréchet topology of uniform convergence on compact sets. For every $\varepsilon > 0$ the space

$$S_\varepsilon = \{(u, v) \in \mathcal{E}_{A_0}(\boldsymbol{R}^n) \times \mathcal{O}(U(\varepsilon)) \mid u = v \text{ on } U(\varepsilon) \cap \boldsymbol{R}^n\}$$

is a closed subspace of $\mathcal{E}_{A_0}(\boldsymbol{R}^n) \times \mathcal{O}(U(\varepsilon))$ and therefore it is a Fréchet space. Let $\pi\colon S_\varepsilon \to \mathcal{E}_{A_0}(\boldsymbol{R}^n)$ be the natural projection on the first factor. It is a continuous linear map. By the assumption that $\mathcal{E}_{A_0}(\boldsymbol{R}^n) = \mathcal{A}_{A_0}(\boldsymbol{R}^n)$, each element $u \in \mathcal{E}_{A_0}(\boldsymbol{R}^n)$ admits a holomorphic continuation to $U(\varepsilon)$ for some $\varepsilon > 0$. Therefore

$$\mathcal{E}_{A_0}(\boldsymbol{R}^n) = \bigcup_{n=1}^{\infty} \operatorname{Im}\{\pi\colon S_{1/n} \to \mathcal{E}_{A_0}(\boldsymbol{R}^n)\}\,.$$

Since $\mathcal{E}_{A_0}(\boldsymbol{R}^n)$ is of second category at least one of the sets on the right hand side, say $\operatorname{Im}\{\pi\colon S_{1/n_0} \to \mathcal{E}_{A_0}(\boldsymbol{R}^n)\}$ is of second category. But then, by the

Banach theorem,

$$\pi: S_{1/n_0} \to \mathcal{E}_{A_0}(\boldsymbol{R}^n)$$

is surjective and open. For $u \in \mathcal{E}_{A_0}(\boldsymbol{R}^n)$ let us denote by $\tilde{u}$ its holomorphic extension to $U(1/n_0)$, let $0 < \varepsilon < 1/n_0$, and set

$$\|\tilde{u}\|_\varepsilon = \sup_{z \in U(\varepsilon)} |\tilde{u}(z)|,$$

$$\|u\|_{m,r} = \sup_{|\alpha| \leq m} \sup_{|x| < r} |D^\alpha u(x)|.$$

As π is open from S_{1/n_0} to $\mathcal{E}_{A_0}(\boldsymbol{R}^n)$, for given ε we can find m and r and $\sigma > 0$ such that

$$\pi\{u \in S_{1/n_0} | \|\tilde{u}\|_\varepsilon < 1\} \supset \{u \in \mathcal{E}_A(\boldsymbol{R}^n) | \|u\|_{m,r} < \sigma\}.$$

From this we deduce that for any $u \in \mathcal{E}_{A_0}(\boldsymbol{R}^n)$ we have the inequality

$$\|\tilde{u}\|_\varepsilon \leq C \|u\|_{m,r}$$

where $C = 1/\sigma$.

Now, given $\xi \in V$, $A_0(\xi)$ has rank $< p_0$ and we can find $\chi \subset \boldsymbol{C}^{p_0}$ with $|\chi| = 1$ such that

$$A_0(\xi)\chi = 0.$$

Set $u = \chi \exp[\langle \xi, x \rangle]$, where

$$\langle \xi, x \rangle = \sum_{i=1}^{n} \xi_i x_i.$$

We have $A_0(D)u = \exp[\langle \xi, x \rangle] A_0(\xi)\chi = 0$ so that $u \in \mathcal{E}_{A_0}(\boldsymbol{R}^n)$. The above inequality yields then an inequality of the form

$$\exp[\varepsilon|\xi|] \leq C(1 + |\xi|^m) \exp[r|\mathrm{Re}\,\xi|].$$

For $C_1 > 0$ large enough, we have $1 + |\xi|^m < C_1 \exp[\varepsilon|\xi|/2]$, $\forall \xi \in \boldsymbol{C}^n$, thus we obtain

$$\exp[\varepsilon|\xi|/2] \leq CC_1 \exp[r|\mathrm{Re}\,\xi|].$$

From this, taking logarithms, we get for every $\xi \in V$ the inequality

$$\varepsilon|\xi|/2 \leq r|\mathrm{Re}\,\xi| + \log CC_1$$

which is an inequality of type i) considered in proposition 3. Therefore $A_0(D)$ is elliptic.

β) We divide the proof in several steps. We set for $r > 0$

$$B(r) = \{x \in \boldsymbol{R}^n \big| |x| < r\} \qquad \text{where } |x| = \Big(\sum_1^n x_j^2\Big)^{\frac{1}{2}}$$

and for $u \in \mathcal{E}(B(r)) = \{C^\infty \text{ functions on } B(r)\}$ we set

$$\|u\|_{W^k(B(r))} = \Big\{\sum_{|\alpha| \leq k} \int_{B(r)} |D^\alpha u|^2 dx\Big\}^{\frac{1}{2}}.$$

STEP i) (Sobolev lemma). *Let k be an integer, $k > n/2$. Given $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ such that*

$$|u(0)| \leq C_\varepsilon \|u\|_{W^k(B(\varepsilon/2))} \qquad \forall u \in \mathcal{E}(B(\varepsilon)) .$$

For any χ C^∞ with compact support in $\boldsymbol{R}^n$ ($\chi \in \mathcal{D}(\boldsymbol{R}^n)$) we can consider its Fourier transform $\hat{\chi}(\xi) = \int \exp[-i\langle x, \xi\rangle]\chi(x)\,dx$.

Then $\chi(0) = (2\pi)^{-n}\int \hat{\chi}(\xi)\,d\xi$ and we have

$$|\chi(0)| \leq (2\pi)^{-n}\int_{\boldsymbol{R}^n} |\hat{\chi}(\xi)|\,d\xi \leq (2\pi)^{-n}\Big\{\int_{\boldsymbol{R}^n}(1 + |\xi|^2)^{-k} d\xi\Big\}^{\frac{1}{2}}\Big\{\int_{\boldsymbol{R}^n} (1 + |\xi|^2)^k |\hat{\chi}(\xi)|^2 d\xi\Big\}^{\frac{1}{2}}.$$

Now $\chi \to \Big\{\int_{\boldsymbol{R}^n}(1 + |\xi|^2)^k|\hat{\chi}(\xi)|^2 d\xi\Big\}^{\frac{1}{2}}$ and $\chi \to \|\chi\|_{W^k(\boldsymbol{R}^n)}$ are equivalent norms while, since $2k > n$, the integral $\int_{\boldsymbol{R}^n}(1 + |\xi|^2)^{-k} d\xi$ is convergent. Thus

$$|\chi(0)| \leq C\|\chi\|_{W^k(\boldsymbol{R}^n)} .$$

Now given $\varepsilon > 0$ let us choose $\varphi \in \mathcal{D}(\boldsymbol{R}^n)$ with $\operatorname{supp}\varphi \subset B(\varepsilon/2)$ and $\varphi(0) = 1$. Then for $u \in \mathcal{E}(B(\varepsilon))$ we have $\varphi u \in \mathcal{D}(\boldsymbol{R}^n)$ and

$$|u(0)| = |\varphi(0)u(0)| \leq C\|\varphi u\|_{W^k(\boldsymbol{R}^n)} \leq C_\varepsilon \|u\|_{W^k(B(\varepsilon/2))}$$

for some $C_\varepsilon > 0$.

STEP ii). *Let $p(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a differential operator with constant coefficients and of order m (≥ 1). Assume that for C_1, $C_2 > 0$ we have*

$$|p(i\xi)| \geq C_1|\xi|^m - C_2 \qquad \forall \xi \in \boldsymbol{R}^n .$$

There exists a constant $C > 0$ depending only on p such that, if $r > 0$ and $u \in \mathcal{E}(B(r))$ satisfies $p(D)u = 0$, then for any $\varepsilon > 0$, $1 \geq \delta > 0$ with $\varepsilon + \delta < r$ we have

$$\|u\|_{W^{(m)}(B(\varepsilon))} \leq C \sum_{j=1}^{m} \delta^{-j} \|u\|_{W^{(m-j)}(B(\varepsilon+\delta))}.$$

Let $\vartheta(t)$ be a real-valued C^∞ function on $\boldsymbol{R}$ with

$$\vartheta(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2. \end{cases}$$

For every ε, $\delta > 0$ set

$$\varphi_{\varepsilon,\delta}(x) = \vartheta\left(\frac{|x| - \varepsilon + \delta}{\delta}\right) = \begin{cases} 1 & \text{if } |x| \leq \varepsilon \\ 0 & \text{if } |x| \geq \varepsilon + \delta \end{cases}$$

and note that for $0 < \delta \leq 1$ we have, with a constant C_α independent of ε and δ,

$$|D^\alpha \varphi_{\varepsilon,\delta}| \leq C_\alpha \delta^{-|\alpha|}.$$

Set $p^{(\alpha)}(\xi) = \partial^{|\alpha|} p(\xi)/\partial \xi^\alpha$. We have for $u \in \mathcal{E}(B(r))$ satisfying $p(D)u = 0$:

$$(*) \qquad \left(\int_{B(\varepsilon+\delta)} |p(D)(\varphi_{\varepsilon,\delta} u)|^2 dx \right)^{\frac{1}{2}} = \left(\int_{B(\varepsilon+\delta)} \left| \sum_{\alpha \neq 0} \frac{1}{\alpha!} D^\alpha \varphi_{\varepsilon,\delta} p^{(\alpha)}(D) u \right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \sum_{\alpha \neq 0} \frac{1}{\alpha!} \left(\int_{B(\varepsilon+\delta)} \left| D^\alpha \varphi_{\varepsilon,\delta} p^{(\alpha)}(D) u \right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C \sum_{j=1}^{m} \delta^{-j} \|u\|_{W^{m-j}(B(\varepsilon+\delta))}$$

for some constant $C > 0$ independent of ε and δ.

On the other hand

$$(2\pi)^{-n} \int_{B(\varepsilon+\delta)} |p(D)(\varphi_{\varepsilon,\delta} u)|^2 dx = \int_{\boldsymbol{R}^n} |p(i\xi) \widehat{\varphi_{\varepsilon,\delta} u}(\xi)|^2 d\xi$$

$$\geq C_1^2 \int_{\boldsymbol{R}^n} |\xi|^{2m} |\widehat{\varphi_{\varepsilon,\delta} u}(\xi)|^2 d\xi - 2C_2 \int_{\boldsymbol{R}^n} |p(i\xi)| |\widehat{\varphi_{\varepsilon,\delta} u}(\xi)|^2 d\xi - C_2^2 \int_{\boldsymbol{R}^n} \left| \widehat{\varphi_{\varepsilon,\delta} u}(\xi) \right|^2 d\xi.$$

From this inequality we deduce an estimate of type

$$(**) \qquad \|\varphi_{\varepsilon,\delta} u\|_{W^{(m)}(\boldsymbol{R}^n)} \leq C' \left(\int_{B(\varepsilon,\delta)} \left| p(D)(\varphi_{\varepsilon,\delta} u) \right|^2 dx \right)^{\frac{1}{2}} + C'' \|\varphi_{\varepsilon,\delta} u\|_{W^{(m-1)}(\boldsymbol{R}^n)}$$

with constants C', $C''>0$ independent of ε and δ. Now

$$\|\varphi_{\varepsilon,\delta}u\|_{W^{(m-1)}(\mathbf{R}^n)} \leq C''' \sum_{0}^{m-1} \delta^{-j}\|u\|_{W^{(m-j-1)}(B(\varepsilon+\delta))}$$

with $C'''>0$ independent of ε and δ, while

$$\|\varphi_{\varepsilon,\delta}u\|_{W^{(m)}(\mathbf{R}^n)} \geq \|u\|_{W^{(m)}(B(\varepsilon))},$$

as $\varphi_{\varepsilon,\delta}=1$ on $B(\varepsilon)$. From these inequalities and from (*) and (**) we deduce the desired estimate.

STEP iii). *With the same assumptions of the previous step if $\varepsilon+(l+1)\delta<r$ we have*

$$\|u\|_{W^{(m+l)}(B(\varepsilon))} \leq C(C+1)^l(1+n)^l \sum_{j=1}^{m} \delta^{-j-l}\|u\|_{W^{(m-j)}(B(\varepsilon+(l+1)\delta))}.$$

This inequality reduces to the one of step ii) for $l=0$. We can thus prove the statement by induction on l, assuming the statement true for l and proving it for $l+1$.

We have

$$\|u\|_{W^{(m+l+1)}(B(\varepsilon))} \leq \sum_{h=1}^{n} \|D_h u\|_{W^{(m+l)}(B(\varepsilon))}.$$

Since $p(D)$ has constant coefficients, $v=D_h u$ is also a solution of $p(D)v=0$ on $B(r)$. Therefore by the inductive hypothesis we have

$$\begin{aligned}\|D_h u\|_{W^{(m+l)}(B(\varepsilon))} &\leq C(C+1)^l(1+n)^l \sum_{j=1}^{m} \delta^{-j-l}\|Du\|_{W^{(m-j)}(B(\varepsilon+(l+1)\delta))}\\ &\leq C(C+1)^l(1+n)^l \sum_{j=2}^{m} \delta^{-l-j}\|u\|_{W^{(m-j+1)}(B(\varepsilon+(l+1)\delta))}\\ &\qquad + C(C+1)^l(1+n)^l\delta^{-l-1}\|u\|_{W^{(m)}(B(\varepsilon+(l+1)\delta))}.\end{aligned}$$

We estimate the last term using step ii). We obtain

$$\begin{aligned}\|D_h u\|_{W^{(m+l)}(B(\varepsilon))} &\leq C(C+1)^l(1+n)^l\Big\{\sum_{j=2}^{m} \delta^{-j-l}\|u\|_{W^{(m-j+1)}(B(\varepsilon+(l+1)\delta))} +\\ &\qquad + \delta^{-l-1}C\sum_{j=1}^{m}\delta^{-j}\|u\|_{W^{(m-j)}(B(\varepsilon+(l+2)\delta))}\Big\}\\ &\leq C(C+1)^l(1+n)^l(C+1)\sum_{j=1}^{m}\delta^{-j-l-1}\|u\|_{W^{(m-j)}(B(\varepsilon+(l+2)\delta))}.\end{aligned}$$

These inequalities for $h = 1, \dots, n$, summed up term by term, give the desired estimate.

STEP iv). *Let now* $\varepsilon > 0$, $r_1 > 0$, $r_2 > 0$, $\varepsilon + r_1 + r_2 < r$.

We apply the previous inequality with $\delta = r_1/(l+1)$ and $\varepsilon + r_2$ instead of ε so that we obtain

$$\|u\|_{W^{(m+l)}(B(\varepsilon+r_2))} \le mC(C+1)^l(1+n)^l(l+1)^{m+l} r_1^{-m-l} \|u\|_{W^{(m-1)}(B(\varepsilon+r_1+r_2))}.$$

By the Sobolev lemma, if $k > n/2$, we obtain

$$\sup_{|x|\le r_2} |D^\alpha u(x)| \leqslant C_{2\varepsilon}\|D^\alpha u\|_{W^{(k)}(B(\varepsilon+r_2))} \le C_{2\varepsilon}\|u\|_{W^{(k+|\alpha|)}(B(\varepsilon+r_2))}$$
$$\le C_{2\varepsilon} mC[(C+1)(n+1)]^{k+|\alpha|-m}(k+|\alpha|-m+1)^{k+|\alpha|} r_1^{-k-|\alpha|}\|u\|_{W^{(m-1)}(B(\varepsilon+r_1+r_2))}$$

for $m + l = k + |\alpha|$ and $|\alpha| \ge m$. In particular, taking into account that

$$|\alpha|! \le n^{|\alpha|}\alpha! \quad \left(n^{|k|} = \sum_{|\alpha|=k} \frac{|k|!}{\alpha!}\right) \quad \text{and that} \quad \sqrt[|k|]{|k|!} \simeq |k|/e$$

we obtain from the previous estimate an inequality of type

$$\sup_{|x|\le r_2} \frac{|D^\alpha u(x)|}{\alpha!} \le C\sigma^{|\alpha|}$$

with convenient $C > 0$ and $\sigma > 0$.

Let now $R_h(x)$ denote the remainder of the Taylor expansion centered at the origin of $u(x)$ up to order h. We have for $|x| \le r_2$ that

$$|R_h(x)| \le C\sigma^{h+1}|x|^{h+1}n^{h+1}$$

as there are $\binom{n+h}{h+1} \le n^{h+1}$ terms in the Lagrange expression of the remainder.

From this it follows that in a neighborhood of 0 the Taylor series of $u(x)$ converges to $u(x)$.

What we have said for $x = 0$ can be repeated for any other point. Thus u is real analytic.

STEP v). *If* $f \in \mathcal{E}^{p_0}(\Omega)$ *and* $A_0(D)f = 0$ *then for every* $p(\xi) \in \mathfrak{b}$ *and every component* f_i *of* f $1 \le i \le p_0$ *we have* $p(D)f_i = 0$. *As* A_0 *is elliptic there exists a* $p \in \mathfrak{b}$ *with* $|p(i\xi)| \ge C_1|\xi|^{\deg p} - C_2$, $\forall \xi \in \boldsymbol{R}^n$. *Thus each component* f_i *of* f *is real analytic.*

c) We end this section by remarking that in the Hilbert complex of the previous section $(\mathcal{E}^*(\Omega), A_*)$, if the first operator $A_0(D)$ is elliptic then we have

$$\mathcal{E}_{A_0} = \mathcal{A}_{A_0}$$

as every germ of C^∞ solution is also real analytic. It follows that in this case notions of C^∞-convexity and analytic convexity coincide.

We mention here the following lemma that will be used later on (cf. [4] theorem 2).

LEMMA 1. *Let $A_0(D)\colon \mathcal{E}^{p_0}(\Omega) \to \mathcal{E}^{p_1}(\Omega)$ be an elliptic operator with constant coefficients. Consider $\boldsymbol{R}^n$ imbedded in $\boldsymbol{C}^n$ in the natural way. For any open set $\Omega \subset \boldsymbol{R}^n$ there exists an Ω-connected neighborhood $\tilde{\Omega} \subset \boldsymbol{C}^n$ such that any $u \in \mathcal{E}^{p_0}(\Omega)$ with $A_0(D)u = 0$ has a unique holomorphic extension $\tilde{u}$ to $\tilde{\Omega}$.*

3. – Formal Cauchy problem.

a) Let $\xi_1, \dots, \xi_n$, $\eta_1, \dots, \eta_k$ be $N = n + k$ indeterminates and set $\mathcal{P}_N = \boldsymbol{C}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_k]$, $\mathcal{P}_n = \boldsymbol{C}[\xi_1, \dots, \xi_n]$ and identify $\mathcal{P}_n$ to the subring of $\mathcal{P}_N$ of those polynomials independent of $\eta_1, \dots, \eta_k$. By the inclusion $\mathcal{P}_n \subset \mathcal{P}_N$ every $\mathcal{P}_N$-module F can be considered as a $\mathcal{P}_n$-module; we denote by $(F)_n$ the module F with its structure as a $\mathcal{P}_n$-module.

Let us consider in $\boldsymbol{R}^N$, where $x_1, \dots, x_n$, $y_1, \dots, y_k$ denote cartesian coordinates, a Hilbert complex

$$(1)\qquad (\mathcal{E}^*(\Omega), S_*) \equiv \left\{\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \dots\right\}$$

defined on all open sets $\Omega \subset \boldsymbol{R}^n$. It is obtained from a Hilbert resolution

$$(2)\qquad 0 \longleftarrow M \longleftarrow \mathcal{P}_N^{s_0} \xleftarrow{{}^tS_0(\xi,\eta)} \mathcal{P}_N^{s_1} \xleftarrow{{}^tS_1(\xi,\eta)} \mathcal{P}_N^{s_2} \longleftarrow \dots$$

of the $\mathcal{P}_N$-module $M = \operatorname{Coker}\{{}^tS_0(\xi,\eta)\colon \mathcal{P}_N^{s_1} \to \mathcal{P}_N^{s_0}\}$.

We set $\boldsymbol{R}^n = \{(x, y) \in \boldsymbol{R}^N | y = 0\}$. We will say that $\boldsymbol{R}^n$ *is* (*algebraically*) *non characteristic for the complex* $(\mathcal{E}^*, S_*)$ *if* $(M)_n$ *is a* $\mathcal{P}_n$*-module of finite type.*

Assume that $\boldsymbol{R}^n$ is algebraically non characteristic for the complex (1). Then we can consider a Hilbert resolution of the $\mathcal{P}_n$-module $(M)_n$:

$$(3)\qquad 0 \longleftarrow (M)_n \longleftarrow \mathcal{P}_n^{r_0} \xleftarrow{{}^tR_0(\xi)} \mathcal{P}_n^{r_1} \xleftarrow{{}^tR_1(\xi)} \mathcal{P}_n^{r_2} \longleftarrow \dots .$$

Setting $\omega = \Omega \cap \boldsymbol{R}^n$ we can consider the Hilbert complex on $\boldsymbol{R}^n$,

$$(4) \qquad (\mathcal{E}^*(\omega), R_*) \equiv \left\{\mathcal{E}^{r_0}(\omega) \xrightarrow{R_0(D_x)} \mathcal{E}^{r_1}(\omega) \xrightarrow{R_1(D_x)} \mathcal{E}^{r_2}(\omega) \longrightarrow \ldots\right\}$$

associated to (3). We will call (4) *a complex of Cauchy data* for the given complex (1) on $\boldsymbol{R}^N$ and (1) *a complex suspending the complex* (4) from $\boldsymbol{R}^n$ to $\boldsymbol{R}^N$. Now we can consider $\mathcal{P}_N$ as an infinite free $\mathcal{P}_n$-module (with generators η^α, $\alpha \in \boldsymbol{N}^k$) and therefore the resolution (2) can be viewed as another (infinite) free resolution of the $\mathcal{P}_n$-module $(M)_n$. It follows that there exist $\mathcal{P}_n$-homomorphisms

$$^t\tau_j\colon \mathcal{P}_n^{r_j} \to \mathcal{P}_N^{s_j}\,; \qquad ^t\varrho_j\colon \mathcal{P}_N^{s_j} \to \mathcal{P}_n^{r_j}$$

($j = 0, 1, 2, \ldots$) so that (2) and (3) factor each one through the other.

This means that in the diagram of $\mathcal{P}_n$-homomorphisms,

$$(*) \qquad \begin{array}{ccccccccc}
0 \longleftarrow & M & \longleftarrow & \mathcal{P}_N^{s_0} & \xleftarrow{^tS_0(\xi,\eta)} & \mathcal{P}_N^{s_1} & \xleftarrow{^tS_1(\xi,\eta)} & \mathcal{P}_N^{s_2} & \longleftarrow \ldots \\
 & \Vert & & {\scriptstyle ^t\tau_0}\uparrow\downarrow{\scriptstyle ^t\varrho_0} & & {\scriptstyle ^t\tau_1}\uparrow\downarrow{\scriptstyle ^t\varrho_1} & & {\scriptstyle ^t\tau_2}\uparrow\downarrow{\scriptstyle ^t\varrho_2} & \\
0 \longleftarrow & (M)_n & \longleftarrow & \mathcal{P}_n^{r_0} & \xleftarrow{^tR_0(\xi)} & \mathcal{P}_n^{r_1} & \xleftarrow{^tR_1(\xi)} & \mathcal{P}_n^{r_2} & \longleftarrow \ldots
\end{array}$$

erasing the homomorphisms $^t\varrho$ or $^t\tau$, we get commutativity. In particular the collection of homomorphisms $\{^t\varrho_j \circ {}^t\tau_j\}$ gives a factorization of the resolution (3) through itself, and similarly $\{^t\tau_j \circ {}^t\varrho_j\}$ gives a factorization of the resolution (2) through itself as a resolution of $(M)_n$ by infinite free $\mathcal{P}_n$-modules. This implies that these maps must be homotopic as $\mathcal{P}_n$-homomorphisms to the identity map.

Therefore there exist $\mathcal{P}_n$-homomorphisms

$$^t\gamma_j\colon \mathcal{P}_N^{s_j} \to \mathcal{P}_N^{s_{j+1}} \quad \text{and} \quad ^t\nu_j\colon \mathcal{P}_n^{r_j} \to \mathcal{P}_n^{r_{j+1}}$$

for $j = 0, 1, 2, \ldots$, such that

$$^t\varrho_0 \circ {}^t\tau_0 = \mathrm{id}_{\mathcal{P}_n^{r_0}} + {}^tR_0(\xi)\, {}^t\nu_0$$

and

$$^t\varrho_j \circ {}^t\tau_0 = \mathrm{id}_{\mathcal{P}_n^{r_j}} + {}^t\nu_{j-1}\, {}^tR_{j-1}(\xi) + {}^tR_j(\xi)\, {}^t\nu_j \quad \text{for } j \geqq 1$$

and similarly

$$^t\tau_0 \circ {}^t\varrho_0 = \mathrm{id}_{\mathcal{P}_N^{s_0}} + {}^tS_0(\xi, \eta) \circ {}^t\gamma_0$$

and

$$ {}^t\tau_j \circ {}^t\varrho_j = \mathrm{id}_{\mathcal{T}_N^{s_j}} + {}^t\nu_{j-1}{}^tS_{j-1}(\xi, \eta) + {}^tS_j(\xi, \eta)\, {}^t\nu_j \quad \text{for } j \geqq 1 . $$

b) Let F be a left $\mathcal{T}_n$-module and let us denote by $p(D_x)f$ the action of the polynomial $p(\xi) \in \mathcal{T}_n$ on F. We can take for instance for F any one of the following spaces

$\mathcal{E}(\omega)$ = space of C^∞ functions on an open set $\omega \subset \boldsymbol{R}^n$

$\mathcal{A}(\omega)$ = space of real analytic functions on an open set $\omega \subset \boldsymbol{R}^n$

$\mathcal{E}_x$ = space of germs of C^∞ functions on $\boldsymbol{R}^n$ at $x \in \boldsymbol{R}^n$

$\mathcal{A}_x$ = space of germs of real analytic functions on $\boldsymbol{R}^n$ at $x \in \boldsymbol{R}^n$

$\boldsymbol{C}\{\{x\}\}$ = space of formal power series in $x = (x_1, \ldots, x_n)$

the polynomial $p(\xi)$ operating on $f \in F$ as a differential polynomial: $p(\xi)\cdot f = p(D_x)f$.

Consider the space $\mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F)$. This has a structure of a $\mathcal{T}_N$-module if we define for any $\tilde{f} \in \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F)$

$$ \big(p(\xi, \eta)\cdot \tilde{f}\big)(X) = \tilde{f}\big(p(\xi, \eta)X\big) \qquad \forall X \in \mathcal{T}_N . $$

The element $\tilde{f} \in \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F)$ is known as soon as we know the values

$$ \tilde{f}(\eta^\alpha) = f_\alpha \in F \quad \forall \alpha \in \boldsymbol{N}^k. $$

We can therefore represent the element $\tilde{f}$ with the formal power series with coefficients in F $\sum (1/\alpha!) f_\alpha y^\alpha = f(y)$. We have thus defined an identification

$$ \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F) \cong F\{\{y\}\} \tag{5} $$

of $\mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F)$ with the space $F\{\{y\}\}$ of formal power series with coefficients in F in the indeterminates $y = (y_1, \ldots, y_k)$.

