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The characterization of strictly parabolic manifolds

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The Characterization of Strictly Parabolic Manifolds (*).

WILHELM STOLL

1. – Introduction.

A non-negative function τ of class C^∞ on a connected, complex manifold M of dimension m with $\Delta = \sup \sqrt{\tau} \leqslant \infty$ is said to be a *strictly parabolic exhaustion* of M and (M, τ) is said to be a *strictly parabolic manifold* if for every $r \in \boldsymbol{R}$ with $0 \leqslant r < \Delta$ the pseudoball

$$M[r] = \{x \in M \,|\, \tau(x) \leqslant r^2\}$$

is compact, if $\tau < \Delta^2$ on M, if $dd^c\tau > 0$ on M and if

$$dd^c \log \tau \geqslant 0 \qquad (dd^c \log \tau)^m \equiv 0 \tag{1.1}$$

on $M_* = M - M[0]$ where $d^c = (i/4\pi)(\bar{\partial} - \partial)$. Here Δ is called the *maximal radius* of the exhaustion τ. On $\boldsymbol{C}^m$ define τ_0 by $\tau_0(z) = |z|^2$ for all $z \in \boldsymbol{C}^m$. For each r with $0 < r \leqslant +\infty$ let $\boldsymbol{C}^m(r) = \{z \in \boldsymbol{C}^m | \tau_0(z) < r^2\}$ be the open ball of radius r centered at 0. Then $(\boldsymbol{C}^m(r), \tau_0)$ is an example of a strictly parabolic manifold of dimension m and maximal radius r.

THEOREM. *If (M, τ) is a strictly parabolic manifold of dimension m with maximal radius Δ, then there exists a biholomorphic map $h\colon \boldsymbol{C}^m(\Delta) \to M$ with $\tau_0 = \tau \circ h$.*

Thus h is an isometry of exhaustions $\tau_0 = \tau \circ h$ and an isometry of Kaehler metrics $h^*(dd^c\tau) = dd^c\tau_0$. Up to isomorphism the balls $(\boldsymbol{C}^m(r), \tau_0)$ if $0 < r < \infty$ and the euclidean space $(\boldsymbol{C}^m, \tau_0)$ if $r = \infty$ are the only strictly parabolic manifolds. In this respect, the characterization theorem resembles

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the Riemann mapping theorem. I am greatly indebted to Daniel Burns for remarks which improved the weaker results announced in [17] and simplified the proof. More details on his improvements will be provided further down in the introduction. Also in [17], only the case $\Delta = \infty$ was considered, the extension to the case $\Delta \leqslant \infty$ posed no problem.

R. E. Greene and H. Wu have developed an extensive theory of non-compact Kaehler manifolds. See the survey [9]. Also Siu and Yau [15] established a uniformization theorem. Although the characterization theorem fits well into this circle of results, the proof proceeds along different lines.

If the condition $dd^c\tau > 0$ is replaced by the condition $(dd^c\tau)^m \not\equiv 0$, the manifold (M, τ) is said to be parabolic. On parabolic manifolds with $\Delta = \infty$ a successful theory of value distribution has been established by Griffiths and King [10], Stoll [16] and Wong [18]. Each affine algebraic manifold is parabolic with $\Delta = \infty$. If $\varphi\colon \tilde{M} \to M$ is a surjective, proper holomorphic map with $\dim \tilde{M} = \dim M$, then $\tilde{M}$ is parabolic. The cartesian product of parabolic manifolds is parabolic [16]. A non-compact Riemann surface is parabolic with $\Delta = \infty$ if and only if every subharmonic function bounded above is constant. The strictly parabolic case had special advantages in value distribution theory, which inspired these investigations. The result explains the advantages.

The function $\log \tau$ is a plurisubharmonic solution of the complex Monge-Ampère equation (1.1). The complex Monge-Ampère equation $(dd^c u)^m = \Omega$ has been investigated intensively in recent years and has led to a number of important applications. Yau [21] solved the Calabi conjecture. The pertinent remark in [17] has been made obsolete by Burns' contribution. Other applications were in the theory of biholomorphic mappings. The PDE of Monge-Ampère equation is difficult because of the non-linearity of the equation. The characterization theorem is a surprisingly smooth result on the Monge-Ampère equation. The proof uses a foliation method already investigated by others, for instance Bedford and Kalka [4].

Let (M, τ) be a strictly parabolic manifold of dimension m. Then the *center* $M[0]$ consists of one and only one point denoted by $0 = 0_M$. (Theorem 2.4). The maximal integral curves of the vector field $\operatorname{grad}\sqrt{\tau}$ on M_* are bijectively parameterized by the unit sphere S in $\boldsymbol{C}^m$ (Section 4d). The integral curve assigned to $\xi \in S$ is complexified to a complex submanifold $L(\xi)$ of dimension 1 in M_*. The foliation $\{L(\xi)\}_{\xi\in S}$ of M_* coincides with the foliation of M_* by the annihilator of $dd^c \log \tau$. The vector field f dual to $\bar{\partial}\tau$ under the Kaehler metric $\varkappa$ defined by $dd^c\tau > 0$ is tangent to the leaves $L(\xi)$ and holomorphic on the leaves $L(\xi)$. The center piece of the proof is the determination of the leaf space of this foliation (Theorem 5.14). This leaf space is the complex projective space $\boldsymbol{P}_{m-1}$ obtained as the quotient space of the Hopf fibration $\boldsymbol{P}\colon S \to \boldsymbol{P}_{m-1}$. If $\xi_1 \in S$ and $\xi_2 \in S$, then $L(\xi_1) = L(\xi_2) = L[w]$ if and only if $\boldsymbol{P}(\xi_1) = \boldsymbol{P}(\xi_2) = w$. As a consequence, each leaf $L[w]$ carries the induced topology and is a closed, smooth complex submanifold of M_*. Also the closure $\bar{L}[w] = L[w] \cup \{0\}$ of $L[w]$ in M is a closed, smooth complex submanifold of M with the induced topology. Also the $\xi \in S$ with $\boldsymbol{P}(\xi) = w$ are the tangents of length one to $L[w]$ at 0 (Theorem 6.1). From here the results as announced in [17] can be easily established. A homeomorphism $h\colon \boldsymbol{C}^m(\Delta) \to M$ exists with $\tau_0 = \tau \circ h$ such that $h\colon \boldsymbol{C}^m(\Delta) - \{0\} \to M_*$ is a diffeomorphism. Also $h\colon \boldsymbol{C}(\Delta)\xi \to \bar{L}[w]$ is biholomorphic if $\xi \in S$ and $w = \boldsymbol{P}(\xi)$. If f is holomorphic, then $h\colon \boldsymbol{C}^m(\Delta) \to M$ is biholomorphic. Also if h is differentiable at 0, then h is biholomorphic.

Here Daniel Burns showed that each $\bar{L}[w]$ is totally geodesic. Since the integral curves of grad $(\sqrt{\tau}\,|L(\xi))$ are trivially geodesic, the integral curves of grad $\sqrt{\tau}$ are geodesic and $h = \exp_0$ is the exponential map if $\boldsymbol{C}^m(\Delta)$ is interpreted as a ball in the tangent-space at the center point 0. Hence h is differentiable at 0 and consequently biholomorphic. Burns constructs a special coordinate system around $\bar{L}(w)$ for his proof. Here an intrinsic variation of his proof is given (Lemma 3.2 and Proposition 3.8). Originally, I failed to establish Lemma 3.7. However once the lemma was proved, it is easy to compute directly that the integral curves of grad $\sqrt{\tau}$ are geodesic.

I want to thank Daniel Burns for the interest in this problem and I acknowledge the considerable improvement of the results and the simplifications in the proof which are due to his contribution. Originally, I had assumed that $M[0]$ consists of one and only one point. I want to thank Alan Huckleberry for suggesting that this assumption may be a consequence of the other axioms of a strictly parabolic exhaustion, which turned out to be so.

2. – Parabolic manifolds.

a) Definitions.

Let S^n be the n-fold cartesian product of the set S. Let $\#S$ be the cardinality of S. Let δ be the Kronecker symbol on S. Thus $\delta_{xy} = 0$ and $\delta_{xx} = 1$ if $x \in S$ and $x \neq y \in S$. If S is partially ordered, define

$$S[a, b] = \{x \in S \,|\, a \leqslant x \leqslant b\} \qquad S(a, b) = \{x \in S \,|\, a < x < b\}$$

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where a and b may belong to a larger partially ordered set. For instance

$$\boldsymbol{R}^+ = \boldsymbol{R}(0, +\infty) = \{x \in \boldsymbol{R} | x > 0\} \qquad \boldsymbol{R}_+ = \boldsymbol{R}[0, +\infty) = \{x \in \boldsymbol{R} | x \geqslant 0\}$$
$$\boldsymbol{R}^- = \boldsymbol{R}(-\infty, 0) = \{x \in \boldsymbol{R} | x < 0\} \qquad \boldsymbol{R}_- = \boldsymbol{R}(-\infty, 0] = \{x \in \boldsymbol{R} | x \leqslant 0\}\,.$$

Let τ be a non-negative function of class C^∞ on a connected, complex manifold M of dimension m. For $0 \leqslant r \leqslant +\infty$ define

$$M[r] = \{x \in M | \tau(x) \leqslant r^2\} \qquad M(r) = \{x \in M | \tau(x) < r^2\}$$
$$M\langle r\rangle = \{x \in M | \tau(x) = r^2\} \qquad M_* = M - M[0]\,.$$

Then $M[r]$ and $M(r)$ are called the *closed* and *open pseudoballs* of radius r for τ respectively and $M\langle r\rangle$ is said to be the *pseudosphere* of radius r for τ. Also $M[0]$ is called the *center* of τ. Define $\Delta = \sup \sqrt{\tau} \leqslant +\infty$. Then τ is said to be an *exhaustion*, if $0 \leqslant \tau < \Delta^2$ on M and if $M[r]$ is compact for all $r \in \boldsymbol{R}[0, \Delta)$. Here Δ is called the maximal radius of the exhaustion and the exaustion is said to be bounded if $\Delta < +\infty$.

The non-negative function τ of class C^∞ on M is said to be *semi-parabolic* if

$$dd^c \log \tau \geqslant 0 \qquad (dd^c \log \tau)^m = 0 \tag{2.1}$$

on M_*. If $k \in \boldsymbol{N}[1, m]$, we have

$$dd^c \tau \geqslant 0 \qquad \text{on } M \tag{2.2}$$

$$\tau^2 dd^c \log \tau = \tau dd^c \tau - d\tau \wedge d^c \tau \qquad \text{on } M_* \tag{2.3}$$

$$\tau^{k+1}(dd^c \log \tau)^k = \tau(dd^c \tau)^k - k\, d\tau \wedge d^c \tau \wedge (dd^c \tau)^{k-1} \qquad \text{on } M_* \tag{2.4}$$

$$\tau(dd^c \tau)^m = m\, d\tau \wedge d^c \tau \wedge (dd^c \tau)^{m-1} \qquad \text{on } M\,. \tag{2.5}$$

A semi-parabolic function τ is said to be *parabolic* if $(dd^c\tau)^m \not\equiv 0$ on M and if $M[0]$ has measure zero. A semi-parabolic function τ is said to be *strictly parabolic* if $dd^c\tau > 0$ on M. If τ is an exhaustion and if τ is semi-parabolic or parabolic or strictly parabolic, then τ is said to be a *semi-parabolic*, resp. *parabolic*, resp. *strictly parabolic exhaustion* and (M, τ) is called a *semi-parabolic*, resp. *parabolic*, resp. *strictly parabolic manifold*. This conceptual structure permits us to separate the local properties of the solutions of (2.1) from the exhaustion properties.

Let τ be a semi-parabolic exhaustion. As shown in [16] a constant $\varsigma \geqslant 0$ exists such that

$$\int_{M[r]} (dd^c \tau)^m = \int_{M(r)} (dd^c \tau)^m = \varsigma r^{2m} \tag{2.6}$$

for all $r \geqslant 0$. Define

$$\mathfrak{E}_\tau = \{r \in \boldsymbol{R}^+ | d\tau(x) \neq 0,\ \forall x \in M\langle r\rangle\}. \tag{2.7}$$

Then $\boldsymbol{R}^+ - \mathfrak{E}_\tau$ has measure zero. If $r \in \mathfrak{E}_\tau$, then $M\langle r\rangle$ is a smooth oriented submanifold of M with $\partial M(r) = M\langle r\rangle$. If $M\langle r\rangle$ is oriented to the exterior of $M(r)$, then

$$\varsigma = \int_{M\langle r\rangle} d\ \log\tau \wedge (dd^c \log\tau)^{m-1} \tag{2.8}$$

for all $r \in \mathfrak{E}_\tau$. The semi-parabolic exhaustion τ is parabolic if and only if $\varsigma > 0$. If the exhaustion τ is strictly parabolic, then M is Stein and $\mathfrak{E}_\tau = \boldsymbol{R}^+$ by (2.5).

In the case $\Delta = +\infty$, parabolic manifolds were introduced by Griffiths and King [10] and studied in [16]. Any affine algebraic manifold is parabolic with $\Delta = \infty$ [10]. The product of parabolic manifolds is parabolic [16]. Parabolic manifolds are important in value distribution theory [10], [16], [18]. For $z \in \boldsymbol{C}^m$ define $\tau_0(z) = |z|^2$; then τ_0 is a strictly parabolic exhaustion of $\boldsymbol{C}^m$.

b) Properties of the center and the sphere volume.

A biholomorphic map $\mathfrak{z} = (z^1, \dots, z^m)\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ of an open subset $U_\mathfrak{z}$ of M onto an open subset $U'_\mathfrak{z}$ of $\boldsymbol{C}^m$ is said to be a *chart*. If $a \in U_\mathfrak{z}$, then $\mathfrak{z}$ is called a *chart at a*. Let τ be a semiparabolic function on M. On $U_\mathfrak{z}$ abbreviate

$$\tau_{\mu_1\dots\mu_p\bar\nu_1\dots\bar\nu_p} = \frac{\partial}{\partial z^{\mu_1}}\cdots\frac{\partial}{\partial z^{\mu_p}}\frac{\partial}{\partial \bar z^{\nu_1}}\cdots\frac{\partial}{\partial \bar z^{\nu_p}}\tau\,. \tag{2.9}$$

We shall use the Einstein summation convention. Then

$$\partial\tau = \tau_\mu dz^\mu \qquad \bar\partial\tau = \tau_{\bar\nu} d\bar z^\nu \tag{2.10}$$

$$d\tau\wedge d^c\tau = \frac{i}{2\pi}\,\partial\tau\wedge\bar\partial\tau = \frac{i}{2\pi}\,\tau_\mu\tau_{\bar\nu}dz^\mu\wedge dz^{\bar\nu} \tag{2.11}$$

$$dd^c\tau = \frac{i}{2\pi}\,\partial\bar\partial\tau = \frac{i}{2\pi}\,\tau_{\mu\bar\nu}dz^\mu\wedge d\bar z^\nu\,. \tag{2.12}$$

Let $H = (\tau_{\mu\bar\nu})$ be the associated matrix and $H^{\mu\bar\nu}$ be the matrix H without the μ-th row and the ν-th column. Define $T = \det H \geqslant 0$ and

$$T^{\mu\nu} = (-1)^{\mu+\nu}\det H^{\mu\bar\nu}\,. \tag{2.13}$$

Define $\tau_0 \colon \boldsymbol{C}^m \to \boldsymbol{R}_+$ by $\tau_0(z) = |z|^2$ for all $z \in \boldsymbol{C}^m$ and define

$$v_0 = \mathfrak{z}^*(dd^c \tau_0) = \frac{i}{2\pi} \sum_{\mu=1}^{m} dz^\mu \wedge d\bar{z}^\mu . \tag{2.14}$$

Then

$$v_0^m = \left(\frac{i}{2\pi}\right)^m m!\, dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^m \wedge d\bar{z}^m \tag{2.15}$$

$$(dd^c \tau)^m = T v_0^m . \tag{2.16}$$

Define

$$\zeta_{\mu\bar{\nu}} = dz^{\bar{\mu}} \wedge dz^\nu \wedge \bigwedge_{\nu \neq \lambda \neq \mu} dz^\lambda \wedge d\bar{z}^\lambda \qquad \text{if } \mu \neq \nu$$

$$\zeta_{\mu\bar{\mu}} = \bigwedge_{\lambda \neq \mu} dz^\lambda \wedge d\bar{z}^\lambda .$$

Then

$$(dd^c\tau)^{m-1} = \left(\frac{i}{2\pi}\right)^{m-1} (m-1)!\, T^{\mu\bar{\nu}} \zeta_{\mu\bar{\nu}} \tag{2.17}$$

$$m\, d\tau \wedge d^c\tau \wedge (dd^c\tau)^{m-1} = T^{\mu\bar{\nu}} \tau_\mu \tau_{\bar{\nu}} v_0^m . \tag{2.18}$$

If τ is strictly parabolic on the open subset $U \neq \emptyset$ of M, then $dd^c\tau > 0$ defines a Kaehler metric $\varkappa$ on U. The matrix H is invertible on $U \cap U_\mathfrak{z}$. Let $H^{-1} = (\tau^{\bar{\nu}\mu})$ be the inverse matrix. Then $T > 0$ and $\tau^{\bar{\nu}\mu} = T^{\mu\bar{\nu}}/T$ on $U \cap U_\mathfrak{z}$. Now (2.5), (2.17) and (2.18) imply

LEMMA 2.1. *Let τ be a semi-parabolic function on M. Let $\mathfrak{z}$ be a chart on M. Then*

$$\tau T = \tau_\mu T^{\mu\bar{\nu}} \tau_{\bar{\nu}} \qquad \textit{on } U_\mathfrak{z} . \tag{2.19}$$

If τ is strictly parabolic on the open subset U of M with $U \cap U_\mathfrak{z} \neq \emptyset$, then

$$\tau = \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu} \tau_\mu \qquad \textit{on } U \cap U_\mathfrak{z} . \tag{2.20}$$

In particular $\sqrt{\tau}$ is the length of $\partial\tau$ and $\bar{\partial}\tau$ in respect to the Kaehler metric $\varkappa$ on U. Also (2.20) is the fundamental identity concerning strictly parabolic functions. The identities (2.5), (2.16) and (2.18) prove Lemma 2.1 trivially.

Now, the center shall be investigated. Let $A(U)$ be the algebra of all complex valued functions of class C^∞ on the open subset of M. Take $a \in U$. Then $\mathfrak{m}_a = \mathfrak{m}_a(U) = \{f \in A(U) | f(a) = 0\}$ is an ideal in $A(U)$. If $\mathfrak{z} \colon U_\mathfrak{z} \to U'_\mathfrak{z}$ is a chart at a with $\mathfrak{z}(a) = 0$ and with $U \subseteq U_\mathfrak{z}$ such that $\mathfrak{z}(U)$ is convex, then $z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m$ generate $\mathfrak{m}_a$ in $A(U)$. If $0 \leqslant q \leqslant p$ are integers and if $f \in \mathfrak{m}_a^p$ then the q-th order derivatives of f belong to $\mathfrak{m}_a^{p-q}$. If K is compact in U and if $f \in \mathfrak{m}_a^p$, then there exists a constant $c > 0$ such that $|f| \leqq c|\mathfrak{z}|^p$ on K. If B is a matrix, let ${}^tB$ be the transposed matrix.