Let $f(y) \in F\{\{y\}\}$ be the element corresponding to $\tilde{f} \in \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{T}_N, F)$. We have

$$ \tilde{f}(\eta^\alpha) = D_y^\alpha f(y)|_{y=0} \, . $$

Given $p(\xi, \eta) \in \mathcal{T}_N$, $p(\xi, \eta) = \sum p_\beta(\xi)\eta^\beta$ we have

$$ \begin{aligned} \big(p(\xi, \eta)\tilde{f}\big)(\eta^\alpha) &= \tilde{f}\big(p(\xi, \eta)\eta^\alpha\big) \\ &= \sum \tilde{f}\big(p_\beta(\xi)\eta^{\beta+\alpha}\big) \\ &= \sum p_\beta(D_x) f(\eta^{\beta+\alpha}) \\ &= \sum p_\beta(D_x) D_y^{\alpha+\beta} f(y)|_{y=0} \\ &= D_y^\alpha\big(p(D_x, D_y) f(y)\big)|_{y=0} \end{aligned} $$

therefore in the correspondence (5) we have

$$p(\xi, \eta)\cdot f = p(D_x, D_y)f(y)$$

which exhibits the structure of $F\{\{y\}\}$ as a $\mathcal{S}_N$-module:

$$p(\xi, \eta)\cdot f(y) = p(D_x, D_y)f(y)\,. \tag{6}$$

Let ${}^t\tau\colon \mathcal{S}_n^r \to \mathcal{S}_N^s$ be a $\mathcal{S}_n$-homomorphism. Applying the functor $\mathrm{Hom}_{\mathcal{S}_n}(\cdot, F)$ we obtain a homomorphism

$$\tau\colon F\{\{y\}\}^s \to F^r\,. \tag{7}$$

To describe τ we proceed as follows. Let $\tilde{f} \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}_N^s, F)$ and let $\tilde{g} = \tilde{f}\circ {}^t\tau$ denote the corresponding element by τ. Let $e_1 = {}^t(1, \ldots, 0), \ldots, e_r = {}^t(0, \ldots, 1)$ be the canonical basis of $\mathcal{S}_n^r$ and let $\sigma_1 = {}^t(1, \ldots, 0), \ldots, \sigma_s = {}^t(0, \ldots, 1)$ denote the canonical basis of $\mathcal{S}_N^s$ so that ${}^t\tau(e_i) = \sum \tau_{ij}(\xi, \eta)\sigma_j$. We set $\tau_{ij}(\xi, \eta) = \sum_\beta \tau_{ij}^\beta(\xi)\eta^\beta$ and define

$$f_j(y) = \sum \frac{1}{\alpha!}\tilde{f}(\eta^\alpha\sigma_j)y^\alpha\,.$$

Then

$$\begin{aligned} g_i = \tilde{g}(e_i) = \tilde{f}({}^t\tau(e_i)) &= \tilde{f}\big(\sum \tau_{ij}(\xi, \eta)\sigma_j\big) \\ &= \tilde{f}\big(\sum \tau_{ij}^\beta(\xi)\eta^\beta\sigma_j\big) \\ &= \sum \tau_{ij}^\beta(D_x)\tilde{f}(\eta^\beta\sigma_j) \\ &= \sum \tau_{ij}^\beta(D_x)D_y^\beta f_j(y)\big|_{y=0} \\ &= \sum \tau_{ij}(D_x, D_y)f_j(y)\big|_{y=0}\,. \end{aligned}$$

Setting $\tau(D_x, D_y) = \big(\tau_{ij}(D_x, D_y)\big)$ we thus have for $f(y) \in F\{\{y\}\}^s$

$$\tau(f) = \tau(D_x, D_y)f(y)\big|_{y=0}\,. \tag{8}$$

Similarly let ${}^t\varrho\colon \mathcal{S}_N^S \to \mathcal{S}_n^r$ be a $\mathcal{S}_n$-homomorphism. Applying the functor $\mathrm{Hom}_{\mathcal{S}_n}(\cdot, F)$ we obtain a homomorphism

$$\varrho\colon F^r \to F\{\{y\}\}^s\,. \tag{9}$$

With the same choice of canonical bases in $\mathcal{T}_N^s$ and $\mathcal{T}_n^r$ as before we will have ${}^t\varrho(\eta^\alpha\sigma_i) = \sum \varrho_{ij}^{(\alpha)}(\xi)e_j$. Given $\tilde{g} \in \operatorname{Hom}_{\mathcal{T}_n}(\mathcal{T}_n^r, F)$ we set $\tilde{f} = \tilde{g} \circ {}^t\varrho$ for the corresponding element by ϱ. Then, setting

$$f_{i\alpha} = \tilde{f}(\eta^\alpha\sigma_i)\,, \qquad g_j = \tilde{g}(e_j)\,,$$

we have,

$$\begin{aligned} f_{i\alpha} = \tilde{f}(\eta^\alpha\sigma_i) &= \tilde{g}\big(\textstyle\sum \varrho_{ij}^{(\alpha)}(\xi)e_j\big) \\ &= \textstyle\sum \varrho_{ij}^{(\alpha)}(D_x)\tilde{g}(e_j) \\ &= \textstyle\sum \varrho_{ij}^{(\alpha)}(D_x)g_j\,. \end{aligned}$$

It follows that, setting $\varrho^{(\alpha)}(D_x) = \big(\varrho_{ij}^{(\alpha)}(D_x)\big)$, we have for $g \in F^r$

$$\varrho(g) = \sum \frac{1}{\alpha!}\big(\varrho^{(\alpha)}(D_x)g\big)y^\alpha\,. \tag{10}$$

c) Applying the functor $\operatorname{Hom}_{\mathcal{T}_n}(\cdot, F)$ to the diagram $(*)$ we get from the top line the complex

$$(F^*\{\{y\}\}, S_*(D)) \equiv \Big\{F\{\{y\}\}^{s_0} \xrightarrow{S_0(D)} F\{\{y\}\}^{s_1} \xrightarrow{S_1(D)} F\{\{y\}\}^{s_2} \longrightarrow \dots\Big\}$$

where $D = (D_x, D_y) = (D_{x_1}, \dots, D_{x_n}, D_{y_1}, \dots, D_{y_k})$. Its cohomology will be denoted by $H^j(F^*\{\{y\}\}, S_*(D))$, $j \geq 0$. As the top line of $(*)$ can be considered as a resolution of $(M)_n$ by infinite free $\mathcal{T}_n$ modules $\mathcal{T}_N^{s_j}$ (and hence by projective modules) we must have

$$H^j(F^*\{\{y\}\}, S_*(D)) \simeq \operatorname{Ext}^j_{\mathcal{T}_n}\big((M)_n, F\big)\,.$$

From the bottom line of diagram $(*)$ we obtain the complex

$$(F^*, R_*) \equiv F^{r_0} \xrightarrow{R_0(D)} F^{r_1} \xrightarrow{R_1(D)} F^{r_2} \longrightarrow \dots$$

where here $D = D_x$. Its cohomology will be denoted by $H^j(F^*, R_*)$, $j \geq 0$, and we must have

$$H^j(F^*, R_*) \simeq \operatorname{Ext}^j_{\mathcal{T}_n}\big((M)_n, F\big)\,.$$

In particular we obtain the following

PROPOSITION 4 (Formal Cauchy problem). *For every $j \geq 0$ we have a natural isomorphism*

$$H^j(F^*\{\{y\}\}, S_*) \simeq H^j(F^*, R_*)\,.$$

This is induced by the maps

$$\tau_j : F\{\{y\}\}^{s_j} \to F^{r_j}$$

and

$$\varrho_j : F^{r_j} \to F\{\{y\}\}^{s_j}$$

given by the formulae (8) *and* (10) *given above.*

d) Let $x \in \boldsymbol{R}^n$ and take $F = \mathcal{A}_x$ and set for $j = 0$

$$\left(\mathcal{A}_x\{\{y\}\}\right)_{S_0} = H^0\left(\mathcal{A}_x\{\{y\}\}^*, S_*\right), \quad (\mathcal{A}_x)_{R_0} = H^0(\mathcal{A}_x^*, R_*) .$$

According to the previous proposition τ_0 induces an isomorphism

$$\tau_0 : \left(\mathcal{A}_x\{\{y\}\}\right)_{S_0} \xrightarrow{\sim} (\mathcal{A}_x)_{R_0} .$$

Let $\mathcal{A}_{S_0}$ denote the sheaf of germs $f \in \mathcal{A}^{s_0}$ of real analytic (complex valued) functions on $\boldsymbol{R}^N$ satisfying $S_0(D)f = 0$. Similarly let $\mathcal{A}_{R_0}$ denote the sheaf of germs $f \in \mathcal{A}^{r_0}$ of real analytic (complex valued) functions on $\boldsymbol{R}^n$ satisfying $R_0(D)f = 0$. For $x \in \boldsymbol{R}^n$ and $(x, 0) \in \boldsymbol{R}^N$ we have $(\mathcal{A}_x)_{R_0} = (\mathcal{A}_{R_0})_x$ and we have an inclusion

$$(\mathcal{A}_{S_0})_{(x,0)} \hookrightarrow \left(\mathcal{A}_x\{\{y\}\}\right)_{S_0} .$$

In particular τ_0 induces an injective map

$$\tau_0 : (\mathcal{A}_{S_0})_{(x,0)} \hookrightarrow (\mathcal{A}_{R_0})_x$$

and therefore, for every open set $\omega \subset \boldsymbol{R}^n$, induces natural homomorphisms

$$\tau_* : H^j(\omega, \mathcal{A}_{S_0}) \to H^j(\omega, \mathcal{A}_{R_0}) \qquad \forall j \geq 0 .$$

If Ω describes a fundamental system of open neighborhoods of ω in $\boldsymbol{R}^N$ we have

$$H^j(\omega, \mathcal{A}_{S_0}) = \varinjlim_{\Omega \supset \omega} H^j(\Omega, \mathcal{A}_{S_0}) .$$

We want to investigate under which conditions the map $\tau_0 : (\mathcal{A}_{S_0})_{(x,0)} \to$ $\to (\mathcal{A}_{R_0})_x$ is an isomorphism for every $x \in \boldsymbol{R}^n$. Since the operators S_0 and R_0 have constant coefficients this is the case if it is so for $x = 0 \in \boldsymbol{R}^n$.

Associated to the given differential operator $S_0(D) = S_0(D_x, D_y)$ we can consider both the characteristic variety V and the asymptotic variety V_0.

If $\mathfrak{b}$ is the ideal of minor determinants of order s_0 of the matrix ${}^tS_0(\xi, \eta)$, $(\xi, \eta) = (\xi, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$, and if $\mathfrak{b}'$ is the ideal $\mathfrak{b}' = \{p \in \boldsymbol{C}[\xi, \eta] | pM = 0\}$ we have that $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{b}'}$ is the ideal of polynomials vanishing on V. By the same type of argument used in the proof of proposition 3 one establishes the following.

PROPOSITION 5. *Let V and V_0 denote the characteristic and the asymptotic variety associated to the differential operator $S_0(D)$. The following conditions are equivalent*

i) *for some constants $C_1, C_2 > 0$ we have*

$$|\eta| \leq C_1|\xi| + C_2 \qquad \forall (\xi, \eta) \in V$$

ii) *for some constant $C > 0$ we have*

$$|\eta| \leq C|\xi| \qquad \forall (\xi, \eta) \in V_0$$

iii) *if $(0, \eta) \in V_0$ then necessarily $\eta = 0$* (i.e. $(0, \eta) \notin V_0$ if $\eta \neq 0$).

PROOF. The implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) are straightforward.

For the implication iii) $\Rightarrow$ i) we remark that if i) does not hold, then for every $\nu = 1, 2, 3, \ldots$ we can find $(\xi_\nu, \eta_\nu) \in V$ with

$$|\eta_\nu| > \nu|\xi_\nu| + \nu .$$

We have $\eta_\nu \neq 0$. Thus we can consider the sequence $(\xi_\nu/|\eta_\nu|, \eta_\nu/|\eta_\nu|)$. By passing to a subsequence $\{k_\nu\}$ we may assume that

$$\left(\frac{\xi_{k_\nu}}{|n_{k_\nu}|}, \frac{\eta_{k_\nu}}{|\eta_{k_\nu}|}\right) \to (0, \sigma) \qquad \text{with } |\sigma| = 1 \text{ (thus } \sigma \neq 0) .$$

As $|\eta_\nu| > \nu$, we have $(0, \sigma) \in V_0$. This contradicts iii). It is worth noticing that the characteristic variety W of the operator $R_0(D_x)$ is the variety of the ideal $\mathfrak{b}''$ where

$$\mathfrak{b}'' = \{p \in \mathcal{T}_n | p(M)_n = 0\} = \mathcal{T}_n \cap \mathfrak{b}'$$

therefore

$$W = \{\xi \in \boldsymbol{C}^n | p(\xi) = 0 \;\; \forall p \in \mathfrak{b}''\} =$$
$$= \{\xi \in \boldsymbol{C}^n | \exists \eta \in \boldsymbol{C}^k \text{ such that } (\xi, \eta) \in V\} .$$

THEOREM 2 (of Cauchy-Kowalewska). *The necessary and sufficient condition for*

$$\tau_0\colon (\mathcal{A}_{S_0})_0 \to (\mathcal{A}_{R_0})_0$$

to be an isomorphism is that the equivalent conditions of proposition 5 *be satisfied.*

PROOF. *Necessity.* Let $(z, w) = (x + is, y + it)$ be holomorphic coordinates in C^N; we identify C^n with the subspace $w = 0$ and we set

$$|(z, w)| = \left\{\sum_1^n |z_j|^2 + \sum_1^k |w_k|^2\right\}^{\frac{1}{2}}, \quad |z| = \left\{\sum_1^n |z_j|^2\right\}^{\frac{1}{2}}, \quad |w| = \left\{\sum_1^k |w_j|^2\right\}^{\frac{1}{2}}.$$

Let $B = \left\{z \in C^n \middle| |z| < 1\right\}$, and denote by $\mathcal{A}_{R_0}(B)$ the space of holomorphic functions u in B, with values in C^{r_0} satisfying the equation $R_0(D_x)u = 0$. For $\varepsilon > 0$ let $U(\varepsilon) = \left\{(z, w) \in C^N \middle| |(z, w)| < \varepsilon\right\}$ and let $\mathcal{A}_{S_0}(U(\varepsilon))$ denote the space of holomorphic functions v on $U(\varepsilon)$, with values in C^{s_0} satisfying $S_0(D_x, D_y)v = 0$. Let

$$\mathcal{S}_\varepsilon = \{(u, v) \in \mathcal{A}_{R_0}(B) \times \mathcal{O}_{S_0}(U(\varepsilon))\, |u = \tau_0(v) \text{ on } U(\varepsilon) \cap C^n\}.$$

The spaces $\mathcal{A}_{R_0}(B)$ and $\mathcal{O}_{S_0}(U(\varepsilon))$ are Fréchet spaces with the topology of uniform convergence on compact sets. The space $\mathcal{S}_\varepsilon$ is a closed subspace of a Fréchet space and thus it is a Fréchet space. Let $\pi\colon \mathcal{S}_\varepsilon \to \mathcal{A}_{R_0}(B)$ denote the natural projection into the first factor; it is a continuous linear map.

By the assumption we must have

$$\mathcal{A}_{R_0}(B) = \bigcup_{n=1}^{\infty} \operatorname{Im}\{\pi\colon \mathcal{S}_{1/n} \to \mathcal{A}_{R_0}(B)\}.$$

Therefore for some value of n, say n_0, the set $\operatorname{Im}\pi\colon S_{1/n_0} \to \mathcal{A}_{R_0}(B)$ must be of second category. But then by Banach theorem

$$\pi\colon \mathcal{S}_{1/n_0} \to \mathcal{A}_{R_0}(B)$$

must be surjective and open. Given $u \in \mathcal{A}_{R_0}(B)$ let us denote by $v \in \mathcal{O}_{S_0}(U(1/n_0))$ any element such that $\tau_0(v) = u$ on $U(1/n_0) \cap C^n$. Let $0 < \varepsilon < 1/n_0$ and let K denote a compact set in B and set

$$\|v\|_\varepsilon = \sup_{U(\varepsilon)} |v|, \qquad \|u\|_K = \sup_K |u|.$$

Given ε, we can find K and $\sigma > 0$ such that

$$\pi\Big\{(u, v) \in \mathcal{S}_{1/n_0} \Big| \|v\|_\varepsilon < 1\Big\} \supset \Big\{u \in \mathcal{A}_{R_0}(B) \Big| \|u\|_K < \sigma\Big\}.$$

From this we deduce the inequality

$$(*) \qquad \|v\|_\varepsilon \leq \frac{1}{\sigma}\, \|u\|_K$$

with $u = \tau_0(v)$.

Given $(\xi, \eta) \in V$ we can find $X \in \boldsymbol{C}^{s_0}$ with $|X| = 1$ such that $S_0(\xi, \eta)X = 0$. Set

$$v = \exp\,[\langle \xi, z\rangle + \langle \eta, w\rangle] X$$

so that $S_0(D_x, D_y)v = \exp\,[\langle \xi, z\rangle + \langle \eta, w\rangle] S_0(\xi, \eta)X = 0$.

The inequality $(*)$ yields an inequality $(c = 1/\sigma)$

$$\begin{aligned}\exp\big[\varepsilon\big(|\xi|^2 + |\eta|^2\big)^{\frac{1}{2}}\big] &\leq c\,\|\tau_0(D_x, D_y)v\|_K \\ &\leq c\exp\big[C_1|\xi|\big]\,|\tau_0(\xi, \eta)X|\,.\end{aligned}$$

Now $|\tau_0(\xi, \eta)X|$ for $|X| = 1$ grows polynomially in ξ, η. Therefore we get an estimate

$$\exp\big[(\varepsilon/2)\big(|\xi|^2 + |\eta|^2\big)^{\frac{1}{2}}\big] \leq c' \exp\big[C_1|\xi|\big]$$

or

$$\exp\big[(\varepsilon/2\sqrt{2})\big(|\xi| + |\eta|\big)\big] \leq c' \exp\big[C_1|\xi|\big]\,.$$

From this we deduce an inequality of the form

$$|\eta| \leq c_1|\xi| + c_2$$

with c_1, $c_2 > 0$ for any $(\xi, \eta) \in V$.

Sufficiency. Let us assume first that $k = 1$ so that $N = n + 1$. Given $u \in (\mathcal{A}_{R_0})_0$ there exists a formal power series v in y with coefficients real analytic in x in a neighborhood of the origin such that $\tau_0(v) = u$.

Because of the assumption there exists in the ideal $\mathfrak{a}$ a homogeneous polynomial $q(\xi, \eta)$ of the form

$$q(\xi, \eta) = \eta^m + \sum_{j=1}^{m} q_j(\xi)\eta^{m-j}$$

Therefore in the ideal $\mathfrak{b}$ there exists a polynomial

$$p(\xi, \eta) = \eta^m + \sum_{j=1}^{m} p_j(\xi)\eta^{m-j}$$

with principal part $q(\xi, \eta)$ and thus with $\deg p_j(\xi) \le j$.

Now each component v_i of v satisfies the equation $p(D_x, D_y)v_i = 0$ while the initial conditions $D_y^k v_i|_{y=0}$ are all real analytic. It follows that the germ of v_i at the origin in C^{n+1} must be real analytic by virtue of the existence and unicity theorem of Cauchy-Kowalewska.

We can then proceed by induction on k. Let

$$\mathcal{P}_{N-1} = C[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_{k-1}] .$$

If M as a $\mathcal{P}_n$-module is finitely generated so it is as a $\mathcal{P}_{N-1}$ module. We can thus construct a commutative diagram of the form

$$\begin{array}{ccccccccccc}
0 \leftarrow & M & \leftarrow & \mathcal{P}_N^{s_0} & \xleftarrow{{}^tS_0} & \mathcal{P}_N^{s_1} & \xleftarrow{{}^tS_1} & \mathcal{P}_N^{s_2} & \leftarrow & \cdots \\
 & \Vert & & \uparrow {}^t\mu_0 & & \uparrow {}^t\mu_1 & & \uparrow {}^t\mu_2 & & \\
0 \leftarrow & (M)_{N-1} & \leftarrow & \mathcal{P}_{N-1}^{u_0} & \xleftarrow{{}^tT_0} & \mathcal{P}_{N-1}^{u_1} & \xleftarrow{{}^tT_1} & \mathcal{P}_{N-1}^{u_2} & \leftarrow & \cdots \\
 & \Vert & & \uparrow {}^t\lambda_0 & & \uparrow {}^t\lambda_1 & & \uparrow {}^t\lambda_2 & & \\
0 \leftarrow & (M)_n & \leftarrow & \mathcal{P}_n^{r_0} & \xleftarrow{{}^tR_0} & \mathcal{P}_n^{r_1} & \xleftarrow{{}^tR_1} & \mathcal{P}_n^{r_2} & \leftarrow & \cdots
\end{array}$$

We may assume that ${}^t\tau_j = {}^t\mu_j \circ {}^t\lambda_j$ (as the action of τ_0 on $\mathcal{A}_{S_0}$ is independent of the choice of the representative because of the homotopy relations). Let V_{N-1} denote the characteristic variety for the differential operator $T_0(D)$ in $N-1$ variables.

We have

$$V_{N-1} = \{(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_{k-1}) \in C^{N-1} | \exists \eta_k \text{ such that } (\xi_1, \dots, \eta_k) \in V\} .$$

Let $(\mathcal{A}_{T_0})_0$ be the set of germs u of analytic functions on C^{N-1} at the origin with values in C^{u_0} such that $T_0(D)u = 0$.

As V_{N-1} satisfies the conditions of proposition 5 we have by the inductive hypothesis an isomorphism

$$\lambda_0\colon (\mathcal{A}_{T_0})_0 \xrightarrow{\sim} (\mathcal{A}_{R_0})_0 .$$

Similarly by the case $k = 1$ treated above we have an isomorphism

$$\mu_0\colon (\mathcal{A}_{S_0})_0 \xrightarrow{\sim} (\mathcal{A}_{T_0})_0 .$$

Since on $\mathcal{A}_{S_0}$ $\tau_0 = \lambda_0 \circ \mu_0$ we get thus an isomorphism

$$\tau_0\colon (\mathcal{A}_{S_0})_0 \xrightarrow{\sim} (\mathcal{A}_{R_0})_0 ;$$

this proves our contention.

COROLLARY. *If the differential operator $S_0(D)$ satisfies the equivalent conditions of proposition* 5 *then the natural map, for ω open in $\boldsymbol{R}^n$,*

$$\tau_*\colon H^j(\omega, \mathcal{A}_{S_0}) \to H^j(\omega, \mathcal{A}_{R_0})$$

is an isomorphism.

Given in $\boldsymbol{R}^N = \boldsymbol{R}^n \times \boldsymbol{R}^k$, where (x, y) are cartesian coordinates, a differential operator

$$S_0(D_x, D_y)\colon \mathcal{E}^{s_0}(\Omega) \to \mathcal{E}^{s_1}(\Omega) \qquad (\Omega \subset \boldsymbol{R}^N)$$

with constant coefficients, we will say that $S_0(D)$, $D = (D_x, D_y)$, is an *operator of Cauchy-Kowalewska with respect to* $\boldsymbol{R}^n$ if the matrix of polynomials $S_0(\xi, \eta)$ satisfies with its characteristic and asymptotic variety the equivalent conditions of proposition 5.

We denote by $\mathcal{E}_{S_0}$ the sheaf of germs of elements $f \in \mathcal{E}^{s_0}$ such that $S_0(D_x, D_y)f = 0$. As a consequence of the previous corollary and of theorem 1 we have the following.

PROPOSITION 6. *Assume that the complex* (1) *is a suspension of the complex* (4). *Assume that the first operator $S_0(D_x, D_y)$ of* (1) *is elliptic and of Cauchy-Kowalewska with respect to $\boldsymbol{R}^n$. Then we have for every open set $\omega \subset \boldsymbol{R}^n$ a natural isomorphism*

$$H^j(\omega, \mathcal{E}_{S_0}) \to H^j(\omega, \mathcal{A}_{R_0})$$

where, for Ω describing a fundamental system of neighborhoods of ω in $\boldsymbol{R}^N$,

$$H^j(\omega, \mathcal{E}_{S_0}) = \varinjlim_{\Omega \supset \omega} H^j(\omega, \mathcal{E}_{S_0}) .$$

4. – Suspending a complex from $\boldsymbol{R}^n$ to $\boldsymbol{R}^N$.

a) Let us consider $\boldsymbol{R}^n$ where $(x_1, \dots, x_n)$ are cartesian coordinates as a subspace of $\boldsymbol{R}^N$ where $(x_1, \dots, x_n, y_1, \dots, y_k)$, $N = n + k$, are cartesian coordinates; $\boldsymbol{R}^n = \{(x, y) \in \boldsymbol{R}^N | y = 0\}$.

Let us consider a Hilbert complex of differential operators on $\boldsymbol{R}^n$

$$(\mathcal{E}^*, A_*) \equiv \mathcal{E}^{p_0}(\omega) \xrightarrow{A_0(D_x)} \mathcal{E}^{p_1}(\omega) \xrightarrow{A_1(D_x)} \mathcal{E}^{p_2}(\omega) \longrightarrow \dots, \qquad \omega \subset \boldsymbol{R}^n,$$

associated to the Hilbert resolution of a certain $\mathcal{T}_n$-module N:

$$0 \longleftarrow N \longleftarrow \mathcal{T}_n^{p_0} \xleftarrow{{}^tA_0(\xi)} \mathcal{T}_n^{p_1} \xleftarrow{{}^tA_1(\xi)} \mathcal{T}_n^{p_2} \longleftarrow \dots$$

where $\mathcal{T}_n = \boldsymbol{C}[\xi_1, \dots, \xi_n]$.

We give now a Hilbert complex of differential operators on $\boldsymbol{R}^N$

$$(\mathcal{E}^*, B_*) \equiv \left\{\mathcal{E}^{q_0}(\Omega) \xrightarrow{B_0(D_x, D_y)} \mathcal{E}^{q_1}(\Omega) \xrightarrow{B_1(D_x, D_y)} \mathcal{E}^{q_2}(\Omega) \longrightarrow \dots\right\}, \qquad \Omega \subset \boldsymbol{R}^N,$$

associated to the Hilbert resolution of a certain $\mathcal{T}_N$-module M:

$$0 \longleftarrow M \longleftarrow \mathcal{T}_N^{q_0} \xleftarrow{{}^tB_0(\xi, \eta)} \mathcal{T}_N^{q_1} \xleftarrow{{}^tB_1(\xi, \eta)} \mathcal{T}_N^{q_2} \longleftarrow \dots$$

where $\mathcal{T}_N = \boldsymbol{C}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_k]$.

We consider M as a $\mathcal{T}_n$-module via the natural inclusion $\mathcal{T}_n \hookrightarrow \mathcal{T}_N$, $M = (M)_n$. We make the following

Assumption: *M as a $\mathcal{T}_n$-module $(M)_n$ is finitely generated and free*:

$$(M)_n \simeq \mathcal{T}_n^{\nu}.$$

This means that $\boldsymbol{R}^n$ is algebraically non characteristic for the complex $(\mathcal{E}^*, B_*)$ and that a complex of Cauchy data for it on $\boldsymbol{R}^n$ reduces to the trivial complex

$$\mathcal{E}^{\nu}(\omega) \to 0.$$

In other words we have a set of free Cauchy data for $B_0(D_x, D_y)$.