PROPOSITION 2.2. *Let τ be a semi-parabolic function on M. Take $a \in M[0]$ and assume that $dd^c\tau(a) > 0$. Let $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart at a with $\mathfrak{z}(a) = 0$. Assume that $U'_\mathfrak{z}$ is convex. Define $\mathfrak{m}_a = \mathfrak{m}_a(U_\mathfrak{z})$. Then there exists a function $R \in \mathfrak{m}_a^3$ such that*

$$\tau = \tau_{\mu\bar{\nu}}(a) z^\mu \bar{z}^\nu + R \qquad \text{on } U_\mathfrak{z}. \tag{2.21}$$

PROOF. Define $H_0 = H(a)$. Consider $z = (z^1, \ldots, z^m)$ as a matrix. Identify $U_\mathfrak{z} = U'_\mathfrak{z}$ such that $\mathfrak{z}$ is the identity. Then τ has a minimum at a with $\tau(a) = 0$. Hence $d\tau(a) = 0$. A constant symmetric matrix B over $\boldsymbol{C}$ and a function $R \in \mathfrak{m}_a^3$ exist such that Taylor's formula at a is given by

$$\tau(z) = \tfrac{1}{2} zB({}^tz) + \tfrac{1}{2}\bar{z}\bar{B}({}^t\bar{z}) + zH_0({}^t\bar{z}) + R(z)$$

for all $z \in U'_\mathfrak{z}$. Define $\mathfrak{e}_\mu = (\delta_{\mu 1}, \ldots, \delta_{\mu m}) \in \boldsymbol{C}^m$. Then

$$\tau_\mu(z) = \mathfrak{e}_\mu B\,{}^tz + \mathfrak{e}_\mu H_0{}^t\bar{z} + R_{z^\mu}(z)$$

$$\tau_{\mu\bar{\nu}}(z) = \tau_{\mu\bar{\nu}}(a) + R_{z^\mu \bar{z}^\nu}(z)$$

for all $z \in U'_\mathfrak{z}$. Hence

$$\lim_{0<\lambda\to 0} \lambda^{-2}\tau(\lambda z) = \tfrac{1}{2}zB({}^tz) + \tfrac{1}{2}\bar{z}\bar{B}({}^t\bar{z}) + zH_0({}^t\bar{z})$$

$$\lim_{0<\lambda\to 0} \lambda^{-1}\tau_\mu(\lambda z) = \mathfrak{e}_\mu B({}^tz) + \mathfrak{e}_\mu H_0({}^t\bar{z})$$

$$\lim_{0<\lambda\to 0} \tau_{\mu\bar{\nu}}(\lambda z) = \tau_{\mu\bar{\nu}}(a)$$

$$\lim_{0<\lambda\to 0} \tau^{\bar{\nu}\mu}(\lambda z) = \tau^{\bar{\nu}\mu}(a)\,.$$

Therefore (2.20) implies

$$\begin{aligned}
&\tfrac{1}{2}zB({}^tz) + \tfrac{1}{2}\bar{z}B({}^t\bar{z}) + zH_0({}^t\bar{z}) \\
&= (\bar{z}\bar{B} + z{}^t\bar{H}_0)H_0^{-1}\big(B({}^tz) + H_0({}^t\bar{z})\big) \\
&= \bar{z}\bar{B}H_0^{-1}B({}^tz) + \bar{z}\bar{B}({}^t\bar{z}) + zB({}^tz) + zH_0({}^t\bar{z})\,.
\end{aligned}$$

Hence $\tfrac{1}{2}B = B$ and $\bar{B}H_0^{-1}B = 0$. Therefore $B = 0$; q.e.d.

Hence the Levi form of τ at a coincides with the Hessian of τ at a.

PROPOSITION 2.3. *Let τ be a semi-parabolic function on M. Take $a \in M[0]$ and assume that $dd^c\tau(a) > 0$. Then a is an isolated point of $M[0]$.*

PROOF. Let $\mathfrak{z}: U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart at a with $\mathfrak{z}(a) = 0$. Then (2.21) holds. Because $dd^c\tau(a) > 0$, a constant $c > 0$ exists such that

$$\tau_{\mu\bar{\nu}}(a) z^\mu \bar{z}^\nu \geqslant 2c|\mathfrak{z}|^2 \qquad \text{on } U_\mathfrak{z}\,.$$

Let U be an open neighborhood of a such that $\overline{U}$ is compact and contained in $U_\mathfrak{z}$. A constant $q > 0$ exists such that $|R| \leqslant q|\mathfrak{z}|^3$ on U. An open neighborhood V of a with $V \subseteq U$ exists such that $q|\mathfrak{z}| < c$ on V. Hence

$$\tau \geqslant 2c|\mathfrak{z}|^2 - q|\mathfrak{z}|^3 > c|\mathfrak{z}|^2 > 0$$

on $V - \{a\}$. Therefore $M[0] \cap V = \{a\}$; q.e.d.

In particular, a strictly parabolic function is parabolic.

LEMMA 2.4. *Let S be a compact subset of a connected manifold M. Then there exists a connected, compact subset K of M with $S \subseteq K$.*

PROOF. For each $a \in S$ select a connected, open neighborhood $U(a)$ of a in M such that $\overline{U}(a)$ is compact. Finitely many points $a_1, \ldots, a_p$ exist in S such that $S \subseteq U(a_1) \cup \ldots \cup U(a_p) = U$. Then $\overline{U}$ is compact. Let L_j be the trace of a curve from a_1 to a_j in M. Then L_j and $L = L_1 \cup \ldots \cup L_p$ are compact. The union $K = L \cup \overline{U}$ is connected and compact with $S \subseteq K \subseteq M$; q.e.d.

THEOREM 2.5. *Let (M, τ) be a semi-parabolic manifold. Assume that $dd^c\tau > 0$ on M_*. Then the center $M[0]$ is connected and not empty. Also $M(r)$ is connected for each $r \in \mathbf{R}(0, \Delta)$.*

PROOF. For each $r \in \mathbf{R}(0, \Delta)$ let $\mathfrak{N}(r)$ be the set of connectivity components of $M(r)$. Define $n(r) = \#\mathfrak{N}(r)$.

1. CLAIM: *If $r \in \mathbf{R}(0, \Delta)$ and $N \in \mathfrak{N}(r)$, then $N \cap M[0] \neq \emptyset$.*

Proof of the 1. *Claim:* Define $m = \dim M$. If $m > 1$, define $g = \tau^{1-m}$. If $m = 1$, define $g = \log \tau$. If $m = 1$, then $dd^c g = 0$ on M_*. If $m > 1$, then

$$dd^c g \wedge (dd^c\tau)^{m-1} = (m-1)\,\tau^{-m-1}\big(m\, d\tau \wedge d^c\tau \wedge (dd^c\tau)^{m-1} - \tau(dd^c\tau)^m\big) = 0$$

on M_*. Hence g is a solution of an elliptic differential equation on M_*. If $M[0] \cap N = \emptyset$, then $\bar{N}$ is a compact subset of M_* with $g = r^{2-2m}$ on ∂N if $m > 1$ respectively $g = \log r^2$ on ∂N if $m = 1$. By the maximum principle g is constant on N. Hence τ is constant on N which contradicts $dd^c \tau > 0$ on N. Therefore $N \cap M[0] \neq \emptyset$. The 1. Claim is proved. In particular, $M[0] \neq \emptyset$.

2. Claim: *If $r \in \mathbf{R}(0, \Delta)$, then $1 \leqslant n(r) < \infty$.*

Proof of the 2. *Claim*: Since $\emptyset \neq M[0] \subset M(r)$, the set $\mathfrak{N}(r)$ is not empty. Hence $n(r) \geqslant 1$. Since $\mathfrak{N}(r)$ is an open covering of $M[0]$, finitely many elements $N_1, \ldots, N_p$ of $\mathfrak{N}(r)$ cover $M[0] \subseteq N_1 \cup \ldots \cup N_p$. If $N \in \mathfrak{N}(r)$, then $N \cap M[0] \neq \emptyset$ implies $N = N_\lambda$ for some λ. Hence $\mathfrak{N}(r)$ is finite. The 2. Claim is proved.

3. Claim: The function n is constant on $\mathbf{R}(0, \Delta)$.

Proof of the 3. *Claim*: Take $0 < r < s < \Delta$. Define $j\colon \mathfrak{N}(r) \to \mathfrak{N}(s)$ by $j(N) \supseteq N$ for all $N \in \mathfrak{N}(r)$. Take $Q \in \mathfrak{N}(s)$. Then $a \in Q \cap M[0]$ exists. Take $N \in \mathfrak{N}(r)$ with $a \in N$. Then $a \in j(N) \cap Q$. Therefore $j(N) = Q$. The map j is surjective. Hence $n(r) \geqslant n(s)$. The function n decreases.

Take $r \in \mathbf{R}(0, \Delta)$. We shall show that n is constant in a neighborhood of r. Let $N_1, \ldots, N_{n(r)}$ be the different connectivity components of $M(r)$. Take $\lambda \in \mathbf{N}[1, n(r)]$. Since $M[0] \cap N_\lambda = M[0] \cap \bar{N}_\lambda$ is compact, a connected, compact subset K_λ of N_λ exists by Lemma 2.4 such that $M[0] \cap N_\lambda \subseteq K_\lambda$. Then $K = K_1 \cup \ldots \cup K_{n(r)}$ is a compact subset of $M(r)$. A number $s_0 \in \mathbf{R}(0, r)$ exists such that $\tau \leqslant s_0^2$ on K. Take any $s \in \mathbf{R}(s_0, r)$. Take $\lambda \in \mathbf{N}[1, n(r)]$. Then $K_\lambda \subseteq M(s)$. One and only one $N'_\lambda \in \mathfrak{N}(s)$ exists such that $K_\lambda \subset N'_\lambda$. Take $N \in \mathfrak{N}(s)$. Then $a \in N \cap M[0]$ exists. Also $a \in N_\lambda \cap M[0]$ for some λ. Hence $a \in K_\lambda \subset N'_\lambda$. Therefore $N \cap N'_\lambda \neq \emptyset$ which implies $N = N'_\lambda$. Consequently, $\mathfrak{N}(s) = \{N'_\lambda | \lambda = 1, \ldots, n(r)\}$ and $n(s) \leqslant n(r)$. Because n decreases $n(r) \leqslant n(s)$, hence $n(r) = n(s)$. The function n is constant on $\mathbf{R}(s_0, r]$.

If $n(r) = 1$, then $1 \leqslant n(s) \leqslant n(r) \leqslant 1$ for all $s \in \mathbf{R}[r, \Delta)$. Then n is constant on $\mathbf{R}(r_0, \Delta)$. Consider the case $n(r) > 1$. Take integers j and k with $1 \leqslant j < k \leqslant n(r)$. Assume that $\bar{N}_j \cap \bar{N}_k \neq \emptyset$. Then $x \in \partial N_j \cap \partial N_k$ exists. Since $\partial N_\lambda \subseteq M\langle r\rangle$ and since $M\langle r\rangle$ is the smooth boundary of $M(r)$ and of $M - M[r]$ an open neighborhood U of x exists such that $U \cap N_j = U \cap M(r) = U \cap N_k \neq \emptyset$. Hence $N_j \cap N_k \neq \emptyset$ which contradicts $j \neq k$. Therefore $\bar{N}_j \cap \bar{N}_k = \emptyset$.

Open sets U_λ with $\bar{N}_\lambda \subset U_\lambda$ exist such that $\bar{U}_\lambda$ is compact and such that $\bar{U}_\lambda \cap \bar{U}_\mu = \emptyset$ if $\lambda \neq \mu$. Then $U = U_1 \cup \ldots \cup U_{n(r)}$ is an open neighborhood of $M[r]$ and $\partial U = \partial U_1 \cup \ldots \cup \partial U_{n(r)}$ is compact. A number $s_1 > r$ with

$s_1 < \Delta$ exists, such that $\tau \geqslant s_1^2$ on ∂U. Take any $s \in \boldsymbol{R}(r, s_1)$. The map $j: \mathfrak{N}(r) \to \mathfrak{N}(s)$ defined by $j(N) \supseteq N$ is surjective. Define $P_\lambda = j(N_\lambda)$. Then $\mathfrak{N}(s) = \{P_\lambda | \lambda = 1, \ldots, n(r)\}$. Since $\tau \geqslant s_1^2 > s^2$ on ∂U_λ, the sets $P_\lambda - U_\lambda = = P_\lambda - \overline{U}_\lambda$ and $P_\lambda \cap U_\lambda$ are open with $\emptyset \neq N_\lambda \subseteq P_\lambda \cap U_\lambda$. Hence $P_\lambda - U_\lambda = \emptyset$ and $P_\lambda \subseteq U_\lambda$. Therefore $P_\lambda \neq P_\mu$ if $\lambda \neq \mu$, which shows $n(s) = n(r)$. The function n is constant on $\boldsymbol{R}(s_0, s_1)$. The locally constant function n is constant on $\boldsymbol{R}(0, \Delta)$. The 3. Claim is proved.

4. Claim: *$M(r)$ is connected for each $r \in \boldsymbol{R}(0, \Delta)$.*

Proof of the 4. *Claim*: By Lemma 2.4 a connected, compact subset K of M contains $M[0]$. A number $s \in \boldsymbol{R}(0, \Delta)$ exists such that $K \subset M(s)$. One and only one $N \in \mathfrak{N}(s)$ exists such that $K \subset N$. Then $M[0] \subset N$. If $P \in \mathfrak{N}(s)$, then $P \cap M[0] \neq \emptyset$. Hence $P \cap N \neq \emptyset$. Therefore $P = N$. Consequently $n(r) = n(s) = 1$ for all $r \in \boldsymbol{R}(0, \Delta)$. The 4. Claim is proved.

5. Claim: *$M[0]$ is connected.*

Proof of the 5. *Claim*: Let U and V be open subsets of M such that $M[0] \subseteq U \cup V$, such that $U \cap V = \emptyset$ and such that $M[0] \cap U \neq \emptyset$. We have to show that $M[0] \subset U$. Since $M[0] \cap U = M[0] - V$ is compact, an open neighborhood W of $M[0] \cap U$ exists such that $\overline{W}$ is compact and contained in U. Then $M[0] \cap \partial W = \emptyset$. A number $r \in R(0, \Delta)$ exists such that $r^2 < \tau$ on ∂W. Then $M(r) - W = M(r) - \overline{W}$ and $M(r) \cap W$ are open with $M(r) \cap W \supseteq M[0] \cap U \neq \emptyset$. Because $M(r)$ is connected, we conclude $M[0] \subset M(r) \subseteq W \subset U$. Hence $M[0]$ is connected. The 5. Claim is proved; q.e.d.

Theorem 2.6. *The center $M[0]$ of a strictly parabolic manifold (M, τ) consists of one and only one point denoted by $0 = O_M$. For each $r \in \boldsymbol{R}(0, \Delta)$ the pseudoball $M(r)$ is connected.*

Proof. By Proposition 2.3, each point of the compact set $M[0]$ is an isolated point. Hence $M[0]$ is a finite set. By Theorem 2.5, $M[0] \neq \emptyset$ is connected. Therefore $M[0]$ consists of one and only one point; q.e.d.

Proposition 2.7. *If (M, τ) is a strictly parabolic manifold, then $\varsigma = 1$.*

Proof. Take a chart $\mathfrak{z}: U_\mathfrak{z} \to U'_\mathfrak{z}$ at O_M such that $\mathfrak{z}(O_M) = 0$, such that $U'_\mathfrak{z}$ is convex and such that $\tau_{\mu\bar{\nu}}(O_M) = \delta_{\mu\nu}$. Then $D = \boldsymbol{C}^m[r_0] \subset U'_\mathfrak{z}$ for some $r_0 > 0$. Identify $U_\mathfrak{z} = U'_\mathfrak{z}$ such that $\mathfrak{z}$ becomes the identity. Then $O_M = 0 \in \boldsymbol{C}^m$. Define $\tau_0: \boldsymbol{C}^m \to \boldsymbol{R}_+$ by $\tau_0(z) = |z|^2$. For each $r \in \boldsymbol{R}(0, r_0)$ define $\boldsymbol{C}^m\langle r\rangle = \{z \in \boldsymbol{C}^m | \, |z| = r\}$. A constant $s(r) > 0$ exists such that $\tau > s(r)^2$

on $C^m\langle r\rangle$. If $0 < t < s(r)$, then $M(t) \cap C^m\langle r\rangle = \emptyset$ and $O \in M(t)$. Since $M(t)$ is connected, we conclude that $M(t) \subseteq C^m(r) \subset D$.

Let $A(U_3)$ be the ring of functions of class C^∞ on U_3. Let $\mathfrak{m}_0$ be the ideal in $A(U_3)$ generated by $z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m$. Then $R = \tau - \tau_0 \in \mathfrak{m}_0^3$ by Proposition 2.2. A constant $c > 0$ exists such that $|R(z)| \leqslant c|z|^3$ for all $z \in D$. Abbreviate $A = C^m(1)$ and $B = C^m[2]$. Take $r_1 > 0$ with $2r_1 < r_0$ and $2cr_1 < 1$. Define $r_2 = \operatorname{Min}(r_1, s(r_1))$. Take $r \in \mathbf{R}(0, r_2)$. Then $M(r) \subset C^m(r_1) \subset \subset D \subset U_3$. Define λ_r on C^m by $\lambda_r(z) = 1$ if $z \in M(r)$ and $\lambda_r(z) = 0$ if $z \in C^m - M(r)$. A biholomorphic map $\mu_r\colon C^m \to C^m$ is defined by $\mu_r(z) = rz$ for all $z \in C^m$. Then $\mu_r(B) \subset D$. If $\lambda_r(rz) = 1$ with $|z| \geqslant 2$, then $rz \in M(r)$. Hence $|rz| < r_1$ and

$$r^2 > \tau(rz) = r^2|z|^2 + R(rz) \geqslant r^2|z|^2(1 - cr|z|) > r^2$$

which is impossible. Hence $\lambda_r(rz) = 0$ if $|z| \geqslant 2$. Therefore

$$\varsigma r^{2m} = \int_{M(r)} (dd^c\tau)^m = \int_D \lambda_r (dd^c\tau)^m = \int_B (\lambda_r \circ \mu_r)\mu_r^*(dd^c\tau)^m .$$

On B define χ_r by

$$\chi_r(z) = \frac{i}{2\pi} R_{z^\mu \bar{z}^\nu}(rz)\, dz^\mu \wedge d\bar{z}^\nu .$$

With $v_0 = dd^c\tau_0$ on C^m we obtain $\mu_r^*(dd^c\tau) = r^2(v_0 + \chi_r)$. Because $R \in \mathfrak{m}_0^3$, a constant $c_1 > 0$ exists such that

$$|R_{z^\mu \bar{z}^\nu}(z)| \leqslant c_1|z| \qquad \forall z \in D .$$

Hence there exists a function g_r of class C^∞ on B and a constant $c_2 > 0$ such that $|g_r| \leqslant C$ on B uniformly for all $r \in \mathbf{R}(0, r_2)$ such that

$$\mu_r^*(dd^c\tau)^m = r^{2m}(1 + rg_r)v_0^m \quad \text{and} \quad \varsigma = \int_B (\lambda_r \circ \mu_r)(1 + rg_r)v_0^m$$

for all $r \in \mathbf{R}(0, r_2)$.

Take $z \in A$. Take $r \in \mathbf{R}(0, r_2)$ with $c|z|^3 r < 1 - |z|^2$. Then

$$\tau(rz) = r^2|z|^2 + R(rz) \leqslant r^2|z|^2 + c|z|^3 r^3 < r^2 .$$

Hence $rz \in M(r)$ and $\lambda_r(\mu_r|z|) = \lambda_r(rz) = 1$. Therefore $\lambda_r \circ \mu_r \to 1$ for $r \to 0$ on A. Take $z \in B - \bar{A}$. Take $r \in \mathbf{R}(0, r_2)$ with $c|z|^3 r < |z|^2 - 1$. Then

$$\tau(rz) = r^2|z|^2 + R(rz) \geqslant r^2|z|^2 - c|z|^3 r^3 > r^2 .$$

Hence $rz \notin M(r)$ and $\lambda_r(rz) = 0$. Therefore $\lambda_r \circ \mu_r \to 0$ for $r \to 0$ on $B - \bar{A}$. The bounded convergence theorem implies

$$\varsigma = \int_A v_0^m = 1 \qquad \text{q.e.d.}\,.$$

3. – Local parabolic geometry.

a) *Local Kaehler geometry.*

Some remarks on local Kaehler geometry shall clarify the notation. Let M be a complex manifold of pure dimension m. Let $T(M)$ and $T^c(M)$ be the real and complexified tangent bundles of M respectively. Then $T(M)$ is a real subbundle of $T^c(M)$. Let $\mathfrak{T}(M)$ and $\bar{\mathfrak{T}}(M)$ be the holomorphic and antiholomorphic tangent bundles respectively. Then $T^c(M) = \mathfrak{T}(M) \oplus \bar{\mathfrak{T}}(M)$. Let $\eta_0\colon T^c(M) \to \mathfrak{T}(M)$ and $\eta_1\colon T^c(M) \to \bar{\mathfrak{T}}(M)$ be the projections. They restrict to bundle isomorphisms $\eta_0\colon T(M) \to \mathfrak{T}(M)$ and $\eta_1\colon T(M) \to \bar{\mathfrak{T}}(M)$ over $\boldsymbol{R}$. The complex structure on M defines a bundle isomorphism $J\colon T^c(M) \to T^c(M)$ over $\boldsymbol{C}$ called the associated almost complex structure such that $-J \circ J$ is the identity and such that $J|\mathfrak{T}(M)$ is the multiplication by i and $J|\bar{\mathfrak{T}}(M)$ is the multiplication by $-i$. Also the restriction $J\colon T(M) \to T(M)$ is a bundle isomorphism over $\boldsymbol{R}$. If $p \in M$ and $u \in T^c_p(M)$, then

$$\eta_0(u) = \tfrac{1}{2}(u - iJu) \qquad \eta_1(u) = \tfrac{1}{2}(u + iJu)\,. \tag{3.1}$$

Hence $\eta_0 \circ J = i\eta_0$ and $\eta_1 \circ J = -i\eta_1$. The sections of $T(M)$, $T^c(M)$, $\mathfrak{T}(M)$ and $\bar{\mathfrak{T}}(M)$ are called *real vector fields*, *complex vector fields*, *vector fields of type* (1, 0), *vector fields of type* (0, 1) respectively. The sections of $\bigwedge_n T(M)$, $\bigwedge_n T^c(M)$, $\bigwedge_p \mathfrak{T}(M) \wedge \bigwedge_q \bar{\mathfrak{T}}(M)$ are called *real vector fields of type* n, *complex vector fields of type* n *and vector fields of type* (p, q) *respectively.* The holomorphic sections of $\bigwedge_p \mathfrak{T}(M)$ are called *holomorphic vector fields.*

The cotangent bundles $T(M)^*$, $T^c(M)^*$, $\mathfrak{T}(M)^*$, $\bar{\mathfrak{T}}(M)^*$ are dual to $T(M)$, $T^c(M)$, $\mathfrak{T}(M)$ and $\bar{\mathfrak{T}}(M)$ respectively with $T^c(M)^* = \mathfrak{T}(M)^* \oplus \bar{\mathfrak{T}}(M)^* \supseteq T(M)^*$. As usual denote

$$T^n(M) = \bigwedge_n T^c(M)^* \qquad T^{p,q}(M) = \bigwedge_p \mathfrak{T}(M)^* \wedge \bigwedge_q \bar{\mathfrak{T}}(M)^*\,.$$

The sections of $T^n(M)$ and $T^{p,q}(M)$ are *called differential forms of degree* n *respectively of bidegree* (p, q). The holomorphic sections of $\bigwedge_p \mathfrak{T}(M)^* =$

$= T^{p,0}(M)$ are called *holomorphic forms of degree p or bidegree* $(p, 0)$. Let $\langle \square, \square \rangle$ be the inner product. If ω is a differential form of degree n and if $X \in \bigwedge_n T^c_p(M)$ with $p \in M$, denote $\langle \omega(p), X \rangle = \omega(p, X)$. If X is a vector field of type n, a function $\omega(X)$ is defined by $\omega(X)(p) = \omega(p, X(p))$. If $X = X_1 \wedge \ldots \wedge X_n$ write also $\omega(p, X) = \omega(p, X_1, \ldots, X_n)$ respectively $\omega(X) = \omega(X_1, \ldots, X_n)$. If X is a vector field of type 1 and if f is a function of class C^1, then X acts on f by $Xf = df(X)$.