Let

$$ {}^t\tau_0 \colon \mathcal{F}_n^{\nu} \to \mathcal{F}_N^{a_0} $$

be a $\mathcal{F}_n$-homomorphism such that the diagram

$$
\begin{array}{ccc}
M & \longleftarrow & \mathcal{F}_N^{a_0} \\
\Vert & & \uparrow {\scriptstyle {}^t\tau_0} \\
(M)_n & \longleftarrow & \mathcal{F}_n^{\nu}
\end{array}
$$

commutes, then the Taylor series in y along $y = 0$ of an element $f \in \mathcal{E}^{a_0}(\Omega)$ with $B_0(D_x, D_y)f = 0$, $f(y) = \sum (1/\alpha!) f_\alpha(x) y^\alpha$, is uniquely determined by the Cauchy data

$$ \tau_0(f) = \tau_0(D_x, D_y) f(y)\big|_{y=0} $$

where $\tau_0(\xi, \eta) = \big(\tau_{0ij}(\xi, \eta)\big)_{1 \le j \le a, 1 \le i \le \nu}$ (cf. n. 3 *b*)).

b) We can consider the complex (4) as a complex of $\mathcal{F}_n$-modules and we can take the tensor product, over $\mathcal{F}_n$, of the complex (4) with the complex (2). We obtain as associated simple complex the complex

$$
(5) \qquad 0 \leftarrow M \otimes_{\mathcal{F}_n} N \leftarrow \mathcal{F}_N^{p_0 a_0}
\begin{array}{c}
\xleftarrow{I \otimes {}^tB_0} \mathcal{F}_N^{p_0 a_1} \\
\oplus \\
\xleftarrow{{}^tA_1 \otimes I} \mathcal{F}_N^{p_1 a_0}
\end{array}
\begin{array}{c}
\xleftarrow{I \otimes {}^tB_1} \mathcal{F}_N^{p_0 a_2} \\
\nwarrow {\scriptstyle {}^tA_0 \otimes I} \\
\oplus \\
\swarrow {\scriptstyle -I \otimes {}^tB_0} \\
\xleftarrow{{}^tA_1 \otimes I} \mathcal{F}_N^{p_2 a_0}
\end{array}
\begin{array}{c}
\\ \\ \mathcal{F}_N^{p_1 a_1} \leftarrow \ldots \\ \\ \\
\end{array}
$$

The same complex could have been obtained by taking first the tensor product of (2) with $\mathcal{F}_N$ over $\mathcal{F}_n$ to obtain the (exact) complex (as $\mathcal{F}_N$ is flat over $\mathcal{F}_n$)

$$
(2') \qquad 0 \longleftarrow N \otimes_{\mathcal{F}_n} \mathcal{F}_N \longleftarrow \mathcal{F}_N^{p_0} \xleftarrow{{}^tA_0(\xi)} \mathcal{F}_N^{p_1} \xleftarrow{{}^tA_1(\xi)} \mathcal{F}_N^{p_2} \longleftarrow \ldots
$$

and then taking the tensor product of (2') with the complex (4).

The complex (5) is associated to the complex of differential operators with constant coefficients in $\boldsymbol{R}^N$

$$
(6)\qquad
\begin{array}{ccccc}
 & & & \xrightarrow{I\otimes B_1(D)} & \mathcal{E}^{p_0q_2}(\Omega) \\
 & \xrightarrow{I\otimes B_0(D)} & \mathcal{E}^{p_0q_1}(\Omega) & & \\
\mathcal{E}^{p_0q_0}(\Omega) & & \oplus & \xrightarrow{\substack{A_0(D)\otimes I \\ I\otimes B_0(D)}} & \mathcal{E}^{p_1q_1}(\Omega)\to\dots \\
 & \xrightarrow{A_0(D)\otimes I} & \mathcal{E}^{p_1q_0}(\Omega) & & \\
 & & & \xrightarrow{A_1(D)\otimes I} & \mathcal{E}^{p_2q_0}(\Omega)
\end{array}
$$

THEOREM 3. *Under the assumption* $(M)_n \cong P_n^\nu$ *we have that*

a) The complex (6) *is the Hilbert complex associated to a Hilbert resolution* (5) *of* $M\otimes_{\mathcal{J}_n} N$.

b) The complex of differential operators on $\boldsymbol{R}^n$

$$
(\mathcal{E}^{q_0}(\omega))^\nu \xrightarrow{\overset{\nu}{\underset{1}{\oplus}} A_0(D_x)} (\mathcal{E}^{q_1}(\omega))^\nu \xrightarrow{\overset{\nu}{\underset{1}{\oplus}} A_1(D_x)} (\mathcal{E}^{q_2}(\omega))^\nu \longrightarrow \dots
$$

is a complex of Cauchy data for the complex (6) *on* $\boldsymbol{R}^n$.

PROOF. *a)* We have the commutative diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longleftarrow & M\otimes_{\mathcal{J}_n} N & \longleftarrow & \mathcal{J}_N^{q_0}\otimes_{\mathcal{J}_n} N & \longleftarrow & \mathcal{J}_N^{q_1}\otimes_{\mathcal{J}_n} N \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longleftarrow & M\otimes_{\mathcal{J}_n} \mathcal{J}_n^{p_0} & \longleftarrow & \mathcal{J}_N^{p_0q_0} & \longleftarrow & \mathcal{J}_N^{p_0q_1} \\
 & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & M\otimes_{\mathcal{J}_n} \mathcal{J}_n^{p_1} & \longleftarrow & \mathcal{J}_N^{p_0q_1} & &
\end{array}
$$

with exact rows and columns. Now one has

$$
M\otimes_{\mathcal{J}_n} N = \operatorname{Tor}^0_{\mathcal{J}_n}(M, N) = \operatorname{Coker}\left\{\begin{array}{c}\mathcal{J}_N^{p_0q_1}\\ \oplus \\ \mathcal{J}_N^{p_1q_0}\end{array} \to \mathcal{J}_N^{p_0q_0}\right\}
$$

and we deduce exactness of (5) in the first two places. The exactness of (5) in the other places follows from the assumption, as the homology of complex (5) is given by the module

$$\operatorname{Tor}^j_{\mathcal{S}_n}(M, N) = \operatorname{Tor}^j_{\mathcal{S}_n}(\mathcal{S}_n^\nu, N) = 0\,, \qquad j \geq 1\,.$$

Note that one has

$$\operatorname{Tor}^j_{\mathcal{S}_n}(M, N \otimes_{\mathcal{S}_n} \mathcal{S}_N) = \operatorname{Tor}^j_{\mathcal{S}_n}((M)_n, N)\,.$$

b) We have $(M \otimes_{\mathcal{S}_n} N)_n \simeq \mathcal{S}_n^\nu \otimes_{\mathcal{S}_n} N \simeq N^\nu$ and this module has a $\mathcal{S}_n$-resolution

$$0 \longleftarrow N^\nu \longleftarrow \mathcal{S}_n^{p_0\nu} \xleftarrow{\bigoplus_1^\nu {}^tA_0(\xi)} \mathcal{S}_n^{p_1\nu} \xleftarrow{\bigoplus_1^\nu {}^tA_1(\xi)} \mathcal{S}_p^{n_2\nu} \longleftarrow \cdots$$

Therefore we obtain the statement *b*) of the theorem.

We note that the first homomorphism in the Hilbert resolution (5)

$$\begin{matrix} \mathcal{S}_N^{p_1q_0} & & \\ & \searrow^{I \otimes {}^tB_0} & \\ \oplus & & \mathcal{S}_N^{p_0q_0} \\ & \nearrow_{{}^tA_0 \otimes I} & \\ \mathcal{S}_N^{p_0q_1} & & \end{matrix}$$

is represented by a matrix ${}^tC = (U, V)$ with p_0q_0 rows and $p_0q_1 + p_1q_0$ columns where $U(V)$ represent the first (last) p_0q_1 (p_1q_0) columns. By a suitable arrangement of the rows we have

$$U = \begin{pmatrix} {}^tB_0 & \ddots & 0 \\ 0 & \ddots & {}^tB_0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} {}^tA_0 & \ddots & 0 \\ 0 & \ddots & {}^tA_0 \end{pmatrix}.$$

It follows that if M_j is a minor determinant of the matrix ${}^tB_0$ of order q_0 then $M_j^{p_0}$ is a minor determinant of the matrix ${}^tC$ of order p_0q_0. Similarly if N_j is a minor determinant of the matrix ${}^tA_0$ of order p_0 then $N_j^{q_0}$ is a minor determinant of the matrix ${}^tC$ of order p_0q_0. We deduce from this the following

PROPOSITION 7. *Let Λ denote the characteristic variety of the operator $A_0(D_x)$ in $\mathbf{C}^n$. Let W denote the characteristic variety of the operator $B_0(D_x, D_y)$*

in $\boldsymbol{C}^N = \boldsymbol{C}^n \times \boldsymbol{C}^k$. *Let* Z *denote the characteristic variety of the operator*

$$C_0(D) = I \otimes B_0(D) \oplus A_0(D) \otimes I$$

in $\boldsymbol{C}^N$.

We have

$$Z \subset (\Lambda \times \boldsymbol{C}^k) \cap W .$$

COROLLARY 1. *If the operator* $B_0(D)$ *is elliptic or of Cauchy-Kowalewska with respect to* $\boldsymbol{R}^n \subset \boldsymbol{R}^N$, *then also the first operator*

$$C_0(D) = I \otimes B_0(D) \oplus A_0(D) \otimes I$$

of the suspended complex (6) *is elliptic or respectively of Cauchy-Kowalewska with respect to* $\boldsymbol{R}^n \subset \boldsymbol{R}^N$.

Indeed if for every $\zeta \in W$ $\big((\xi, \eta) \in W\big)$ we have an inequality

$$|\zeta| \leq C_1 |\mathrm{Re}\,\zeta| + C_2 \quad \big(|\eta| < c_1|\xi| + c_2\big) ,$$

the same is true for every point of $Z \subset W$.

COROLLARY 2. *Let the first operator* $B_0(D)$ *of the « suspending » complex* (3) *be elliptic and of Cauchy-Kowalewska with respect to* $\boldsymbol{R}^n \subset \boldsymbol{R}^N$. *Let* $\mathcal{E}_{C_0}$ *denote the sheaf of germs* $f \in \mathcal{E}^{p_0 q_0}$ *such that* $C_0(D) f = 0$.

For every open set $\omega \subset \boldsymbol{R}^n$ *we have, for every* $j \geq 0$

$$H^j(\omega, \mathcal{E}_{C_0}) \simeq \bigoplus_{1}^{v} H^j(\omega, \mathcal{A}_{A_0})$$

where if Ω *describes a fundamental system of neighborhoods of* ω *in* $\boldsymbol{R}^N$ *we have*

$$H^j(\omega, \mathcal{E}_{C_0}) = \varinjlim_{\Omega \supset \omega} H^j(\omega, \mathcal{E}_{C_0}) .$$

PROOF. Since $B_0(D)$ is elliptic $C_0(D)$ is also elliptic so that with obvious notations we have $\mathcal{E}_{C_0} = \mathcal{A}_{C_0}$. Since $B_0(D)$ is of Cauchy-Kowalewska with respect to $\boldsymbol{R}^n$ so is $C_0(D)$ and therefore we have an isomorphism

$$\tau_* \colon \mathcal{A}_{C_0} \xrightarrow{\sim} \mathcal{A}^v_{A_0}$$

where τ_* is induced by $\tau_0 \otimes I \colon \mathcal{E}_N^{p_0 q_0} \to (\mathcal{E}_n^{p_1})^v$, $\mathcal{E}_N$ denoting the sheaf of germs of C^∞ functions in $N = n + k$ variables and similarly for $\mathcal{E}_n$.

Combining these two facts we deduce a natural isomorphism

$$\tau_*: \mathcal{E}_{C_0} \xrightarrow{\sim} \mathcal{A}^{\nu}_{A_0};$$

this gives the isomorphism of cohomology groups stated in the corollary.

REMARK. For the validity of the isomorphism ($j \geq 0$)

$$H^j(\omega, \mathcal{E}_{C_0}) \simeq \bigoplus_{1}^{\nu} H^j(\omega, \mathcal{A}_{A_0})$$

one actually needs only that $C_0(D)$ be an elliptic operator and of Cauchy-Kowalewska with respect to $\boldsymbol{R}^n \subset \boldsymbol{R}^N$.

Note that the above isomorphism reduces the study of the analytic convexity with respect to the complex (1) to the study of the C^∞ convexity of a suspension (6) of the given complex (1).

c) Let $\mathcal{E}_{B_0}$ be the sheaf of germs of functions $f \in \mathcal{E}^{q_0}$ such that $B_0(D)f = 0$. Consider the double complex

$$K^{r,s} = \mathcal{E}^{p_r q_s}(\Omega)$$

with the operators induced by

$$B_s(D): \mathcal{E}^{q_s}(\Omega) \to \mathcal{E}^{q_{s+1}}(\Omega)$$

and

$$A_r(D): \mathcal{E}^{p_r}(\Omega) \to \mathcal{E}^{p_{r+1}}(\Omega).$$

The simple complex associated to it is the suspended complex (6). By considering the spectral sequence of this double complex we obtain in particular the following

PROPOSITION 8. *Let Ω be an open set which is C^∞ convex for the suspending complex* (3) *(i.e. such that $(\mathcal{E}^*(\Omega), B_*)$ is exact).*

Then the cohomology on Ω of the suspended complex (6) *is naturally isomorphic to the cohomology of the complex*

$$\mathcal{E}_{B_0}(\Omega)^{p_0} \xrightarrow{A_0(D_x)} \mathcal{E}_{B_0}(\Omega)^{p_1} \xrightarrow{A_1(D_x)} \mathcal{E}_{B_0}(\Omega)^{p_2} \longrightarrow \dots$$

d) *Examples* (α). Let $N = n + 1$ and consider on $\boldsymbol{R}^{n+1}$ as a suspending complex (3) the complex

(3) $$\mathcal{E}(\Omega) \xrightarrow{\Delta} \mathcal{E}(\Omega) \longrightarrow 0$$

where $\Delta = -\sum_{1}^{n} \partial^2/\partial x_i^2 - \partial/\partial y^2$ is the Laplace operator in $n+1$ variables. This corresponds to the Hilbert resolution

$$0 \longleftarrow M \longleftarrow \mathcal{T}_{n+1} \xleftarrow{\sum_1^n \xi_i^2 + \eta^2} \mathcal{T}_{n+1} \longleftarrow 0 \tag{4}$$

where

$$M = \mathcal{T}_{n+1} \Big/ \Big(\sum_1^n \xi_i^2 + \eta^2\Big) \mathcal{T}_{n+1} \simeq \{\alpha(\xi) + \eta\beta(\xi),\ \alpha, \beta \in \mathcal{T}_n\}$$

so that $(M)_n \simeq \mathcal{T}_n^2$.

Suspending the complex (1) we obtain the complex

$$\begin{array}{ccccccc} & & \mathcal{E}^{p_0}(\Omega) & \xrightarrow{A_0} & \mathcal{E}^{p_1}(\Omega) & \xrightarrow{A_1} & \\ & \nearrow^{\Delta} & \oplus & \nearrow^{-\Delta} & \oplus & \nearrow^{\Delta} & \cdots \\ \mathcal{E}^{p_0}(\Omega) & \xrightarrow[A_0]{} & \mathcal{E}^{p_1}(\Omega) & \xrightarrow[A_1]{} & \mathcal{E}^{p_2}(\Omega) & \xrightarrow[A_2]{} & \end{array} \tag{6}$$

Since (3) is acyclic on any open set Ω the cohomology of (6) is isomorphic on any open set Ω to the cohomology of the complex

$$\boldsymbol{H}(\Omega)^{p_0} \xrightarrow{A_0} \boldsymbol{H}(\Omega)^{p_1} \xrightarrow{A_1} \boldsymbol{H}(\Omega)^{p_2} \longrightarrow \ldots$$

where $\boldsymbol{H}(\Omega)$ denotes the space of harmonic functions on Ω.

The suspending operator Δ is elliptic and Cauchy-Kowalewska with respect to $\boldsymbol{R}^n$. Thus for any $\omega \subset \boldsymbol{R}^n$ we have

$$H^j(\omega, \mathcal{E}_{C_0}) \xrightarrow{\sim} \bigoplus_{\nu=1}^{2} H^j(\omega, \mathcal{A}_{A_0})$$

where C_0 is the first operator of the complex (6).

(β) Let $N = 2n$ and identify $\boldsymbol{R}^{2n}$ with $\boldsymbol{C}^n$ where $\xi_1 + i\eta_1, \ldots, \xi_n + i\eta_n$ are complex coordinates so that $\boldsymbol{R}^n = \{(\xi + i\eta) \in \boldsymbol{C}^n | \eta = 0\}$. Consider on $\boldsymbol{C}^n$ as suspending complex (3) the Dolbeault complex

$$C^{00}(\Omega) \xrightarrow{\bar\partial} C^{01}(\Omega) \xrightarrow{\bar\partial} C^{02}(\Omega) \longrightarrow \ldots \tag{3}$$

where $C^{0i}(\Omega)$ denotes the space of C^∞ forms of type $(0, i)$ on $\boldsymbol{C}^n$. This corresponds to the Hilbert resolution

$$(4) \qquad 0 \to \mathcal{T}^{(0)} \xrightarrow{\wedge\alpha} \mathcal{T}^{(1)} \xrightarrow{\wedge\alpha} \dots \xrightarrow{\wedge\alpha} \mathcal{T}^{(n)} \longrightarrow M \longrightarrow 0$$

where $\mathcal{T}^{(i)}$ denotes the space of differential forms in $dt_1, \dots, dt_n$ with coefficients in $\mathcal{T}_{2n} = \boldsymbol{C}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]$ and where

$$\alpha = (\xi_1 + i\eta_1)dt_1 + \dots + (\xi_n + i\eta_n)dt_n .$$

If we identify $\mathcal{T}^{(n)}$ with $\mathcal{T}^{(0)}$ the image of the last homomorphism in the sequence (4) is the ideal of $\mathcal{T}^{(0)}$ generated by $(\xi_1 + i\eta_1, \dots, \xi_n + i\eta_n)$.

Therefore we have

$$\begin{aligned} M &\simeq \boldsymbol{C}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]/(\xi_1 + i\eta_1, \dots, \xi_n + i\eta_n) \\ &\simeq \boldsymbol{C}[\xi_1, \dots, \xi_n] = \mathcal{T}_n . \end{aligned}$$

Suspending by means of (3) complex (1) we obtain the complex

$$(6) \qquad \begin{array}{ccccc} & & & \overset{\bar\partial}{\nearrow} & (C^{02}(\Omega))^{p_0} \\ & \overset{\bar\partial}{\nearrow} & (C^{01}(\Omega))^{p_0} & \underset{A_0}{\searrow} & \oplus \\ (C^{00}(\Omega))^{p_0} & & \oplus & & (C^{01}(\Omega))^{p_1} \longrightarrow \dots \\ & \underset{A_0}{\searrow} & (C^{00}(\Omega))^{p_1} & \overset{\bullet\,\bar\partial}{\nearrow} & \oplus \\ & & & \underset{A_1}{\searrow} & (C^{00}(\Omega))^{p_2} \end{array}$$

If Ω is an open set of holomorphy, its cohomology is isomorphic to the cohomology of the complex

$$(7) \qquad \Gamma(\Omega, \mathcal{O})^{p_0} \xrightarrow{A_0(\partial/\partial z)} \Gamma(\Omega, \mathcal{O})^{p_1} \xrightarrow{A_1(\partial/\partial z)} \Gamma(\Omega, \mathcal{O})^{p_2} \longrightarrow \dots$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\boldsymbol{C}^n$ and where the operators $A_j(D_x)$ have been replaced by the operators $A_j(\partial/\partial z)$ as they have the same effect on holomorphic functions. In particular the sequence (7) is exact if Ω is open and convex. We derive therefore the following

PROPOSITION 9. *Let ω be open in $\boldsymbol{R}^n$ and let $f \in \mathcal{A}^{p_h}(\omega)$ be such that $A_h(D_x)f = 0$. For any open relatively compact convex subset $\omega_1 \subset\subset \omega$ we can find $u \in \mathcal{A}^{p_{h-1}}(\omega_1)$ such that*

$$A_{h-1}(D_x)u = f , \qquad \text{on } \omega_1 .$$

Indeed we can find an open neighborhood Ω of ω in C^n and $F \in \Gamma(\Omega, \mathcal{O})^{p_h}$ with $F|\omega = f$. Let $\Omega_1 \subset \Omega$ be a convex neighborhood of ω_1. There exists $U \in \Gamma(\Omega, \mathcal{O})^{p_{h-1}}$ such that $A_{h-1}(\partial/\partial z)U = F$. It is enough to set $u = U|\omega_1$.

Let $C_0(D)$ denote the first operator of the complex (6). Since the first operator $\bar\partial$ of the suspending complex is elliptic and of Cauchy-Kowalewska with respect to R^n, we have for any open set $\omega \subset R^n$

$$H^j(\omega, \mathcal{E}_{C_0}) \simeq H^j(\omega, \mathcal{A}_{A_0}).$$

(γ) Consider for $n \geq 2$ $R^{n+k} \subset C^n$, the space C^n being the minimal complex subspace of C^n containing R^{n+k}. If $z_j = x_j + iy_j$, $1 \leq j \leq n$ are complex coordinates in C^n we may assume

$$R^{n+k} = C^k \times R^{n-k} = \{y_{k+1} = \ldots = y_n = 0\}.$$

Set $z = (z', z'')$ with $z' = (z_1, \ldots, z_k)$, $z'' = (z_{k+1}, \ldots, z_n)$ and let $\bar\partial'$, $\bar\partial''$ denote, respectively, exterior differentiation with respect to the antiholomorphic coordinates $\bar z'$ or $\bar z''$.

We take as complex to be suspended the complex of the $\bar\partial'$ along the fibers C^k of R^{n+k} (so that now R^n is replaced by R^{n+k}).

$$(1) \qquad C^{00}(\omega) \xrightarrow{\bar\partial'} C^{01}(\omega) \xrightarrow{\bar\partial'} C^{02}(\omega) \longrightarrow \ldots \longrightarrow C^{0k}(\omega) \longrightarrow 0.$$

We take as suspending complex in $R^{2n} = C^n$ $(N = 2n)$ the complex

$$(3) \qquad C^{00}(\Omega) \xrightarrow{\bar\partial''} C^{01}(\Omega) \xrightarrow{\bar\partial''} C^{02}(\Omega) \longrightarrow \ldots \longrightarrow C^{0n-k}(\Omega) \longrightarrow 0$$

where $C^{0l}(\Omega)$ denotes the space of C^∞ forms of type $(0, l)$ on Ω in the differentials $d\bar z_{k+1}, \ldots, d\bar z_n$.

This last complex corresponds to a Hilbert resolution of a $\mathcal{P}_{2n}$-module $M \simeq C[\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k, \xi_{k+1}, \ldots, \xi_n] \simeq \mathcal{P}_{n+k}$. Indeed M is isomorphic to $\mathcal{P}_{2n}$ modulo the ideal generated by $\xi_{k+1} + i\eta_{k+1}, \ldots, \xi_n + i\eta_n$.

Suspending the complex (1) by (3) we obtain the Dolbeault complex in C^n. This has its first operator elliptic and of Cauchy-Kowalewska with respect to R^{n+k}. In particular we will obtain for $\omega \subset R^{n+k}$

$$H^j(\omega, \mathcal{O}) \simeq H^j(\omega, \mathcal{A}_{\bar\partial'})$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions in C^n.

5. – Sufficient conditions for analytic convexity.

a) Let a Hilbert complex be given on $\boldsymbol{R}^n$, where $x = (x_1, \ldots, x_n)$ are coordinates,

$$(1) \qquad (\mathcal{E}^*(\omega), A_*) \equiv \left\{\mathcal{E}^{s_0}(\omega) \xrightarrow{A_0(D_x)} \mathcal{E}^{s_1}(\omega) \xrightarrow{A_1(D_x)} \mathcal{E}^{s_2}(\omega) \longrightarrow \ldots\right\}$$

where ω is open in $\boldsymbol{R}^n$. Let us consider in $\boldsymbol{R}^N$, $N = n + k$, where $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k)$ are cartesian coordinates an *elliptic and Cauchy-Kowalewska* suspension of the complex (1);

$$(2) \quad (\mathcal{E}^*(\Omega), S_*) \equiv \left\{\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D_x, D_y)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D_x, D_y)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \ldots\right\}.$$

By this we mean that (2) is a suspension of (1) and that the first operator $S_0(D)$ of (2) is an elliptic operator and of Cauchy-Kowalewska with respect to $\boldsymbol{R}^n$. Here Ω is open in $\boldsymbol{R}^N$.

If we set $\mathcal{P}_N = \boldsymbol{C}[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k]$, $\mathcal{P}_n = \boldsymbol{C}[\xi_1, \ldots, \xi_n]$ the above situation arises from a commutative diagram of $\mathcal{P}_n$-homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
0 \longleftarrow & M & \longleftarrow & \mathcal{P}_N^{s_0} & \xleftarrow{{}^tS_0(\xi,\eta)} & \mathcal{P}_N^{s_1} & \xleftarrow{{}^tS_1(\xi,\eta)} & \mathcal{P}_N^{s_2} & \longleftarrow \ldots \\
 & \| & & \downarrow {}^t\tau_0 & & \downarrow {}^t\tau_1 & & \downarrow {}^t\tau_2 & \\
0 \longleftarrow & (M)_n & \longleftarrow & \mathcal{P}_n^{p_0} & \xleftarrow{{}^tA_0(\xi)} & \mathcal{P}_n^{p_1} & \xleftarrow{{}^tA_1(\xi)} & \mathcal{P}_n^{p_2} & \longleftarrow \ldots
\end{array}$$

Here the maps ${}^t\tau_j$ define linear maps, for $\omega = \Omega \cap \boldsymbol{R}^n$,

$$\tau_j(D_x, D_y) \colon \mathcal{E}^{s_j}(\Omega) \to \mathcal{E}^{p_j}(\omega) \qquad j = 0, 1, \ldots$$

by

$$f \to \left(\tau_j(D_x, D_y) f\right)\big|_{y=0}\,.$$

We consider $\omega \subset \boldsymbol{R}^n$, $\Omega \subset \boldsymbol{R}^N$ with $\Omega \cap \boldsymbol{R}^n = \omega$, and the Cauchy problem

$$(*) \qquad \begin{cases} S_0(D)u = 0 & u \in \mathcal{A}^{s_0}(\Omega) \\ \tau_0(u) \quad = v & \\ A_0(D)v = 0 & v \in \mathcal{A}^{p_0}(\omega)\,. \end{cases}$$

and consider $\boldsymbol{R}^n$, $\boldsymbol{R}^N$ naturally imbedded in $\boldsymbol{C}^n$, $\boldsymbol{C}^N$ respectively.

The following lemma is a consequence of the inductive argument given for the sufficiency part of theorem 2 and of the argument developed for instance in [4].

LEMMA 2. *Let $S_0(D)$ be of Cauchy-Kowalewska with respect to $\boldsymbol{R}^n$. Let $\tilde{\omega}$ be a neighborhood of ω in $\boldsymbol{C}^n$ and let v be defined and holomorphic in $\tilde{\omega}$. There exists a neighborhood $\tilde{\Omega}$ of ω in $\boldsymbol{C}^N$, depending only on $\tilde{\omega}$ and not on v, such that the solution u of the Cauchy problem* (*) *is defined and holomorphic in $\tilde{\Omega}$.*

b) For an *elliptic and Cauchy-Kowalewska suspension* (2) of (1), for any open set $\omega \subset \boldsymbol{R}^n$, we have

$$H^j(\omega, \mathcal{E}_{S_0}) \simeq H^j(\omega, \mathcal{A}_{A_0})$$

and if Ω describes a fundamental system of open neighborhoods of ω in $\boldsymbol{R}^N$ we have

$$H^j(\omega, \mathcal{E}_{S_0}) = \varinjlim_{\Omega \supset \omega} H^j(\Omega, \mathcal{E}_{S_0}) .$$

We deduce from this the following proposition.