Let $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart on M. Then $\mathfrak{z} = (z^1, \ldots, z^n)$ with $x^\mu = \operatorname{Re} z^\mu$ and $y^\mu = \mathfrak{Im}\, z^\mu$. Then $\partial/\partial x^1, \partial/\partial y^1, \ldots, \partial/\partial x^m, \partial/\partial y^m$ is a real analytic frame of $T(M)$ over $\boldsymbol{R}$ and of $T^c(M)$ over $\boldsymbol{C}$ on $U_\mathfrak{z}$ and $dx^1, dy^1, \ldots, dx^m, dy^m$ is the dual frame. Also $\partial/\partial z^1, \ldots, \partial/\partial z^m$ is a holomorphic frame of $\mathfrak{T}(M)$ and $\partial/\partial \bar{z}^1, \ldots, \partial/\partial \bar{z}^m$ is an antiholomorphic frame of $\overline{\mathfrak{T}}(M)$ over $U_\mathfrak{z}$ with $dz^1, \ldots, dz^m$ and $d\bar{z}, \ldots, d\bar{z}^m$ as the dual frames respectively. We have

$$J\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu} \qquad J\left(\frac{\partial}{\partial z^\mu}\right) = i\,\frac{\partial}{\partial z^\mu} \qquad J\left(\frac{\partial}{\partial \bar{z}^\mu}\right) = -\,i\,\frac{\partial}{\partial \bar{z}^\mu} \tag{3.2}$$

$$\eta_0\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial z^\mu} \qquad \eta_0\left(\frac{\partial}{\partial y^\mu}\right) = i\,\frac{\partial}{\partial z^\mu}\,. \tag{3.3}$$

A vector field X and a differential form ψ on $U_\mathfrak{z}$ are written as

$$X = X^\mu \frac{\partial}{\partial z^\mu} + X^{\bar{\mu}} \frac{\partial}{\partial \bar{z}^\mu}, \qquad \psi = \psi_\mu dz^\mu + \psi_{\bar{\mu}} d\bar{z}^\mu\,. \tag{3.4}$$

These index and baring conventions extended to higher types and degrees.

A hermitian metric $\varkappa$ on M is a function $\varkappa\colon \mathfrak{T}(M) \oplus \mathfrak{T}(M) \to \boldsymbol{C}$ of class C^∞ which restricts to a positive definite hermitian form $\varkappa_p = \varkappa | \mathfrak{T}_p(M) \oplus \mathfrak{T}_p(M)$ for each $p \in M$. The associated differential form ω of bidegree $(1, 1)$ is defined by $\omega(p, X, \bar{Y}) = (i/2\pi)\varkappa_p(X, Y)$ for $X \in \mathfrak{T}_p(M)$ and $Y \in \mathfrak{T}_p(M)$ and $p \in M$. Then $\omega > 0$ and $\varkappa$ is a Kaehler metric if and only if $d\omega = 0$.

The Kaehler metric $\varkappa$ defines a Riemannian metric g on $T(M)$ by

$$g_p(X, Y) = \operatorname{Re} \varkappa_p(\eta_0(X), \eta_0(Y)) \tag{3.5}$$

for X and Y in $T_p(M)$. Then $g_p(JX, JY) = g_p(X, Y)$ and $g_p(X, JX) = 0$. Also g extends to $T^c_p(M)$ such that $g_p\colon T^c_p(M) \oplus T^c_p(M) \to \boldsymbol{C}$ is complex bilinear with $g_p(X, Y) = 0$ if X and Y are in $\mathfrak{T}_p(M)$ or if X and Y are in $\overline{\mathfrak{T}}_p(M)$. The Kaehler metric $\varkappa$ induces a dual hermitian metric along the fibers of $\mathfrak{T}(M)^*$.

Let $\mathfrak{z} \in U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ be a chart. Take $p \in U_{\mathfrak{z}}$. Take X and Y in $\mathfrak{T}_p(M)$ and ψ and χ in $\mathfrak{T}_p(M)^*$. Then

$$\varkappa_p(X, Y) = h_{\mu\bar{\nu}}(p) X^\mu \overline{Y}^\nu \tag{3.6}$$

$$\omega = \frac{i}{2\pi} h_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu \tag{3.7}$$

$$\varkappa_p^*(\psi, \chi) = h^{\bar{\nu}\mu}(p) \psi_\mu \bar{\chi}_\nu \tag{3.8}$$

where $(h^{\bar{\nu}\mu})$ is the inverse matrix to $H = (h_{\mu\bar{\nu}}) = {}^tH$. The Kaehler metric $\varkappa$ defines a *connection* in $\mathfrak{T}(M)$ given by

$$\Gamma = (\partial H) H^{-1} = \Gamma_\mu dz^\mu \tag{3.9}$$

where $\Gamma_\mu = (\Gamma^\lambda_{\mu\nu})$ is a matrix with

$$\Gamma^\lambda_{\mu\nu} = h_{\nu\bar{\alpha}z^\mu} h^{\bar{\alpha}\lambda} = - h_{\nu\bar{\alpha}} h^{\bar{\alpha}\lambda}_{z^\mu}\,. \tag{3.10}$$

Also Γ is the Riemannian connection of the associated Riemannian metric and is given as such by $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = \overline{\Gamma}^\lambda_{\mu\nu}$ and by $\Gamma^{\bar{\lambda}}_{\mu\nu} = \Gamma^\lambda_{\bar{\mu}\nu} = \Gamma^\lambda_{\mu\bar{\nu}} = \Gamma^{\bar{\lambda}}_{\bar{\mu}\nu} = \Gamma^{\bar{\lambda}}_{\mu\bar{\nu}} = \Gamma^\lambda_{\bar{\mu}\bar{\nu}} = 0$ in respect to the frame $\partial/\partial z^1, \ldots, \partial/\partial z^m, \partial/\partial \bar{z}^1, \ldots, \partial/\partial \bar{z}^m$ Because $\varkappa$ is a Kaehler metric,

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}\,. \tag{3.11}$$

The *curvature tensor* of $\varkappa$ is defined by

$$R^\alpha_{\beta\bar{\mu}\nu} = \Gamma^\alpha_{\beta\nu\bar{z}^\mu}\,, \qquad R_{\bar{\alpha}\beta\bar{\mu}\nu} = h_{\lambda\bar{\alpha}} R^\lambda_{\beta\bar{\mu}\nu}\,. \tag{3.12}$$

The *Ricci curvature* and the *Ricci form* ϱ are defined by

$$R_{\bar{\nu}\mu} = R^\beta_{\beta\bar{\nu}\mu} = (\log \det H)_{z^\mu \bar{z}^\nu}\,, \tag{3.13}$$

$$\varrho = dd^c \log H = \frac{i}{2\pi} R_{\bar{\nu}\mu} dz^\mu \wedge d\bar{z}^\nu\,. \tag{3.14}$$

For each $p_0 \in M$, there exists a chart $\mathfrak{z}\colon U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ *normal at* $p_0 \in U_{\mathfrak{z}}$ such that

$$h_{\mu\bar{\nu}}(p_0) = \delta_{\mu\nu} = h^{\bar{\nu}\mu}(p_0) \tag{3.15}$$

$$h_{\mu\bar{\nu}z^\lambda}(p_0) = h_{\mu\bar{\nu}\bar{z}^\lambda}(p_0) = h^{\bar{\nu}\mu}_{z^\lambda}(p_0) = h^{\bar{\nu}\mu}_{\bar{z}^\lambda}(p_0) = 0 \tag{3.16}$$

$$\Gamma^\lambda_{\mu\nu}(p_0) = 0\,. \tag{3.17}$$

The connection Γ defines a *covariant derivative* ∇. If X and Y are vector fields on M, if Y has class C^1 and if $\mathfrak{z}\colon U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ is a chart, then in the notation of (3.4) we have

$$\nabla_X Y = (Y^\lambda_{z^\mu} X^\mu + Y^\lambda_{\bar{z}^\mu} X^{\bar{\mu}} + \Gamma^\lambda_{\nu\mu} X^\mu Y^\nu)\frac{\partial}{\partial z^\lambda} + (Y^{\bar{\lambda}}_{z^\mu} X^\mu + Y^{\bar{\lambda}}_{\bar{z}^\mu} X^{\bar{\mu}} + \Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} X^{\bar{\mu}} Y^{\bar{\nu}})\frac{\partial}{\partial \bar{z}^\lambda}. \tag{3.18}$$

In particular if Y is of type $(1, 0)$ and holomorphic and if X is of type $(0, 1)$, then $\nabla_X Y = 0$.

Assume that $-\infty \leqslant \alpha < \beta \leqslant +\infty$. A map $\varphi\colon \boldsymbol{R}(\alpha, \beta) \to M$ of class C^1 is called a curve and

$$\dot{\varphi}(t) = d\varphi\left(t, \frac{\partial}{\partial t}\right) \in T_{\varphi(t)}(M)$$

is called the tangent vector at $\varphi(t)$ for each $t \in \boldsymbol{R}(\alpha, \beta)$. The curve is said to be smooth if $\dot{\varphi}(t) \neq 0$ for all $t \in \boldsymbol{R}(\alpha, \beta)$. Let $\varphi\colon \boldsymbol{R}(\alpha, \beta) \to M$ be a smooth curve. Take $\alpha < \alpha' < \beta' < \beta$ such that $\varphi|\boldsymbol{R}[\alpha', \beta']$ is injective. A vector field X of class C^1 on M exists such that $X \circ \varphi = \dot{\varphi}$ on $\boldsymbol{R}[\alpha', \beta']$. Then $(\nabla_X X) \circ \varphi$ is independent of the choice of X on $\boldsymbol{R}[\alpha', \beta']$. The curve φ is said to be *geodesic* if $(\nabla_X X) \circ \varphi = 0$ on $\boldsymbol{R}[\alpha', \beta']$ for any such choice. A geodesic is of class C^∞.

Let $\varphi\colon \boldsymbol{R}(\alpha, \beta) \to M$ be a smooth curve. Take a chart $\mathfrak{z}\colon U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ and an interval $\boldsymbol{R}(\alpha_0, \beta_0)$ with $\alpha \leqslant \alpha_0 < \beta_0 \leqslant \beta$ such that $\varphi(t) \in U_{\mathfrak{z}}$ for all $t \in \boldsymbol{R}(\alpha_0, \beta_0)$. Define $\mathfrak{z} \circ \varphi = (\varphi^1, \ldots, \varphi^m)$ and

$$J^\mu = \ddot{\varphi}^\mu + \Gamma^\mu_{\nu\lambda} \dot{\varphi}^\nu \dot{\varphi}^\lambda \tag{3.19}$$

on $\boldsymbol{R}(\alpha_0, \beta_0)$. Then φ is geodesic if and only if $J^\mu = 0$ for $\mu = 1, \ldots, m$ and all possible choices of α_0, β_0 and $\mathfrak{z}$.

If $-\infty \leqslant \tilde{\alpha} \leqslant \alpha < \gamma < \beta \leqslant \tilde{\beta}$ if $\varphi\colon \boldsymbol{R}(\alpha, \beta) \to M$ and $\psi\colon \boldsymbol{R}(\tilde{\alpha}, \tilde{\beta}) \to M$ are geodesics with $\varphi(\gamma) = \psi(\gamma)$ and $\dot{\varphi}(\gamma) = \dot{\psi}(\gamma)$, then $\psi|\boldsymbol{R}(\alpha, \beta) = \varphi$ and ψ is called an extension of φ. There exists one and only one maximal extension. Take $p \in M$ and $0 \neq X \in T_p(M)$. Then one and only one maximal geodesic

$$\varphi_{Xp}\colon \boldsymbol{R}(\alpha_{Xp}, \beta_{Xp}) \to M \qquad \text{with } 0 \in \boldsymbol{R}(\alpha_{Xp}, \beta_{Xp})$$

exists such that $\varphi_{X,p}(0) = p$ and $\dot{\varphi}_{X,p}(0) = X$. A geodesic $\varphi\colon \boldsymbol{R}(\alpha, \beta) \to M$ is said to be *complete* if $\alpha = -\infty$ and $\beta = +\infty$. The Kaehler manifold M is said to be *complete* if every maximal geodesic is complete, which is the case if and only if $p \in M$ exists such that $\beta_{X,p} = +\infty$ for all $X \in T_p(M)$. If $X = 0$, define $\beta_{X,p} = +\infty$ and $\alpha_{X,p} = -\infty$.

Take $p \in M$. Then $E = \{X \in T_p(M) | \beta_{X,p} > 1\}$ is an open neighborhood of $0 \in T_p(M)$. The *exponential map* $\exp_p: E \to M$ is defined by $\exp_p X = \varphi_{X,p}(1)$ if $0 \neq X \in E$ and $\exp_p(0) = p$. Then $\exp_p(tX) = \varphi_{X,p}(t)$ for all $X \in E$ and all $t \in \boldsymbol{R}(\alpha_{X,p}, \beta_{X,p})$. Open neighborhoods U of 0 in E and N of p in M exist such that $\exp_p: U \to N$ is a diffeomorphism. Also $\exp_p: E \to M$ is of class C^∞. If M is complete, then $\exp_p: T_p(M) \to M$ is surjective.

Let N be a connected, complex *submanifold* of dimension n of M. This means that N is a connected, complex manifold and a subset of N not necessarily carrying the induced topology and that the inclusion map $\iota: N \to M$ is differentiable and that the differential $d\iota(x): \mathfrak{T}_x(N) \to \mathfrak{T}_x(M)$ is injective. The *normal bundles* $T(N)^\perp$, $T^c(N)^\perp$ and $\mathfrak{T}(N)^\perp$ are defined such that

$$T_p(N)^\perp = \{x \in T_p(M) | g(X, Y) = 0 \ \ \forall Y \in T_p(N)\}$$

$$T_p^c(N)^\perp = \{x \in T_p^c(M) | g(X, Y) = 0 \ \ \forall Y \in T_p^c(N)\}$$

$$\mathfrak{T}(N)^\perp = \{x \in \mathfrak{T}_p(M) | \varkappa(X, Y) = 0 \ \ \forall Y \in \mathfrak{T}_p(N)\}$$

$$T(M)|N = T(N) \oplus T(N)^\perp \qquad T^c(M)|N = T^c(N) \oplus T^c(N)^\perp$$

$$\mathfrak{T}(M)|N = \mathfrak{T}(N) \oplus \mathfrak{T}(N)^\perp \qquad \eta_0(T(N)^\perp) = \mathfrak{T}(N)^\perp.$$

The Kaehler metric $\varkappa$ on M restricts to a Kaehler metric $\acute{\varkappa}$ on N. Let $\acute{\Gamma}$ and $\acute{\nabla}$ be the Riemannian connection and the covariant derivative associated to $\acute{\varkappa}$. Let B be the associated *second fundamental form*. If X and Y are vector fields of class C^∞ on N, then $B(X, Y) = B(Y, X)$ is a section of class C^∞ of $T^c(N)^\perp$ which is in $T(N)^\perp$ if X and Y are real and in $\mathfrak{T}(N)^\perp$ if Y is of type $(1, 0)$. Also B is bilinear over $\boldsymbol{C}$ for complex vector fields and bilinear over $\boldsymbol{R}$ for real vector fields and $B(fX, Y) = B(X, fY) = fB(X, Y)$ if f is a function of class C^∞. If U is open in M with $N \cap U \neq \emptyset$ and if X and Y are vector fields of class C^∞ on N and if $\tilde{X}$, $\tilde{Y}$ are vector fields of class C^∞ on M with $\tilde{X}|N \cap U = X|N \cap U$ and $\tilde{Y}|N \cap U = Y|N \cap U$, then $B(X, Y) = \nabla_{\tilde{X}} \tilde{Y} - \acute{\nabla}_X Y$ on $N \cap U$. If X is a vector field of type $(0, 1)$ on N and if Y is a holomorphic vector field on N, then $B(X, Y) = 0$.

The submanifold N is said to be *totally geodesic*, if each geodesic $\varphi: \boldsymbol{R}(\alpha, \beta) \to M$ such that there exists $\gamma \in \boldsymbol{R}(\alpha, \beta)$ with $\varphi(\gamma) \in N$ and $\dot{\varphi}(\gamma) \in T_{\varphi(\gamma)}(N)$ is a curve in N. A submanifold N is totally geodesic if $B(X, Y) = 0$ for all vector fields of class C^∞ on N. If there exist holomorphic vector fields $Z_1, \ldots, Z_n$ on N such that $Z_1(p), \ldots, Z_n(p)$ are linearly independent over $\boldsymbol{C}$ for at least one point $p \in N$ and such that $B(Z_\mu, Z_\nu) = 0$ for all $\mu, \nu = 1, \ldots, n$, then N is totally geodesic.

b) The complex gradient vector field.

Let M be a connected complex manifold of dimension m. Let τ be a strictly parabolic function on M. Let $\varkappa$ be the Kaehler metric defined by $dd^c\tau > 0$. Then there exists one and only one vector field $\mathfrak{f}$ of type $(1, 0)$ and class C^∞ on M such that $\varkappa(\mathfrak{f}, X) = \bar{\partial}\tau(\bar{X})$ for every vector field X of type $(1, 0)$ and class C^∞ on M. The vector field $\mathfrak{f}$ is said to be the complex gradient vector field of τ. If $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ is a chart, then

$$(3.20) \qquad \mathfrak{f} = \mathfrak{f}^\mu \frac{\partial}{\partial z^\mu} \qquad \text{on } U_\mathfrak{z}\,.$$

If X is any vector field of type $(1, 0)$ on M, then

$$\mathfrak{f}^\mu \tau_{\mu\bar{\nu}} \bar{X}^\nu = \varkappa(\mathfrak{f}, X) = \bar{\partial}\tau(\bar{X}) = \tau_{\bar{\nu}} \bar{X}^\nu$$

on $U_\mathfrak{z}$. Therefore we have

$$(3.21) \qquad \mathfrak{f}^\mu \tau_{\mu\bar{\nu}} = \tau_{\bar{\nu}} \qquad \mathfrak{f}^\mu = \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu}$$

$$(3.22) \qquad \mathfrak{f} = \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu} \frac{\partial}{\partial z^\mu}$$

on $U_\mathfrak{z}$ and (2.20) implies

$$(3.23) \qquad \tau = \mathfrak{f}^\mu \tau_\mu = \bar{\mathfrak{f}}^\nu \tau_{\bar{\nu}} = \mathfrak{f}^\mu \tau_{\mu\bar{\nu}} \bar{\mathfrak{f}}^\nu = \varkappa(\mathfrak{f}, \mathfrak{f})\,.$$

Therefore $\sqrt{\tau}$ is the length of $\mathfrak{f}$ and $\mathfrak{f}(p) \neq 0$ if $p \in M_*$.