PROPOSITION 10. *Let ω be open in $\boldsymbol{R}^n$. Assume that for some $j \geq 1$*

(*A*) *for any open neighborhood Ω of ω in $\boldsymbol{R}^N$ we can find an open neighborhood A of ω in Ω such that the restriction map*

$$r_A^\Omega : H^j(\Omega, \mathcal{E}_{S_0}) \to H^j(A, \mathcal{E}_{S_0})$$

has zero image.

Then $H^j(\omega, \mathcal{A}_{A_0}) = 0$.

In particular when $\Omega = A$ we get the

COROLLARY. *Let ω be open in $\boldsymbol{R}^n$. If ω admits a fundamental system of open neighborhoods Ω of ω in $\boldsymbol{R}^N$ such that $H^j(\Omega, \mathcal{E}_{S_0}) = 0$ then $H^j(\omega, \mathcal{A}_{A_0}) = 0$.*

Let $j \geq 1$. A pair of open sets $A \subset \Omega$ in $\boldsymbol{R}^N$ is called *j-compatible* with respect to the suspended complex (2) if

$$\operatorname{Im} \{ r_A^\Omega : H^j(\Omega, \mathcal{E}_{S_0}) \to H^j(A, \mathcal{E}_{S_0}) \} = 0$$

so that condition (*A*) of proposition 10 can be stated as « *ω admits a fundamental system of j-compatible pairs of open neighborhoods in $\boldsymbol{R}^N$ for the suspended complex* (2) ».

Consider now another elliptic and Cauchy-Kowalewska suspension in some space $\boldsymbol{R}^H \supset \boldsymbol{R}^n$ of the complex (1) where $(x_1, \dots, x_n, t_1, \dots, t_h)$ are cartesian coordinates in $\boldsymbol{R}^H$, $H = n + h$,

$$(3) \qquad (\mathcal{E}^*(G), L_*) \equiv \left\{\mathcal{E}^{l_0}(G) \xrightarrow{L_0(D_x, D_t)} \mathcal{E}^{l_1}(G) \xrightarrow{L_1(D_x, D_t)} \mathcal{E}^{l_2}(G) \longrightarrow \dots\right\}$$

where G is open in $\boldsymbol{R}^H$. Similar to the map τ_0 on the Cauchy data we choose a map

$$\lambda_0(D_x, D_t)\colon \mathcal{E}^{l_0}(G) \to \mathcal{E}^{p_0}(\omega) \qquad \omega = G \cap \boldsymbol{R}^n$$

so that the corresponding first Cauchy problem for the complex (3) is given by

$$(**) \qquad \begin{cases} L_0(D)w = 0 & w \in \mathcal{A}^{l_0}(G) \\ \lambda_0(w) \quad = v & \\ A_0(D)v = 0 & v \in \mathcal{A}^{p_0}(\omega)\,. \end{cases}$$

PROPOSITION 11. *Let ω be open in $\boldsymbol{R}^n$. If ω satisfies condition (A) with respect to an elliptic and Cauchy-Kowalewska suspension of the given complex* (1) *then ω satisfies condition (A) with respect to any other elliptic and Cauchy-Kowalewska suspension of* (1).

PROOF. We assume that ω satisfies condition (A) with respect to the suspension (2) of (1). We want to show that it satisfies condition (A) with respect to the suspension (3) of (1).

We choose a countable locally finite covering of ω by convex open sets

$$\omega = \bigcup_{i \in \boldsymbol{N}} \omega_i = \bigcup_{i \in \boldsymbol{N}} \omega_i' = \bigcup_{i \in \boldsymbol{N}} \omega_i''$$

with

$$\omega_i'' \subset\subset \omega_i' \subset\subset \omega_i \subset\subset \omega \qquad \forall i \in \boldsymbol{N}.$$

For $\varepsilon = \{\varepsilon_i\}_{i \in \boldsymbol{N}}$, $\varepsilon_i > 0$ $\forall i$, we set

$$U_i^{(\nu)}(\varepsilon) = \{x \in \omega_i^{(\nu)}, |t| < \varepsilon_i\}, \qquad \mathfrak{U}^{(\nu)}(\varepsilon) = \{U_i^{(\nu)}(\varepsilon)\}_{i \in \boldsymbol{N}}$$

$$W_i^{(\nu)}(\varepsilon) = \{x \in \omega_i^{(\nu)}, |y| < \varepsilon_i\}, \qquad \mathfrak{W}^{(\nu)}(\varepsilon) = \{W_i^{(\nu)}(\varepsilon)\}_{i \in \boldsymbol{N}}$$

where $\omega_i^{(\nu)}$ $\nu = 0, 1, 2$ denote respectively ω_i, ω_i', ω_i''. We also set $U_{i_0 \dots i_l}^{(\nu)} = U_{i_0}^{(\nu)} \cap \dots \cap U_{i_l}^{(\nu)}$ and similarly for $W^{(\nu)}$ and $\omega^{(\nu)}$.

Let G be an open neighborhood of ω in $\boldsymbol{R}^H$. By replacing G by a possibly smaller open set, we may assume that $G = \bigcup_{i \in \boldsymbol{N}} U_i(\varepsilon)$ so that $\mathcal{U}(\varepsilon)$ is a Leray covering of G for the sheaf $\mathcal{E}_{L_0}$. We want to show that there is an open neighborhood B of ω in G so that

$$\operatorname{Im}\{H^q(G, \mathcal{E}_{L_0}) \to H^q(B, \mathcal{E}_{L_0})\} = 0 .$$

Let $\{f_{i_0 \dots i_q}\} \in H^q(G, \mathcal{E}_{L_0}) \simeq H^q(\mathcal{U}(\varepsilon), \mathcal{E}_{L_0})$ be a cohomology class represented by a cocycle $f_{i_0 \dots i_q}$ with $f_{i_0 \dots i_q} \in \Gamma(U_{i_0 \dots i_q}(\varepsilon), \mathcal{E}_{L_0})$ and (with loose notations, suppressing the restriction maps) $\sum (-1)^h f_{i_0 \dots \hat{i}_h \dots i_{q+1}} = 0$.

Because of lemma 1, since $L_0(D)$ is an elliptic operator, $f_{i_0 \dots i_q}$ is defined and holomorphic in a neighborhood $\tilde{U}_{i_0 \dots i_q}(\varepsilon)$ of $U_{i_0 \dots i_q}(\varepsilon)$ in $\boldsymbol{C}^H$, which is independent of $\{f_{i_0 \dots i_q}\}$ but depends only on $U_{i_0 \dots i_q}(\varepsilon)$. Therefore $\lambda_0(f_{i_0 \dots i_q})$ is defined and holomorphic in a neighborhood $\tilde{\omega}_{i_0 \dots i_q}$ of $\omega_{i_0 \dots i_q}$ in $\boldsymbol{C}^n$ which is independent of $\{f_{i_0 \dots i_q}\}$ but depends only on $U_{i_0 \dots i_q}(\varepsilon)$. We have $A_0(D)\lambda_0(f_{i_0 \dots i_q}) = 0$.

For every $(i_0, \dots, i_q)$ we consider the Cauchy problem

$$(*) \qquad \begin{cases} S_0(D) s_{i_0 \dots i_q} = 0 \\ \tau_0(s_{i_0 \dots i_q}) = \lambda_0(f_{i_0 \dots i_q}) \\ A_0(D)\lambda_0(f_{i_0 \dots i_q}) = 0 . \end{cases}$$

This can be solved with

$$s_{i_0 \dots i_q} \in \Gamma(\{x \in \omega'_{i_0 \dots i_q}, |y| < \sigma_{i_0 \dots i_q}\}, \mathcal{E}_{S_0})$$

with $\sigma_{i_0 \dots i_q} > 0$ that can be chosen, by virtue of lemma 2 independent of $\{f_{i_0 \dots i_q}\}$ and depending only on $U_{i_0 \dots i_q}(\varepsilon)$. Since the covering $\{\omega'_i\}_{i \in \boldsymbol{N}}$ is locally finite we have

$$\sigma_i = \inf_{(i_1 \dots i_q)} \sigma_{i i_1 \dots i_q} > 0 .$$

Therefore we have found a sequence $\sigma = \{\sigma_i\}_{i \in \boldsymbol{N}}$, with $\sigma_i > 0$, $\forall i$, independent of $\{f_{i_0 \dots i_q}\}$ and depending only on $\mathcal{U}(\varepsilon)$ and $s_{i_0 \dots i_q} \in \Gamma(W'_{i_0 \dots i_q}(\sigma), \mathcal{E}_{S_0})$ solving the Cauchy problem $(*)$. With loose notations, we must have $\tau_0(\sum (-1)^h s_{i_0 \dots \hat{i}_h \dots i_q}) = 0$ because $\{f_{i_0 \dots i_q}\}$ is a cocycle, therefore also we must have $\sum (-1)^h s_{i_0 \dots \hat{i}_h \dots i_q}) = 0$.

Let $\Omega = \bigcup W'_i(\sigma)$. By assumption there exists an open neighborhood A of ω in Ω such that

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(A, \mathcal{E}_{S_0})\} = 0 .$$

Without loss of generality we may assume $A = \bigcup W_i'(\eta)$ for some sequence $\eta = \{\eta_i\}_{i\in N}$ with $\eta_i > 0$, $\forall i$. As $\mathcal{W}'(\eta)$ is a Leray covering of A for the sheaf $\mathcal{E}_{S_0}$ and as $\{s_{i_0\ldots i_q}\}$ is given on the covering $\mathcal{W}'(\sigma)$ we can find $\{v_{i_0\ldots i_{q-1}}\}$ with

$$v_{i_0\ldots i_{q-1}} \in \Gamma(W'_{i_0\ldots i_{q-1}}(\eta), \mathcal{E}_{S_0})$$

such that $\delta v = s|\mathcal{W}'(\eta)$ i.e., with loose notations,

$$\sum (-1)^h v_{i_0\ldots \hat{i}_h \ldots i_q} = s_{i_0\ldots i_q} \quad \text{ on } W'_{i_0\ldots i_q}(\eta)\,.$$

Because of lemma 1 $v_{i_0\ldots i_{q-1}}$ is defined and holomorphic in a neighborhood $\tilde{W}'_{i_0\ldots i_{q-1}}(\eta)$ of $W'_{i_0\ldots i_{q-1}}(\eta)$ in C^N which is independent of $v = \{v_{i_0\ldots i_{q-1}}\}$ and depends only on $\mathcal{W}'(\eta)$.

Therefore $\tau_0(v_{i_0\ldots i_{q-1}})$ is defined and holomorphic in a neighborhood $\tilde{\omega}'_{i_0\ldots i_{q-1}}$ of $\omega'_{i_0\ldots i_{q-1}}$ in C^n which is independent of v and depends only on $\mathcal{W}'(\eta)$. We have $A_0(D)\tau_0(v_{i_0\ldots i_{q-1}}) = 0$. For every $(i_0, \ldots, i_{q-1})$ we consider the Cauchy problem

$$(**) \qquad \begin{cases} L_0(D)\, w_{i_0\ldots i_{q-1}} = 0 \\ \lambda_0(w_{i_0\ldots i_{q-1}}) = \tau_0(v_{i_0\ldots i_{q-1}}) \\ A(D)\,\tau_0(v_{i_0\ldots i_{q-1}}) = 0\,. \end{cases}$$

As before we realize that we can find a sequence $\mu = \{\mu_i\}_{i\in N}$ with $\mu_i > 0$, $\forall i$, which (by lemma 2) is independent of v and depends only on $W'(\eta)$ such that the Cauchy problem $(**)$ has a solution

$$w_{i_0\ldots i_{q-1}} \in \Gamma\big(U''_{i_0\ldots i_{q-1}}(\mu), \mathcal{E}_{L_0}\big)\,.$$

We set $B = \bigcup U_i''(\mu)$; it is a neighborhood of ω in G. With loose notations we have

$$\lambda_0(f) = \tau_0(s)\,, \qquad \lambda_0(w) = \tau_0(v)\,, \qquad s = \delta v\,,$$

thus

$$\lambda_0\big(\delta w - f|\mathcal{U}''(\mu)\big) = 0\,.$$

Because of the unicity of the Cauchy problem $(**)$ we deduce that $\delta w = f|\mathcal{U}''(\mu)$ i.e., with loose notations,

$$\sum (-1)^h w_{\ldots \hat{i}_h \ldots i_q} = f_{i_0\ldots i_q} \quad \text{ on } \; U''_{i_0\ldots i_q}(\mu)\,.$$

But this shows that $\operatorname{Im}\{H^q(G, \mathcal{E}_{L_0}) \to H^q(B, \mathcal{E}_{L_0})\} = 0$ as we wanted to prove.

c) An application. Let $\xi = (\xi_1, \dots, \xi_n)$ be n indeterminates and let us consider a homogeneous ideal $\mathfrak{b}$ in the graded ring $\boldsymbol{C}_0[\xi_1, \dots, \xi_n]$ of homogeneous polynomials in n variables. Let $\psi_1(\xi), \dots, \psi_l(\xi)$ be a set of (homogeneous) generators of $\mathfrak{b}$. Let

$$W(\mathfrak{b}) = \{\xi \in \boldsymbol{P}_{n-1}(\boldsymbol{C}) | \psi_1(\xi) = \dots = \psi_l(\xi) = 0\}$$

be the projective variety associated to $\mathfrak{b}$. We will make the following assumptions:

i) $W(\mathfrak{b})$ is 0-dimensional so that it consists of finite many points

$$a^{(s)} = (a_1^{(s)}, \dots, a_n^{(s)}) \quad 1 \leq s \leq \mu$$

ii) each one of the points $a^{(s)}$ is simple for $W(\mathfrak{b})$ i.e.

$$\operatorname{rank}\left\{\frac{\partial(\psi_1(\xi), \dots, \psi_l(\xi))}{\partial(\xi_1, \dots, \xi_n)}\right\}_{\xi = a^{(s)}} = n - 1 .$$

For each point $a^{(s)}$ we define a linear map $\pi_s \colon \boldsymbol{C}^n \to \boldsymbol{C}$ by $z = (z_1, \dots, z_n) \to \to \pi_s(z) = \sum_{j=1}^{n} a_j^{(s)} z_j$. This is defined up to multiplication by a complex number different from zero. We will call the projection π_s *real* or *complex* according whether $a^{(s)} \in \boldsymbol{P}_{n-1}(\boldsymbol{R})$ or not respectively.

Set ${}^tA_0(\xi) = (\psi_1(\xi), \dots, \psi_l(\xi))$, $\mathcal{T}_n = \boldsymbol{C}[\xi_1, \dots, \xi_n]$ and consider a Hilbert resolution of the $\mathcal{T}_n$-linear map $\mathcal{T}_n^l \xrightarrow{{}^tA_0(\xi)} \mathcal{T}_n$ defined by ${}^tA_0(\xi)$:

$$(4) \qquad 0 \longrightarrow \mathcal{T}_n^t \longrightarrow \dots \longrightarrow \mathcal{T}_n^l \xrightarrow{{}^tA_0(\xi)} \mathcal{T}_n \longrightarrow N \longrightarrow 0 .$$

To this corresponds a complex of differential operators

$$(5) \qquad \mathcal{E}(\omega) \xrightarrow{A_0(D)} \mathcal{E}^l(\omega) \longrightarrow \dots \longrightarrow \mathcal{E}^t(\omega) \longrightarrow 0$$

for any ω open in $\boldsymbol{R}^n$. Here, if $x_1, \dots, x_n$ are cartesian coordinates in $\boldsymbol{R}^n$,

$$A_0(D) = \begin{pmatrix} \psi_1(D) \\ \vdots \\ \psi_l(D) \end{pmatrix} \qquad D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

We consider now the $\bar{\partial}$-suspension in $\boldsymbol{C}^n$ of the complex (5). We obtain in this way a Hilbert complex for any Ω open in $\boldsymbol{C}^n$

$$\mathcal{E}(\Omega) \xrightarrow{C_0(D)} \mathcal{E}(\Omega)^{l+n} \longrightarrow \ldots \tag{6}$$

where, if $z_1, \ldots, z_n$ denote complex coordinates in $\boldsymbol{C}^n$, we may assume that

$$C_0(D) = \begin{pmatrix} \psi_1\left(\dfrac{\partial}{\partial z}\right) \\ \vdots \\ \psi_l\left(\dfrac{\partial}{\partial z}\right) \\ \dfrac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \dfrac{\partial}{\partial \bar{z}_n} \end{pmatrix}, \qquad D = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right)$$

We denote by $\mathcal{O}_{A_0}$ the sheaf of germs u of (holomorphic) solutions of the equation $C_0(D)u = 0$. We denote by $\mathcal{O}_s$ the sheaf of germs of holomorphic functions in the variable $t = \sum a_j^{(s)} z_j$. We have a natural linear map

$$\coprod_{s=1}^{\mu} \mathcal{O}_s \to \mathcal{O}_A$$

given by $(\alpha_1, \ldots, \alpha_\mu) \to \sum_{s=1}^{\mu} \alpha_s$. If Ω is open in $\boldsymbol{C}^n$ and $H^j(\Omega, \boldsymbol{C}) = 0$, $\forall j > 0$ then we have ([4] Corollary 2 to proposition 14)

$$H^j(\Omega, \mathcal{O}_A) = \bigoplus_{s=1}^{\mu} H^j(\Omega, \mathcal{O}_s) \qquad (j > 0)\,.$$

Let ω be an open bounded set in $\boldsymbol{R}^n$ and let $\varrho: \boldsymbol{R}^n \to \boldsymbol{R}$ be a C^∞ defining function for ω, i.e. a C^∞ function such that $\omega = \{x \in \boldsymbol{R}^n | \varrho(x) < 0\}$. It will be convenient to assume that $\varrho(x) = 0$ for $x \in \boldsymbol{R}^n - \omega$.

Set

$$\tilde{\omega} = \{(x_1, \ldots, x_n, \theta) \in \boldsymbol{R}^{n+1} | \varrho(x) < -\theta^2\}$$

and let α denote the projection $(x_1, \ldots, x_n, \theta) \to (\theta)$ of $\boldsymbol{R}^{n+1}$ on the θ-axis $\boldsymbol{R}$.

We will make on ω the following assumptions

i)$_\omega$ for any real projection $\pi_s: (z_1, \ldots, z_n) \to \sum_{j=1}^{n} a_j^{(s)} z_j$, $a_j^{(s)} \in \boldsymbol{R}$, $1 \le j \le n$, consider the linear map $\beta_s\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ defined by

$$(x_1, \dots, x_n, \theta) \to \sum_{j=1}^{n} a_j^{(s)} x_j$$

then $(\tilde{\omega}, \alpha \times \beta_s | \tilde{\omega}, \alpha \times \beta_s(\tilde{\omega}))$ is a differentiable fiber space with convex fibers for a convenient choice $\varrho = \varrho^{(s)}$ of the defining function of ω.

ii)$_\omega$ for any complex projection $\pi_s\colon (z_1, \dots, z_n) \to \sum_{j=1}^{n} a_j^{(s)} z_j$, $a^{(s)} \notin \boldsymbol{P}_{n-1}(\boldsymbol{R})$, $(\omega, \pi_s|\omega, \pi_s(\omega))$ is a differentiable fiber space with convex fibers.

iii)$_\omega$ $H^j(\omega, \boldsymbol{C}) = 0$, $\forall j > 0$.

Set $\omega(\theta) = \{x \in \boldsymbol{R}^n | \varrho(x) < -\theta^2\}$. Condition i)$_\omega$ implies that for any real π_s (and for a convenient choice of ϱ), $\pi_s|\omega(\theta)$ has convex fibers for any θ. Condition ii)$_\omega$ implies that for any complex π_s, $\pi_s|\omega$ has also convex fibers. By a differentiable fibre space we mean a fibre space which is locally differentiably trivial.

PROPOSITION 12. *Consider the Hilbert complex* (2) *associated to a homogeneous polynomial ideal* $\mathfrak{b}$ *on which we make the assumptions* i) *and* ii).

Let ω *be an open set in* $\boldsymbol{R}^n$*, bounded and verifying the assumptions* i)$_\omega$, ii)$_\omega$, iii)$_\omega$.

Then ω *is analytically convex i.e. for any* $j > 0$ *we have*

$$H^j(\omega, \mathcal{A}_{A_0}) = 0\,.$$

PROOF. Let $h\colon \boldsymbol{R} \to \boldsymbol{R}$ be a C^∞ function with $h(t) < 0$ if $t < 0$, $h(t) = 0$ if $t \geq 0$ and which is strictly monotone increasing for $t < 0$. Set $z = x + iy$ in $\boldsymbol{C}^n$ and consider the sets

$$\Omega = \left\{z = x + iy \in \boldsymbol{C}^n \,|\, \sum_{1}^{n} y_j^2 + h(\varrho(x)) < 0\right\}.$$

These describe a fundamental system of neighborhoods of ω in $\boldsymbol{C}^n$. Each Ω is fibered in n-dimensional balls over ω and thus is contractible onto ω. Because of the assumption iii)$_\omega$ we have thus $H^j(\Omega, \boldsymbol{C}) = 0$ for every $j > 0$; therefore $H^j(\Omega, \mathcal{O}_A) = \bigoplus_{s=1}^{\mu} H^j(\Omega, \mathcal{O}_s)$, $(j > 0)$. It will be therefore sufficient to show that for every s, $1 \leq s \leq \mu$, there exists an open neighborhood A_s of ω in Ω such that

$$\operatorname{Im}\{H^j(\Omega, \mathcal{O}_s) \to H^j(A_s, \mathcal{O}_s)\} = 0 \qquad (j > 0)\,.$$

Indeed taking $A = \bigcap_{s=1}^{\mu} A_s$ we will have

$$\operatorname{Im}\{H^j(\Omega, \mathcal{O}_A) \to H^j(A, \mathcal{O}_A)\} = 0 \qquad (j > 0)$$

therefore by proposition 10 it will follow that $H^j(\omega, \mathcal{A}_{A_0}) = 0$ for every $j > 0$.

We distinguish the case whether π_s is a complex or real projection. To simplify notations we will denote the function $h(\varrho)$ by ϱ itself so that

$$\Omega = \left\{x + iy \in \boldsymbol{C}^n \mid \sum_1^n y_i^2 + \varrho(x) < 0\right\}.$$

Let π_s be a complex projection. Let $a^{(s)} = \alpha + i\beta$ and let $(\cdot,\cdot)$ denote the euclidean scalar product in $\boldsymbol{R}^n$. We can always multiply $a^{(s)}$ by a non zero complex number to have $(\alpha, \beta) = 0$. We may also assume $(\alpha, \alpha)^{\frac{1}{2}} = 1$, $(\beta, \beta)^{\frac{1}{2}} = k$, as π_s being complex, α and β are linearly independent. We can find a real orthogonal matrix M so that the change of coordinates $z = Mz'$ in $\boldsymbol{C}^n$ brings $a^{(s)}$ into the point ${}^t(1, ik, 0, \ldots, 0) \in \boldsymbol{C}^n$. We will continue to denote by $\varrho(x)$ the new defining function $\varrho(Mx)$ for ω. Denoting by $\sigma + i\theta$ the complex coordinate in the target space of π_s, in these new coordinates π_s will take the equations

$$\begin{cases} x_1 - ky_2 = \sigma \\ y_1 + kx_2 = \theta. \end{cases}$$

Let $\Sigma_{(\sigma,\theta)} = \pi_s^{-1}(\sigma + i\theta)$. Then $y_1, \ldots, y_n, x_3, \ldots, x_n$ can be taken as affine coordinates on the complex hyperplane $\Sigma_{(\sigma,\theta)}$ for any choice of σ and θ and in these coordinates the open set $\Sigma_{(\sigma,\theta)} \cap \Omega$ will be given by the condition

$$(*) \qquad \sum_1^n y_j^2 + \varrho\left(\sigma + ky_2, \frac{\theta - y_1}{k}, x_3, \ldots, x_n\right) < 0\,.$$

The left hand side of this inequality is given by a function $\psi(y_1, \ldots, y_n)$ for any fixed choice of $x_3, \ldots, x_n$. Replacing ϱ by $\varepsilon\varrho$ with $\varepsilon < 0$ and sufficiently small we may suppose that the Hessian of ψ with respect to $y_1, \ldots, y_n$ is positive definite so that $\psi(y_1, \ldots, y_n)$ is a C^∞ strictly convex function.

If

$$P \equiv \left(\sigma + ky_2 + iy_1, \frac{\theta - y_1}{k} + iy_2, x_3 + iy_3, \ldots, x_n + iy_n\right) \in \Sigma_{(\sigma,\theta)} \cap \Omega$$

then

$$\tau(P) \equiv \left(\sigma, \frac{\theta}{k}, x_3, \ldots, x_n\right) \in \Sigma_{(\sigma,\theta)} \cap \omega = \omega_1(\sigma, \theta)\,.$$

Indeed if this last point is not in ω, then all derivatives of ϱ must vanish (by the way the defining function has been chosen). Therefore the gradient of $\psi(y)$ for $y=0$ must vanish. Hence $y=0$ must be the absolute minimum of ψ so that $\psi(y)\geq\psi(0)=0$. Hence no point P could exist over $\tau(P)$, with $P\in\Omega$.

It follows that $\Sigma_{(\sigma,\theta)}\cap\Omega$ is fibered by τ over $\omega_1(\sigma,\theta)$ with convex fibers defined by $(*)$ for $\sigma,\theta,x_3,\ldots,x_n$ given. By assumption ii)$_\omega$, $\omega_1(\sigma,\theta)$ is convex, therefore $\Sigma_{(\sigma,\theta)}\cap\Omega$ is contractible for any choice of (σ,θ) provided $\Sigma_{(\sigma,\theta)}\cap\Omega\neq\emptyset$.

From proposition 16 of [4] we derive then that, for the given choice of Ω, we will have

$$H^j(\Omega,\mathcal{O}_s)=0 \quad \text{for any } j>0\,,$$

provided we prove the following contention.

Let (σ_0,θ_0) be such that $\Sigma_{(\sigma_0,\theta_0)}\cap\Omega\neq\emptyset$ and let M be a sufficiently small open spherical neighborhood of (σ_0,θ_0) so that for $(\sigma,\theta)\in M$, $\Sigma_{(\sigma,\theta)}\cap\Omega\neq\emptyset$. Then $\{\Sigma_{(\sigma,\theta)}\cap\Omega\}_{(\sigma,\theta)\in M}$ is a differentiably trivial fiber space over M.

Let us denote by $\nabla\psi$ the gradient of ψ with respect to the coordinates $(y_1,\ldots,y_n)\in\boldsymbol{R}^n$. Let $y^{(1)}$, $y^{(2)}$ be two points in $\boldsymbol{R}^n$. We claim that

$$\big(\nabla\psi(y^{(1)})-\nabla\psi(y^{(2)})\,,\;y^{(1)}-y^{(2)}\big)>0$$

if $y^{(1)}\neq y^{(2)}$. Indeed set $f(t)=\big(\nabla\psi(y^{(1)}+t(y^{(2)}-y^{(1)}))-\nabla\psi(y^{(1)}),y^{(2)}-y^{(1)}\big)$. We have $f(0)=0$ and $f'(t)>0$ if $y^{(1)}\neq y^{(2)}$ because ψ is strictly convex. Therefore $f(1)>0$ as we wanted to prove.

This shows that $\nabla\psi\colon\boldsymbol{R}^n\to\boldsymbol{R}^n$ is an injective map.