Let g be the Riemannian metric associated to $\varkappa$. A real vector field field grad τ of class C^∞ called the *gradient* of τ is defined by $g(X, \operatorname{grad}\tau) = d\tau(X)$ for all real vector fields X. Then (3.5) implies

$$\begin{aligned} \operatorname{Re}\varkappa\big(\eta_0(X), \eta_0(\operatorname{grad}\tau)\big) &= g(X, \operatorname{grad}\tau) = d\tau(X) \\ &= 2\operatorname{Re}\partial\tau\big(\eta_0(X)\big) = 2\operatorname{Re}\varkappa\big(\eta_0(X), \mathfrak{f}\big)\,. \end{aligned}$$

Hence $2\mathfrak{f} = \eta_0(\operatorname{grad}\tau)$ and $\operatorname{grad}\tau = 2\mathfrak{f} + 2\bar{\mathfrak{f}}$. Take $X = \operatorname{grad}\tau$, then

$$(3.24) \qquad d\tau(\operatorname{grad}\tau) = g(\operatorname{grad}\tau, \operatorname{grad}\tau) = 4\varkappa(\mathfrak{f}, \mathfrak{f}) = 4\tau\,.$$

Now some local properties of the vector field $\mathfrak{f}$ shall be proven. If X is a vector field of type $(1, 0)$ and class C^1 on M, then $\bar{\partial}X$ is a section of $\bar{\mathfrak{T}}(M)^* \otimes \mathfrak{T}(M)$. The Kaehler metric $\varkappa$ is an hermitian metric along the fibers of $\mathfrak{T}(M)$. Let $\bar{\varkappa}^*$ be the conjugate dual hermitian metric along the fibers of $\mathfrak{T}(M)^*$. The tensor product hermitian metric $\bar{\varkappa}^* \otimes \varkappa$ along the fibers of $\overline{\mathfrak{T}}(M)^* \otimes \mathfrak{T}(M)$ shall be denoted by $\varkappa$ again. Then the length $\|\bar{\partial} X\|_\varkappa$ is defined. Let ϱ be the Ricci form associated to the Kaehler metric $\varkappa$.

THEOREM 3.1.

$$\varrho(f, \bar{f}) = \frac{i}{2\pi} \|\bar{\partial} f\|_\varkappa^2 .$$

PROOF. Take any chart $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$. For any differentiable function g use the convention (2.9). Then

$$\tau^{\bar{\alpha}\mu}{}_{\bar{\lambda}} \tau_{\mu\bar{\beta}} + \tau^{\bar{\alpha}\mu} \tau_{\mu\bar{\beta}\bar{\lambda}} = 0 , \quad \tau^{\bar{\alpha}\mu}{}_{\bar{\lambda}} = - \tau^{\bar{\beta}\mu} \tau_{\nu\bar{\beta}\bar{\lambda}} \tau^{\bar{\alpha}\nu} . \tag{3.25}$$

Hence (3.21) implies

$$f^\mu_{\bar{\lambda}} = \tau_{\overline{\alpha\nu}} \tau^{\bar{\alpha}\mu} - \tau_{\bar{\alpha}} \tau^{\bar{\alpha}\nu} \tau_{\nu\overline{\beta\lambda}} \tau^{\bar{\beta}\mu} \tag{3.26}$$

$$f^\mu_{\bar{\lambda}} = \tau_{\overline{\alpha\lambda}} \tau^{\bar{\alpha}\mu} - f^\nu \tau_{\nu\overline{\lambda\beta}} \tau^{\bar{\beta}\mu} \tag{3.27}$$

$$f^\mu_{\bar{\lambda}\sigma} = \tau_{\overline{\alpha\lambda}\sigma} \tau^{\bar{\alpha}\mu} + \tau_{\overline{\alpha\lambda}} \tau^{\bar{\alpha}\mu}_{\sigma} - f^\nu_\sigma \tau_{\nu\overline{\beta\lambda}} \tau^{\bar{\beta}\mu} - f^\nu \tau_{\nu\overline{\beta\lambda}\sigma} \tau^{\bar{\beta}\mu} - f^\nu \tau_{\nu\overline{\beta\lambda}} \tau^{\bar{\alpha}\mu}_{\sigma} \tag{3.28}$$

$$f^\mu_\lambda = \tau_{\bar{\alpha}\lambda} \tau^{\bar{\alpha}\mu} + \tau_{\bar{\alpha}} \tau^{\bar{\alpha}\mu}_{\lambda} = \delta_{\lambda\mu} + \tau_{\bar{\alpha}} \tau^{\bar{\alpha}\mu}_{\lambda} . \tag{3.29}$$

The identity (3.23) implies

$$\tau_{\bar{\lambda}} = f^\mu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu + f^\mu \tau_{\mu\overline{\nu\lambda}} \bar{f}^\nu + f^\mu \tau_{\mu\bar{\nu}} \bar{f}^\nu_{\bar{\lambda}} \tag{3.30}$$

$$\begin{aligned} \tau_{\bar{\lambda}\sigma} = {} & f^\mu_{\bar{\lambda}\sigma} \tau_{\mu\bar{\nu}} \bar{f}^\nu + f^\mu_{\bar{\lambda}} \tau_{\nu\bar{\mu}\sigma} \bar{f}^\nu + f^\mu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu_\sigma + f^\mu_\sigma \tau_{\mu\overline{\nu\lambda}} \bar{f}^\nu + f^\mu \tau_{\mu\overline{\nu\lambda}\sigma} \bar{f}^\nu + f^\mu \tau_{\mu\overline{\nu\lambda}} \bar{f}^\nu_\sigma \\ & + f^\mu_\sigma \tau_{\mu\bar{\nu}} f^\nu_{\bar{\lambda}} + f^\mu \tau_{\mu\bar{\nu}\sigma} \bar{f}^\nu_{\bar{\mu}} + f^\mu \tau_{\mu\bar{\nu}} f^\nu_{\bar{\lambda}\sigma} . \end{aligned} \tag{3.31}$$

Take $p_0 \in M$. Then there exists a chart $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ at p_0 which is normal at p_0 for $\varkappa$. Hence (3.15)-(3.17) hold for $\tau_{\mu\bar{\nu}} = h_{\mu\bar{\nu}}$ and $\tau^{\bar{\nu}\mu} = h^{\bar{\nu}\mu}$. At p_0 we have the following identities.

$$f^\mu = \tau_{\bar{\mu}} \qquad \bar{f}^\mu = \tau_\mu \qquad f^\mu_{\bar{\lambda}} = \tau_{\overline{\mu\lambda}} \qquad \bar{f}^\mu_\lambda = \tau_{\mu\lambda} \tag{3.32}$$

$$f^\mu_\lambda = \delta_{\mu\lambda} = \bar{f}^\mu_{\bar{\lambda}} \tag{3.33}$$

$$f^\mu_{\bar{\lambda}\sigma} = - f^\nu \tau_{\nu\overline{\mu\lambda}\sigma} \qquad \bar{f}^\mu_{\bar{\lambda}\sigma} = - \bar{f}^\varrho \tau_{\bar{\varrho}\nu\bar{\lambda}\sigma} \tag{3.34}$$

$$R_{\bar{\alpha}\beta\bar{\mu}\nu} = \tau_{\bar{\alpha}\beta\bar{\mu}\nu} \qquad R_{\bar{\mu}\nu} = \sum_{\beta=1}^{m} \tau_{\bar{\beta}\beta\bar{\nu}\mu} \tag{3.35}$$

$$\delta_{\lambda\sigma} = - f^\nu \tau_{\nu\overline{\lambda\mu}\sigma} \bar{f}^\mu + f^\mu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu_\sigma + f^\mu \tau_{\mu\overline{\lambda\nu}\sigma} \bar{f}^\nu + \delta_{\sigma\lambda} - f^\nu \bar{f}^\varrho \tau_{\bar{\varrho}\nu\bar{\lambda}\sigma} \tag{3.36}$$

which implies

$$f^\mu \tau_{\mu\bar{\nu}\lambda\bar{\sigma}} \bar{f}^\nu = f^\mu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu_{\bar{\sigma}} . \tag{3.37}$$

We obtain

$$2\pi\varrho(f, \bar{f}) = iR_{\mu\bar{\nu}} f^\mu \bar{f}^\nu = i \sum_{\lambda=1}^{m} f^\mu \tau_{\mu\bar{\nu}\lambda\bar{\lambda}} \bar{f}^\nu \tag{3.38}$$

$$= i \sum_{\lambda=1}^{m} f^\mu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu_{\bar{\lambda}} = i f^\nu_{\bar{\lambda}} \tau_{\mu\bar{\nu}} \bar{f}^\nu_{\varrho} \tau^{\bar{\lambda}\varrho} \tag{3.38}$$

$$= i \|\bar{\partial} f\|^2_\varkappa \qquad \text{q.e.d.}$$

Hence f is holomorphic if and only if $\varrho(f, \bar{f}) = 0$ on M.

LEMMA 3.2. *Let ∇ be the covariant differentiation defined by $\varkappa$. Then*

$$\nabla_f f = f . \tag{3.39}$$

PROOF. Take any chart $\mathfrak{z}$. Then

$$\nabla f = (f^\beta f^\lambda_{z^\beta} + f^\beta f^\alpha \Gamma^\lambda_{\alpha\beta}) \frac{\partial}{\partial z^\lambda}$$

$$= (f^\beta \tau_{\bar{\nu}\beta} \tau^{\bar{\nu}\lambda} + f^\beta \tau_{\bar{\nu}} \tau^{\bar{\nu}\lambda}_{z^\beta} - f^\beta \tau_{\bar{\mu}} \tau^{\bar{\mu}\alpha} \tau_{\bar{\nu}\alpha} \tau^{\bar{\nu}\lambda}_{z^\beta}) \left(\frac{\partial}{\partial z^\lambda}\right)$$

$$= (f^\lambda + f^\beta \tau_{\bar{\nu}} \tau^{\bar{\nu}\lambda}_{z^\beta} - f^\beta \tau_{\bar{\mu}} \tau^{\bar{\mu}\lambda}_{z^\beta}) \left(\frac{\partial}{\partial z^\lambda}\right) = f . \qquad \text{q.e.d.}$$

Let ψ be a differential form of bidegree $(1, 1)$ on M. Take $p \in M$. Then

$$\mathfrak{A}_p(\psi) = \{X \in \mathfrak{T}_p(M) | \psi(X, \bar{Y}) = 0 \ \ \forall Y \in \mathfrak{T}_p(M)\} \tag{3.40}$$

is called the *annihilator* of ψ of type $(1, 0)$ at p. Clearly $\mathfrak{A}_p(\psi)$ is a linear subspace of $\mathfrak{T}_p(M)$ over $\boldsymbol{C}$. If $\psi \geqslant 0$, then

$$\mathfrak{A}_p(\psi) = \{X \in \mathfrak{T}_p(M) | \psi(X, \bar{X}) = 0 \ \ \forall X \in \mathfrak{T}_p(M)\} . \tag{3.41}$$

LEMMA 3.3. *If $p \in M_*$, then $\mathfrak{A}_p(dd^c \log \tau) = \boldsymbol{C} f(p)$.*

PROOF. Abbreviate $\omega = dd^c \log \tau$. Let $\mathfrak{z}: U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart at p. Then (2.3) and (3.23) imply

$$2\pi\tau^2 \omega(f, \bar{f}) = i(\tau f^\mu \tau_{\mu\bar{\nu}} \bar{f}^\nu - \tau_\mu f^\mu \tau_{\bar{\nu}} \bar{f}^\nu) = i(\tau^2 - \tau^2) = 0 .$$

Hence $f(p) \in \mathfrak{A}_p(\omega)$. In fact $\mathfrak{z}$ can be taken such that

$$2\pi\tau(p)^2\omega(p) = i\sum_{\mu=1}^{n} e_\mu dz^\mu(p)\wedge d\bar{z}^\mu(p)\,.$$

Here $n < m$ since $\omega^m = 0$. Also (2.4) and (2.5) imply

$$(3.42)\qquad 0 < (dd^c\tau)^m = m\tau^{m-1}d\tau\wedge d^c\tau\wedge(dd^c\log\tau)^{m-1}\,.$$

Hence $\omega(p)^{m-1} \neq 0$. Therefore $n \geqslant m-1$. Hence $n = m-1$ with $e_\mu > 0$ for $\mu = 1, \ldots, m-1$. Consequently $\mathfrak{A}_p(\omega)$ has complex dimension 1. Since $f(p) \neq 0$, we conclude that $\mathfrak{A}_p(\omega) = \boldsymbol{C}f(p)$, q.e.d.

c) *The complex gradient foliation.*

Let M be a connected, differentiable manifold of dimension m with real tangent bundle $T(M)$. A subset $\mathfrak{D}$ of $T(M)$ is said to be a *distribution* on M if $\mathfrak{D}_p = \mathfrak{D} \cap T_p(M) \neq \emptyset$ is a linear subspace of $T_p(M)$ for each $p \in M$. If $\mathfrak{D}_p$ is r-dimensional for each $p \in M$, then r is called the *rank* of $\mathfrak{D}$. The distribution $\mathfrak{D}$ is said to be *differentiable*, if $\mathfrak{D}$ has rank r for some $r \in \boldsymbol{Z}[0, n]$ and if $\mathfrak{D}$ is a subbundle of class C^∞ of $T(M)$. A vector field X on M *belongs* to $\mathfrak{D}$ if $X(p) \in \mathfrak{D}_p$ for all $p \in M$. A distribution is *involutive* if $[X, Y]$ belongs to $\mathfrak{D}$ for all vector fields X and Y of class C^1 which belong to $\mathfrak{D}$.

Let N be a submanifold of M. This means that N is a subset of M and N is a differentiable manifold of class C^∞ and of pure dimension not necessarily with the induced topology and that the inclusion map $\iota: N \to M$ is of class C^∞ and such that $d\iota(p): T_p(N) \to T_p(M)$ is injective; then $d\iota(p)$ is considered an inclusion such that $T(N)$ is a subbundle of $T(M)|N$. If N carries the induced topology, N is said to be a *proper* submanifold, which is the case, if and only if there exists an open neighborhood U of N such that $\iota: N \to U$ is proper. Let $\mathfrak{D}$ be a distribution of rank r on M. A connected submanifold N of M is said to be an *integral manifold* of $\mathfrak{D}$ if $T_p(N) = \mathfrak{D}_p$ for all $p \in N$. An integral manifold $\tilde{N}$ of $\mathfrak{D}$ is an extension of an integral manifold N of $\mathfrak{D}$ if N is open in the manifold $\tilde{N}$. An integral manifold N of $\mathfrak{D}$ is said to be *maximal* if N is the only extension of N.

A differentiable distribution $\mathfrak{D}$ of rank r is said to be *completely integrable* if for each $p \in M$ there exists an open neighborhood p of U and a map $F: U \to \boldsymbol{R}^{m-r}$ of class C^∞ and rank $dF(p) = m - r$ for each $p \in U$ such that $F^{-1}(F(p))$ is an integral manifold of $\mathfrak{D}$ for each $p \in U$. By Frobenius, a differentiable distribution is completely integrable if and only if $\mathfrak{D}$ is involutive. If $\mathfrak{D}$ is a completely integrable, differentiable distribution on M and if $p \in M$, then one and only one maximal integral manifold L_p

of $\mathfrak{D}$ with $p \in L_p$ exists. If $p \neq q$, then $L_p \neq L_q$. Here L_p is called the *leaf* of $\mathfrak{D}$ through p and $\mathfrak{L} = \{L_p\}_{p\in M}$ is called the *foliation* defined by $\mathfrak{D}$. The *leaf space* $\Lambda = \{L_p | p \in M\}$ carries the quotient topology and the residual map $\lambda\colon N \to \Lambda$ is defined by $\lambda(p) = L_p$.

Let Ω be a differential form of degree 2 and class C^∞ on M. The annihilator $\mathfrak{A}[\Omega]$ is a distribution on M defined by

$$\mathfrak{A}_p[\Omega] = \{u \in T_p(M) | \Omega(u, v) = 0 \ \ \forall v \in T_p(M)\} . \tag{3.43}$$

LEMMA 3.4. *If $d\Omega = 0$, then $\mathfrak{A}[\Omega]$ is involutive.*

PROOF. Let X and Y be vector fields of class C^1 which belong to $\mathfrak{A}[\Omega]$. If $p \in M$ and $v \in T_p(M)$, then $\Omega([X, Y](p), v) = 0$ has to be shown. Since for each $v \in T_p(M)$, there exists a vector field V of class C^1 on M with $V(p) = v$, only $\Omega([X, Y], Z) = 0$ for all vector fields Z of class C^1 on M has to be demonstrated. Because X and Y belong to $\mathfrak{A}[\Omega]$ the function $\Omega(Y, Z)$, $\Omega(X, Z)$, $\Omega(X, Y)$, $\Omega(X, [Y, Z])$ and $\Omega(Y, [X, Z])$ vanish identically on M. We have

$$\begin{aligned} 0 = d\Omega(X, Y, Z) = X\Omega(Y, Z) - Y\Omega(X, Z) + Z\Omega(X, Y) \\ - \Omega([X, Y], Z) + \Omega(X, [Y, Z]) - \Omega(Y, [X, Z]) . \end{aligned}$$

Hence $\Omega([X, Y], Z) = 0$ on M; q.e.d.

Now let M be a connected complex manifold of complex dimension m. Let Ω be a real form of bidegree $(1, 1)$ on M. Take $p \in M$. Then $\eta_0\colon T_p(M) \to \to \mathfrak{T}_p(M)$ is a linear isomorphism over $\boldsymbol{R}$. Define $\mathfrak{A}_p(\Omega)$ by (3.40) and $\mathfrak{A}_p[\Omega]$ by (3.43). Then we see easily that

$$\eta_0(\mathfrak{A}_p[\Omega]) = \mathfrak{A}_p(\Omega) . \tag{3.44}$$

Let τ be a strictly parabolic function on M. Define a real vector field

$$F = f + \bar{f} = \tfrac{1}{2} \operatorname{grad} \tau \tag{3.45}$$

on M. Then $\eta_0(F) = f$. Abbreviate $\omega = dd^c \log \tau \geqslant 0$ on M_*. Then $\mathfrak{A}[\omega]$ is an involutive distribution on M_*. Lemma 3.3 and (3.44) imply

$$\mathfrak{A}[\omega] = \boldsymbol{R}F \oplus \boldsymbol{R}JF \tag{3.46}$$

on M_*. Hence $\mathfrak{A}[\omega]$ is an involutive, differentiable distribution of rank 2 on M_* since $F(p) \neq 0$ for each $p \in M_*$. The foliation $\mathfrak{L} = \{L_p\}_{p\in M_*}$ and the leaf space $\Lambda = \{L_p | p \in M_*\}$ defined by $\mathfrak{A}[\omega]$ on M_* are called the foliation and the leaf space associated to τ. Take $p \in M_*$. Then L_p is a connected submanifold of class C^2 of M_*. If $q \in L_p$, then $T_q(L_p) = \mathfrak{A}_p[\omega]$ is invariant under $J: T_q(M) \to T_q(M)$ by (3.46). Therefore J restricts to an almost complex structure without torsion on L_p. Hence J defines a complex structure on L_p, such that L_p is a complex submanifold of M_* with

$$T^c(L(p)) = \mathbf{C}f \oplus \mathbf{C}\bar{f} \qquad \mathfrak{T}(L(p)) = \mathbf{C}f \tag{3.47}$$

on $L(p)$.

PROPOSITION 3.5. *If $p \in M_*$, then $f|L_p$ is a holomorphic vector field on L_p. Also* $\log \tau|L_p$ *is harmonic on L_p.* (Compare Bedford and Kalka [4], Theorem 2.4.)

PROOF. Take $q \in L_p$. Let $\mathfrak{z}: U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ be a chart of L_p at q. Then $\mathfrak{v} = \mathfrak{z}^{-1}: U'_{\mathfrak{z}} \to U_{\mathfrak{z}}$ is biholomorphic and $\mathfrak{v}: U'_{\mathfrak{z}} \to M_*$ is holomorphic. Here $U'_{\mathfrak{z}}$ is open in $\mathbf{C}$. If $z \in U'_{\mathfrak{z}}$, then $\mathfrak{v}'(z) \in \mathfrak{T}_{\mathfrak{v}(z)}(L_p)$. Because $f|L_p$ is a differentiable frame of $\mathfrak{T}(L_p)$, a function $h: U'_{\mathfrak{z}} \to \mathbf{C}$ of class C^∞ exists such that $\mathfrak{v}'(z) = h(z)f(\mathfrak{v}(z)) \neq 0$. Hence

$$dd^c \log \tau \circ \mathfrak{v} = \mathfrak{v}^*(\omega) = \frac{i}{2\pi}\,\omega(\mathfrak{v}, \bar{\mathfrak{v}}')dz \wedge d\bar{z} = \frac{i}{2\pi}\,|h(z)|^2 \omega(f, \bar{f})\,dz \wedge d\bar{z} = 0$$

on $U'_{\mathfrak{z}}$. Hence $\log \tau \circ \mathfrak{v}$ is harmonic on $U_{\mathfrak{z}}$. Consequently, $\log \tau|L_p$ is harmonic.

Without loss of generality we can assume that $U_{\mathfrak{z}} \subseteq U_{\mathfrak{w}}$ where $\mathfrak{w}: U_{\mathfrak{w}} \to U'_{\mathfrak{w}}$ is a chart on M_*. Then $\mathfrak{w} \circ \mathfrak{v} = (v^1, \ldots, v^m)$ and $\mathfrak{v}' = (v^\mu)'(\partial/\partial w^\mu)$ and $f = f^\mu(\partial/\partial w^\mu)$. Hence $(v^\mu)' = hf^\mu \circ \mathfrak{v}$ on $U'_{\mathfrak{z}}$. Therefore

$$\frac{\partial}{\partial z} \log \tau \circ \mathfrak{v} = h(f^\mu \circ \mathfrak{v})(\tau_\mu \circ \mathfrak{v})\,\frac{1}{\tau \circ \mathfrak{v}} = h$$

$$h_{\bar{z}} = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} \log \tau \circ \mathfrak{v} = 0$$

since $\log \tau \circ \mathfrak{v}$ is harmonic. Therefore h is holomorphic. Hence $f \circ \mathfrak{v}$ is holomorphic on $U'_{\mathfrak{z}}$. Consequently $f|L_p$ is holomorphic; q.e.d.

LEMMA 3.6. *Take $p \in M_*$ and $q \in L_p$. Take $a \in \mathbf{C}$. Then there exists a biholomorphic map $\mathfrak{v}: U' \to U$ of an open neighborhood U' of a in $\mathbf{C}$ onto an open neighborhood U of q in L_p such that $\mathfrak{v}(a) = q$ and $\mathfrak{v}' = f \circ \mathfrak{v}$ on U'.*

PROOF. Let $\mathfrak{z}: U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ be a chart of L_p at q with $\mathfrak{z}(q) = a$. Then $\mathfrak{y} = \mathfrak{z}^{-1}: U'_{\mathfrak{z}} \to U_{\mathfrak{z}}$ is biholomorphic. A holomorphic function h exists on $U'_{\mathfrak{z}}$

such that $\mathfrak{y}' = hf\circ\mathfrak{y}$ on $U'_\mathfrak{z}$. Since $\mathfrak{y}'(z) \neq 0$, we have $h(z) \neq 0$ for all $z \in U'_\mathfrak{z}$. Without loss of generality we can assume that $U'_\mathfrak{z}$ is convex. A holomorphic function H exists on $U'_\mathfrak{z}$ such that $H' = 1/h$ on $U'_\mathfrak{z}$ and such that $H(a) = a$. Open neighborhoods U' and U'' of a in $U'_\mathfrak{z}$ exist such that $H: U' \to U''$ is biholomorphic. Then $U = \mathfrak{y}(U'')$ is open in L_p and $\mathfrak{v} = \mathfrak{y}\circ H: U' \to U$ is biholomorphic with $\mathfrak{v}(a) = q$. Also $\mathfrak{v}' = (\mathfrak{y}'\circ H)\cdot H' = h(f\circ\mathfrak{y}\circ H)H' = f\circ\mathfrak{v}$ on U'; q.e.d.

LEMMA 3.7. *Let* $\mathfrak{z}: U_\mathfrak{z} \to U'_\mathfrak{z}$ *be a chart on* M. *Then* $f^\lambda_{\bar z^\mu}\bar f^\mu = 0$ *on* $U_\mathfrak{z}$.

PROOF. Since $f(p) = 0$ for all $p \in M[0]$, we have $f^\lambda_{\bar z^\mu}\bar f^\mu = 0$ on $M[0] \cap U_\mathfrak{z}$. Take $p \in M_* \cap U_\mathfrak{z}$. A biholomorphic map $\mathfrak{v}: U' \to U$ of an open neighborhood U' of 0 in $\boldsymbol{C}$ onto an open neighborhood U of p in L_p exists such that $\mathfrak{v}(0) = p$ and $\mathfrak{v}' = f\circ\mathfrak{v}$ on U' and $U \subseteq U_\mathfrak{z} \cap M_*$ Define $\mathfrak{z}\circ\mathfrak{v} = (v^1, \ldots, v^m)$ where each v^λ is holomorphic on U' with $(v^\lambda)' = f^\lambda\circ\mathfrak{v}$ on U'. Hence

$$0 = \frac{\partial}{\partial \bar z}(f^\lambda\circ\mathfrak{v}) = (f^\lambda_{z^\mu}\circ\mathfrak{v})v^\mu_{\bar z} + (f^\lambda_{\bar z^\mu}\circ\mathfrak{v})(\bar v^\mu)_{\bar z} = (f^\lambda_{\bar z^\mu}\circ\mathfrak{v})\operatorname{conj}(v^\mu)' = (f^\lambda_{\bar z^\mu}\circ\mathfrak{v})(\bar f^\mu\circ\mathfrak{v}).$$

Thus $f^\lambda_{\bar z^\mu}\cdot\bar f^\mu = 0$ at p; q.e.d.

If X is a real vector field on M, a curve $\varphi: \boldsymbol{R}(\alpha, \beta) \to M$ is said to be an integral curve of X, if $\dot\varphi = X\circ\varphi$ on $\boldsymbol{R}(\alpha, \beta)$.

THEOREM 3.8. *The integral curves of the vector field* grad $\sqrt{\tau}$ *on* M_* *are geodesic.*

PROOF. Abbreviate $\mathfrak{q} = \operatorname{grad}\sqrt{\tau} = (1/\sqrt{\tau})(f + \bar f)$. Let $\varphi: \boldsymbol{R}(\alpha, \beta) \to M_*$ be an integral curve of $\mathfrak{q}$. Then $\dot\varphi = \mathfrak{q}\circ\varphi$ on $\boldsymbol{R}(\alpha, \beta)$. Because $\mathfrak{q}(p) \neq 0$ for all $p \in M_*$, the curve φ is smooth. Also φ is of class C^∞. Take $\gamma \in \boldsymbol{R}(\alpha, \beta)$ and let $\mathfrak{z}: U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart of M_* at $\varphi(\gamma)$. Take α_0 and β_0 in $\boldsymbol{R}$ with $\alpha < \alpha_0 < \gamma < \beta_0 < \beta$ such that $\varphi(t) \in U_\mathfrak{z}$ for all $t \in \boldsymbol{R}(\alpha_0, \beta_0)$. Define $\mathfrak{z}\circ\varphi = (\varphi^1, \ldots, \varphi^m)$. Then $\dot\varphi = \dot\varphi^\lambda(\partial/\partial z^\lambda) + \dot{\bar\varphi}^\lambda(\partial/\partial \bar z^\lambda)$ on $\boldsymbol{R}(\alpha_0, \beta_0)$ and $\mathfrak{q} = q^\lambda_\mu(\partial/\partial z^\mu) + \bar q^\lambda_\mu(\partial/\partial \bar z^\mu)$ on $U_\mathfrak{z}$ with $q^\lambda = (1/\sqrt{\tau})f^\lambda$ and $\dot\varphi^\lambda = q^\lambda\circ f$. Define J^μ by (3.19). We have to show $J^\lambda = 0$ on $\boldsymbol{R}(\alpha_0, \beta_0)$ for $\lambda = 1, \ldots, m$. Here $\dot\varphi^\lambda = q^\lambda\circ\varphi$ and

$$\ddot\varphi^\lambda = (q^\lambda_{z^\mu}\circ\varphi)\dot\varphi^\mu + (q^\lambda_{\bar z^\mu}\circ\varphi)\dot{\bar\varphi}^\mu = (q^\lambda_{z^\mu}\circ\varphi)(q^\mu\circ\varphi) + (q^\lambda_{\bar z^\mu}\circ\varphi)(\bar q^\mu\circ\varphi).$$

Define $H = H^\lambda(\partial/\partial z^\lambda)$ on $U_\mathfrak{z}$ with

$$H^\lambda = q^\lambda_{z^\mu}q^\mu + q^\lambda_{\bar z^\mu}\bar q^\mu + \Gamma^\lambda_{\mu\nu}q^\mu q^\nu.$$

Then $H^\lambda \circ \varphi = J^\lambda$. It suffices to show that $H = 0$ on U_3. Now, (3.23), Lemma 3.7 and Lemma 3.2 imply

$$\tau H = \left(f^\lambda_{z^\mu} f^\mu - \frac{1}{2\tau} f^\lambda \tau_\mu f^\mu + f^\lambda_{\bar{z}^\mu} \bar{f}^\mu - \frac{1}{2\tau} f^\lambda \tau_{\bar{\mu}} \bar{f}^\mu + \Gamma^\lambda_{\mu\nu} f^\mu f^\nu \right) \frac{\partial}{\partial z^\lambda}$$
$$= \nabla_f f - f = 0 \,. \qquad \text{q.e.d.}$$

PROPOSITION 3.9 (D. Burns). *The leaves of the foliation associated to τ are totally geodesic.*

PROOF. Take $p \in M_*$. Let L_p be the leaf of $\mathfrak{L}$ with $p \in L_p$. Then $f|L_p$ is a holomorphic frame of $\mathfrak{T}(L(p))$ with $\nabla_f f = f$. Let $\overset{\prime}{\nabla}$ be the covariant derivative of the restriction of $\varkappa$ to L_p. Let B be the second fundamental form. Then $B(f, f) = \nabla_f f - \overset{\prime}{\nabla}_f f = f - \overset{\prime}{\nabla}_f f$ is a section in $\mathfrak{T}(L_p)$ and in $\mathfrak{T}(L_p)^\perp$. Hence $B(f, f) = 0$. As mentioned at the end of section 3*a*, this shows that L_p is totally geodesic; q.e.d.

This result of D. Burns permits an essential improvement and a substantial simplification of the results as announced in [17]. The intrinsic proof given here differs from the proof provided by D. Burns. We shall use Theorem 3.8 instead of Proposition 3.9 to obtain the afore-named improvement and simplification.

4. – The flow of the gradient vector field.

a) The foliation defined by a vector field.

If not stated otherwise, differentiability means differentiability of class C^∞. Let M be a connected, differentiable manifold of dimension m with real tangent bundle $T(M)$. A *patch* on M is a diffeomorphism $\mathfrak{x}\colon U_\mathfrak{x} \to U'_\mathfrak{x}$ of an open subset $U_\mathfrak{x}$ of M onto an open subset $U_\mathfrak{x}$ of $\mathbf{R}^m$. Then $\mathfrak{x} = (x^1, \ldots, x^m)$ and $\partial/\partial x^1, \ldots, \partial/\partial x^m$ is a frame of $T(M)$ over $U_\mathfrak{x}$. A differentiable curve $\varphi\colon \mathbf{R}(\alpha, \beta) \to M$ is said to be an *integral curve* of the differentiable vector field Y on M if $\dot{\varphi} = Y \circ \varphi$ on $\mathbf{R}(\alpha, \beta)$. If $t_0 \in \mathbf{R}(\alpha, \beta)$ and $q = \varphi(t_0)$, then φ is called an *integral curve through* q. If $\tilde{\varphi}\colon \mathbf{R}(\tilde{\alpha}, \tilde{\beta}) \to M$ is an integral curve through $q = \tilde{\varphi}(t_0)$ with $t_0 \in \mathbf{R}(\tilde{\alpha}, \tilde{\beta})$, then $\varphi = \tilde{\varphi}$ on $\mathbf{R}(\alpha, \beta) \cap \mathbf{R}(\tilde{\alpha}, \tilde{\beta})$. Given $q \in M$ and $t_0 \in \mathbf{R}$, there exists one and only one maximal interval $\mathbf{R}(\alpha, \beta)$ and one and only one integral curve $\varphi\colon \mathbf{R}(\alpha, \beta) \to M$ of Y such that $t_0 \in \mathbf{R}(\alpha, \beta)$ and $\varphi(t_0) = q$. This curve φ is called the *maximal integral curve* of Y through q for t_0.

Again, let Y be a differentiable vector field on M. Then Y defines a *local one parameter group of diffeomorphisms* at any $q \in M$, which means:

An open neighborhood U of q and $0 < \varepsilon \leqslant \infty$ and a map

$$\varphi: \boldsymbol{R}(-\varepsilon, \varepsilon) \times U \to M$$

of class C^∞ exist such that

1°) $\varphi(0, p) = p$ and $\dot{\varphi}(t, p) = Y(\varphi(t, p))$ $\forall p \in U$ and $t \in \boldsymbol{R}(-\varepsilon, \varepsilon)$.

2°) The map $\varphi_t: U \to \varphi_t(U)$ defined by $\varphi_t(p) = \varphi(t, p)$ is a diffeomorphism of U onto the open image $\varphi_t(U)$.

3°) If $p \in U$, if t, s, $t + s$ belong to $\boldsymbol{R}(-\varepsilon, \varepsilon)$ and if $\varphi_s(p) \in U$, then

$$\varphi_{t+s}(p) = \varphi_t(\varphi_s(p)) .$$

If $\tilde{\varphi}$, $\tilde{U}$, $\tilde{\varepsilon}$ is another choice, then $\varphi(t, p) = \tilde{\varphi}(t, p)$ for all $t \in \boldsymbol{R}(-\varepsilon, \varepsilon) \cap \cap \boldsymbol{R}(-\tilde{\varepsilon}, \tilde{\varepsilon})$ and $p \in U \cap \tilde{U}$. Also φ is called *a (global) one parameter group of diffeomorphisms* of Y if $\varepsilon = +\infty$ and $U = M$. If Y admits a global one parameter group Y is said to be *complete*. If M is compact, then Y is complete.

Let $\delta: M \to \boldsymbol{R}$ be a differentiable function. If $-\infty \leqslant \alpha \leqslant \beta \leqslant +\infty$ define

$$M[\alpha, \beta] = \{x \in M | \alpha \leqslant \delta(x) \leqslant \beta\} \qquad M(\alpha, \beta) = \{x \in M | \alpha < \delta(x) < \beta\}$$
$$M[\alpha, \beta) = \{x \in M | \alpha \leqslant \delta(x) < \beta\} \qquad M(\alpha, \beta] = \{x \in M | \alpha < \delta(x) \leqslant \beta\}$$
$$M\langle\alpha\rangle = M\langle\alpha, \alpha\rangle = \{x \in M | \delta(x) = \alpha\} .$$

THEOREM 4.1. *Let M be a connected, differentiable manifold of dimension m. Take $0 < \Delta \leqslant +\infty$. Let $\delta: M \to \boldsymbol{R}(-\infty, \Delta)$ be a function of class C^∞. Assume that $M[0, \Delta)$ is not empty, not compact and connected. Suppose that $M[0, t]$ is compact for each $t \in \boldsymbol{R}[0, \Delta)$. Then $S = M\langle 0\rangle$ is not empty. Further assume that a vector field Y of class C^∞ is given on M such that $d\delta(p, Y(p)) = 1$ for all $p \in M[0, \Delta)$. Then there exists $\eta > 0$ and a map*

$$\psi: \boldsymbol{R}(-\eta, \Delta) \times S \to M \tag{4.1}$$

of class C^∞ such that

(1) *If $p \in S$, then $\psi(0, p) = p$.*

(2) *If $p \in S$ and $t \in \boldsymbol{R}(-\eta, \Delta)$, then $\dot{\psi}(t, p) = Y(\psi(t, p))$.*

(3) *The map $\psi: \boldsymbol{R}[0, \Delta) \times S \to M[0, \Delta)$ is a diffeomorphism.*

(4) *If $p \in S$ and $t \in \boldsymbol{R}[0, \Delta)$, then $\delta(\psi(t, p)) = t$.*

(5) *If* $t \geqslant 0$, *define* $\psi_t: S \to M$ *by* $\psi_t(p) = \psi(t, p)$ *for all* $p \in S$. *Then* $\psi_t: S \to M\langle t\rangle$ *is a diffeomorphism.*

REMARK 1. *If* η *is given, then* ψ *is uniquely defined by* (1) *and* (2).

REMARK 2. *The number* $\eta > 0$ *can be taken so small that* ψ *in* (4.1) *is a diffeomorphism onto its open image.*

The proof is delegated to the appendix.

b) The double at a point.

Let N be a differentiable manifold of dimension $m > 1$. Let L be a proper differentiable submanifold of dimension $m-1$. Take $a \in L$. Let $h: N \to \boldsymbol{R}$ be a function of class C^∞ with $h|L = 0$. An open neighborhood U of a and functions $t: U \to \boldsymbol{R}$ and $h_0: U \to \boldsymbol{R}$ of class C^∞ exist such that $t^{-1}(0) = L \cap U$, such that $dt(x) \neq 0$ for all $x \in U$ and such that $h = th_0$ on U. Then h is said to *vanish* of order p at a on L, if an open neighborhood U_1 of a in U and a function $h_1: U_1 \to \boldsymbol{R}$ of class C^∞ exist such that $h = t^p h_1$ on U_1 and $h_1(a) \neq 0$. Of course, h may not vanish of any order at a on L.

Let M be a differentiable manifold of dimension m. Let $f: N \to M$ be a differentiable map. Assume that $b \in M$ exists such that $f|L = b$. Take $a \in L$. Let $\mathfrak{x}: U_{\mathfrak{x}} \to U'_{\mathfrak{x}}$ and $\mathfrak{y}: U_{\mathfrak{y}} \to U'_{\mathfrak{y}}$ be patches at a respectively b with $\mathfrak{x}(a) = 0$ and $\mathfrak{y}(b) = 0$. Then $\mathfrak{x} = (x^1, \ldots, x^m)$ and $\mathfrak{y} \circ f = (f^1, \ldots, f^m)$ and

$$df^1 \wedge \ldots \wedge df^m = \Delta\, dx^1 \wedge \ldots \wedge dx^m$$

with $\Delta|L = \mathbf{0}$. Then f is said to *branch* of order $p - m + 1$ at a on L if Δ vanishes of order p at a on L. Since $f^j|L = 0$, we have $f^j = tg^j$ in a neighborhood U of a where g^j is of class C^∞ on U. Hence $\Delta = t^{m-1}\Delta_0$ on U where Δ_0 is of class C^∞ on U. Therefore $p - m + 1 \geqslant 0$. This definition is independent of the choice of the patches $\mathfrak{x}$ and $\mathfrak{y}$.

Let $S = \{\mathfrak{x} \in \boldsymbol{R}^m \mid |\mathfrak{x}| = 1\} = \boldsymbol{R}^m\langle 1\rangle$ be the unit sphere in $\boldsymbol{R}^m$. Let M be a connected, differentiable manifold of dimension $m > 1$. Assume that a base point $0 = O_M$ in M is given and define $M_* = M - \{0\}$. Then (N, ϱ) is said to be a *double of M at* 0 (connected sum of M and M at 0) (Kervaire-Milnor [13]) if and only if

(1) *A connected differentiable manifold* N *of dimension* m *is given.*

(2) *The map* $\varrho: N \to M$ *is proper, surjective and of class* C^∞.

(3) *The inverse image* $S_0 = \varrho^{-1}(0)$ *is a proper, compact, differentiable submanifold of dimension* $m-1$ *of* N. *Moreover* S_0 *is diffeomorphic to the sphere* S.

(4) *The map* ϱ *branches of order* 0 *at every point of* S_0.

(5) *Open subsets* $N_j \neq 0$ *of* N *exist for* $j = 1, 2$ *such that* $N = N_1 \cup \cup N_2 \cup S$ *is a disjoint union and such that*

$$\varrho_j = \varrho | N_j : N_j \to M_*$$

is a diffeomorphism.

Since N is connected, we conclude that $S_0 = \partial N_j$ is the smooth boundary of N_j with $\bar{N}_j = S_0 \cup N_j$.

Let M and $\tilde{M}$ be differentiable manifolds of dimension $m > 1$ with base points $0 \in M$ and $\tilde{0} \in \tilde{M}$. Let $\varphi : M \to \tilde{M}$ be a diffeomorphism with $\tilde{0} = \varphi(0)$. Let (N, ϱ) and $(\tilde{N}, \tilde{\varrho})$ be doubles of M at 0 respectively $\tilde{M}$ at $\tilde{0}$. A diffeomorphism $f : N \to \tilde{N}$ is said to be an *isomorphism of doubles over* φ if and only if $\tilde{\varrho} \circ f = \varphi \circ \varrho$ and $f(N_j) = \tilde{N}_j$ for $j = 1, 2$. Then $f : S_0 \to \tilde{S}_{\tilde{0}}$ is a diffeomorphism. If $M = \tilde{M}$ and if φ is the identity, an isomorphism of doubles over the identity is also called an *isomorphism of doubles.*

Define $\mathfrak{r} : \boldsymbol{R} \times S \to \boldsymbol{R}^m$ by $\mathfrak{r}(t, \mathfrak{x}) = t\mathfrak{x}$ for all $t \in \boldsymbol{R}$ and $\mathfrak{x} \in S$. Then $(\boldsymbol{R} \times S, \mathfrak{r})$ is a double of $\boldsymbol{R}^m$ at 0 such that $(\boldsymbol{R} \times S)_2 = \boldsymbol{R}^+ \times S$.

The proofs of the following statements on doubles are found in the appendix. Here M and $\tilde{M}$ denote connected differentiable manifolds of dimension $m > 1$ with base points $0 \in M$ respectively $\tilde{0} \in \tilde{M}$. Also (N, ϱ) and $(\tilde{N}, \tilde{\varrho})$ are doubles of M at 0 respectively of $\tilde{M}$ at $\tilde{0}$.

(6) If U is open in M with $0 \in U$ and if $V = \varrho^{-1}(U)$, then $(V, \varrho | V)$ *is a double of* U *with* $V_j = N_j \cap V$. *Denote* $(V, \varrho | V) = (N, \varrho) | U$.