Let us now choose a C^∞ function $h(t)$ defined for $t<0$ which is convex and increasing, such that

$$h(t)>0\,,\qquad h'(t)\geq1\,,\qquad h''(t)>0\,,\qquad h(t)\to+\infty \text{ for } t\to0^-\,.$$

Replace the function ψ by $\exp[h(\psi)]$ and consider the map defined on the fibers of τ over $\omega_1(\sigma,\theta)$ by $\nabla\exp[h(\psi)]=h'(\psi)\nabla\psi\exp[h(\psi)]$. One then verifies that this defines a diffeomorphism of the fibers of τ onto $\boldsymbol{R}^n$, so that we get a differentiable isomorphism of fibered spaces

$$\nabla e^\psi\colon\big(\Sigma_{(\sigma,\theta)}\cap\Omega,\,\tau,\,\omega_1(\sigma,\theta)\big)\xrightarrow{\sim}\big(\omega_1(\sigma,\theta)\times\boldsymbol{R}^n,\,pr_{\omega_1(\sigma,\theta)},\,\omega_1(\sigma,\theta)\big)\,.$$

This diffeomorphism depends differentiably on the parameters σ, θ. By assumption ii)$_\omega$, for M small, we have that $\{\omega_1(\sigma,\theta)\}_{(\sigma,\theta)\in M}$ is diffeomorphic to a trivial fiber-space over M, with typical convex fiber F, $F\times M$. Combining this fact with the previous diffomorphism we get a differentiable isomorphism

$$\{\Sigma_{(\sigma,\theta)} \cap \Omega\}_{(\sigma,\theta)\in M} \xrightarrow{\sim} \boldsymbol{R}^n \times F \times M .$$

This establishes our contention.

Let π_s be a real projection. By an orthogonal real change of coordinates we may now assume $a^{(s)} = {}^t(1, 0, \ldots, 0)$ so that π_s has the equations $x_1 = \sigma$, $y_1 = \theta$, $\sigma + i\theta$ being the complex coordinate on the target space of π_s. Let $\Sigma_{(\sigma,\theta)} = \pi_s^{-1}(\sigma + i\theta)$; on it $y_2, \ldots, y_n, x_2, \ldots, x_n$ can be chosen as affine coordinates and $\Sigma_{(\sigma,\theta)} \cap \Omega$ is given by

$$\sum_2^n y_j^2 + \varrho(\sigma, x_2, \ldots, x_n) < -\theta^2 .$$

Set $\omega_1(\sigma, \theta) = \{(x_2, \ldots, x_n) \in \boldsymbol{R}^{n-1} | \varrho(\sigma, x_2, \ldots, x_n) < -\theta^2\}$. Then $\Sigma_{(\sigma,\theta)} \cap \Omega$ is fibered over $\omega_1(\sigma, \theta)$ with fibers $(n-1)$-dimensional balls. By the assumption i)$_\omega$, $\omega_1(\sigma, \theta)$ is convex, therefore $\Sigma_{(\sigma,\theta)} \cap \Omega$ is contractible for any choice of (σ, θ) provided $\Sigma_{(\sigma,\theta)} \cap \Omega \neq \emptyset$.

From proposition 16 of [4] we derive then, that for the given choice of Ω, we will have

$$H^j(\Omega, \mathcal{O}_s) = 0 \qquad \text{for any } j > 0$$

provided a statement on local differentiable triviality similar to the previous one can be established.

By the previous remark $\Sigma_{(\sigma,\theta)} \cap \Omega$ as a fiber-space with ball-fibers is differentiably isomorphic to the trivial fiber-space $\omega_1(\sigma, \theta) \times \boldsymbol{R}^{n-1}$ over $\omega_1(\sigma, \theta)$ with a diffeomorphism which depends differentiably on σ and θ.

By assumption i)$_\omega$ $\{\omega_1(\sigma, \theta)\}_{(\sigma,\theta)\in M}$ is differentiably isomorphic to the trivial fiber-space with convex typical fiber F, $F \times M$ over M. It follows that $\{\Sigma_{(\sigma,\theta)} \cap \Omega\}_{(\sigma,\theta)\in M}$ is differentiably isomorphic to the trivial fiber-space over M, $\boldsymbol{R}^{n-1} \times F \times M$.

This establishes our contention also in this case.

d) Analytic convexity on convex open sets. We will call an open set $\Omega \subset \boldsymbol{R}^N$ a *staircase* if Ω is the union of a countable family $\mathcal{U} = \{U_i\}_{i=1,2,..}$ of convex open sets such that

$$U_i \cap U_j \subset U_h \cap U_k \qquad \text{whenever } i \leq h \leq k \leq j .$$

For instance let $N = n + k$ and let $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k)$ be cartesian coordinates in $\boldsymbol{R}^N$; let $\boldsymbol{R}^n = \{y = 0\}$ be considered as a subspace of $\boldsymbol{R}^N$. Let ω be an open convex set in $\boldsymbol{R}^n$ and let $\{\omega_i\}_{i=1,2,\ldots}$ be an increasing sequence of open convex subsets of ω such that

i) $\omega_i \subset\subset \omega$ for every $i = 1, 2, \ldots$;

ii) $\omega = \bigcup_{i=1}^{\infty} \omega_i$.

Choose a decreasing sequence $\{T_i\}_{i=1,2,\ldots}$ of strictly positive numbers and set

$$C(\omega_i, T_i) = \{(x, y) \in \boldsymbol{R}^N | x \in \omega_i, |y| < T_i\}$$

where $|y| = (\Sigma y_j^2)^{\frac{1}{2}}$. Then $U_i = C(\omega_i, T_i)$ is an open convex set in $\boldsymbol{R}^N$. Set

$$\Omega = \bigcup_{i=1}^{\infty} C(\omega_i, T_i).$$

Then Ω is a staircase. Indeed for $i \leq h \leq k \leq j$ we have $\omega_i \subset \omega_h \subset \omega_k \subset \omega_j$, $T_i \geq T_h \geq T_k \geq T_j$ so that

$$C(\omega_i, T_i) \cap C(\omega_j, T_j) = C(\omega_i, T_j) \subset C(\omega_h, T_k) = C(\omega_h, T_h) \cap C(\omega_k, T_k).$$

Clearly Ω is an open neighborhood of ω in $\boldsymbol{R}^N$ and when the sequence $\{T_i\}_{i=1,2,\ldots}$ varies Ω describes a fundamental system of open neighborhoods of Ω in $\boldsymbol{R}^N$.

Consider any Hilbert complex in $\boldsymbol{R}^N$

$$\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \ldots \tag{7}$$

for Ω open in $\boldsymbol{R}^N$. Let $\mathcal{E}_{S_0}$ denote the sheaf of germs $u \in \mathcal{E}^{s_0}$ such that $S_0(D)u = 0$. We have the following

PROPOSITION 13. *For any staircase $\Omega \subset \boldsymbol{R}^N$ and for any Hilbert complex* (7) *we have*

$$H^j(\Omega, \mathcal{E}_{S_0}) = 0 \quad \text{for } j \geq 2.$$

PROOF. Let us consider a flabby resolution of the sheaf $\mathcal{E}_{S_0}$,

$$0 \longrightarrow \mathcal{E}_{S_0} \longrightarrow \mathcal{C}^0 \xrightarrow{\delta_0} \mathcal{C}^1 \xrightarrow{\delta_1} \mathcal{C}^2 \longrightarrow \ldots$$

so that

$$H^j(\Omega, \mathcal{E}_{S_0}) = \frac{\operatorname{Ker}\left\{\Gamma(\Omega, \mathcal{C}^j) \xrightarrow{\delta_j} \Gamma(\Omega, \mathcal{C}^{j+1})\right\}}{\operatorname{Im}\left\{\Gamma(\Omega, \mathcal{C}^{j-1}) \xrightarrow{\delta_{j-1}} \Gamma(\Omega, \mathcal{C}^j)\right\}}.$$

Let $\xi \in H^j(\Omega, \mathcal{E}_{S_0})$ be represented by $f_j \in \Gamma(\Omega, \mathcal{C}^j)$ with $\delta_j f = 0$, and let $j \geq 2$.

Let $\Omega = \bigcup_{i=1}^{\infty} U_i$ be a staircase, with U_i convex and open and $U_i \cap U_j \subset \subset U_h \cap U_K$ whenever $i \leq h \leq k \leq j$.

As the complex (7) is a Hilbert complex and as the U_i's are open convex we have (since $j \geq 1$) that

$$f|U_1 = \delta_{j-1} u_1, \quad f|U_2 = \delta_{j-1} u_2, \quad f|U_3 = \delta_{j-1} u_3, \ \ldots$$

with $u_i \in \Gamma(U_i, \mathcal{C}^{j-1})$, $i = 1, 2, 3, \ldots$.

On $U_1 \cap U_2$ we have $\delta_{j-1}(u_1 - u_2) = 0$. Thus as $U_1 \cap U_2$ is convex and $j \geq 2$ we can find $v_2 \in \Gamma(U_1 \cap U_2, \mathcal{C}^{j-2})$ such that

$$u_1 - u_2 = \delta_{j-2} v_2 \quad \text{on } U_1 \cap U_2 .$$

Since $\mathcal{C}^{j-2}$ is a flabby sheaf we can extend v_2 to

$$\hat{v}_2 \in \Gamma(U_2, \mathcal{C}^{j-2}), \quad \hat{v}_2|U_1 \cap U_2 = v_2 .$$

Set $g_1 = u_1$ on U_1. Set

$$g_2 = \begin{cases} u_1 & \text{on } U_1 \\ u_2 + \delta_{j-2} \hat{v}_2 & \text{on } U_2 . \end{cases}$$

Then $g_2 = g_1$ on U_1 and $g_2 \in \Gamma(U_1 \cup U_2, \mathcal{C}^{j-1})$ with

$$f = \delta_{j-1} g_2 \quad \text{on } U_1 \cup U_2 .$$

On $(U_1 \cup U_2) \cap U_3 = U_2 \cap U_3$ we have $\delta_{j-1}(g_2 - u_3) = 0$ so that we can find $v_3 \in \Gamma(U_2 \cap U_3, \mathcal{C}^{j-2})$ with

$$g_2 - u_3 = \delta_{j-2} v_3 \quad \text{on } U_2 \cap U_3 .$$

We can extend v_3 to $\hat{v}_3 \in \Gamma(U_3, \mathcal{C}^{j-2})$; $\hat{v}_3|U_2 \cap U_3 = v_3$.

Set

$$g_3 = \begin{cases} g_2 & \text{on } U_1 \cup U_2 \\ u_3 + \delta_{j-2} \hat{v}_3 & \text{on } U_3 . \end{cases}$$

Then g_3 is defined on $U_1 \cup U_2 \cup U_3$ and $g_3 = g_2$ on $U_1 \cup U_2$ and

$$f = \delta_{j-1} g_3 \quad \text{on } U_1 \cup U_2 \cup U_3 .$$

Proceeding in this way we define

$$g_\nu \in \Gamma(U_1 \cup U_2 \cup \ldots U_\nu, \mathcal{C}^{j-1})$$

with

$$g_\nu = g_{\nu-1} \quad \text{on } U_1 \cup \ldots \cup U_{\nu-1}$$

and such that

$$f = \delta_{j-1} g_\nu \quad \text{on } U_1 \cup \ldots \cup U_\nu .$$

The collection of the g_ν's defines a $g \in \Gamma(\Omega, \mathcal{C}^{j-1})$ with $f = \delta_{j-1} g$ on Ω. This proves that $\xi = 0$.

If we apply the previous proposition to the case where the complex (7) is an elliptic and Cauchy-Kowalewska suspension of a complex (1) in $\boldsymbol{R}^n$ we obtain the following

THEOREM 4. *For any Hilbert complex* (1) *in* $\boldsymbol{R}^n$ *and for any open convex set* $\omega \subset \boldsymbol{R}^n$ *we have*

$$H^j(\omega, \mathcal{A}_{A_0}) = 0 \quad \text{for } j \geq 2 .$$

PROOF. Indeed ω has a fundamental system of neighborhoods in $\boldsymbol{R}^N$ (where the complex (1) is suspended by the complex (2)) which are staircases. By the previous proposition each staircase is selfcompatible for $j \geq 2$. Thus we can apply the corollary to proposition 10 for $j \geq 2$.

6. – Some lemmas on staircases.

a) Consider in $\boldsymbol{R}^N$ a Hilbert complex

$$\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \ldots \tag{1}$$

for Ω open in $\boldsymbol{R}^N$. Let $\mathcal{E}_{S_0}$ denote the sheaf of germs of functions $f \in \mathcal{E}^{s_0}$ such that $S_0(D) f = 0$. For Ω open we endow the space $\Gamma(\Omega, \mathcal{E}_{S_0})$ with the topology of uniform convergence of the functions and their partial derivatives on compact subsets of Ω (Schwartz topology). Then $\Gamma(\Omega, \mathcal{E}_{S_0})$ becomes a Fréchet space.

We consider now an open set $\Omega \subset \boldsymbol{R}^N$ which is a staircase i.e. $\Omega = \bigcup_{i=1}^{\infty} U_i$ with U_i open and convex and such that

$$U_i \cap U_j \subset U_h \cap U_k \quad \text{whenever } i \leq h \leq k \leq j .$$

Then $\mathcal{U} = \{U_i\}_{i=1,2,\dots}$ is a Leray covering of Ω for the sheaf $\mathcal{E}_{S_0}$. We denote by $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$ the space of the alternate 1-cocycles of the covering $\mathcal{U}$ in the sheaf $\mathcal{E}_{S_0}$. Let $f_{ij} \in \Gamma(U_i \cap U_j, \mathcal{E}_{S_0})$ be such that $\{f_{ij}\}$ is a 1-cocycle on the covering $\mathcal{U}$ i.e.

$$f_{ij} + f_{jk} = f_{ik} \quad \text{on } U_i \cap U_j \cap U_k .$$

If $i \leq j \leq k$ then $U_i \cap U_j \cap U_k = U_i \cap U_k$ as Ω is a staircase. We deduce then that for $i < j$ we must have

$$f_{ij} = \sum_{h=i}^{j-1} f_{h\,h+1} \quad \text{on } U_i \cap U_j$$

as each $f_{h\,h+1}$ for $i \leq h \leq j-1$ is defined in $U_h \cap U_{h+1} \supset U_i \cap U_j$.

One deduces from this the following

LEMMA 3. *The linear map*

$$\prod_{h=1}^{\infty} (U_h \cap U_{h+1}, \mathcal{E}_{S_0}) \to Z^1(\mathcal{U}, \mathcal{E}_{S_0})$$

defined by

$$\{f_{h\,h+1}\} \mapsto \left\{ f_{ij} = \sum_{h=i}^{j-1} f_{h\,h+1} \text{ for } i < j \right\}$$

is a topological isomorphism.

An element $\{f_{h,h+1}\} \in \prod_{h=1}^{\infty} \Gamma(U_h \cap U_{h+1}, \mathcal{E}_{S_0})$ *represents a coboundary if and only if one can find* $g_h \in \Gamma(U_h, \mathcal{E}_{S_0})$ $h = 1, 2, \dots$, *such that, for any* h, *we have*

$$f_{h\,h+1} = g_h - g_{h+1} .$$

PROOF. Indeed the alternate Čech 1-cocycles in $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$ are represented by the cochains $\{f_{ij}\} \in \prod_{i<j} \Gamma(U_i \cap U_j, \mathcal{E}_{S_0})$ such that for $i \leq j \leq k$ we have $f_{ij} + f_{jk} = f_{ik}$ on $U_i \cap U_k$. Thus the defined linear map is injective and surjective. It is also continuous for the Fréchet topologies of source and target space. By Banach theorem it is a topological isomorphism. The last part of the lemma is straightforward.

LEMMA 4. *Consider the map*

$$\prod_{h=1}^{\infty} \Gamma(\boldsymbol{R}^N, \mathcal{E}_{S_0}) \xrightarrow{\beta} \prod_{h=1}^{\infty} \Gamma(U_h \cap U_{h+1}, \mathcal{E}_{S_0})$$

defined by $\{F_h\} \xrightarrow{\beta} \{f_{h\,h+1}\}$ *with*

$$f_{h\,h+1} = r^{\boldsymbol{R}^N}_{U_h \cap U_{h+1}} F_h$$

where r denotes restriction map.

Then Im β *is dense in the target space and consists of coboundaries.*

PROOF. For every $h = 1, 2, \ldots,$ the map

$$r^{\boldsymbol{R}^N}_{U_h \cap U_{h+1}} : \Gamma(\boldsymbol{R}^N, \mathcal{E}_{S_0}) \to \Gamma(U_h \cap U_{h+1}, \mathcal{E}_{S_0})$$

has a dense image as $U_h \cap U_{h+1}$ is convex and as $S_0(D)$ is an operator with constant coefficients (Theorem 7.6.14 of [10]). Then β as the product map of the above restrictions has dense image. Set now

$$g_1 = 0\,, \quad g_2 = -F_1\,, \quad g_3 = -F_1 - F_2\,, \quad g_4 = -F_1 - F_2 - F_3, \ldots$$

Then g_h is defined on U_h and

$$g_1 - g_2 = F_1 = f_{12}\,, \quad g_2 - g_3 = F_2 = f_{23}\,, \quad g_3 - g_4 = F_3 = f_{34}, \ldots$$

Therefore $\beta(\{F_h\})$ is a coboundary.

LEMMA 5. *Let* $\Omega = \bigcup_{i=1}^{\infty} U_i$ *be a staircase and let* $B \subset \Omega$ *be open. If*

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})\} < \infty$$

then necessarily

$$\operatorname{Im}\{H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})\} = 0\,.$$

PROOF. Set $V_i = U_i \cap B$. Then $\mathcal{V} = \{V_i\}_{i=1,2,\ldots}$ is a covering of B. The natural map $H^1(\mathcal{V}, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})$ is an injective map (by Leray theorem). The covering $\mathcal{U} = \{U_i\}_{i=1,2,\ldots}$ of Ω is a Leray covering; thus $H^1(\Omega, \mathcal{E}_{S_0}) = H^1(\mathcal{U}, \mathcal{E}_{S_0})$. If an element $\xi \in H^1(\mathcal{U}, \mathcal{E}_{S_0})$ vanishes in $H^1(B, \mathcal{E}_{S_0})$, it must vanish in $H^1(\mathcal{V}, \mathcal{E}_{S_0})$ already.

We set $C^0(\mathcal{V}, \mathcal{E}_{S_0}) = \prod_{i=1}^{\infty} \Gamma(V_i, \mathcal{E}_{S_0})$ and consider the space

$$G = \{(u, v) \in Z^1(\mathcal{U}, \mathcal{E}_{S_0}) \times C^0(\mathcal{V}, \mathcal{E}_{S_0}) \big| \, u|\mathcal{V} = \delta_{\mathcal{V}} v\}$$

where $\delta_{\mathcal{V}}$ represents the coboundary map $\delta_{\mathcal{V}}: C^0(\mathcal{V}, \mathcal{E}_{S_0}) \to C^1(\mathcal{V}, \mathcal{E}_{S_0})$ in Čhech cohomology $(C^1(\mathcal{V}, \mathcal{E}_{S_0}) = \prod \Gamma(V_h \cap V_k, \mathcal{E}_{S_0}))$. This is a closed subspace of the space $Z^1(\mathcal{U}, \mathcal{E}_{S_0}) \times C^0(\mathcal{V}, \mathcal{E}_{S_0})$ with its natural Fréchet topology. Thus G is a Fréchet space.

Set

$$W = \mathrm{pr}_{Z^1(\mathcal{U}, \mathcal{E}_{S_0})} G .$$

Then W represents the subspace of $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$ of those cocycles which become coboundaries when restricted to B. By construction W is a continuous image of a Fréchet space.

The assumption of the lemma states that

$$\dim_{\mathbf{C}} \frac{Z^1(\mathcal{U}, \mathcal{E}_{S_0})}{W} < \infty$$

i.e. that W is of finite codimension in $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$. By the above remark we must have that W is a closed subspace of $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$.

By lemma 4 W, containing the coboundaries of $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$, must be dense in $Z^1(\mathcal{U}, \mathcal{E}_{S_0})$. Therefore we must have $W = Z^1(\mathcal{U}, \mathcal{E}_{S_0})$ and this proves that $\mathrm{Im}\,\{H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})\} = 0$.

b) Set $\mathbf{R}^N = \mathbf{R}^n \times \mathbf{R}^k$ with $N = n + k$ and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k)$ as cartesian coordinates in $\mathbf{R}^N$. Let ω be an open set in $\mathbf{R}^n$ and let us assume that

$$\dim_{\mathbf{C}} H^q(\omega, \mathcal{E}_{S_0}) = d < \infty .$$

Then there exists an open neighborhood Ω_0 of ω in $\mathbf{R}^N$ and d cohomology classes

$$\xi_j \in H^q(\Omega_0, \mathcal{E}_{S_0}) \quad \text{for } 1 \le j \le d,$$

such that, under the natural restriction map

$$r_\omega^{\Omega_0} : H^q(\Omega_0, \mathcal{E}_{S_0}) \to H^q(\omega, \mathcal{E}_{S_0}) ,$$

the classes $r_\omega^{\Omega_0}(\xi_j)$ for $1 \le j \le d$ form a basis over $\mathbf{C}$ of $H^q(\omega, \mathcal{E}_{S_0})$. Each class ξ_j will be represented by a function $f_j \in \mathcal{E}^{s_q}(\Omega_0)$ with $S_q(D) f_j = 0$, $\xi_j = \{f_j\}$.

Let Ω be any open neighborhood of ω in $\mathbf{R}^N$ and let ϕ denote the class of closed subsets $C \subset \Omega$ with $C \cap \omega = \emptyset$. We have then the exact cohomology sequence

$$\ldots \to H^q_\phi(\Omega, \mathcal{E}_{S_0}) \to H^q(\Omega, \mathcal{E}_{S_0}) \xrightarrow{r_\omega^\Omega} H^q(\omega, \mathcal{E}_{S_0}) \to \ldots$$

where r_ω^Ω denotes the natural restriction map and where the suffix ϕ denotes cohomology with supports in ϕ.

Let $q \geq 1$ and let $\{f\} \in H^q(\Omega, \mathcal{E}_{S_0})$ with $f \in \mathcal{E}^{s_q}(\Omega)$, $S_q(D)f = 0$. We can find d complex numbers $\lambda_j(f)$, $1 \leq j \leq d$, such that

$$r_\omega^\Omega\{f\} = \sum_{j=1}^{d} \lambda_j(f) r_\omega^{\Omega_0}\{f_j\} .$$

We deduce then from the above cohomology sequence that

« *There exists an open neighborhood B_f of ω in Ω_0 and a function $u \in \mathcal{E}^{s_{q-1}}(B_f)$ such that we have on B_f*

$$f - \sum_{j=1}^{d} \lambda_j(f) f_j = S_{q-1}(D)u . \text{»}$$

For ω open in $\boldsymbol{R}^n$ and any $T > 0$ we set, as usual,

$$C(\omega, T) = \{(x, y) \in \boldsymbol{R}^N | x \in \omega,\ |y| < T\}$$

where $|y| = (\sum y_j^2)^{\frac{1}{2}}$.

In the following lemma ω is not supposed to be convex. We will write $\omega = \bigcup_{j=1}^{\infty} \omega_j$ with ω_j open and relatively compact in ω and $\omega_j \subset \omega_{j+1}$, for $j = 1, 2, \ldots$.

LEMMA 6. *Let $\Omega = \bigcup_{j=1}^{\infty} \omega_j$ be an open set in $\boldsymbol{R}^n$ as above. Let Ω be an open neighborhood of ω in $\boldsymbol{R}^N$.*

Assume that

$$\dim_{\boldsymbol{C}} H^q(\Omega, \mathcal{E}_{S_0}) = d < \infty .$$

We can find a sequence of positive numbers $\{T_j\}$, $T_j > 0$, $\forall j$, such that, if we set

$$B_h = \bigcup_{j=1}^{h} C(\omega_j, T_j)$$

we have:

$$B_h \subset \Omega \quad \textit{for every } h = 1, 2, \ldots$$

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} \leq d , \quad \textit{for every } h = 1, 2, \ldots .$$

PROOF. We use the notations and remarks made above. We set

$$Z^q(\Omega, \mathcal{E}_{S_0}) = \{f \in \mathcal{E}^{s_q}(\Omega) | S_q(D)f = 0\} .$$

We denote by V_n the subset of $Z^q(\Omega, \mathcal{E}_{S_0})$ of those $f \in Z^q(\Omega, \mathcal{E}_{S_0})$ such that there exist an open neighborhood B_f of ω in $\Omega_0 \cap \Omega$ with $C(\omega_1, 1/n) \subset B_f$

and a function $u \in \mathcal{E}^{S_{q-1}}(B_f)$ such that

$$f - \sum_{j=1}^{d} \lambda_j(f) f_j = S_{q-1}(D) u \quad \text{on } B_f .$$

We have

$$Z^q(\Omega, \mathcal{E}_{S_0}) = \bigcup_{n=1}^{\infty} V_n .$$

Therefore for some n_1 V_{n_1} will be a set of second category. We denote by $V_{n_1 n}$ the subset of $Z^q(\Omega, \mathcal{E}_{S_0})$ of those $f \in Z^q(\Omega, \mathcal{E}_{S_0})$ such that

there exist an open neighborhood B_f of ω in $\Omega_0 \cap \Omega$ with

$$C\left(\omega_1, \frac{1}{n_1}\right) \cup C\left(\omega_2, \frac{1}{n}\right) \subset B_f$$

and a function $u \in \mathcal{E}^{S_{q-1}}(B_f)$ such that

$$f - \sum_{j=1}^{d} \lambda_j(f) f_j = S_{q-1}(D) u \text{ on } B_f .$$

We have $V_n = \bigcup_{n=1}^{\infty} V_{n_1 n}$. Therefore, for some integer n_2, $V_{n_1 n_2}$ will be of second category.

Proceeding in this way we define for every integer $h > 0$ subsets $V_{n_1 n_2 \ldots n_h}$ of $Z^q(\Omega, \mathcal{E}_{S_0})$ of second category such that if $f \in V_{n_1 n_2 \ldots n_h}$ there exist an open neighborhood B_f of ω in $\Omega_0 \cap \Omega$ with

$$C\left(\omega_1, \frac{1}{n_1}\right) \cup C\left(\omega_2, \frac{1}{n_2}\right) \cup \ldots \cup C\left(\omega_h, \frac{1}{n_h}\right) \subset B_f$$

and a function $u \in \mathcal{E}^{S_{q-1}}(B_f)$ such that

$$f - \sum_{j=1}^{d} \lambda_j(f) f_j = S_{q-1}(D) u \quad \text{on } B_f .$$

Let $G = \left(\sum_{j=1}^{d} \lambda_j f_j\right)$ be the vector space generated by the functions f_j on $\mathcal{E}^{S_q}(\Omega_0)$. We have $G \simeq \boldsymbol{C}^d$.

We choose $T_j = 1/n_j$ and set $B_h = \bigcup_{j=1}^{h} C(\omega_j, 1/n_j)$. For every integer $h > 0$, we consider the space

$$E(h) = \{(f, u, g) \in Z^q(\Omega, \mathcal{E}_{S_0}) \times \mathcal{E}^{S_{q-1}}(B_h) \times G \,|\, f - g = S_{q-1}(D) u \text{ on } B_h\} .$$

This is a closed subspace of the product space $Z^q(\Omega, \mathcal{E}_{S_0}) \times \mathcal{E}^{s_{q-1}}(B_h) \times G$ and therefore it is a Fréchet space. The projection of $E(h)$ into $Z^q(\Omega, \mathcal{E}_{S_0})$ is a continuous linear map whose image contains the set $V_{n_1 n_2 \dots n_h}$ which is of second category. It follows from Banach theorem that the projection must be surjective. This means that for every $f \in Z^q(\Omega, \mathcal{E}_{S_0})$ there exist

an open neighborhood B_f of ω in $\Omega_0 \cap \Omega$ with $B_f \supset B_h$
a function $u \in \mathcal{E}^{s_{q-1}}(B_f)$
an element $g = \sum \lambda_i(f) f_i \in G$

such that

$$f - \sum \lambda_i(f) f_i = S_{q-1}(D) u \quad \text{on } B_f .$$

This shows that $\dim_{\mathbf{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} \le d$ because that image is generated by the classes $\{r^{\Omega}_{B_h} f_j\}$ for $1 \le j \le d$.