(7) *If* $\mathfrak{x} : U_{\mathfrak{x}} \to U'_{\mathfrak{x}}$ *is a patch of* M *at* 0 *with* $\mathfrak{x}(0) = 0$, *then there exists one and only one isomorphism of doubles from* $(N, \varrho) | U_{\mathfrak{x}}$ *to* $(\boldsymbol{R} \times S, \mathfrak{r}) | \; U'_{\mathfrak{x}}$ *over* $\mathfrak{x}$.

(8) *If* $\varphi : M \to \tilde{M}$ *is a diffeomorphism with* $\varphi(0) = \tilde{0}$, *then there exists one and only one isomorphism of doubles from* (N, ϱ) *to* $(\tilde{N}, \tilde{\varrho})$ *over* φ.

(9) *There exists one and up to isomorphism only one double of* M *at* 0.

Now, the behavior of functions and vector fields lifted to the double shall be studied.

LEMMA 4.2. *Let* U *be an open, connected neighborhood of* $0 \in \boldsymbol{R}^m$. *Define* $V = \mathfrak{r}^{-1}(U)$ *and* $V_1 = (\boldsymbol{R}^- \times S) \cap V$ *and* $V_2 = (\boldsymbol{R}^+ \times S) \cap V$. *Define* $U_* = U - \{0\}$. *Let* $h : U \to \boldsymbol{R}_+$ *be a function of class* C^∞ *with* $h(0) = 0 < h(x)$ *for all* $x \in U_*$. *Assume that the Hessian* H *of* h *at* 0 *is positive definite. Then there exist uniquely determined functions* $g : V \to \boldsymbol{R}$ *and* $G : V \to \boldsymbol{R}$ *of* *class* C^∞ *such that* $g^2 = h\circ\mathfrak{r}$ *on* V *with* $g|V_1 < 0$ *and* $g|V_2 > 0$ *and such that*

$$g(t, x) = t\sqrt{H(x)} + t^2 G(t, x) \qquad \forall (t, x) \in V .$$

PROOF. A function $R\colon U \times \boldsymbol{R}^m \to \boldsymbol{R}$ of class C^∞ exists such that $R(x, ty) = t^3 R(x, y)$ for all $t \in \boldsymbol{R}$, $x \in U$, and $y \in \boldsymbol{R}^m$ such that

$$h(x) = H(x) + R(x, x) \qquad \forall x \in U .$$

$$h(\mathfrak{r}(t, x)) = t^2 H(x) + t^3 R(tx, x)$$

for all $(t, x) \in V$. Hence $H(x) + tR(tx, x) > 0$ for all $(t, x) \in V$. Also $H|S > 0$. A function $G_1\colon V \to \boldsymbol{R}$ of class C^∞ exists such that

$$G_1(t, x) = \big(H(x) + tR(tx, x)\big)^{\frac{1}{2}} - \big(H(x)\big)^{\frac{1}{2}} .$$

Then $G_1(0, x) = 0$ for all $x \in S$. Therefore a function $G\colon V \to \boldsymbol{R}$ of class C^∞ exists such that $G_1(t, x) = tG(t, x)$ for all $(t, x) \in V$. A function $g\colon V \to \boldsymbol{R}$ of class C^∞ is defined by

$$g(t, x) = t\big(H(x)\big)^{\frac{1}{2}} + t^2 G(t, x) = t\big(H(x) + tR(tx, x)\big)^{\frac{1}{2}} .$$

Hence $g|V_1 < 0$ and $g|V_2 > 0$. Also $g^2 = h\circ\mathfrak{r}$; q.e.d.

We obtain an immediate consequence:

COROLLARY 4.3. *Let M be a connected differentiable manifold of dimension $m > 1$ with $0 \in M$. Let (N, ϱ) be the double of M at 0. Let $h\colon M \to \boldsymbol{R}$ be a function of class C^∞ such that $h(0) = 0 < h(x)$ if $0 \neq x \in M$. Assume that h has a positive definite Hessian at zero. Then one and only one function $g\colon N \to \boldsymbol{R}$ of class C^∞ exists such that $g^2 = h\circ\varrho$ and such that $g|N_1 < 0$ and $g|N_2 > 0$ with $g|S_0 = 0$.*

In order to study vector fields on (N, ϱ), it suffices to investigate a neighborhood of S. Hence we can restrict ourself to the double $(\boldsymbol{R}\times S, \mathfrak{r})$ of $\boldsymbol{R}^m$ with $\mathfrak{r}(t, x) = tx$ for all $t \in \boldsymbol{R}$ and $x \in S$. The tangent bundle splits

$$T(\boldsymbol{R}\times S) = T(\boldsymbol{R}) \oplus T(S) = \boldsymbol{R}\frac{\partial}{\partial t} \oplus T(S) \tag{4.1}$$

where $\partial/\partial t$ is a frame of $T(\boldsymbol{R})$ over $\boldsymbol{R}$. Let $\langle\cdot,\cdot\rangle$ be the inner product on $\boldsymbol{R}^m$. If $x \in S$, then

$$T_x(S) = \{y \in \boldsymbol{R}^m | \langle x, y\rangle = 0\} \subseteq \boldsymbol{R}^m \tag{4.2}$$

we can consider $T(S)$ as a subbundle of the trivial bundle $T(\boldsymbol{R}^m)|S$. Also we can identify $\boldsymbol{R}(\partial/\partial t) = \boldsymbol{R}$ at any given point $t \in \boldsymbol{R}$ such that $\partial/\partial t = 1$. Then

$$T_{(t,x)}(\boldsymbol{R} \times S) = \boldsymbol{R} \oplus T_x(S) \subseteq \boldsymbol{R}^{m+1} \tag{4.3}$$

is a linear subspace of $\boldsymbol{R}^{m+1}$. If $(t, x) \in \boldsymbol{R} \times S$, then the linear map

$$d\mathfrak{r}(t, x) \colon T_{(t,x)}(\boldsymbol{R} \times S) \to \boldsymbol{R}^{m+1} \tag{4.4}$$

is given by

$$d\mathfrak{r}(t, x, u, y) = ux + ty \tag{4.5}$$

for all $(t, x) \in \boldsymbol{R} \times S$ and $(u, y) \in \boldsymbol{R} \oplus T_x(S)$. If $t \neq 0$, (4.4) is an isomorphism, the inverse map is given by

$$d\mathfrak{r}(t, x)^{-1}(z) = \left(\langle x, z\rangle, \frac{z}{t} - \frac{\langle x, z\rangle}{t}\, x\right). \tag{4.6}$$

Now consider the complex case. We identify $\boldsymbol{R}^{2m} = \boldsymbol{C}^m$ such that

$$(x_1, y_1, \ldots, x_m, y_m) = (x_1 + iy_1, \ldots, x_m + iy_m) \tag{4.7}$$

$$(z|w) = \sum_{\mu=1}^{m} z_\mu \overline{w}_\mu \qquad \langle z, w\rangle = \mathrm{Re}\,(z|w)\,. \tag{4.8}$$

At each point p of $\boldsymbol{R}^{2m} = \boldsymbol{C}^m$, we identify tangent spaces

$$T_p(\boldsymbol{C}^m) = \mathfrak{T}_p(\boldsymbol{C}^m) = \overline{\mathfrak{T}}_p(\boldsymbol{C}^m) = \boldsymbol{C}^m \tag{4.9}$$

such that η_0 is the identity and such that J is multiplication with i and such that conjugation is obtained by conjugation of the coordinates. Then

$$\frac{\partial}{\partial z^\mu} = (\delta_{1\mu}, \ldots, \delta_{m\mu}) = \mathfrak{e}_\mu \in \boldsymbol{C}^m \tag{4.10}$$

defines an orthogonal base over $\boldsymbol{C}$ with $\bar{\mathfrak{n}}_\mu = \bar{\mathfrak{e}}_\mu$.

Now, S is the unit sphere in $\boldsymbol{C}^m$ with tangent space $T_x(S) \subseteq \boldsymbol{C}^m$ for each $x \in S$. The holomorphic tangent space at $x \in S$ is given by

$$\begin{aligned} \mathfrak{T}_x(S) &= \{y \in T_x(S) \,|\, iy \in T_x(S)\} \\ &= \{y \in T_x(S) \,|\, \langle y, ix\rangle = 0\} \\ &= \{y \in \boldsymbol{C} \,|\, (x|y) = 0\}\,. \end{aligned} \tag{4.11}$$

$$T_x(S) = \boldsymbol{R}ix + \mathfrak{T}_x(S) \qquad \boldsymbol{C}^m = \boldsymbol{C}x \oplus \mathfrak{T}_x(S) \tag{4.12}$$

are direct sums. If $u \in \boldsymbol{R}$ and $v \in \boldsymbol{R}$ and $w \in \mathfrak{T}_x(S)$, then

$$(4.13) \qquad d\mathfrak{r}(t, x, u, ivx + w) = (u + itv)x + tw$$

reflects the splitting (4.12). Hence

$$(4.14) \qquad d\mathfrak{r}(t, x)^{-1}(z) = \left(\operatorname{Re}\,(z|x), \frac{\mathfrak{Im}(z|x)}{t}\, ix + \frac{z}{t} - \frac{(z|x)}{t}\, x\right)$$

with $(z - (z|x)x) \in \mathfrak{T}_x(S)$. Define $\boldsymbol{R}_* = \boldsymbol{R} - \{0\}$. The surjective local diffeomorphism $\mathfrak{r}\colon \boldsymbol{R}_* \times S \to \boldsymbol{C}^m - \{0\}$ define a complex structure on $\boldsymbol{R}_* \times S$. Let $\hat{J}$ be the associated almost complex structure. Then

$$(4.15) \qquad \hat{J}_{t,x} = d\mathfrak{r}(t, x)^{-1} \circ J_{tx} \circ d\mathfrak{r}(t, x)$$

$$(4.16) \qquad \hat{J}_{t,x}(u, ivx + w) = \left(-tv, \frac{u}{t}\, ix + iw\right).$$

The almost complex structure breaks down if $t \to 0$.

c) The gradient vector field near the center.

Let (M, τ) be a strictly parabolic manifold of dimension m. The center $M[0]$ consists of one and only one point $0 = O_M$. Let (N, ϱ) be the double of M at 0. By Corollary 4.3, there exists one and only one function $\delta\colon N \to \boldsymbol{R}$ of class C^∞ such that $\delta^2 = \tau \circ \varrho$ and such that $\delta | N_1 < 0$ and $\delta | N_2 > 0$. Also the vector fields $\mathfrak{f}$ and $\mathfrak{q} = \operatorname{grad} \sqrt{\tau}$ lift to vector fields $\hat{\mathfrak{f}}$ and $\mathfrak{g}$ on $N - S_0$ by $\varrho_j^{-1}\colon M_* \to N_j$ for $j = 1, 2$. In order to study the behavior at 0 respectively S_0, local coordinates at 0 are chosen.

Let $\mathfrak{z}\colon U_\mathfrak{z} \to U'_\mathfrak{z}$ be a chart at $0 \in M$ such that $\mathfrak{z}(0) = 0$. We can assume that $U_\mathfrak{z}$ is connected and that $\mathfrak{z}$ is normal at 0 in respect to the Kaehler metric $\varkappa$ defined by $dd^c\tau$. Then

$$(4.17) \qquad \tau_{\mu\bar{\nu}}(0) = \delta_{\mu\nu} = \tau^{\bar{\nu}\mu}(0)$$

$$(4.18) \qquad \tau_{\mu\bar{\nu}\lambda}(0) = \tau_{\mu\bar{\nu}\bar{\lambda}}(0) = \tau^{\bar{\nu}\mu}_{\lambda}(0) = \tau^{\bar{\nu}\mu}_{\bar{\lambda}}(0) = 0\,.$$

A number $t_0 > 0$ exists such that the closure of the ball $B = \boldsymbol{C}^m(t_0)$ of radius t_0 and center 0 is contained in $U'_\mathfrak{z}$. We identify $U_\mathfrak{z} = U'_\mathfrak{z}$ such that $\mathfrak{z}$ becomes the identity. A number $r_0 > 0$ exists such that $\tau(x) > r_0^2$ if $x \in \partial B$. Because $M[r_0]$ is connected and contains 0, we have

$$(4.19) \qquad M(r_0) \subset M[r_0] \subset B \subset \bar{B} \subset U_\mathfrak{z} = U'_\mathfrak{z}\,.$$

There exists one and only one isomorphism of doubles from $(N, \varrho)|U_{\mathfrak{z}}$ to $(\boldsymbol{R}\times S, \mathfrak{r})|U'_{\mathfrak{z}}$ over $\mathfrak{z}$. This isomorphism is used as identification map such that $(N, \varrho)|B$ identifies with $(\boldsymbol{R}(-t_0, t_0)\times S, \mathfrak{r})$. On $\boldsymbol{C}^m$ we define

$$\tau_0(z) = (z|z) = \langle z, z\rangle = \varkappa_0(z, z)\,. \tag{4.20}$$

According to Proposition 2.2, the function τ has the Hessian τ_0 at 0. By Lemma 4.2, there exist functions δ, δ_0 on $\boldsymbol{R}(-t_0, t_0)\times S$ of class C^∞ such that $\delta^2 = \tau\circ\mathfrak{r}$ and

$$\delta(t, x) = t + t^2\delta_0(t, x) \tag{4.21}$$

if $-t_0 < t < t_0$ and $x\in S$. By (4.18) and (4.19) there exists a holomorphic, homogeneous polynomial $P\colon \boldsymbol{C}^m\to\boldsymbol{C}$ of degree 3 and a function $\tilde{R}\colon B\times\boldsymbol{C}^m\to\boldsymbol{R}$ of class C^∞ such that

$$\tilde{R}(z, tw) = t^4\tilde{R}(z, w) \quad \text{if } (t, z, w)\in\boldsymbol{R}\times B\times\boldsymbol{C}^m \tag{4.22}$$

$$\tau = \tau_0 + P + \overline{P} + R \quad \text{with } R(z) = \tilde{R}(z, z)\,. \tag{4.23}$$

Define $P_\mu = P_{z^\mu}$ and $Q = (P_1, \ldots, P_m)$. Also define

$$W = \{(t, z)\in\boldsymbol{R}\times\boldsymbol{C}^m | tz\in B\}\,. \tag{4.24}$$

The vector field $f = f^\mu(\partial/\partial z^\mu)$ can be considered as a vector function $f = (f^1, \ldots, f^m)\colon B\to\boldsymbol{C}^m$.

LEMMA 4.4. *There exists a vector function $K\colon W\to\boldsymbol{C}^m$ of class C^∞ such that*

$$f(tz) = tz + t^2\overline{Q}(z) + t^3K(t, z) \qquad \forall (t, z)\in W\,.$$

PROOF. Define $\hat{R}\colon W\to\boldsymbol{R}$ by $\hat{R}(t, z) = R(tz, z)$. Use the convention (4.9) for τ, P and $\hat{R}$. Then

$$\tau(tz) = t^2|z|^2 + t^3P(z) + t^3\overline{P}(z) + t^4\hat{R}(t, z)$$

$$\tau_{\bar{\nu}}(tz) = tz^\nu + t^2\overline{P}_{\bar{\nu}}(z) + t^3\hat{R}^4_{\bar{\nu}}(t, z)$$

$$\tau_{\mu\bar{\nu}}(tz) = \delta_{\mu\nu} + t^2\hat{R}_{\mu\bar{\nu}}(t, z)\,.$$

Function $\hat{R}^{\bar{\nu}\mu}\colon W\to\boldsymbol{C}$ of class C^∞ exist such that

$$\tau^{\bar{\nu}\mu}(tz) = \delta_{\mu\nu} + t^2\hat{R}^{\bar{\nu}\mu}(t, z)\,.$$

Define $K_\mu \colon W \to \boldsymbol{C}$ by

$$K_\mu(t, z) = \hat{R}_{\bar{\mu}}(t, z) + \sum_{\nu=1}^{m} \left(z^\nu + t\bar{P}_\nu(z) + t^2 \hat{R}_\nu(t, z)\right) \hat{R}^{\bar{\nu}\mu}(t, z)\right) .$$

Then $K = (K_1, \ldots, K_m) \colon W \to \boldsymbol{C}$ is of class C^∞ with

$$\begin{aligned} f^\mu(tz) &= \tau_{\bar{\nu}}(tz)\tau^{\bar{\nu}\mu}(tz) = tz^\mu + t^2\bar{P}_\mu(z) + t^3 K_\mu(t, z) \\ f(tz) &= tz + t^2\bar{Q}(z) + t^3 K(t, z) , \qquad \text{q.e.d.} \end{aligned}$$

Therefore, (4.5) implies

$$\hat{f}(t, x) = \left(\operatorname{Re}\left(f(tx)|x\right), \frac{f(tx)}{t} - \operatorname{Re}\left(f(tx)|x\right)\frac{x}{t}\right)$$

for $-t_0 < t < t_0$ and $x \in S$. Functions of class C^∞ are defined by

$$\begin{aligned} Q_0(x) &= \operatorname{Re}\left(\bar{Q}(x)|x\right) \in \boldsymbol{R} \\ Q_1(x) &= \operatorname{Im}\left(\bar{Q}(x)|x\right) \in \boldsymbol{R} \\ Q_2(x) &= \bar{Q}(x) - Q_0(x)x \in T_x(S) \subseteq \boldsymbol{C}^m \\ K_0(t, x) &= \operatorname{Re}\left(K(t, x)|x\right) \in \boldsymbol{R} \\ K_1(t, x) &= \operatorname{Im}\left(K(t, x)|x\right) \in \boldsymbol{R} \\ K_2(t, x) &= K(t, x) - K_0(t, x)x \in T_x(S) \subseteq \boldsymbol{C}^m . \end{aligned}$$

If $-t_0 < t < t_0$ and $x \in S$, then

$$\left(f(tx)|x\right) = t + t^2\left(\bar{Q}(x)|x\right) + t^3\left(K(t, x)|x\right) \tag{4.25}$$

$$\operatorname{Re}\left(f(tx)|x\right) = t + t^2 Q_0(x) + t^3 K_0(t, x) = tF_1(t, x) \tag{4.26}$$

$$\operatorname{Im}\left(f(tx)|x\right) = t^2 Q_1(x) + t^3 K_1(t, x) = t^2 F_2(t, x) \tag{4.27}$$

$$f(tx) - \operatorname{Re}\left(f(tx)|x\right)x = t^2 Q_2(x) + t^3 K_2(t, x) . \tag{4.28}$$

A vector field F of class C^∞ on $\boldsymbol{R}(-t_0, t_0) \times S$ is defined by

$$F(t, x) = \left(1 + tQ_0(x) + t^2 K_0(t, x), Q_2(x) + tK_2(t, x)\right) \tag{4.29}$$

such that

$$\hat{f}(t, x) = tF(t, x) \quad \text{if } (t, x) \in \boldsymbol{R}(-t_0, t_0) \times S . \tag{4.30}$$

We see that the vector field $\hat{f}$ originally only defined on $N - S_0$ extends to a vector field of class C^∞ on N by (4.29) and (4.30). The vector field $f_i = if$ lifts to a vector field $\hat{f}_i$ of class C^∞ on $N - S$. Also this vector field will extend over S_0. By (4.14), we have

$$\hat{f}(t, x) = \left(\operatorname{Re}\,(f(t, x)|x),\ \operatorname{Im}\,(f(t, x)|x)\,\frac{ix}{t} + \frac{f(tx)}{t} - (f(tx)|x)\,\frac{x}{t}\right).$$

Define functions of class C^∞ by

$$(4.31)\qquad \begin{cases} Q_3(x) = \bar{Q}(x) - (\bar{Q}(x)|x)\,x & \in \mathfrak{T}_x(S) \\ K_3(t, x) = K(t, x) - (K(t, x)|x)\,x & \in \mathfrak{T}_x(S) \\ F_3(t, x) = Q_3(x) + tK_3(t, x) & \in \mathfrak{T}_x(S)\,. \end{cases}$$

Then

$$(4.32)\qquad \hat{f}(t, x) = t\big(F_1(t, x), F_2(t, x)\,ix + F_3(t, x)\big) = tF(t, x)$$

$$(4.33)\qquad Q_2(x) = Q_1(x)\,ix + Q_3(x)\,.$$

Since $\hat{f}_i(t, x) = \hat{J}_{t,x}\hat{f}(t, x)$, the identity (4.16) implies

$$(4.34)\qquad \hat{f}_i(t, x) = \big(-t^2F_2(t,x), F_1(t, x)\,ix + itF_3(t, x)\big)$$

for $(t, x) \in \mathbf{R}(-t_0, t_0) \times S$ if $t \neq 0$. However (4.34) extends the vector field $\hat{f}_i$ to a vector field of class C^∞ on N with

$$(4.35)\qquad \hat{f}_i(0, x) = \big(0, F_1(0, x)\,ix\big) = (0, ix) \qquad \forall x \in S\,.$$

Now consider the real vector fields of class C^∞

$$(4.36)\qquad \mathfrak{q} = \operatorname{grad} \sqrt{\tau} = \frac{1}{\sqrt{\tau}}\,(f + \bar{f}) \qquad \text{on } M_*$$

$$(4.37)\qquad \mathfrak{g} = \frac{1}{\delta}\,(\hat{f} + \bar{\hat{f}}) \qquad \text{on } N - S_0\,.$$

Then

$$(4.38)\qquad d\varrho(p, \mathfrak{g}(p)) = -\,\mathfrak{q}(\varrho(p)) \qquad \text{for } p \in N_1$$

$$(4.39)\qquad d\varrho(p, \mathfrak{g}(p)) = \mathfrak{q}(\varrho(p)) \qquad \text{for } p \in N_2\,.$$

By (4.21), (4.29) and (4.30), the vector field $\mathfrak{g}$ extends across S_0 to a vector field $\mathfrak{g}$ of class C^∞ on M_* such that

$$(4.40)\qquad \eta_0(\mathfrak{g}(0, x)) = F(0, x) = (1, Q_2(x)) = (1, Q_1(x)ix + Q_3(x)) .$$

Since $\eta_0(\hat{J}\mathfrak{g}) = (1/\delta)\hat{f}_i$, the identities (4.21), (4.26), (4.31) and (4.34) imply

$$(4.41)\qquad \eta_0(\hat{J}\mathfrak{g}(t, x)) - (1/t)(0, ix) \to Q_0(x)ix + iQ_3(x)$$

for $t \to 0$ if $x \in S$.

For each $z \in M_*$ we have

$$(d\sqrt{\tau})(z, \mathfrak{q}(z)) = (1/4\tau(z))\, d\tau(z, \operatorname{grad} \tau(z)) = 1 .$$

If $p \in N_2$, abbreviate $z = \varrho(p)$. Then

$$d\delta(p, \mathfrak{g}(p)) = d\sqrt{\tau}(z, d\varrho(p, \mathfrak{g}(p))) = d\sqrt{\tau}(z, \mathfrak{q}(z)) = 1 .$$

If $p \in N_1$, abbreviate $z = \varrho(p)$. Then

$$d\delta(p, \mathfrak{g}(p)) = - d\sqrt{\tau}(z, d\varrho(p, \mathfrak{g}(p))) = - d\sqrt{\tau}(z, - \mathfrak{q}(z)) = 1 .$$

Hence $d\delta(p, \mathfrak{g}(p)) = 1$ for all $p \in N - S_0$. Also

$$\sup \{\delta(p) \,|\, p \in N_1\} = \sup \{\sqrt{\tau(z)} \,|\, z \in M_*\} = \Delta$$

is the maximal radius of the exhaustion τ. We have

$$N[0, \Delta) = \{x \in N \,|\, 0 \leqslant \delta(x) < \Delta\} = N_1 .$$

If $t \in \mathbf{R}[0, \Delta)$, then

$$N[0, t] = \varrho^{-1}(M[t]) \cap \bar{N}_1$$

is compact since ϱ is proper. The assumptions of Theorem 4.1 are satisfied.

d) The gradient lines.

Again let (M, τ) be a strictly parabolic manifold of dimension m. The center $M[0]$ consists of one and only one point $0 = O_M$: Let (N, ϱ) be the double of M at 0. We use the same notation as in the previous section *c*). The results obtained in *c*) show that the assumptions of Theorem 4.1 are satisfied, which gives the following result:

THEOREM 4.5. *There exists a number* $\eta > 0$ *and a map*

$$\psi\colon \boldsymbol{R}(-\eta, \Delta) \times S \to N \tag{4.42}$$

of class C^∞ *such that*

(1) *If* $\xi \in S$, *then* $\psi(0, \xi) = \xi$.

(2) *If* $\xi \in S$ *and* $t \in \boldsymbol{R}(-\eta, \Delta)$, *then* $\dot\psi(t, \xi) = \mathfrak{g}(\psi(t, \xi))$.

(3) *The map* $\psi\colon \boldsymbol{R}[0, \Delta) \times S \to N[0, \Delta)$ *is a diffeomorphism.*

(4) *If* $\xi \in S$ *and* $t \in \boldsymbol{R}[0, \Delta)$, *then* $\delta(\psi(t, \xi)) = t$.

(5) *If* $t \geqslant 0$, *define* $\psi_t\colon S \to N$ *by* $\psi_t(\xi) = \psi(t, \xi)$. *Then* $\psi_t\colon S \to N\langle t\rangle$ *is a diffeomorphism.*

Observe that $S = S_0$ is the unit sphere in $\boldsymbol{C}^m$. Also η can be taken so small that $0 < \eta < r_0$ and that the map ψ in (4.42) is a diffeomorphism onto the image and that $\varrho(N[-\eta, \eta]) \subset B$. Also observe that $N[0, \Delta) = \bar N_2 = N_2 \cup S$. If $t \in \boldsymbol{R}(0, \Delta)$ then

$$\varrho(N\langle t\rangle) = M\langle t\rangle \qquad \varrho(N[0, t)) = M(t) \tag{4.43}$$

and the maps

$$\varrho\colon N\langle t\rangle \to M\langle t\rangle \qquad \varrho\colon N(0, t) \to M(t) \cap M_* \tag{4.44}$$

are diffeomorphic.

Now, we define a map φ of class C^∞ called the *flow* of τ by

$$\varphi = \varrho\circ\psi\colon \boldsymbol{R}(-\eta, \Delta) \times S \to M\,. \tag{4.45}$$

Then φ satisfies the following conditions.

(1′) *If* $\xi \in S$, *then* $\varphi(0, \xi) = 0$.

(2′) *If* $\xi \in S$ *and* $t \in \boldsymbol{R}(0, \Delta)$, *then* $\dot\varphi(t, \xi) = \mathfrak{q}(\varphi(t, \xi))$.

(3′) *The map* $\boldsymbol{\varphi}\colon \boldsymbol{R}(0, \Delta) \times S \to M_*$ *is a diffeomorphism.*

(4′) *If* $\xi \in S$ *and* $t \in \boldsymbol{R}[0, \Delta)$, *then* $\tau(\varphi(t, \xi)) = t^2$.

(5′) *If* $0 < t < \Delta$, *define* $\varphi_t\colon S \to M$ *by* $\varphi_t(\xi) = \varphi(t, \xi)$. *Then* $\varphi_t\colon S \to M\langle t\rangle$ *is a diffeomorphism.*

(6′) *If* $\xi \in S$ *and* $t \in \boldsymbol{R}(0, \Delta)$, *then* $t\eta_0(\dot{\varphi}(t, \xi)) = f(\varphi(t, \xi))$.

(7′) *If* $\xi \in S$, *the curve* $\varphi(\square, \xi): \boldsymbol{R}(0, \Delta) \to M_*$ *is geodesic in respect to the Kaehler metric* $\varkappa$. (See Theorem 3.8, and (4.36)).

Define τ_0 as in (4.20).

THEOREM 4.6. *A homeomorphism* $h: \boldsymbol{C}^m(\Delta) \to M$ *is defined by* $h(0) = 0$ *and*

$$h(z) = \varphi\left(|z|, \frac{z}{|z|}\right) \quad \text{if } 0 \neq z \in \boldsymbol{C}^m(\Delta)\,.$$

Moreover $h: \boldsymbol{C}^m - \{0\} \to M_*$ *is a diffeomorphism. Moreover* $\tau_0 = \tau \circ h$ *on* $\boldsymbol{C}^m(\Delta)$.

PROOF. The map $\mathfrak{r}_2: \boldsymbol{R}(0, \Delta) \times S \to \boldsymbol{C}^m(\Delta)$ defined by $\mathfrak{r}_2(t, \xi) = t\xi$ is a diffeomorphism with $\mathfrak{r}_2^{-1}(z) = (|z|, z/|z|)$. Hence

$$h = \varphi \circ \mathfrak{r}_2^{-1}: \boldsymbol{C}^m(\Delta) - \{0\} \to M_*$$

is a diffeomorphism. If $0 \neq z \in \boldsymbol{C}^m$, define $\xi = z/|z|$. Then $\tau(h(z)) = \tau(\varphi(|z|, \xi)) = |z|^2 = \tau_0(z)$. Hence $h(z) \to 0$ for $z \to 0$ and $h^{-1}(p) \to 0$ for $p \to 0$. Therefore $h: \boldsymbol{C}^m \to M$ is a homeomorphism, q.e.d.

Later we shall show, that h is biholomorphic.

5. – The leaf space of a strictly parabolic exhaustion.

a) A parameterization of the leaf space.

Let (M, τ) be a strictly parabolic manifold of dimension m with maximal exhaustion radius Δ and with center $M[0] = \{0\}$. Then τ is a strictly parabolic function on M, which determines a foliation $\mathfrak{L} = \{L_p\}_{p \in M_*}$ with leaf space $\Lambda = \{L_p | p \in M_*\}$ on M_* as described in (3.45) to (3.47) and Propositions 3.5 to 3.9. We will show that the unit sphere S in $\boldsymbol{C}^m$ provides a natural parameter space for Λ. First some preparations are needed.

REMARK 1. *Let* $\varphi: \boldsymbol{R}(-\eta, \Delta) \times S \to M$ *be the flow of* τ *as defined in* (4.45). *Put* $\Delta_0 = \log \Delta$ *and* $I_0 = \boldsymbol{R}(-\infty, \Delta_0)$. *Define a diffeomorphism*

$$\chi: I_0 \times S \to M_* \quad \text{by } \chi(t, \xi) = \varphi(e^t, \xi)\,. \tag{5.1}$$

Then (1′)-(7′) *in section* 4*d*) *imply*

(1°) *If* $\xi \in S$ *and* $t \in I_0$, *then* $\eta_0(\dot{\chi}(t, \xi)) = f(\chi(t, \xi))$.

(2°) *If* $\xi \in S$ *and* $t \in I_0$, *then* $\tau(\chi(t, \xi)) = e^{2t}$.

(3°) *If* $t \in I_0$, *define* $\chi_t: S \to M_*$ *by* $\chi_t(\xi) = \chi(t, \xi)$. *Then* $\chi_t: S \to M\langle e^t \rangle$ *is a diffeomorphism.*

REMARK 2. Let U be an open subset of $\boldsymbol{C}$ with $a = \alpha + i\beta \in U$ where α and β are real. Let $\mathfrak{v}: U \to M$ be a holomorphic map. Define $b = \mathfrak{v}(a)$. The differential $d\mathfrak{v}(a)$ is a $\boldsymbol{C}$-linear map of $T_a^c(\boldsymbol{C})$ into $T_b^c(M)$ which maps $T_a(\boldsymbol{C})$, $\mathfrak{T}_a(\boldsymbol{C})$, $\overline{\mathfrak{T}}_a(\boldsymbol{C})$ into $T_b(M)$, $\mathfrak{T}_b(M)$, $\overline{\mathfrak{T}}_b(M)$ respectively. Denote

$$\mathfrak{v}'(a) = d\mathfrak{v}\left(a, \frac{\partial}{\partial z}(a)\right) \in \mathfrak{T}_b(M)$$

$$\mathfrak{v}_x(a) = d\mathfrak{v}\left(a, \frac{\partial}{\partial x}(a)\right) \in T_b(M)$$

$$\mathfrak{v}_y(a) = d\mathfrak{v}\left(a, \frac{\partial}{\partial y}(a)\right) \in T_b(M).$$

Then

$$\eta_0(\mathfrak{v}_x(a)) = \mathfrak{v}'(a) \qquad \eta_0(\mathfrak{v}_y(a)) = i\mathfrak{v}'(a)$$

$$\mathfrak{v}_y(a) = J_b\,\mathfrak{v}_x(a) \qquad \mathfrak{v}'(a) = d\mathfrak{v}\left(a, \frac{\partial}{\partial \bar{z}}(a)\right).$$

The sets $I(\beta) = \{t \in \boldsymbol{R} | t + i\beta \in U\}$ and $I[\alpha] = \{t \in \boldsymbol{R} | \alpha + it \in U\}$ are open with $\alpha \in I(\beta)$ and $\beta \in I[\alpha]$. Define $\zeta: I(\beta) \to M$ and $\lambda: I[\alpha] \to M$ by $\zeta(t) = \mathfrak{v}(t + i\beta)$ for $t \in I(\beta)$ and $\lambda(t) = \mathfrak{v}(\alpha + it)$ for $t \in I[\alpha]$. Then $\dot{\zeta}(\alpha) = \mathfrak{v}_x(a)$ and $\dot{\lambda}(\beta) = \mathfrak{v}_y(a)$. Therefore, we also write

$$\dot{\mathfrak{v}}(a) = \left(\frac{\partial}{\partial x}\mathfrak{v}\right)(a) = \mathfrak{v}_x(a)$$

$$J\dot{\mathfrak{v}}(a) = \left(\frac{\partial}{\partial y}\mathfrak{v}\right)(a) = \mathfrak{v}_y(a)\,.$$

LEMMA 5.1. *For each* $\xi \in S$, *there exists one and only one leaf* $L(\xi) \in \Lambda$ *of* $\mathfrak{L}$ *such that* $\chi(t, \xi) \in L(\xi)$ *for all* $t \in I_0$. *Moreover*, $\Lambda = \{L(\xi) | \xi \in S\}$.

PROOF. Take $r \in I_0$: For $\xi \in S$ define $L(\xi) = L_{\chi(r,\xi)}$. For each $L \in \Lambda$ define

$$I(L, \xi) = \{t \in I_0 | \chi(t, \xi) \in L\}\,.$$

If $s \in I(L, \xi)$, then $p = \chi(s, \xi) \in L$. Hence $L = L_p$. By Lemma 3.6 there exists a rectangle

$$Q(s, \xi) = \boldsymbol{R}(a_s, b_s) \times \boldsymbol{R}(-c_s, c_s) \tag{5.1}$$

with $a_s < s < b_s < \Delta_0$ and $c_s > 0$ and a biholomorphic map

$$w(\square, s, \xi): Q(s, \xi) \to U_s \tag{5.2}$$

onto an open subset U_s of L_p such that

$$w(s, s, \xi) = \chi(s, \xi) = p \tag{5.3}$$

$$w'(z, s, \xi) = f(w(z, s, \xi)) \qquad \forall z \in Q(s, \xi) . \tag{5.4}$$

Now, (5.4) gives

$$\eta_0(\dot{w}(t, s, \xi)) = f(w(t, s, \xi)) \qquad \forall t \in \mathbf{R}(a_s, b_s) . \tag{5.5}$$

Hence (5.3), (5.5) and (1°) imply

$$\chi(t, \xi) = w(t, s, \xi) \in L_p = L \qquad \forall t \in \mathbf{R}(a_s, b_s) . \tag{5.6}$$

Therefore $s \in \mathbf{R}(a_s, b_s) \subseteq I(L, \xi)$. The set $I(L, \xi)$ is open.

If $t \in I_0$, define $p = \chi(t, \xi)$. Then $p \in L_p \in \Lambda$. Hence $t \in I(L_p, \xi)$. If $t \in I(L, \xi) \cap I(L', \chi)$, then $L = L_p = L'$ with $p = \chi(t, \xi)$. The open connected interval I_0 is the disjoint union of the open set $I(L, \xi)$ with $L \in \Lambda$ where $r \in I(L(\xi), \xi)$. Hence $I_0 = I(L(\xi), \xi)$. Consequently $\chi(t, \xi) \in L(\xi)$ for all $t \in I_0$. If $\chi(t, \xi) \in L \in \Lambda$ for all $t \in I_0$, then $L = L(\xi)$ trivially. If $L \in \Lambda$, then $L = L_p$ for some $p \in M_*$. Then $s \in I_0$ and $\xi \in S$ exist such that $p = \chi(s, \xi)$. Then $p \in L(\xi) \cap L_p$. Hence $L(\xi) = L_p = L$; q.e.d.

By (3.23) we have $\partial\tau(f) = \bar{\partial}\tau(\bar{f}) = \tau$. Define $F = f + \bar{f}$ as in (3.45). Then

$$d\tau(JF) = d\tau(if - i\bar{f}) = i\,\partial\tau(f) - i\bar{\partial}\tau(\bar{f}) = i\tau - i\tau = 0 .$$

Take $t \in \mathbf{R}(0, \Delta)$. Then $M\langle t\rangle$ is a proper, compact submanifold of real dimension $2m - 1$ of M_*. Moreover $v \in T_p(M)$ belongs to $T_p(M\langle t\rangle)$ if and only if $d\tau(p, v) = 0$. Hence $JF(p) \in T_p(M\langle t\rangle)$ and $JF|M\langle t\rangle$ is a differentiable vector field on the compact manifold $M\langle t\rangle$. Therefore there exists one and only one parameter group

$$\sigma = \sigma(\square, t): \mathbf{R} \times M\langle t\rangle \to M\langle t\rangle$$

of diffeomorphisms such that

$$\sigma(0, p, t) = p \qquad \forall p \in M\langle t\rangle \tag{5.7}$$

$$\dot{\sigma}(y, p, t) = JF(\sigma(y, p, t)) \qquad \forall (y, p) \in \mathbf{R} \times M\langle t\rangle \tag{5.8}$$

$$\sigma(y_1 + y_2, p, t) = \sigma(y_1, \sigma(y_2, p, t), t) \tag{5.9}$$

for all $y_1 \in \boldsymbol{R}$, $y_2 \in \boldsymbol{R}$ and $p \in M\langle t\rangle$. If $y \in \boldsymbol{R}$, the map

$$\sigma(y, \square, t)\colon M\langle t\rangle \to M\langle t\rangle \tag{5.10}$$

is a diffeomorphism. Consider $D = I_0 \times \boldsymbol{R}$ as an open subset of $\boldsymbol{C}$. Define

$$\mathfrak{w}\colon D \times S \to M_* \tag{5.11}$$

by

$$\mathfrak{w}(x+iy, \xi) = \sigma\big(y, \chi(x, \xi), e^x\big)\ . \tag{5.12}$$

Lemma 5.2. *The map $\mathfrak{w}\colon D\times S \to M_*$ is of class C^∞.*

Proof. Clearly, $\mathfrak{w}$ is of class C^∞ on $\{x\}\times \boldsymbol{R}\times S$ for every fixed $x \in I_0$ but we have to prove more. Take $x_0 \in I_0$ and define $t_0 = e^{x_0}$. Then $t_0 \in \boldsymbol{R}(0, \Delta)$.

The real vector field JF defines a local one parameter group of diffeomorphisms at every $q \in M_*$. An open neighborhood W_q of q and a number $\varepsilon_q > 0$ and a differentiable map

$$\varrho_q\colon \boldsymbol{R}(-\varepsilon_q, \varepsilon_q) \times W_q \to M_*$$

exist such that $\varrho_q(0, p) = p$ and $\dot{\varrho}_q(y, p) = JF\big(\varrho_q(y, p)\big)$ for all $y \in \boldsymbol{R}(-\varepsilon_q, \varepsilon_q)$ and $p \in W_q$. Finitely many points $q_1, \ldots, q_n$ exist in $M\langle t_0\rangle$ such that

$$M\langle t_0\rangle \subset W_{q_1} \cup \ldots \cup W_{q_n} = W\ .$$

Define $\varepsilon = \operatorname{Min}(\varepsilon_1, \ldots, \varepsilon_q)$. Then $\varrho_{q_j}(y, p) = \varrho_{q_k}(y, p)$ if $|y| < \varepsilon$ and $p \in W_{q_j} \cap W_{q_k}$. Therefore a map

$$\varrho\colon \boldsymbol{R}(-\varepsilon, \varepsilon) \times W \to M_*$$

of class C^∞ exists such that $\varrho(y, p) = \varrho_{q_j}(y, p)$ if $|y| < \varepsilon$ and $p \in W_{q_j}$. In particular $\varrho(0, p) = p$ and $\dot{\varrho}(y, p) = JF\big(\varrho(y, p)\big)$ for all $y \in \boldsymbol{R}(-\varepsilon, \varepsilon)$ and $p \in W$.