7. – Necessary conditions for analytic convexity.

a) We consider applications of the last lemma 6 to the following situation.

The Hilbert complex in $\mathbf{R}^N = \mathbf{R}^n \times \mathbf{R}^k$

$$(\mathcal{E}^*(\Omega), S_*) \equiv \left\{\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \dots\right\} \tag{1}$$

in an elliptic and Cauchy-Kowalewska suspension of a Hilbert complex in $\mathbf{R}^n$

$$(\mathcal{E}^*(\omega), A_*) \equiv \left\{\mathcal{E}^{p_0}(\omega) \xrightarrow{A_0(D)} \mathcal{E}^{p_1}(\omega) \xrightarrow{A_1(D)} \mathcal{E}^{p_2}(\omega) \longrightarrow \dots\right\}. \tag{2}$$

Let ω be a given open set in $\mathbf{R}^n$ and let Ω be an open neighborhood of ω in $\mathbf{R}^N$, let $q \ge 1$ be an integer and let us assume that $H^q(\omega, \mathcal{A}_{A_0}) = 0$, where $\mathcal{A}_{A_0}$ as usual denotes the sheaf of germs of analytic functions on $\mathbf{R}^n$, $u \in \mathcal{A}^{p_0}$, such that $A_0(D)u = 0$. Because of the assumption that the suspension complex is elliptic and Cauchy-Kowalewska we have that $H^q(\omega, \mathcal{E}_{S_0}) = H^q(\omega, \mathcal{A}_{A_0})$. Therefore by lemma 6, writing $\omega = \bigcup_{j=1}^{\infty} \omega_j$ with ω_j open, $\omega_j \subset\subset \omega$, $\omega_j \subset \omega_{j+1}$ $\forall j$, we can find a sequence of positive numbers $\{T_j\}_{j=1,2,\dots}$ so that if we set

$$B_h = \bigcup_{j=1}^{h} C(\omega_j, T_j)$$

we have $B_h \subset \Omega$, $\forall h = 1, 2, \ldots$ and

$$\text{(3)} \qquad \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} = 0 .$$

Set

$$B = \bigcup_{j=1}^{\infty} C(\omega_j, T_j) .$$

We can then ask if we can pass to the limit for $h \to \infty$ in (3) to obtain

$$\text{(4)} \qquad \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} = 0 .$$

In that case we will get the following

STATEMENT. *A necessary and sufficient condition for having*

$$H^q(\Omega, \mathcal{A}_{A_0}) = 0$$

is that for any neighborhood Ω of ω in $\boldsymbol{R}^N$ we can find a neighborhood B of ω in Ω such that

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} = 0 .$$

In other words « a necessary and sufficient condition for ω open in $\boldsymbol{R}^N$ to have analytic convexity in dimension q (i.e. that $H^q(\omega, \mathcal{A}_{A_0}) = 0$) with respect to the Hilbert complex (2) is that ω admits a fundamental system of q-compatible pairs of open neighborhoods in $\boldsymbol{R}^N$ for the suspended complex (1) of (2) ».

The possibility to obtain a limit relation (4) from (3) is based on a Runge-type approximation theorem.

b) We consider on $\boldsymbol{R}^N$ a Hilbert complex $(\mathcal{E}^*(\Omega), S_*)$. For Ω open in $\boldsymbol{R}^N$ and for any $j \geq 0$ we set

$$Z^j(\Omega, \mathcal{E}_{S_0}) = \{f \in \mathcal{E}^{s_j}(\Omega) \,|\, S_j(D) f = 0\} .$$

We endow this space with the topology of uniform convergence on compact subsets of Ω of the functions and all their partial derivatives (Schwartz topology). With that topology $Z^j(\Omega, \mathcal{E}_{S_0})$ is a Fréchet space. Given a compact subset K of Ω we denote by $Z^j(\Omega, \mathcal{E}_{S_0})|K$ the space of all restrictions to K of functions of $Z^j(\Omega, \mathcal{E}_{S_0})$.

Let A be another open set in $\boldsymbol{R}^N$ and let $\Omega \subset A$.

We will say that the restriction map

$$Z^j(A, \mathcal{E}_{S_0}) \to Z^j(\Omega, \mathcal{E}_{S_0})|K$$

has a dense image if given $\varepsilon > 0$, for every $f \in Z^j(\Omega, \mathcal{E}_{S_0})$ and for every integer $k \geq 0$ we can find $g_{\varepsilon,k} \in Z^j(A, \mathcal{E}_{S_0})$ such that

$$\sum_{|\alpha| \leq k} \sup_K |D^\alpha g_{\varepsilon,k} - D^\alpha f| < \varepsilon .$$

We set as a notation $\|g_{\varepsilon,k} - f\|_{K,k} = \sum_{|\alpha|\leq k} \sup_K |D^\alpha g_{\varepsilon,k} - D^\alpha f|$.
We have the following

PROPOSITION 14. *Let Ω be open in $\mathbf{R}^N$. We suppose that we can find an increasing sequence of open subsets B_h of Ω, $B_h \subset B_{h+1}$ for every $h \geq 1$, such that, for some integer $q \geq 1$, we have*

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} = 0 .$$

Set $B = \bigcup_{h=1}^{\infty} B_h$ and let K denote a compact subset of B.
We assume that for every K there exists an integer $h(K) \geq 1$ such that

i) $K \subset B_{h(K)}$;

ii) *the restriction map $Z^{q-1}(B, \mathcal{E}_{S_0}) \to Z^{q-1}(B_{h(K)}, \mathcal{E}_{S_0})|K$ has a dense image.*
Then

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} = 0 .$$

PROOF. We set $B = \bigcup_{j=1}^{\infty} K_j$ with K_j compact and $K_j \subset K_{j+1}$ for every j. We have $K_j \subset B_{h(K_j)}$ for every j. By dropping some of the B_h's and renumbering them, we may assume that $K_j \subset B_j$ and that $Z^{q-1}(B, \mathcal{E}_{S_0}) \to$ $\to Z^{q-1}(B_j, \mathcal{E}_{S_0})|K_j$ has a dense image.

Let $f \in Z^q(\Omega, \mathcal{E}_{S_0})$. By the first assumption we can find $u_j \in \mathcal{E}^{s_{q-1}}(B_j)$ such that

$$f = S_{q-1}(D) u_j \quad \text{ on } B_j .$$

We have

$$S_{q-1}(D)(u_2 - u_1) = 0 \quad \text{ on } B_1 , \qquad S_{q-1}(D)(u_3 - u_2) = 0 \quad \text{ on } B_2 , \ldots .$$

By the second assumption we can find $g_1 \in Z^{q-1}(B, \mathcal{E}_{S_0})$ such that

$$\|u_2 - u_1 - g_1\|_{K_1, 1} < \tfrac{1}{2} .$$

Therefore replacing u_2 by $u_2 - g_1$ we may assume that

$$\|u_2 - u_1\|_{K_1,1} < \tfrac{1}{2}\,.$$

Similarly we can find $g_2 \in Z^{q-1}(B, \mathcal{E}_{S_0})$ such that

$$\|u_3 - u_2 - g_2\|_{K_2,2} < \frac{1}{2^2}$$

so that replacing u_3 by $u_3 - g_2$ we may assume that

$$\|u_3 - u_2\|_{K_2,2} < \frac{1}{2^2}\,.$$

Proceeding in this way we see that under our assumptions we may assume that

$$\|u_{h+1} - u_h\|_{K_h,h} < \frac{1}{2^h}$$

for every $h \geq 1$.

We consider the series

$$u_h + (u_{h+1} - u_h) + (u_{h+2} - u_{h+1}) + \ldots$$

This defines a C^∞ function U_h on B_h, because the series converges uniformily with all partial derivatives on every compact set K_l as $\|u_{h+1} - u_h\|_{K_l,m} < 1/2^h$ if $h \geq l + m$. Moreover $U_h = U_{h+1}$ on B_h for every $h \geq 1$. Thus we have defined a function $U \in \mathcal{E}^{s_{q-1}}(B)$ such that $f = S_{q-1}(D)\,U$. This proves our contention.

With few changes in the proof we obtain also the following

PROPOSITION 15. *Let ω be open in $\boldsymbol{R}^n$ and let Ω be an open neighborhood of ω in $\boldsymbol{R}^N$.*

We assume that

$$\dim_{\boldsymbol{C}} H^q(\omega, \mathcal{E}_{S_0}) = d < \infty$$

so that, according to lemma 6, we can find an increasing sequence of open sets B_h in Ω, $B_h \subset B_{h+1}$ for every $h \geq 1$ such that

i) $\dim_{\boldsymbol{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} \leq d$;

ii) for $B = \bigcup_{h=1}^{\infty} B_h$ we have $\omega \subset B$.

We assume that for every compact subset K of B there exists an integer $h(K) \geq 1$ such that

i) $K \subset B_{h(K)}$

ii) *the restriction map* $Z^{q-1}(B, \mathcal{E}_{S_0}) \to Z^{q-1}(B_{h(K)}, \mathcal{E}_{S_0})|K$ *has a dense image.*
Then we also have

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} \leq d\,.$$

PROOF. We use the same notations as in lemma 6 and in the previous proposition. Let $f \in Z^q(\Omega, \mathcal{E}_{S_0})$ and let $\xi_1 = \{f_1\}, \ldots, \xi_d = \{f_d\}$ with $f_j \in Z^q(\Omega_0, \mathcal{E}_{S_0})$, $\Omega_0 \subset \Omega$, represent generators of $H^q(\omega, \mathcal{E}_{S_0})$. Choose $\lambda_j(f) \in \boldsymbol{C}$ so that

$$r^{\Omega}_{\omega}\{f\} = \sum \lambda_j(f) r^{\Omega_0}_{\omega}\{f_j\}\,.$$

Then we can find $u_h \in \mathcal{E}^{s_{q-1}}(B_h)$ such that

$$f - \sum \lambda_j(f) f_j = S_{q-1}(D) u_h \quad \text{ on } B_h\,.$$

By the same argument used in the previous proposition we can choose the elements u_h so that

$$\|u_{h+1} - u_h\|_{K_h, h} < \frac{1}{2^h}\,.$$

The sequence of compact sets $\{K_h\}_{h=1,2,\ldots}$ being so chosen that $B = \bigcup K_h$, $K_h \subset K_{h+1}$, $Z^{q-1}(B, \mathcal{E}_{S_0})$ has dense image in $Z^{q-1}(B_h, \mathcal{E}_{S_0})|K_h$, $\forall h \geq 1$. As in the previous proposition we construct $U \in \mathcal{E}^{s_{q-1}}(B)$ such that

$$f - \sum \lambda_j(f) f_j = S_{q-1}(D)\, U \quad \text{ on } B\,.$$

This achieves the proof.

8. – Analytic convexity on convex open sets.

a) We give a Hilbert complex in $\boldsymbol{R}^n$

$$(\mathcal{E}^*(\omega), A_*) \equiv \left\{\mathcal{E}^{s_0}(\omega) \xrightarrow{A_0(D)} \mathcal{E}^{s_1}(\omega) \xrightarrow{A_1(D)} \mathcal{E}^{s_2}(\omega) \longrightarrow \ldots\right\} \tag{1}$$

for all ω open in $\boldsymbol{R}^n$. By $\mathcal{A}_{A_0}$ we denote as usual the sheaf of germs of real analytic functions with values in $\boldsymbol{C}^{p_0}$, $u \in \mathcal{A}^{p_0}$ such that $A_0(D)u = 0$. We are interested in the groups of analytic cohomology $H^q(\omega, \mathcal{A}_{A_0})$.

If we make the drastic *assumption that* ω *is a convex open set* in $\boldsymbol{R}^n$ then by theorem 4 $H^q(\omega, \mathcal{A}_{A_0}) = 0$ if $q \geq 2$. It remains then to study only the group $H^1(\omega, \mathcal{A}_{A_0})$.

To this end we consider an elliptic Cauchy-Kowalewska suspension of the given complex (1) in $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ (cf. section 4*d*) example (α))

$$(2) \qquad (\mathcal{E}^*(\Omega), S_*) \equiv \left\{\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \ldots\right\},$$

for Ω open in $\boldsymbol{R}^{n+1}$. We choose as coordinates in $\boldsymbol{R}^{n+1}$ $(x, y) = (x_1, \ldots, x_n, y)$. For ω open in $\boldsymbol{R}^n$ we consider the set in $\boldsymbol{R}^{n+1}$

$$\Omega = \{(x, y) \in \boldsymbol{R}^{n+1} | x \in \omega,\ |y| < \varrho(x)\}$$

where $\varrho: \omega \to \boldsymbol{R}$ is positive, $\varrho > 0$, and upper semicontinuous. Then Ω is open and when ϱ varies Ω describes a fundamental system of neighborhoods of ω in $\boldsymbol{R}^{n+1}$.

We denote by $\mathcal{E}_{S_0}$ the sheaf of germs of functions $f \in \mathcal{E}^{s_0}$ such that $S_0(D)f = 0$. For Ω open in $\boldsymbol{R}^{n+1}$ we consider the space $Z^0(\Omega, \mathcal{E}_{S_0}) = \Gamma(\Omega, \mathcal{E}_{S_0})$ endowed with the topology of uniform convergence on compact sets of the functions and all their partial derivatives. We have the following approximation theorem

THEOREM 5. *Let* (2) *be a Hilbert complex in* $\boldsymbol{R}^{n+1}$ *with the first operator* $S_0(D)$ *elliptic.*

Let $\omega \subset \boldsymbol{R}^n$ *be open and convex and for* $\varrho: \omega \to \boldsymbol{R}$ *positive and upper semicontinuous set*

$$\Omega = \{(x, y) \in \boldsymbol{R}^{n+1} | x \in \omega,\ |y| < \varrho(x)\}.$$

Then the restriction map

$$r_\Omega^{\boldsymbol{R}^{n+1}}: \Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega, \mathcal{E}_{S_0})$$

has a dense image.

b) We admit for a moment the previous theorem. We derive then the following consequence

THEOREM 6. *Let* ω *be open and convex. For any Hilbert complex* (1) *we have that*

$$either\ H^1(\omega, \mathcal{A}_{A_0}) = 0 \qquad or \qquad \dim_{\boldsymbol{C}} H^1(\omega, \mathcal{A}_{A_0}) = \infty .$$

PROOF. Write $\omega = \bigcup_{j=1}^{\infty} \omega_j$ with ω_j open convex $\omega_j \subset\subset \omega$ and $\omega_j \subset \omega_{j+1}$ for every j. Let $\{T_j\}$ be a decreasing sequence of positive numbers. Then $B = \bigcup C(\omega_j, T_j)$ (cf. section 6 *b*) for the notation) describes a fundamental system of open neighborhoods of ω in $\boldsymbol{R}^{n+1}$; moreover each B is a staircase as the ω_j are convex. We take now any staircase neighborhood Ω of ω in $\boldsymbol{R}^{n+1}$. By lemma 6 and theorem 5 we can apply proposition 15 for $q = 1$ provided we have

$$\dim H^1(\omega, \mathcal{E}_{S_0}) = \dim H^1(\omega, \mathcal{A}_{A_0}) = d < \infty .$$

We then find a $B \subset \Omega$ such that

$$\dim \operatorname{Im} \{H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})\} \leq d .$$

Since Ω is a staircase we have then by lemma 5 that the restriction map $H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})$ is the zero map.

Therefore

$$H^1(\omega, \mathcal{E}_{S_0}) = \varinjlim_{\Omega} H^1(\Omega, \mathcal{E}_{S_0}) = 0 .$$

This shows that necessarily $d = 0$. This proves the theorem. Assume now that $H^1(\omega, \mathcal{A}_{A_0}) = 0$ i.e. $H^1(\omega, \mathcal{E}_{S_0}) = 0$. Then we can apply proposition 14 to the conclusion of lemma 6 by virtue of theorem 5 for $q = 1$. We obtain then the following

THEOREM 7. *Let* (1) *be any Hilbert complex in* $\boldsymbol{R}^n$ *and let* (2) *be any elliptic Cauchy-Kowalewska suspension in* $\boldsymbol{R}^N$ $(N > n)$. *Let* ω *be open and convex in* $\boldsymbol{R}^n$. *Necessary and sufficient condition for* $H^1(\omega, \mathcal{A}_{A_0}) = 0$ *is that*

(A) *for every open neighborhood* Ω *of* ω *in* $\boldsymbol{R}^N$ *there exists an open neighborhood* B *of* ω *in* Ω *such that*

$$\operatorname{Im} \{H^1(\Omega, \mathcal{E}_{S_0}) \to H^1(B, \mathcal{E}_{S_0})\} = 0 .$$

PROOF. For $N = n + 1$ this is what one obtains from the argument given above. From proposition 11 we deduce then that condition (A) being verified by suspensions from $\boldsymbol{R}^n$ to $\boldsymbol{R}^{n+1}$ must also be verified by any other (elliptic and Cauchy-Kowalewska) suspension from $\boldsymbol{R}^n$ to $\boldsymbol{R}^N$.

c) Proof of theorem 5. (α) Set

$$\Omega^- = \{(x, y) \in \boldsymbol{R}^{n+1} | x \in \omega, y < \varrho(x)\} ; \qquad \Omega^+ = \{(x, y) \in \boldsymbol{R}^{n+1} | x \in \omega, y > -\varrho(x)\} .$$

From the Mayer-Vietoris sequence we deduce the exact sequence:

$$0 \to \Gamma(\Omega^+ \cup \Omega^-, \mathcal{E}_{S_0}) \to \Gamma(\Omega^+, \mathcal{E}_{S_0}) \oplus \Gamma(\Omega^-, \mathcal{E}_{S_0}) \to \Gamma(\Omega, \mathcal{E}_{S_0}) \to H^1(\Omega^+ \cup \Omega^-, \mathcal{E}_{S_0}) .$$

Since ω is convex $\Omega^+ \cup \Omega^-$ is also convex. Therefore $H^1(\Omega^+ \cup \Omega^-, \mathcal{E}_{S_0}) = 0$ and hence the map

$$\Gamma(\Omega^+, \mathcal{E}_{S_0}) \oplus \Gamma(\Omega^-, \mathcal{E}_{S_0}) \to \Gamma(\Omega, \mathcal{E}_{S_0})$$

given by

$$u^+ \oplus u^- \to u^+ - u^-$$

is a surjective map.

It follows that to prove that the restriction map

$$\Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega, \mathcal{E}_{S_0})$$

has a dense image, it is enough to show that the restriction maps

$$\Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega^+, \mathcal{E}_{S_0}) , \quad \Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega^-, \mathcal{E}_{S_0}) ,$$

have dense image and for this it is enough to show that, for any choice of ϱ, the map

$$\Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega^-, \mathcal{E}_{S_0})$$

has a dense image.

(β) We imbed $\boldsymbol{R}^{n+1}$ in $\boldsymbol{C}^{n+1}$ where $z_j = x_j + iy_j$ $1 \leq j \leq n$ and $w = y + it$ are complex coordinates.

Since the operator $S_0(D)$ is elliptic there exists an open neighborhood $\tilde{\Omega}^-$ of Ω^- in $\boldsymbol{C}^{n+1}$ such that any $u \in \Gamma(\Omega^-, \mathcal{E}_{S_0})$ extends to an element $\tilde{u}$ holomorphic in $\tilde{\Omega}$ (such that $S_0(D)\tilde{u} = 0$).

For any compact set $K \subset \boldsymbol{C}^{n+1}$ and any $\varepsilon > 0$ we denote by $K(\varepsilon)$ the ε-neighborhood of K in $\boldsymbol{C}^{n+1}$ i.e. the set of points of $\boldsymbol{C}^{n+1}$ whose polycylindrical distance from K is not greater than ε:

$$K(\varepsilon) = \left\{(z, w) \in \boldsymbol{C}^{n+1} \middle| \exists (z^0, u^0) \in K \text{ with } \sup_{1 \leq \alpha \leq n} |z_\alpha - z_\alpha^0| \leq \varepsilon, \ |w - w^0| \leq \varepsilon \right\}.$$

Let now $K \subset \Omega^-$ be a compact set and let $\lambda_0 > 0$ be so chosen that

$$K \subset \{(x, y) \in \boldsymbol{R}^{n+1} | x \in \omega, y < \lambda_0\} = U .$$

It will be enough to prove the following

STATEMENT. *Given* $u \in \Gamma(\Omega^-, \mathcal{E}_{S_0})$, *given* $\delta > 0$ *and given an integer* $k \geq 0$, *we can find* $v \in \Gamma(U, \mathcal{E}_{S_0})$ *such that*

$$\sum_{|\alpha|+\beta\leq k} \sup_K |D_x^\alpha D_y^\beta (u - v)| < \delta .$$

Indeed as U is a convex set, the restriction map

$$\Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(U, \mathcal{E}_{S_0})$$

has a dense image because S_0 is an operator with constant coefficients ([10] theorem 7.6.14). It follows then that given $u \in \Gamma(\Omega^-, \mathcal{E}_{S_0})$, given a compact set $K \subset \Omega^-$, given $\delta > 0$ and k integer with $k \geq 0$ we can find $v \in \Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0})$ such that

$$\sum_{|\alpha|+\beta\leq k} \sup_K |D_x^\alpha D_y^\beta (u - v)| < \delta .$$

But this means that the restriction map $\Gamma(\boldsymbol{R}^{n+1}, \mathcal{E}_{S_0}) \to \Gamma(\Omega^-, \mathcal{E}_{S_0})$ has a dense image.

(γ) To prove the above statement we proceed as follows. Let $e = {}^t(0, \dots, 0, 1) \in \boldsymbol{C}^{n+1}$ be the unit vector in the y-direction. Given $K \subset \Omega^-$ compact we determine $\lambda_0 > 0$ as before and set

$$F = \bigcup_{0\leq\lambda\leq\lambda_0} (K - \lambda e) = \{(x, y) \in \boldsymbol{R}^{n+1} | (x, y + \lambda) \in K \text{ for some } 0 \leq \lambda \leq \lambda_0\} .$$

Then F is a compact subset of Ω^-.

We can find $\varepsilon > 0$ such that $F(4\varepsilon)$, the 4ε-neighborhood of F in $\boldsymbol{C}^{n+1}$, is contained in $\tilde{\Omega}^-$; $F(4\varepsilon) \subset \tilde{\Omega}^-$. Then every $u \in \Gamma(\Omega^-, \mathcal{E}_{S_0})$ extends holomorphically to a neighborhood of $F(4\varepsilon)$. Let H be a compact set of $\boldsymbol{C}^{n+1}$ and let $u = {}^t(u_1, \dots, u_{s_0})$ be a continuous function defined in a neighborhood of H. We set

$$|u| = \sup_{1\leq\alpha\leq s_0} |u_\alpha| \quad \text{and} \quad \|u\|_H = \sup_H |u| .$$

Let $\alpha \in \boldsymbol{N}^n$, $\beta \in \boldsymbol{N}$, $\varepsilon > 0$ and let $v = {}^t(v_1, \dots, v_{s_0})$ be holomorphic in a neighborhood of $K(\varepsilon)$ (any K compact). From Cauchy integral formula we deduce the estimate

$$(*) \qquad \|D_z^\alpha D_w^\beta v\|_K \leq \alpha!\, \beta!\, \varepsilon^{-|\alpha|-\beta} \|v\|_{K(\varepsilon)} .$$

We deduce then that the above statement is a consequence of the following

LEMMA 7. *Let* K *be compact in* $\boldsymbol{C}^{n+1}$, *let* $\lambda_0 > 0$, *let*

$$F = \bigcup_{0\leq\lambda\leq\lambda_0} (K - \lambda e) .$$

Let $u = {}^t(u_1, \dots, u_{s_0})$ *be holomorphic in a neighborhood of* $F(4\varepsilon)$ *and let us choose an integer* $k > 0$ *such that* $\sigma = (\lambda_0/k) \leq \varepsilon$.

Let $\delta > 0$ *be given. We can find integers* $m_1(\delta) \geq 0, \dots, m_k(\delta) \geq 0$ *such that, setting*

$$u_\delta(z, w) = \sum_{\substack{0 \leq s_1 \leq m_1(\delta) \\ \cdots \\ 0 \leq s_k \leq m_k(\delta)}} \frac{\sigma^{s_1 + \dots + s_k}}{s_1! \dots s_k!} D_w^{s_1 + \dots + s_k} u(z, w - \lambda_0)$$

we have

$$\|u(z, w) - u_\delta(z, w)\|_{K(\varepsilon)} < \delta .$$

Indeed the functions $f = D_w^{s_1 + \dots + s_k} u(z, w - \lambda_0)$ for $u \in \Gamma(\Omega^-, \mathcal{E}_{s_0})$ are defined in $\Omega^- + \lambda_0 e$ which is an open set containing U and moreover they satisfy the equation $S_0(D) f = 0$. The desired estimate with the partial derivatives is derived from the conclusion of the lemma and estimate $(*)$.

(δ) *Proof of the lemma.* Let $f(z, w)$ and $f(z, w - \sigma)$ be both holomorphic in a neighborhood of $K(\varepsilon + r)$ for $\varepsilon < 0$, $r > \sigma$; then we have*

$$\left\| f(z, w) - \sum_{s=0}^{m} \frac{\sigma^s}{s!} D_w^s f(z, w - \sigma) \right\|_{K(\varepsilon)} \leq \left(\frac{\sigma}{r}\right)^m \frac{\sigma}{r - \sigma} \|f(z, w - \sigma)\|_{K(\varepsilon + r)} .$$

In fact one has that the left hand side equals

$$\left\| \sum_{m+1}^{\infty} \frac{\sigma^s}{s!} D_w^s f(z, w - \sigma) \right\|_{K(\varepsilon)} \leq \sum_{m+1}^{\infty} \frac{\sigma^s}{s!} s! \, r^{-s} \|f(z, w - \sigma)\|_{K(\varepsilon + r)}$$

$$\leq \left(\frac{\sigma}{r}\right)^m \frac{\sigma}{r - \sigma} \|f(z, w - \sigma)\|_{K(\varepsilon + r)}$$

by virtue of inequality $(*)$.

We have then

$$\|u(z, w) - u_\delta(z, w)\|_{K(\varepsilon)} \leq \left\| u(z, w) - \sum_{s_1=0}^{m_1} \frac{\sigma^{s_1}}{s_1!} D_w^s u(z, w - \sigma) \right\|_{K(\varepsilon)} +$$

$$+ \sum_{j=1}^{k-1} \left\| \sum_{\substack{0 \leq s_1 \leq m_1 \\ \cdots \\ 0 \leq s_j \leq m_j}} \frac{\sigma^{s_1 + \dots + s_j}}{s_1! \dots s_j!} D_w^{s_1 + \dots + s_j} \left\{ u(z, w - j\sigma) - \sum_{s_{j+1}=0}^{m_{j+1}} \frac{\sigma^{s+1}}{s_{j+1}!} D_w^{s_{j+1}} u(z, w - (j + 1)\sigma) \right\} \right\|_{K(\varepsilon)} \leq$$

$$\leq \sum_{j=0}^{k-1} c_j \left\| u(z, w - j\sigma) - \sum_{s=0}^{m_{j+1}} \frac{\sigma^s}{s!} D_w^s u(z, w - (j + 1)\sigma) \right\|_{K(\varepsilon + \sigma)}$$

where $c_0 = 1$, $c_1 = 1$, and in general

$$c_j = \sum_{\substack{0 \le s_1 \le m_1 \\ \cdots \\ 0 \le s_j \le m_j}} \frac{(s_1 + \ldots + s_j)!}{s_1! \ldots s_j!} \qquad \text{for } 1 \le j \le k-1 .$$

By the previous remark, taking $r = 2\sigma$, we get

$$\|u(z, w) - u_\delta(z, w)\|_{K(\varepsilon)} \le \sum_{j=0}^{k-1} c_j \left(\frac{1}{2}\right)^{m_j+1} \|u(z, w - (j+1)\sigma)\|_{K(\varepsilon+3\sigma)}$$
$$\le \left\{\sum_{j=0}^{k-1} c_j \left(\frac{1}{2}\right)^{m_j+1}\right\} \|u(z, w)\|_{F(4\varepsilon)} .$$

Let ν be a positive integer. We choose $m_1 = \nu + 1$. Then c_1 is defined and we can choose $m_2 = c_1 + \nu$. Then c_2 is defined and we can choose $m_3 = c_2 + \nu$. Proceeding in this way we choose m_j successively so that $m_{j+1} = c_j + \nu$. For every real $\alpha > 0$ we have $\alpha < 2^\alpha$. Therefore with the above choice we have

$$\|u(z, w) - u_\delta(z, w)\|_{K(\varepsilon)} \le \frac{k}{2^\nu} \|u(z, w)\|_{F(4\varepsilon)} .$$

It is enough to choose ν large to get the conclusion of the lemma.

9. – Systems of homogeneous differential operators.

a) Consider a matrix $B(\xi) = (b_{ij}(\xi))_{1 \le j \le p, 1 \le i \le q}$ of type $q \times p$ with polynomial entries in the variables $\xi = (\xi_1, \ldots, \xi_N)$. We will say that the matrix $B(\xi)$ is a *homogeneous matrix* if integers r_i, $1 \le i \le q$ and s_j, $1 \le j \le p$, can be found such that for each choice of i and j, $b_{ij}(\xi)$ is a homogeneous polynomial of degree $r_i - s_j$. We will agree that the zero polynomial is homogeneous of any degree $\gtreqless 0$. The integers r_i, s_j are determined up to an additive constant so that it is not restrictive to assume, if need be, that $r_i \ge 0$, $s_j \ge 0$ for every i and j.

Let $\mathcal{T} = \boldsymbol{C}[\xi_1, \ldots, \xi_N]$ and consider the map

$${}^tB(\xi) \colon \mathcal{T}^q \to \mathcal{T}^p .$$

Let $\nu = {}^t(\nu_1(\xi), \ldots, \nu_q(\xi))$ be an element of $\operatorname{Ker} {}^tB(\xi)$:

$$\sum_{i=1}^{q} b_{ij}(\xi)\nu_i(\xi) = 0 \qquad 1 \le j \le p .$$

If we set $v_i(\xi) = \sum_l v_i^{(l-r_i)}(\xi)$ where $v_i^{(l-r_i)}$ is homogeneous of degree $l - r_i$, we get that, for any l, ${}^t(v_1^{(l-r_1)}, \ldots, v_q^{(l-r_q)})$ is also an element of $\operatorname{Ker} {}^tB(\xi)$. It follows that in a Hilbert resolution

$$0 \to \mathcal{P}^s \xrightarrow{{}^tD(\xi)} \ldots \longrightarrow \mathcal{P}^r \xrightarrow{{}^tC(\xi)} \mathcal{P}^q \xrightarrow{{}^tB(\xi)} \mathcal{P}^p \longrightarrow N \longrightarrow 0$$

of the morphism ${}^tB(\xi)$ ($N = \operatorname{Coker} {}^tB$) all matrices ${}^tC(\xi), \ldots, {}^tD(\xi)$ can be assumed to be homogeneous. If $\mathcal{H} = \boldsymbol{C}_0[\xi_1, \ldots, \xi_N]$ is the graded ring of homogeneous polynomials in the variables $\xi_1, \ldots, \xi_N$, from a « homogeneous » Hilbert resolution of ${}^tB(\xi)$ we obtain therefore an exact sequence of multi-graded $\mathcal{H}$-homomorphisms

$$0 \longrightarrow \mathcal{H}^s \xrightarrow{{}^tD(\xi)} \ldots \longrightarrow \mathcal{H}^r \xrightarrow{{}^tC(\xi)} \mathcal{H}^q \xrightarrow{{}^tB(\xi)} \mathcal{H}^p \longrightarrow N \longrightarrow 0 .$$

b) Let $B(\xi) = \big(b_{ij}(\xi)\big)$ be a homogeneous matrix. Let $x_1, \ldots, x_N$ be cartesian coordinates in $\boldsymbol{R}^N$ and let us consider the system of differential operators with constant coefficients

$$\sum_{j=1}^{p} b_{ij}(D) u_j = f_i , \qquad 1 \le i \le q \tag{1}$$

i.e. $B(D)u = f$, where $D = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$, in matrix notation.

We will call the system (1) *a system of homogeneous differential operators.*

We imbed $\boldsymbol{R}^N$ into $\boldsymbol{C}^N$ and we will consider in a neighborhood U of the origin in $\boldsymbol{R}^N$ a real analytic (valued in $\boldsymbol{C}^p$) solution u of the homogeneous equation

$$B(D)u = 0 . \tag{2}$$

There exists a neighborhood $\tilde{U}$ of U in $\boldsymbol{C}^N$ so that u extends to a holomorphic function (valued in $\boldsymbol{C}^p$) defined on $\tilde{U}$.

LEMMA 8. *Let $u = {}^t(u_1, \ldots, u_p)$ be a germ of holomorphic function in a neighborhood of the origin in $\boldsymbol{C}^N$, solution of the homogeneous equation*

$$B(D)u = 0 \tag{2}$$

where $B(\xi) = \big(b_{ij}(\xi)\big)_{1 \le i \le q, 1 \le j \le p}$ is a homogeneous matrix of type (r_i, s_j).
Let

$$u_j = u_j^{(0)} + u_j^{(1)} + u_j^{(2)} + \ldots \qquad 1 \le j \le p$$

be the Taylor expansion of u_j at the origin, where $u_j^{(k)}$ is a homogeneous polynomial of degree k.

Then for every l integer

$$u^{(l)} = {}^t(u_1^{(l-s_1)}, \ldots, u_p^{(l-s_p)})$$

is a polynomial solution of equation (2).

PROOF. Let $\sum_{j=1}^{p} b_{ij}(D)u_j = \varphi_i^{(0)} + \varphi_i^{(1)} + \ldots$, where $\varphi_i^{(k)}$ is a homogeneous polynomial of degree k, be the Taylor series of the left hand side. For every k we must have

$$\sum_{j=1}^{p} b_{ij}(D)u_j^{(k-s_j+r_i)} = \varphi_i^{(k)} .$$

If u satisfies $B(D)u = 0$ then $\varphi_i^{(k)} = 0$ for every i and every k. Thus choosing $k = k_i$ such that $k_i + r_i = l$ we get

$$\sum_{j=1}^{p} b_{ij}(D)u_j^{(l-s_j)} = 0 .$$

LEMMA 9. *With the same assumption of the previous lemma, set for any α with $0 < \alpha \leq 1$*

$$u_\alpha = \sum_{l=0}^{\infty} \frac{u^{(l)}}{\Gamma(1+\alpha l)} .$$

Then

i) *u_α is an entire function on $\boldsymbol{C}^N$*

ii) *u_α satisfies the equation $B(D)u_\alpha = 0$*

iii) *given $\varepsilon > 0$ and K compact in $\boldsymbol{C}^N$ we can find $l_0 = l_0(\varepsilon, K, \alpha)$ such that*

$$\sup_K \left| u_\alpha(z) - \sum_{l=0}^{l_0} \frac{u^{(l)}(z)}{\Gamma(1+\alpha l)} \right| < \varepsilon .$$

PROOF. Only the first of these statements needs to be proved as the given series is certainly the Taylor series of u_α. Let

$$u = \sum a_\beta z^\beta .$$

Then there are positive constants c, R such that

$$|a_\beta| \leq c R^{|\beta|} \quad \forall \beta \in \boldsymbol{N}^N .$$

We thus have

$$|u_j^{(l-s_j)}(z)| \leq (l - s_j + 1)^N R^{l-s_j} |z|^{l-s_j}.$$

The function $\Gamma(x)$ is an increasing function of x for $x \geq 2$; therefore

$$\Gamma(1 + \alpha l) \geq \Gamma(1 + [\alpha l]) = [\alpha l]!$$

if $\alpha l \geq 2$, and where $[\alpha l]$ denotes the integral part of αl. For fixed α with $0 < \alpha \leq 1$ we thus have

$$\left\{\frac{|u_j^{(l-s_j)}|}{\Gamma(1 + \alpha l)}\right\}^{1/l} \leq c(l - s_j + 1)^{N/l} \frac{R^{1-s_j/l}|z|^{1-s_j/l}}{([\alpha l]!)^{1/l}}.$$

As $l \to \infty$, we have that $c^{1/l}(l - s_j + 1)^{N/l} \to 1$, $R^{1-s_j/l} \to R$ and $|z|^{1-s_j/l} \to |z|$ uniformly on compact subsets K of C^N, while

$$\begin{aligned}([\alpha l]!)^{1/l} &= ([\alpha l]!)^{\alpha/l\alpha} \\ &= \left\{([\alpha l]!)^{1/([\alpha l]+1)+(1/\alpha l - 1/([\alpha l]+1))}\right\}^{\alpha} \\ &\geq \left\{([\alpha l]!)^{1/([\alpha l]+1)}\right\}^{\alpha} \qquad (\text{for } \alpha l \geq 1)\end{aligned}$$

and therefore

$$\lim_{l\to+\infty} ([\alpha l]!)^{1/l} \geq \lim_{l\to+\infty} \left\{([\alpha l]!)^{1/([\alpha l]+1)}\right\}^{\alpha} \geq \lim_{l\to+\infty} \left\{\left(([\alpha l] + 1)!\right)^{1/([\alpha l]+1)}\right\}^{\alpha} = +\infty$$

because

$$\sqrt[[\alpha l]+1]{([\alpha l] + 1)!} \simeq \frac{[\alpha l] + 1}{e}.$$

Thus, uniformly for z in a compact subset K of C^N, we have

$$\lim_{l\to+\infty} \left\{\frac{|u_j^{(l-s_j)}|}{\Gamma(1 + \alpha l)}\right\}^{1/l} = 0.$$

This proves that the series of u_α's converges uniformly on K. As K is arbitrary it follows that for any α, $0 < \alpha \leq 1$, u_α is an entire function.

LEMMA 10. (cf. [1]) *With the same notations as in the previous lemmas, let us assume that u is defined and holomorphic in a starshaped open set E (at the origin) in C^N.*

Let K be a compact subset of E and let $\varepsilon > 0$ be given. We can find $\alpha_0 = \alpha_0(K, \varepsilon) > 0$ such that for $0 < \alpha \leq \inf(1, \alpha_0)$ we have

$$\sup_K |u(z) - u_\alpha(z)| < \varepsilon.$$

PROOF. Let γ be the curve in the plane of the complex variable w consisting of the two segments

$$w = \varrho \exp[\pm i\varphi_0] \qquad 0 \leq \varrho \leq 1$$

where φ_0 is fixed with $\pi/2 < \varphi_0 < \pi$, and the arc

$$w = \exp[i\varphi] \qquad -\varphi_0 \leq \varphi \leq \varphi_0 .$$

We will orient γ counterclockwise. We have the Hankel formula

$$\frac{1}{\Gamma(1+\beta)} = \frac{1}{2\pi i}\int_\gamma w^\beta \exp\left[\frac{1}{w}\right]\frac{dw}{w} .$$

For $\alpha > 0$ sufficiently near to zero we have in E

$$\frac{u_j(z) - \Gamma(1+\alpha s_j)u_{\alpha_j}(z)}{\Gamma(1+\alpha s_j)} = \frac{1}{2\pi i}\int_\gamma w^{\alpha s_j}\exp\left[\frac{1}{w}\right]\frac{dw}{w}\{u_j(z) - u_j(w^\alpha z)\} .$$

Now for $\alpha \to 0^+$, $\Gamma(1+\alpha s_j) \to 1$ and the integrand in the right hand side is well defined and uniformly bounded in module for $z \in K$ and converges almost everywhere (for $w \neq 0$) to zero as $\alpha \to 0^+$. Therefore the left hand side converges for $\alpha \to 0^+$ uniformly to zero for $z \in K$. This shows that for $\alpha < \alpha_0(K, \varepsilon)$ convenient, we have

$$\sup_K |u_j(z) - u_{\alpha_j}(z)| < \varepsilon$$

for $1 \leq j \leq p$.

Let K be compact in E and let $\delta > 0$ be so small that $K(\delta)$ (= the set of points of C^N where polycylindrical distance from K is $\leq \delta$) is contained (and compact) in E. Then for every multiindex β one has for any holomorphic function v in E

$$\|D_z^\beta v\|_K \leq \beta!\, \delta^{-|\beta|}\|v\|_{K(\delta)} ,$$

where the norm is the sup-norm. From this remark and the previous lemmas we deduce the following

PROPOSITION 16. *Let $\sigma \subset R^N$ be an open starshaped domain around the origin. Let u be a function defined in σ, real analytic (complex valued in C^p) solution of the homogeneous system of differential operators*

$$B(D)u = 0 .$$

Let K be a compact of σ, let $\varepsilon > 0$ and let $k \geq 0$ be an integer. We can find an entire function U in C^N (with values in C^p) solution of the equation $B(D)U = 0$ and such that

$$\sum_{|\beta| \leq k} \| D^\beta u - D^\beta U \|_K < \varepsilon .$$

PROOF. Write $\sigma = \bigcup_{h=1}^{\infty} \sigma_h$ with $\sigma_h \subset\subset \sigma$ open and starshaped, $\sigma_h \subset \sigma_{h+1}$ for every h. Let $z_j = x_j + iy_j$ denote the coordinates in C^N $1 \leq j \leq N$ and set for $T_h > 0$

$$C(\sigma_h, T_h) = \{ z \in C^N | x \in \sigma_h, \ |y| < T_h \} .$$

For any choice of T_h, $C(\sigma_h, T_h)$ is starshaped in C^N. For any choice of a sequence $\{T_h\}_{h=1,2,\ldots}$ of positive numbers the set $\bigcup_{h=1}^{\infty} C(\sigma_h, T_h)$ is an open starshaped neighborhood of σ in C^N and when $\{T_h\}$ varies it describes a fundamental system of neighborhoods of σ in C^N. Now u extends as a holomorphic function to an open neighborhood $\tilde{\sigma}$ of σ in C^N. By the above remark we may assume $\tilde{\sigma}$ to be a starshaped open set E in C^N. If $\delta > 0$ is small then $K(\delta) \subset E$. By lemma 10 given $\varepsilon' > 0$ taking $U = u_\alpha$, for α sufficiently near 0, we obtain an entire function, solution of $B(D)U = 0$, and such that

$$\| u - U \|_{K(\delta)} < \varepsilon' .$$

If we choose $\varepsilon' > 0$ such that $\Big(\sum_{|\beta| \leq k} \beta! \, \delta^{-|\beta|} \Big) \varepsilon' < \varepsilon$ we get the desired conclusion.

c) We consider now a Hilbert complex in $\boldsymbol{R}^N$

$$\mathcal{E}^{s_0}(\Omega) \xrightarrow{S_0(D)} \mathcal{E}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathcal{E}^{s_2}(\Omega) \longrightarrow \ldots . \tag{1}$$

We will assume that

i) the first operator $S_0(D)$ is an elliptic operator;

ii) the operators $S_j(D)$ are homogeneous i.e. the corresponding polynomial matrices $S_j(\xi)$ are homogeneous matrices (we will say that (1) is a *homogeneous Hilbert complex*). With the same notation as in section 7 we introduce the spaces $Z^i(\Omega, \mathcal{E}_{S_0})$ with their Schwartz topology.

PROPOSITION 17. *Let E be open and starshaped around the origin in $\boldsymbol{R}^N$. Under the above specified assumptions* i) *and* ii) *the restriction map*

$$r_E^{\boldsymbol{R}^n}: Z^q(\boldsymbol{R}^N, \mathcal{E}_{S_0}) \to Z^q(E, \mathcal{E}_{S_0})$$

has a dense image, for every $q \geq 0$.

PROOF. Let us first assume that $q = 0$. Let $f \in Z^0(E, \mathcal{E}_{S_0})$. As $S_0(D)$ is elliptic, f is real analytic (theorem 1). The assertion follows then from proposition 16.

Let now $q > 0$. Since $\mathcal{E}_{S_0} = \mathcal{A}_{S_0}$ we have an acyclic resolution of $\mathcal{E}_{S_0}$ in the resolution

$$0 \longrightarrow \mathcal{E}_{S_0} \longrightarrow \mathcal{A}^{s_0} \xrightarrow{S_0(D)} \mathcal{A}^{s_1} \xrightarrow{S_1(D)} \mathcal{A}^{s_2} \longrightarrow \dots .$$

Therefore given $f \in Z^q(E, \mathcal{E}_{S_0})$ we can find $g \in \mathcal{E}^{s_{q-1}}(E)$ such that

$$f - S_{q-1}(D)g \in \mathcal{A}^{s_q}(E).$$

Given K compact in E, given $\varepsilon > 0$ and $k \geq 0$ an integer we can find $F \in Z^q(\boldsymbol{R}^N, \mathcal{E}_{S_0})$ such that

$$\sum_{|\beta| \leq k} \|D^\beta (f - S_{q-1}(D)g - F)\|_K < \varepsilon .$$

This by virtue of proposition 16. Let $\chi: E \to \boldsymbol{R}$ be a C^∞ function such that $\chi | K = 1$, supp $\chi \subset\subset E$. Then also

$$\sum_{|\beta| \leq k} \|D^\beta (f - S_{q-1}(D)\chi g - F)\|_K < \varepsilon .$$

This shows that $S_{q-1}(D)\chi g + F \in Z^q(\boldsymbol{R}^N, \mathcal{E}_{S_0})$ ε-approximates f on K with all derivatives up to order k. This proves our contention.

Let $j \geq 1$

$$B^j(\Omega, \mathcal{E}_{S_0}) = \{f \in Z^j(\Omega, \mathcal{E}_{S_0}) | \exists g \in \mathcal{E}^{s_{j-1}}(\Omega) \quad \text{such that } f = S_{j-1}(D)g\}$$

this is the space of j-th coboundaries.

COROLLARY. *Let E be open and starshaped in $\boldsymbol{R}^N$. For any $q \geq 1$ $B^q(E, \mathcal{E}_{S_0})$ is dense in $Z^q(E, \mathcal{E}_{S_0})$.*

PROOF. Since $\boldsymbol{R}^N$ is convex $Z^q(\boldsymbol{R}^N, \mathcal{E}_{S_0}) = B^q(\boldsymbol{R}^N, \mathcal{E}_{S_0})$. By restriction coboundaries go into coboundaries.

d) We now assume that $\boldsymbol{R}^N = \boldsymbol{R}^n \times \boldsymbol{R}^k$ and that the Hilbert complex (1) is an elliptic and Cauchy-Kowalewska suspension of a Hilbert complex (2) in $\boldsymbol{R}^n$:

$$\mathcal{E}^{p_0}(\omega) \xrightarrow{A_0(D)} \mathcal{E}^{p_1}(\omega) \xrightarrow{A_1(D)} \mathcal{E}^{p_2}(\omega) \longrightarrow \dots . \tag{2}$$

We will assume that (2) is a homogeneous Hilbert complex i.e. that the polynomial matrices $A_j(\xi)$ corresponding to the differential operators $A_j(D)$ are homogeneous. Then according to *n.* 4 *d*) examples α) and β), the suspension of a homogeneous Hilbert complex (2) by a «homogeneous» elliptic operator (example α)) or the $\bar{\partial}$-suspension of (2) (example β)) are homogeneous Hilbert complexes, as one verifies directly. Moreover they are Cauchy-Kowalewska and elliptic.

For ω open in $\boldsymbol{R}^n$ we are interested in the groups

$$H^q(\omega, \mathcal{A}_{A_0}) = H^q(\omega, \mathcal{E}_{S_0})$$

where we have used the usual notations.

THEOREM 8. Let (2) be a homogeneous Hilbert complex in $\boldsymbol{R}^n$. *Let σ be open and starshaped in $\boldsymbol{R}^n$. For any $q \geq 1$ we have either $H^q(\sigma, \mathcal{A}_{A_0}) = 0$ or* $\dim_{\boldsymbol{C}} H^q(\sigma, \mathcal{A}_{A_0}) = \infty$.

PROOF. Let $\sigma = \bigcup_{j=1}^{\infty} \sigma_h$ with σ_h open starshaped $\sigma_h \subset\subset \sigma$, $\sigma_h \subset \sigma_{h+1}$, $\forall h$. Let $\boldsymbol{R}^N = \boldsymbol{R}^n \times \boldsymbol{R}^k$ and let $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_k)$ be cartesian coordinates. For $\omega \subset \boldsymbol{R}^n$, $T > 0$, we set

$$C(\omega, T) = \{(x, y) \in \boldsymbol{R}^N \,|\, x \in \omega \quad |y| < T\}$$

for $|y| = \left(\sum_1^k y_j^2\right)^{\frac{1}{2}}$. If ω is starshaped then $C(\omega, T)$ is starshaped. Given a sequence $\{T_h\}$ of positive numbers $T_h > 0$ we have that $\bigcup_{h=1}^{\infty} C(\sigma_h, T_h)$ is a starshaped open neighborhood of σ in $\boldsymbol{R}^N$ and that when the sequence $\{T_h\}$ varies these neighborhoods describe a fundamental system of neighborhoods of σ in $\boldsymbol{R}^N$.

Consider now an elliptic Cauchy-Kowalewska «homogeneous» suspension (1) of the complex (2) to $\boldsymbol{R}^N$ and let Ω denote an open starshaped neighborhood of σ in $\boldsymbol{R}^N$.

Assume that $\dim_{\boldsymbol{C}} H^q(\sigma, \mathcal{A}_{A_0}) = d < \infty$. Then according to lemma 6 we can find a sequence $\{T_h\}_{h=1,2,\dots}$ with $T_h > 0$ such that, if we set

$B_h = \bigcup_{j=1}^{h} C(\sigma_j, T_j)$ we will have $B_h \subset \Omega$, $\forall h$, and

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B_h, \mathcal{E}_{S_0})\} \leq d\,.$$

In view of proposition 17 we derive from proposition 15 that, setting $B = \bigcup_{j=1}^{\infty} C(\sigma_j, T_j)$,

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} \leq d\,.$$

Set

$$G = \{(u, v) \in Z^q(\Omega, \mathcal{E}_{S_0}) \times \mathcal{E}^{s_{q-1}}(B) \mid u = S_{q-1}(D)v \text{ on } B\}\,.$$

Then G is a Fréchet space. Set $W = pr_{Z^q(\Omega,\mathcal{E}_{S_0})}(G)$. Then W is the image of a Fréchet space by a continuous linear map.

By the assumption

$$\dim_{\boldsymbol{C}} \frac{Z^q(\Omega, \mathcal{E}_{S_0})}{W} \leq d$$

so that W must be a closed subspace of $Z^q(\Omega, \mathcal{E}_{S_0})$. But Ω is starshaped and thus, by the corollary to proposition 17, W must be dense in $Z^q(\Omega, \mathcal{E}_{S_0})$. Hence $W = Z^q(\Omega, \mathcal{E}_{S_0})$ and thus

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} = 0\,.$$

Since Ω can describe (remaining starshaped) a fundamental system of open neighborhoods of σ in $\boldsymbol{R}^N$ we conclude that

$$H^q(\sigma, \mathcal{A}_{A_0}) = H^q(\sigma, \mathcal{E}_{S_0}) = \varinjlim_{\Omega \supset \sigma} H^q(\Omega, \mathcal{E}_{S_0}) = 0\,.$$

THEOREM 9. *Let* (2) *be a homogeneous Hilbert complex in* $\boldsymbol{R}^n$ *and let* (1) *be any elliptic Cauchy-Kowalewska suspension of* (2) *in* $\boldsymbol{R}^N$ *which is still a homogeneous Hilbert complex.*

Let σ *be open and starshaped in* $\boldsymbol{R}^n$ *and let* $q \geq 1$. *Necessary and sufficient condition for* $H^q(\sigma, \mathcal{A}_{A_0}) = 0$ *is that*

(*A*) *for every open neighborhood* Ω *of* σ *in* $\boldsymbol{R}^N$ *there exists an open neighborhood* B *of* σ *in* Ω *such that*

$$\operatorname{Im}\{H^q(\Omega, \mathcal{E}_{S_0}) \to H^q(B, \mathcal{E}_{S_0})\} = 0\,.$$

PROOF. We use the same notations as in the previous theorem. The sufficiency of the condition was established in proposition 10.

To establish the necessity of condition (A) we apply lemma 6 and proposition 14 in view of the density theorem given by proposition 17.

10. – Study of some examples.

a) Preliminaries. Consider the complex space $\boldsymbol{C}^2$ where $z_1 = x + iy$, $z_2 = s + it$ are complex coordinates and let

$$\boldsymbol{R}^3 = \{(z_1, z_2) \in \boldsymbol{C}^2 | t = 0\} .$$

LEMMA 11. *Let f be a holomorphic function in the region*

$$\left\{\begin{array}{l} |z_1| < R \\ |s| < \varepsilon \\ t < 0 \end{array}\right. \cup \left\{\begin{array}{l} |z_1| < \varepsilon \\ \\ |z_2| < \varepsilon \end{array}\right. \qquad (\text{for } R > 0,\ 1 > \varepsilon > 0) .$$

Then f is also holomorphic in the region

$$\bigcup_{0 \le \theta \le 1} \left\{ |z_1| < \varepsilon^\theta R^{1-\theta},\ |z_2 + i\varepsilon/2| < \frac{\varepsilon}{2} 2^\theta \right\} .$$

In particular f is holomorphic in the region

$$\Delta_\varepsilon = \left\{\begin{array}{l} |z_1| < \varepsilon^\varepsilon R^{1-\varepsilon} , \\ s = 0 , \\ 0 \le t < \dfrac{\varepsilon}{2} (2^\varepsilon - 1) . \end{array}\right.$$

PROOF. Because of the assumption f is holomorphic in the union of the two polycylinders centered at $(0, -i\varepsilon/2)$

$$\left\{\begin{array}{l} |z_1| < R \\ \left| z_2 + i \dfrac{\varepsilon}{2} \right| < \dfrac{\varepsilon}{2} \end{array}\right. \cup \left\{\begin{array}{l} |z_1| < \varepsilon \\ \left| z_2 + i \dfrac{\varepsilon}{2} \right| < \varepsilon . \end{array}\right.$$

Therefore the Taylor series of f centered at the point $(0, -i\varepsilon/2)$ converges in the region

$$\left\{\begin{array}{l} |z_1| < \varepsilon^\theta R^{1-\theta} \\ \left| z_2 + i \dfrac{\varepsilon}{2} \right| < \varepsilon^\theta \left(\dfrac{\varepsilon}{2}\right)^{1-\theta} = \dfrac{\varepsilon}{2} 2^\theta \end{array}\right.$$

for any θ with $0 \le \theta \le 1$ (cf. [10] pg. 34).

As $0<\varepsilon<1$ we can take in particular $\theta=\varepsilon$ and we get over $s=0$ the region for $t\geq 0$

$$|z_1|<\varepsilon^{\varepsilon}R^{1-\varepsilon}, \qquad t+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}2^{\varepsilon}$$

which is the region Δ_ε of the lemma.

LEMMA 12. *Let ω be a non empty open subset of $\mathbf{R}^3$. No fundamental system of neighborhoods of ω in $\mathbf{C}^2$ can be all of open sets of holomorphy.*

PROOF. α) Without loss of generality we may assume that the origin $0=(0,0)\in\omega$. Let $R>0$ (may be $R=+\infty$) be so chosen that the disc $\{z_2=0, |z_1|<R\}$ is the largest disc of this sort contained in ω.

We can choose sequences of positive numbers $R_n\nearrow R$ and $\varepsilon_n\searrow 0$ $(n=1,2,...)$ (with the precaution to take when $R=+\infty$, $R_n<n$ and $\varepsilon_n<1/n$) such that

$$D_n=\{|z_1|<R_n,\ |s|<\varepsilon_n, t=0\}\subset\omega\,.$$

Passing to a subsequence in the ε_n we may assume that

$$\varepsilon_1^{\varepsilon_1}R_1^{1-\varepsilon_1}<\varepsilon_2^{\varepsilon_2}R_2^{1-\varepsilon_2}<\varepsilon_3^{\varepsilon_3}R_3^{1-\varepsilon_3}<\dots\,.$$

Let $\varphi=\varphi(z_1,s)$ be a continuous function defined in D_n with $\varphi>0$. Set

$$\begin{aligned} A_n &=\{(z_1,s)\in D_n,\ |t|<\varphi(z_1,s)\}\\ A_n^+ &=\{(z_1,s)\in D_n,\ t>-\varphi(z_1,s)\}\\ A_n^- &=\{(z,s)\in D_n,\ t<\varphi(z_1,s)\}\,. \end{aligned}$$

We have $A_n=A_n^+\cap A_n^-$ while $A_n^+\cup A_n^-$ is a convex set and therefore a domain of holomorphy.

Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions in $\mathbf{C}^2$. We claim that: for any $f\in\Gamma(A_n,\mathcal{O})$ there exist $f^+\in\Gamma(A_n^+,\mathcal{O})$ and $f^-\in\Gamma(A_n^-,\mathcal{O})$ such that

$$f=f^+-f^-\quad\text{ on } A_n\,.$$

Indeed from the Mayer-Vietoris sequence we derive the exact sequence

$$0\to H^0(A_n^+\cup A_n^-,\mathcal{O})\to H^0(A_n^+,\mathcal{O})\oplus H^0(A_n^-,\mathcal{O})\to H^0(A_n,\mathcal{O})\to 0\,,$$

the last 0 being given by $H^1(A_n^+\cup A_n^-,\mathcal{O})=0$.

β) Let now Ω be an open neighborhood of ω in $\boldsymbol{C}^2$. We can choose a continuous function φ defined on ω such that, for every n we have

$$A_n = \{(z, s) \in D_n,\ |t| < \varphi(z, s)\} \subset \Omega\,.$$

We can choose an integer $n_0 = n_0(\Omega)$ such that for $n \geq n_0$ we have

$$\left\{\begin{array}{l} |z_1| < R_n \\ |s| < \varepsilon_n \\ t < 0 \end{array}\right. \cup \left\{\begin{array}{l} |z_1| < \varepsilon_n \\ \\ |z_2| < \varepsilon_n \end{array}\right. \quad \text{is contained in } A_n^-$$

and

$$\left\{\begin{array}{l} |z_1| < R_n \\ |s| < \varepsilon_n \\ t > 0 \end{array}\right. \cup \left\{\begin{array}{l} |z_1| < \varepsilon_n \\ \\ |z_2| < \varepsilon_n \end{array}\right. \quad \text{is contained in } A_n^+\,.$$

From the previous lemma and point α) of this proof we deduce that any function f holomorphic in Ω must be holomorphic at all points of the set

$$\Delta_{\varepsilon_n} = \left\{\begin{array}{l} |z| < \varepsilon_n^{\varepsilon_n} R_n^{1-\varepsilon_n}\,, \\ s = 0\,, \\ |t| < \dfrac{\varepsilon_n}{2}(2^{\varepsilon_n} - 1)\,. \end{array}\right.$$

γ) When $R < \infty$ we may assume that on the circle $\{z_2 = 0,\ |z_1| = R\}$ there is a point of $\partial\omega$ and that this point is the point $z_2 = 0$, $z_1 = R$ on the real axis of the z_1-plane.

Consider a function $\psi(x) > 0$ defined and continuous on $0 \leq x < R$ having the property that

$$\text{for } \varepsilon_{n-1}^{\varepsilon_{n-1}} R_{n-1}^{1-\varepsilon_{n-1}} \leq x < \varepsilon_n^{\varepsilon_n} R_n^{1-\varepsilon_n}$$

we have

$$0 < \psi(x) < \tfrac{1}{4}\,\varepsilon_n(2^{\varepsilon_n} - 1)\,.$$

Consider the strip

$$A = \{y = 0,\ 0 \leq x < R,\ s = 0\}$$

and on A the region

$$A_\psi = \{y = 0, 0 \le x < R, \ s = 0, \ |t| < \psi(x)\}.$$

As the segment $y = s = t = 0, \ 0 \le x < R$ goes from the origin to a boundary point of ω, we can choose an open neighborhood Ω of ω in C_z with the property

$$\Omega \cap A \subset A_\psi.$$

Now Ω cannot be an open set of holomorphy. Indeed if Ω is an open set of holomorphy we can find $n_0 = n_0(\Omega)$ integer such that for $n \ge n_0$ Ω contains the region Δ_{ε_n} described above. Now this conclusion is ruled out by the condition $\Omega \cap A \subset A_\psi$. This completes the proof.

COROLLARY. *Let* $\boldsymbol{R}^{n+k} \subset \boldsymbol{C}^n$, $n \ge 2$, $k \ge 1$, *and assume that* $\boldsymbol{C}^n$ *is the minimal complex subspace containing* $\boldsymbol{R}^{n+k}$.

Let ω *be any non empty open subset of* $\boldsymbol{R}^{n+k}$.

No fundamental system of open neighborhoods of ω *in* $\boldsymbol{C}^n$ *can be all of open sets of holomorphy.*

PROOF. Let $z_j = x_j + iy_j$, $1 \le j \le n$ be complex coordinates in $\boldsymbol{C}^n$. We may assume that $\boldsymbol{R}^{n+k} = \{z \in \boldsymbol{C}^n | y_{k+1} = \ldots = y_n = 0\}$. We can also assume that the origin 0 of the coordinates belongs to ω, $0 \in \omega$.

Set $\boldsymbol{C}^2 = \{z_2 = \ldots = z_k = z_{k+2} = \ldots = z_n = 0\}$, so that (z_1, z_{k+1}) are complex coordinates on $\boldsymbol{C}^2$. Then $\boldsymbol{R}^{n+k} \cap \boldsymbol{C}^2 = \{(z_1, z_{k+1}) \in \boldsymbol{C}^2 | y_{k+1} = 0\} = \boldsymbol{R}^3$.

Let $\sigma = \omega \cap \boldsymbol{R}^3$. Then $\sigma \ne \emptyset$. If $\{U_\alpha\}$ is a fundamental system of open neighborhoods of ω in $\boldsymbol{C}^n$ all of open sets of holomorphy then $\{U_\alpha \cap \boldsymbol{C}^2\}$ is a similar system of neighborhoods of σ in $\boldsymbol{C}^2$. This contradicts the previous lemma.

As usual we denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions on $\boldsymbol{C}^2$. We have the following

LEMMA 13. *Let* $B \subset A$ *be open sets in* $\boldsymbol{C}^2$. *Assume that*

$$\dim_{\boldsymbol{C}} \operatorname{Im}\{H^1(A, \mathcal{O}) \to H^1(B, \mathcal{O})\} = d < \infty.$$

Let $\pi: \tilde{B} \to \boldsymbol{C}^2$ *be the envelope of holomorphy of* B.

Then $\pi(\tilde{B}) \subset A$.

PROOF. Let p be a point in the complement of A. We choose complex coordinates z_1, z_2 in $\boldsymbol{C}^2$ with p at the origin. Set $U_i = \{z_i \ne 0\}$ $i = 1, 2$ and $\mathfrak{U} = \{U_1, U_2\}$ as a covering of $\boldsymbol{C}^2 - \{0\}$; we also set $\mathfrak{U} \cap A = \{U_1 \cap A, U_2 \cap A\}$ and $\mathfrak{U} \cap B = \{U_1 \cap B, U_2 \cap B\}$; these are open coverings of A and B as $0 \notin A$.

For $r \geq 1$, $s \geq 1$, $1/z_1^r z_2^s \in Z^1(\mathfrak{U}, \mathcal{O})$ the space of Čhech 1-cocycles on the convering $\mathfrak{U}$ with values in $\mathcal{O}$. By restriction these give cohomology classes in $Z^1(\mathfrak{U} \cap A, \mathcal{O})$ and $Z^1(\mathfrak{U} \cap B, \mathcal{O})$. We note now that the natural map $H^1(\mathfrak{U} \cap B, \mathcal{O}) \to H^1(B, \mathcal{O})$ is injective; according to Leray theorem this is a general property of the first Čhech cohomology group. Choose $r_0 < r_1 < \ldots < r_d$, $s_0 < s_1 < \ldots < s_d$ with r and s positive integers. By the assumption the $d+1$ cohomology classes represented by the $d+1$ cocycles $z_1^{-r_i} z_2^{-s_i}$, $0 \leq i \leq d$, must become linearly dependent on $H^1(B, \mathcal{O})$ and thus on $H^1(\mathfrak{U} \cap B, \mathcal{O})$. This means that there exist constants c_i, $0 \leq i \leq d$ not all zero and holomorphic functions $g_j \in \Gamma(U_j \cap B, \mathcal{O})$ $j = 1, 2$, such that

$$\sum_{i=0}^{d} \frac{c_i}{z_1^{r_i} z_2^{s_i}} = g_1 - g_2$$

on $U_1 \cap U_2 \cap B$. We may assume $c_d \neq 0$ (otherwise we replace d with the maximal integer i for which $c_i \neq 0$). Chasing denominators we get

$$c_d + c_{d-1} z_1^{r_d - r_{d-1}} z_2^{s_d - s_{d-1}} + \ldots = z_2^{s_d}(z_1^{r_d} g_2) - z_1^{r_d}(z_2^{s_d} g_1) .$$

This shows that $z_1^{r_d} g_1$ and $z_2^{s_d} g_2$ are holomorphic on $U_1 \cap B$ and $U_2 \cap B$ and thus are holomorphic on B. These functions therefore extend to holomorphic functions G_1, G_2 respectively on the envelope $\tilde{B}$. We must have moreover on $\hat{B}$

$$c_d + c_{d-1} z_1^{r_d - r_{d-1}} z_2^{s_d - s_{d-1}} + \ldots = z_2^{s_d} G_1 - z_1^{r_d} G_2 .$$

But this shows that $0 \notin \pi(\tilde{B})$ because setting $z_1 = z_2 = 0$ in the above relation one would get $c_d = 0$ and this is impossible.

We have thus proved that $p \notin \pi(\tilde{B})$. This being true for any $p \notin A$ in $\boldsymbol{C}^2$ we deduce that $\pi(\tilde{B}) \subset A$.

b) We consider now on $\boldsymbol{R}^3$ the (homogeneous Hilbert) complex

$$\mathcal{E}(\omega) \xrightarrow{\partial/\partial \bar{z}_1} \mathcal{E}(\omega) \longrightarrow 0 . \tag{1}$$

Suspending it in $\boldsymbol{C}^2 = \boldsymbol{R}^4$ with the complex (homogeneous Hilbert)

$$\mathcal{E}(\Omega) \xrightarrow{\partial/\partial \bar{z}_2} \mathcal{E}(\Omega) \longrightarrow 0 \tag{2}$$

we get as a suspension the Dolbeault complex of $\boldsymbol{C}^2$ (cf. n. 4*d*) example γ)) which is a new homogeneous Hilbert complex:

$$\mathcal{E}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{E}^{01}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{E}^{02} \longrightarrow 0 . \tag{3}$$

For every open set ω in $\boldsymbol{R}^3$ we thus have

$$H^1(\omega, \mathcal{A}_{\partial/\partial\bar{z}_1}) = \frac{\mathcal{A}(\omega)}{(\partial/\partial\bar{z}_1)\,\mathcal{A}(\omega)} = H^1(\omega, \mathcal{O})\,.$$

THEOREM 10. *For ω open non empty and starshaped in $\boldsymbol{R}^3$ we have*

$$\dim_{\boldsymbol{C}} H^1(\omega, \mathcal{A}_{\partial/\partial\bar{z}_1}) = \infty\,.$$

PROOF. Let Ω be any open neighborhood. If $\dim_{\boldsymbol{C}} H^1(\omega, \mathcal{A}_{\partial/\partial\bar{z}_1}) < \infty$ then by theorem 8 we must have $H^1(\omega, \mathcal{A}_{\partial/\partial\bar{z}_1}) = 0$. By theorem 9 we can then find an open starshaped neighborhood B of ω in Ω such that

$$\operatorname{Im}\{H^1(\Omega, \mathcal{O}) \to H^1(B, \mathcal{O})\} = 0\,.$$

The envelope of holomorphy $\tilde{B}$ of B is « shlicht » i.e. it is also an open subset of $\boldsymbol{C}^2$ as B is starshaped. By lemma 13 we should have $\tilde{B} \subset \Omega$. Therefore ω would have a fundamental system of neighborhoods in $\boldsymbol{C}^2$ which are domains of holomorphy. This contradicts lemma 12.

REMARK. Without invoking theorems 8 and 9 one can argue directly through lemma 6, proposition 15 and using as approximation theorem the Runge theorem of Laufer ([12] theorem 4.11) or the theorem of Behnke-Stein [5] that says that an increasing union of domains of holomorphy in $\boldsymbol{C}^n$ is a domain of holomorphy.

c) More generally let us consider $\boldsymbol{R}^{n+k} = \boldsymbol{C}^k \times \boldsymbol{R}^{n-k} \subset \boldsymbol{C}^n$, $n \geq 2$, $k \geq 1$, $\boldsymbol{C}^n$ being the complex span of $\boldsymbol{R}^{n+k}$. As in example γ) of section 4 *d*) we consider the Dolbeault complex along the fibers $\boldsymbol{C}^k$ of $\boldsymbol{R}^{n+k}$

$$C^{00}(\omega) \xrightarrow{\bar{\partial}'} C^{01}(\omega) \xrightarrow{\bar{\partial}'} C^{02}(\omega) \longrightarrow \dots \longrightarrow C^{0k}(\omega) \longrightarrow 0\,.$$

This we suspend in $\boldsymbol{C}^n$ by the Dolbeault complex along the fibers $\boldsymbol{C}^{n-k}$ of $\boldsymbol{C}^n = \boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$. We obtain as a suspension the Dolbeault complex in $\boldsymbol{C}^n$. If $\mathcal{O}$ is the sheaf of germs of holomorphic functions in $\boldsymbol{C}^n$ we will have

$$H^j(\omega, \mathcal{A}_{\bar{\partial}'}) \simeq H^j(\omega, \mathcal{O})\,.$$

We will have the following

THEOREM 11. *For ω open non empty and convex in $\boldsymbol{R}^{n+k}$ we have*

$$\dim_{\boldsymbol{C}} H^1(\omega, \mathcal{A}_{\bar{\partial}'}) = \infty\,.$$

PROOF. Let $\boldsymbol{R}^{n+k} = \{z \in \boldsymbol{C}^n | y_{k+1} = \dots = y_n = 0\}$ with the usual notations. We can proceed by induction on n because of theorem 10. We will assume that the origin of the coordinates 0 is in ω.

If $k = n-1$ we set $\boldsymbol{C}^{n-1} = \{z_1 = 0\}$ and if $k < n-1$ we set $\boldsymbol{C}^{n-1} = \{z_n = 0\}$. Set $\omega' = \omega \cap \boldsymbol{C}^{n-1}$ and let $\xi = z_1$ or $\xi = z_n$ respectively so that we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{C}^n} \xrightarrow{\xi} \mathcal{O}_{\boldsymbol{C}^n} \longrightarrow \mathcal{O}_{\boldsymbol{C}^{n-1}} \longrightarrow 0 .$$

We obtain an exact cohomology sequence

$$H^1(\omega, \mathcal{O}_{\boldsymbol{C}^n}) \to H^1(\omega', \mathcal{O}_{\boldsymbol{C}^{n-1}}) \to H^2(\omega, \mathcal{O}_{\boldsymbol{C}^n}) .$$

By theorem 4 we must have $H^2(\omega, \mathcal{O}_{\boldsymbol{C}^n}) = 0$. Therefore $\dim_{\boldsymbol{C}} H^1(\omega, \mathcal{O}_{\boldsymbol{C}^n})$ is infinite if $\dim_{\boldsymbol{C}} H^1(\omega', \mathcal{O}_{\boldsymbol{C}^{n-1}})$ is infinite. But this is the inductive assumption since $\omega' \neq \emptyset$ is convex in a space $R^{n+l-1} \subset \boldsymbol{C}^{n-1}$ (we have for $k = n-1$ $l = n-2$ and for $1 \leq k < n-1$, $l = k$).

d) We end this section with an example of a non convex open subset of $\boldsymbol{R}^7 \subset \boldsymbol{C}^4$ which presents analytic cohomology in dimension 2 while the C^∞ cohomology in that dimension vanishes.

We consider $\boldsymbol{R}^7 = \boldsymbol{C}^3 \times \boldsymbol{R} \subset \boldsymbol{C}^4$ and the Dolbeault complex along the fibers $\boldsymbol{C}^3$ of $\boldsymbol{R}^7$

$$C^{00}(\omega) \xrightarrow{\bar\partial'} C^{01}(\omega) \xrightarrow{\bar\partial'} C^{02}(\omega) \xrightarrow{\bar\partial'} C^{03}(\omega) \longrightarrow 0 .$$

We denote by $\mathcal{E}_{\bar\partial'}$ the sheaf of germs of C^∞ functions in $\boldsymbol{R}^7$ with $\bar\partial' f = 0$. We denote by $\mathcal{A}_{\bar\partial'}$ the sheaf of germs of real analytic functions on $\boldsymbol{R}^7$ with $\bar\partial' f = 0$. By $\mathcal{O}_{\boldsymbol{C}^3}$ we denote the sheaf of germs of holomorphic functions on $\boldsymbol{C}^3$.

We take $\omega = (\boldsymbol{C}^2 - \{0\}) \times \boldsymbol{C} \times \boldsymbol{R}$. This is open but not convex.

PROPOSITION 18. *We have for all* $j \geq 2$

$$H^j(\omega, \mathcal{E}_{\bar\partial'}) = 0 .$$

PROOF. Set $V = (\boldsymbol{C}^2 - \{0\}) \times \boldsymbol{C}$, $M = \boldsymbol{R}$. Then V is an open set of $\boldsymbol{C}^3$ which has a Stein covering by 2 open sets $U_1 = \{z_1 \neq 0\}$, $U_2 = \{z_2 \neq 0\}$ where z_1, z_2, z_3 denote complex coordinates on $\boldsymbol{C}^3$.

Therefore $H^j(V, \mathcal{O}_{\boldsymbol{C}^3}) = 0$ for $j \geq 2$. We can then apply proposition 7 of [2] p. 208 and conclude that $H^j(V \times M, \mathcal{E}_{\bar\partial'}) = 0$ for $j \geq 2$.

Consider now the natural projection

$$\pi\colon (\boldsymbol{C}^2 - \{0\}) \times (\boldsymbol{C} \times \boldsymbol{R}) \to \boldsymbol{C} \times \boldsymbol{R}$$

and set, for convenience of notation $\sigma = \boldsymbol{C} \times \boldsymbol{R}$. We consider on σ the Dolbeault complex along the fibers $\boldsymbol{C}$ of $\boldsymbol{C} \times \boldsymbol{R}$:

$$C^{00}(\eta) \xrightarrow{\bar{\partial}'} C^{01}(\eta) \longrightarrow 0$$

for η open in $\boldsymbol{C} \times \boldsymbol{R}$. We denote by $z_1, z_2, z_3, z_4 = t + is$ complex coordinates in $\boldsymbol{C}^4$. We are going to define a linear map

$$\pi_*\colon C^{3,s}(\omega) \to C^{1,s-1}(\sigma) \quad \text{for } s = 1, 2$$

as follows.

Set $S = \{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

Let first $s = 2$ and $\mu \in C^{3,2}(\omega)$:

$$\mu = dz_1 dz_2 dz_3 d\bar{z}_3 (a_1(z_1, z_2, z_3, t) d\bar{z}_1 + a_2(z_1, z_2, z_3, t) d\bar{z}_2) + \\ + dz_1 dz_2 dz_3 b(z_1, z_2, z_3, t) d\bar{z}_1 d\bar{z}_2$$

then set

$$\pi_* \mu = \left\{ \int_S a_1(z_1, z_2, z_3, t) dz_1 dz_2 d\bar{z}_1 + a_2(z_1, z_2, z_3, t) dz_1 dz_2 d\bar{z}_2 \right\} dz_3 d\bar{z}_3 .$$

Let $s = 1$ and $\mu \in C^{3,1}(\omega)$:

$$\mu = dz_1 dz_2 dz_3 (\alpha(z_1, z_2, z_3, t) d\bar{z}_1 + \beta(z_1, z_2, z_3, t) d\bar{z}_2) + dz_1 dz_2 dz_3 \gamma(z_1, z_2, z_3, t) d\bar{z}_3$$

then set

$$\pi_* \mu = \left\{ -\int_S \alpha(z_1, z_2, z_3, t) dz_1 dz_2 d\bar{z}_1 + \beta(z_1, z_2, z_3, t) dz_1 dz_2 d\bar{z}_2 \right\} dz_3 .$$

LEMMA 14. *For $\theta^{3,1} \in C^{3,1}(\omega)$ we have*

$$\pi_* \bar{\partial}' \theta^{3,1} = \bar{\partial}' \pi_* \theta^{3,1} .$$

PROOF. Set $\mu = \bar{\partial}' \theta^{3,1}$ and let

$$\mu = dz_1 dz_2 dz_3 (a_{12} d\bar{z}_1 d\bar{z}_2 + a_{13} d\bar{z}_1 d\bar{z}_3 + a_{23} d\bar{z}_2 d\bar{z}_3) .$$

Let

$$\theta = dz_1 dz_2 dz_3 (\alpha d\bar{z}_1 + \beta d\bar{z}_2 + \gamma d\bar{z}_3) .$$

Then

$$\pi_*\mu = \left\{-\int_S (a_{13}dz_1dz_2d\bar{z}_1 + a_{23}dz_1dz_2d\bar{z}_2)\right\}dz_3d\bar{z}_3 .$$

As $\mu = \bar{\partial}'\theta$ we have

$$a_{12} = \frac{\partial\beta}{\partial\bar{z}_1} - \frac{\partial\alpha}{\partial\bar{z}_2}, \qquad a_{13} = \frac{\partial\gamma}{\partial\bar{z}_1} - \frac{\partial\alpha}{\partial\bar{z}_3}, \qquad a_{23} = \frac{\partial\gamma}{\partial\bar{z}_2} - \frac{\partial\beta}{\partial\bar{z}_3} .$$

Thus

$$\pi_*\mu = \left\{\frac{\partial}{\partial\bar{z}_3}\left(\int_S \alpha\, dz_1dz_2d\bar{z}_1 + \beta\, dz_1dz_2d\bar{z}_2\right)\right\}dz_3d\bar{z}_3 - \left\{\int_S \left(\frac{\partial\gamma}{\partial\bar{z}_1}d\bar{z}_1 + \frac{\partial\gamma}{\partial\bar{z}_2}d\bar{z}_2\right)dz_1dz_2\right\}dz_3d\bar{z}_3 .$$

The second integral equals

$$\int_S \bar{\partial}(\gamma\, dz_1dz_2) = \int_S d(\gamma\, dz_1dz_2) = \int_{\partial S} \gamma\, dz_1dz_2 = 0 .$$

Thus

$$\pi_*\mu = \left\{\frac{\partial}{\partial\bar{z}_3}\left(\int_S \alpha\, dz_1dz_2d\bar{z}_1 + \beta\, dz_1dz_2d\bar{z}_2\right)\right\}dz_3d\bar{z}_3 .$$

Now

$$\pi_*\theta = \left\{\int_S \alpha\, dz_1dz_2d\bar{z}_1 + \beta\, dz_1dz_2d\bar{z}_2\right\}dz_3 .$$

Therefore

$$\pi_*\mu = \bar{\partial}'\pi_*\theta .$$

Consider now the form on ω

$$K = \frac{\bar{z}_1d\bar{z}_2 - \bar{z}_2d\bar{z}_1}{(z_1\bar{z}_1 + z_2\bar{z}_2)^2}dz_1dz_2 .$$

We have $\bar{\partial}'K = 0$

Let $\lambda \in C^{1,1}(\sigma)$, $\lambda = f(z_3, t)dz_3d\bar{z}_3$. Consider the form

$$\mu = K\wedge\pi^*(\lambda) = f(z_3, t)\frac{\bar{z}_1d\bar{z}_2 - \bar{z}_2d\bar{z}_1}{(z_1\bar{z}_1 + z_2\bar{z}_2)^2}dz_1dz_2dz_3d\bar{z}_3 .$$

This is an element of $C^{3,2}(\omega)$ and $\bar{\partial}'\mu = 0$. Moreover if $f(z_3, t)$ is real analytic then μ has real analytic coefficients. Also note that $C^{1,s}(\sigma) \cong C^{0,s}(\sigma)dz_3$, $C^{3s}(\omega) \cong C^{0,s}(\omega)dz_1dz_2dz_3$ for any $s \geq 0$ are isomorphisms compatible with the operators $\bar{\partial}'$ on ω and σ. Thus

$$\tau\colon C^{1,1}(\sigma) \to C^{3,2}(\omega)$$

defined by

$$\lambda \to K \wedge \pi^*(\lambda)$$

induces a homomorphism

$$\tau^*\colon H^1(\sigma, \mathcal{A}_{\bar{\partial}'}) \to H^2(\omega, \mathcal{A}_{\bar{\partial}'})$$

where $\mathcal{A}_{\bar{\partial}'}$ denotes the sheaf of germs of real analytic functions f on ω or σ respectively with $\bar{\partial}' f = 0$.

LEMMA 15. *For* $\lambda \in C^{1,1}(\sigma)$ *we have*

$$\pi_*\big(K \wedge \pi^*(\lambda)\big) = (2\pi i)^2 \lambda .$$

PROOF. Indeed the right hand side equals

$$\left\{\int_S f(z_3, t)\, \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{(z_1\bar{z}_1 + z_2\bar{z}_2)^2}\, dz_1 dz_2\right\} dz_3 d\bar{z}_3 = (2\pi i)^2 f(z_3, t)\, dz_3 d\bar{z}_3 .$$

As a corollary we obtain

PROPOSITION 19. *The linear map*

$$\tau^*\colon H^1(\sigma, \mathcal{A}_{\bar{\partial}'}) \to H^2(\omega, \mathcal{A}_{\bar{\partial}'})$$

is an injective map. In particular we have

$$\dim H^2(\omega, \mathcal{A}_{\bar{\partial}'}) = \infty .$$

PROOF. Indeed the map $\pi_*\colon C^{3,2}(\omega) \to C^{1,1}(\sigma)$ induces a homomorphism $\pi_*\colon H^2(\omega, \mathcal{A}_{\bar{\partial}'}) \to H^1(\sigma, \mathcal{A}_{\bar{\partial}'})$. This because it transforms forms with real analytic coefficients into forms with real analytic coefficients; and forms which are $\bar{\partial}'$-closed into forms $\bar{\partial}'$-closed and by lemma 14 it also transforms forms which are $\bar{\partial}'$-coboundaries into forms which are $\bar{\partial}'$-coboundaries.

Because of lemma 15 we have (up to a non zero constant)

$$\pi_* \circ \tau^* = \text{identity} .$$

This proves that τ^* must be injective. Now $\sigma = \boldsymbol{C} \times \boldsymbol{R}$ is starshaped and therefore by theorem 10 $H^1(\sigma, \mathcal{A}_{\bar{\partial}'})$ is infinite-dimensional. From this the second assertion follows.

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