Since $\chi(x_0, \xi) \in M\langle t_0\rangle \subset W$ for all $\xi \in S$. A number $\eta_0 > 0$ exists such that $x_0 + \eta < \Delta_0$ and such that $\chi(x, \xi) \in W$ if $|x - x_0| < \eta_0$ and $\xi \in S$. Then $M\langle e^x\rangle \subset W$ if $|x - x_0| < \eta_0$. Take $p \in M\langle e^x\rangle$. Then

$$\varrho(0, p) = p = \sigma(0, p, e^x)$$

$$\dot{\varrho}(y, p) = JF\big(\varrho(y, p)\big) \qquad \dot{\sigma}(y, p, e^x) = JF\big(\sigma(y, p, e^x)\big)$$

for all $y \in \boldsymbol{R}(-\varepsilon, \varepsilon)$. Therefore $\varrho(y, p) = \sigma(y, p, e^x)$ if $p \in M\langle e^x\rangle$ and $|y| < \varepsilon$ and $|x - x_0| < \eta_0$. If $\xi \in S$, if $|x - x_0| < \eta_0$ and $|y| < \varepsilon$, then $\chi(x, \xi) \in M\langle e^x\rangle$. Hence

$$\mathfrak{w}(x + iy, \xi) = \sigma(y, \chi(x, \xi), e^x) = \varrho(y, \chi(x, \xi)) .$$

Therefore $\mathfrak{w}$ is of class C^∞ on $\boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times \boldsymbol{R}(-\varepsilon, \varepsilon) \times S$. There exists a maximal r with $\varepsilon \leqslant r \leqslant +\infty$, such that $\mathfrak{w}$ is of class C^∞ on

$$\boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times \boldsymbol{R}(-r, r) \times S .$$

Assume that $r < \infty$: Take $s \in \boldsymbol{R}(0, r)$ with $0 < r - s < \varepsilon$. The maps

$$\begin{aligned}
&\mathfrak{w}_1 \colon \boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times S \to W && \text{with } \mathfrak{w}_1(x, \xi) = \mathfrak{w}(x + is, \xi)\\
&\mathfrak{w}_2 \colon \boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times S \to W && \text{with } \mathfrak{w}_2(x, \xi) = \mathfrak{w}(x - is, \xi)\\
&\varrho_1 \colon \boldsymbol{R}(-\varepsilon + s, s + \varepsilon) \times W \to M_* && \text{with } \varrho_1(y, p) = \varrho(y - s, p)\\
&\varrho_2 \colon \boldsymbol{R}(-\varepsilon - s, -s + \varepsilon) \times W \to M_* && \text{with } \varrho_2(y, p) = \varrho(y + s, p)
\end{aligned}$$

are of class C^∞. If $|x - x_0| < \eta_0$ and $\xi \in S$ and $|y - s| < \varepsilon$, then

$$\begin{aligned}
\mathfrak{w}(x + iy, \xi) &= \sigma(y, \chi(x, \xi), e^x) = \sigma(y - s, \sigma(s, \chi(x, \xi), e^x), e^x)\\
&= \varrho(y - s, \mathfrak{w}(x + is, \xi)) = \varrho_1(y, \mathfrak{w}_1(x, \xi)) .
\end{aligned}$$

If $|x - x_0| < \eta$ and $\xi \in S$ and $|y + s| < \varepsilon$, then

$$\begin{aligned}
\mathfrak{w}(x + iy, \xi) &= \sigma(y, \chi(x, \xi), e^x) = \sigma(y + s, \sigma(-s, \chi(x, \xi), e^x), e^x)\\
&= \varrho(y + s, \mathfrak{w}(x - is, \xi)) = \varrho_2(y, \mathfrak{w}_2(x, \xi)) .
\end{aligned}$$

Thus $\mathfrak{w}$ is of class C^∞ on

$$\boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times \boldsymbol{R}(-s - \varepsilon, s + \varepsilon) \times S .$$

The maximality of r implies $s + \varepsilon \leqslant r$ which contradicts $r - s < \varepsilon$. Therefore $r = \infty$. Hence $\mathfrak{w}$ is of class C^∞ on $\boldsymbol{R}(-\eta_0 + x_0, x_0 + \eta_0) \times \boldsymbol{R} \times S$ for each $x_0 \in I_0$. Consequently $\mathfrak{w}$ is of class C^∞ on $D \times S$; q.e.d.

LEMMA 5.3. *For every $\xi \in S$ there exists an open neighborhood U_ξ of I_0 in $\boldsymbol{C}$ of the form*

$$U_\xi = \{x + iy \in \boldsymbol{C} \mid x \in I_0 \text{ and } -c(x, \xi) < y < c(x, \xi)\} \subseteq D$$

where $0 < c(x, \xi) \leqslant +\infty$ *for all* $x \in I_0$ *such that there exists a locally biholomorphic map* $w(\square, \xi)\colon U_\xi \to L(\xi)$ *such that*

$$w(t, \xi) = \chi(t, \xi) \qquad \forall t \in I_0$$
$$w'(z, \xi) = f(w(z, \xi)) \qquad \forall z \in U_\xi .$$

If U_ξ *is given, then* $w(\square, \xi)$ *is unique.*

PROOF. Take $\xi \in S$. Take $s \in I_0$. Then $\chi(s, \xi) \in L(\xi)$ by Lemma 5.1. Construct the rectangle $Q(s, \xi)$ of (5.1) and the map $w(\square, s, \xi)$ of (5.2) and (5.3) to (5.6) where $U_s \subseteq L(\xi)$. For $x \in I_0$, define $c(x, \xi) = \sup \{c_s | x \in \boldsymbol{R}(a_s, b_s)$ and $s \in I_0\}$. Then

$$U_\xi = \bigcup_{s \in I_0} Q(s, \xi) = \{x + iy | x \in I_0 \text{ and } |y| < c(x, \xi)\} \subseteq D$$

is an open, connected neighborhood of I_0 in $\boldsymbol{C}$.

Take $s_1 \in I_0$ and $s_2 \in I_0$ with $Q(s_1, \xi) \cap Q(s_2, \xi) \neq \emptyset$, which is convex. Then $\boldsymbol{R}(a_{s_1}, b_{s_1}) \cap \boldsymbol{R}(a_{s_2}, b_{s_2}) \neq \emptyset$ with

$$w(t, s_1, \xi) = \chi(t, \xi) = w(t, s_2, \xi)$$

for all $t \in \boldsymbol{R}(a_{s_1}, b_{s_1}) \cap \boldsymbol{R}(a_{s_2}, b_{s_2})$ by (5.6). Analytic continuation shows that

$$w(z, s_1, \xi) = w(z, s_2, \xi) \qquad \forall z \in Q(s_1, \xi) \cap Q(s_2, \xi) .$$

Hence a locally biholomorphic map $w(\square, \xi)\colon U_\xi \to L(\xi)$ is defined by $w(z, \xi) = w(z, s, \xi)$ for all $z \in Q(s, \xi)$ and $s \in I_0$. If $t \in I_0$, then $w(t, \xi) = w(t, t, \xi) = \chi(t, \xi)$. If $z \in U_\xi$, then $z \in Q(s, \xi)$ for some $s \in I_0$. Then

$$w'(z, \xi) = w'(z, s, \xi) = f(w(z, s, \xi)) = f(w(z, \xi)) \qquad \text{q.e.d.} .$$

LEMMA 5.4. *If* $z \in U_\xi$, *then* $\mathfrak{w}(z, \xi) = w(z, \xi)$.

PROOF. Take $\xi \in S$ and $x \in I_0$. Then

$$\mathfrak{w}(x, \xi) = \sigma(0, \chi(x, \xi), e^x) = \chi(x, \xi) = w(x, \xi) .$$

If $|y| < c(x, \xi)$, then we have

$$\begin{aligned} \mathfrak{w}_y(x + iy, \xi) = \dot\sigma(y, \chi(x, \xi), e^x) &= JF(\sigma(y, \chi(x, \xi), e^x)) \\ &= JF(\mathfrak{w}(x + iy, \xi)) \end{aligned}$$

$$\eta_0(w_y(x+iy,\xi)) = iw'(x+iy,\xi) = if(w(x+iy,\xi))$$
$$= \eta_0(JF(w(x+iy,\xi))) .$$

Since η_0 is injective, we obtain $w_y(x+iy,\xi) = JF(w(x+iy,\xi))$. Therefore $\mathfrak{w}(x+iy,\xi) = w(x+iy,\xi)$ for all $y \in \boldsymbol{R}(-c(x,\xi), c(x,\xi))$; q.e.d.

Consequently $\mathfrak{w}(\square,\xi)$ is locally biholomorphic on U_ξ and maps U_ξ into $L(\xi)$.

LEMMA 5.5. *If $\xi \in S$ and $z \in D$, then $\mathfrak{w}(z,\xi) \in L(\xi)$.*

PROOF. Take $\xi \in S$ and $x \in I_0$ and keep fixed. For each leaf $L \in \Lambda$ define

$$I(L) = \{y \in \boldsymbol{R} | \mathfrak{w}(x+iy,\xi) \in L\} .$$

Take $b \in I(L)$. Then $p = \mathfrak{w}(x+ib,\xi) \in L$. The vector field JF restricts to a vector field $JF|L$ along L. Hence there exists a curve $\gamma\colon \boldsymbol{R}(\alpha,\beta) \to L$ of class C^∞ with $\alpha < b < \beta$ such that $\gamma(b) = p$ and $\dot\gamma(y) = JF(\gamma(y))$ for all $y \in \boldsymbol{R}(\alpha,\beta)$. As before $\mathfrak{w}_y(x+iy,\xi) = JF(\mathfrak{w}(x+iy,\xi)$ for all $y \in \boldsymbol{R}$. Therefore $\mathfrak{w}(x+iy,\xi) = \gamma(y) \in L$ for all $y \in \boldsymbol{R}(\alpha,\beta)$. Hence $\boldsymbol{R}(\alpha,\beta) \subseteq I(L)$. The set $I(L)$ is open in $\boldsymbol{R}$.

If $y \in \boldsymbol{R}$, then $p = \mathfrak{w}(x+iy,\xi) \in M_*$. Hence $\mathfrak{w}(x+iy,\xi) \in L_p$ and $y \in I(L_p)$. If $y \in I(L) \cap I(L')$, then $\mathfrak{w}(x+iy,\xi) \in L \cap L'$. Therefore $L = L'$. Since $\mathfrak{w}(x,\xi) = \chi(x,\xi) \in L(\xi)$, we have $0 \in I(L(\xi))$. Therefore $\boldsymbol{R}$ is the disjoint union of the open sets $I(L)$ with $L \in \Lambda$ where $I(L(\xi)) \neq 0$. Consequently, $I(L(\xi)) = \boldsymbol{R}$ and $\mathfrak{w}(x+iy,\xi) \in L(\xi)$ for all $y \in \boldsymbol{R}$; q.e.d.

LEMMA 5.6. *If $x \in I_0$ and $y \in \boldsymbol{R}$ and $\xi \in S$, then*

$$\log \tau(\mathfrak{w}(x+iy,\xi)) = 2x . \tag{5.13}$$

PROOF. By (5.12) and (5.10) we have $\mathfrak{w}(x+iy,\xi) \in M\langle e^x\rangle$, which implies (5.13); q.e.d.

LEMMA 5.7. *For each $\xi \in S$, the map $\mathfrak{w}(\square,\xi)\colon D \to L(\xi)$ is surjective.*

PROOF. Take $\xi \in S$ and $x \in I_0$. Define $L(\xi,x) = L(\xi) \cap M\langle e^x\rangle$. Then $L(\xi,x)$ is a closed subset of $L(\xi)$. Moreover $p \in L(\xi)$ belongs to $L(\xi,x)$ if and only if $\tau(p) = e^x$. Let $j\colon L(\xi) \to M_*$ be the inclusion map. Take $p \in L(\xi,x)$. Then $F(p) \in T_p(L(\xi))$ and

$$d(\tau\circ j)(p,F(p)) = d\tau(j(p), d_j(p,F(p))) = d\tau(p,F(p)) =$$
$$= \partial\tau(p,f(p)) + \bar\partial\tau(p,\bar f(p)) = 2\tau(p) = 2e^{2x} \neq 0 .$$

Hence $L(\xi, x)$ is a proper, differentiable submanifold of dimension 1 of $L(\xi)$.

We claim that $L(\xi, x)$ is connected. Define $q = \chi(x, \xi) \in L(\xi, x)$. Pick any point $p \in L(\xi, x)$. Because $L(\xi)$ is a connected manifold there exists a continuous curve $\gamma: \boldsymbol{R}[0, 1] \to L(\xi)$ with $\gamma(0) = q$ and $\gamma(1) = p$. The map $\chi: I_0 \times S \to M_*$ is a diffeomorphism. Therefore

$$\beta = (\beta_1, \beta_2) = \chi^{-1} \circ \gamma: \boldsymbol{R}[0, 1] \to I_0 \times S$$

is a continuous curve in $I_0 \times S$ with $\beta(0) = (x, \xi)$. Hence $\beta_1(0) = x$ and $\beta_2(0) = \xi$. Also $\chi(\beta_1(1), \beta_2(1)) = \gamma(1) = p \in M\langle e^x\rangle$. Hence $\beta_1(1) = x$. Abbreviate $Q = \boldsymbol{R}[0, 1] \times \boldsymbol{R}[0, 1]$. A continuous map $B: Q \to M^*$ is defined by

$$B(y, \lambda) = \chi((1 - \lambda)\beta_1(y) + \lambda x, \beta_2(y)) .$$

Keep $y \in \boldsymbol{R}[0, 1]$ fixed. Then $B(y, \lambda) \in L(\beta_2(y))$ for all $\lambda \in \boldsymbol{R}[0, 1]$. If $\lambda = 0$, then

$$B(y, 0) = \chi(\beta_1(y), \beta_2(y)) = \gamma(y) \in L(\xi) \cap L(\beta_2(y)) .$$

Therefore $L(\beta_2(y)) = L(\xi)$. Hence B maps into $L(\xi)$. Define a continuous curve $\alpha: \boldsymbol{R}[0, 1] \to L(\xi)$ by $\alpha(y) = B(y, 1)$ for all $y \in [0, 1]$. Then

$$\alpha(y) = \chi(x, \beta_2(y)) \in M\langle e^x\rangle \cap L(\xi) = L(\xi, x)$$

with $\alpha(0) = \chi(x, \xi) = q$ and $\alpha(1) = \chi(\beta_1(1), \beta_2(1)) = \gamma(1) = p$. Consequently $L(\xi, x)$ is connected.

Since $L(\xi, x)$ is a connected, differentiable submanifold of dimension 1, there exists a surjective map $\mu: \boldsymbol{R}(-1, 1) \to L(\xi, x)$ of class C^∞ with $\mu(0) = q$ and $0 \neq \dot{\mu}(t) \in T_{\mu(t)}(L(\xi))$. Because F and JF are a frame of the bundle $T(L(\xi))$, there exist real differentiable functions a and b on $\boldsymbol{R}(-1, 1)$ such that

$$\dot{\mu}(t) = a(t)F(\mu(t)) + b(t)JF(\mu(t))$$

for all $t \in \boldsymbol{R}(-1, 1)$. By replacing t by $-t$, if needed, we can assume w.l.o.g. that $b(0) \geqslant 0$. Since $\tau \circ \mu = e^{2x}$ is constant on $\boldsymbol{R}(-1, 1)$ we have

$$0 = d\tau(\mu(t), \dot{\mu}(t)) = a(t) 2\tau(\mu(t)) = a(t) 2e^{2x} .$$

Therefore $a(t) = 0$ and $b(t) \neq 0$ for all $t \in \boldsymbol{R}(-1, 1)$. Hence $b(t) > 0$ for all $t \in \boldsymbol{R}(-1, 1)$. A function $v: \boldsymbol{R}(-1, 1) \to \boldsymbol{R}$ of class C^∞ exists such that $v(0) = 0$ and $v'(t) = b(t) > 0$. Then real numbers v_0 and v_1 with $v_0 < 0 < v_1$ exist such that $v: \boldsymbol{R}(-1, 1) \to \boldsymbol{R}(v_0, v_1)$ is a diffeomorphism. Define $u = v^{-1}$.

Then $\varrho = \mu \circ u \colon \boldsymbol{R}(v_0, v_1) \to L(x, \xi)$ is a surjective map of class C^∞ with $\varrho(0) = \mu(0) = q$.

$$\dot{\varrho}(y) = \dot{\mu}\big(u(y)\big)\,\dot{u}(y) = b\big(u(y)\big)\,\dot{u}(y)\,JF\big(\mu(u(y))\big) = JF\big(\varrho(y)\big)$$

for all $y \in \boldsymbol{R}(v_0, v_1)$. Consequently, (5.7) and (5.8) imply

$$\varrho(y) = \sigma(y, q, e^x) = \sigma\big(y, \chi(x, \xi), e^x\big) = \mathfrak{w}(x + iy, \xi)\,.$$

For all $y \in \boldsymbol{R}(v_0, v_1)$. Since ϱ is surjective, we have $L(\xi, x) \subseteq \mathfrak{w}(D, \xi)$ for all $x \in I_0$. Hence $L(\xi) \subseteq \mathfrak{w}(D, \xi) \subseteq L(\xi)$; q.e.d.

LEMMA 5.8. *For each $\xi \in S$, the map $\mathfrak{w}(\square, \xi) \colon D \to L(\xi)$ is locally biholomorphic. If $z \in L(\xi)$, then $\mathfrak{w}'(z, \xi) = f\big(\mathfrak{w}(z, \xi)\big)$.*

PROOF. According to Proposition 3.5, the function $u = \log \tau | L(\xi)$ is harmonic on $L(\xi)$. Hence $\lambda = \partial u$ is a holomorphic form on $L(\xi)$. Also $f | L(\xi)$ is a holomorphic vector field on $L(\xi)$. Hence a holomorphic function $\Phi \colon L(\xi) \to \boldsymbol{C}$ is defined by $\Phi(p) = \lambda\big(p, f(p)\big)$. For all $x \in I_0$ and $y \in \boldsymbol{R}$ Lemma 5.6 implies $u\big(\mathfrak{w}(x + iy, \xi)\big) = 2x$. Abbreviate $p = \mathfrak{w}(x + iy, \xi)$. Then

$$du\big(p, \mathfrak{w}_x(x + iy, \xi)\big) = 2 \qquad du\big(p, \mathfrak{w}_y(x + iy, \xi)\big) = 0\,.$$

By Lemma 5.3 and Lemma 5.4 $\mathfrak{w}(\square, \xi)$ is locally biholomorphic on U_ξ: Hence $V_\xi = \mathfrak{w}(U_\xi, \xi)$ is open in $L(\xi)$. If $z = x + iy \in U_\xi$, then

$$\partial u\big(p, \mathfrak{w}'(z, \xi)\big) + \bar{\partial} u\big(p, \overline{\mathfrak{w}}'(z, \xi)\big)_s = 2$$

$$i\,\partial u\big(p, \mathfrak{w}'(z, \xi)\big) - i\,\bar{\partial} u\big(p, \overline{\mathfrak{w}}'(z, \xi)\big) = 0$$

where $\mathfrak{w}'(z, \xi) = f(p)$ by Lemma 5.3. Hence $\Phi(p) = \partial u\big(p, f(p)\big) = 1$ for all $p \in V_\xi$. By analytic continuation $\Phi = 1$ on $L(\xi)$.

Let $\pi \colon N \to L(\xi)$ be the universal covering space of $L(\xi)$. Take $x_0 \in I_0$ and define $p_0 = \chi(x_0, \xi) \in L(\xi)$. Pick $q_0 \in N$ with $\pi(q_0) = p_0$. Because π is locally biholomorphic, the form $\pi^*(\lambda)$ is holomorphic on N and nowhere zero. Hence one and only one locally biholomorphic function $H \colon N \to \boldsymbol{C}$ exists such that $dH = \pi^*(\lambda)$ and $H(q_0) = x_0$. Observe that $\mathfrak{w}(x_0, \xi) = \chi(x_0, \xi) = p_0$ and that D is simple connected. Hence one and only one differentiable map $W \colon D \to N$ exists such that $W(x_0) = q_0$ and $\pi\big(W(z)\big) = \mathfrak{w}(z, \xi)$ for all $z \in D$. For $x \in I_0$ abbreviate $W(x) = q$ and $\chi(x, \xi) =$

$= \mathfrak{w}(x, \xi) = p$. Then $\pi(q) = p$ and

$$\begin{aligned}\frac{\partial}{\partial x} H(W(x)) &= dH(q, W_x(x)) = \pi^*(\lambda)(q, W_x(x)) \\ &= \lambda(p, d\pi(q, W_x(x))) = \lambda(p, \mathfrak{w}_x(x, \xi)) \\ &= \lambda(p, \dot\chi(x, \xi)) = \lambda(p, \eta_0(\dot\chi(x, \xi))) \\ &= \lambda(p, f(\chi(x, \xi))) = \lambda(p, f(p)) = 1 .\end{aligned}$$

Since $H(W(x_0)) = H(q_0) = x_0$, we obtain $H(W(x)) = x$ for all $x \in I_0$.

Fix $x \in I_0$. For any $y \in \boldsymbol{R}$, denote $p = \mathfrak{w}(x + iy, \xi)$ and $q = W(x + iy)$. Then (5.12) and (5.8) imply $\mathfrak{w}_y(x + iy, \xi) = JF(p)$. Therefore

$$\begin{aligned}\frac{\partial}{\partial y} H(W(x + iy)) &= dH(q, W_y(x + iy)) = \pi^*(\lambda)(q, W_y(x + iy)) \\ &= \lambda(p, d\pi(p, W_y(x + iy))) = \lambda(p, \mathfrak{w}_y(x + iy, \xi)) \\ &= \lambda(p, JF(p)) = \lambda(p, \eta_0(JF(p))) \\ &= \lambda(p, if(p)) = i\lambda(p, f(p)) = i .\end{aligned}$$

Therefore $H(W(x + iy)) = H(W(x)) + iy = x + iy$. The map $H \circ W: D \to D$ is the identity.

Take $a \in D$. Define $b = W(a)$ and $c \in \pi(b)$. Open connected neighborhoods U_a of a in D and U_b of b in N and U_c of c in $L(\xi)$ exist such that $W(U_a) \subseteq U_b$ and such that $H: U_b \to U_a$ and $\pi: U_b \to U_c$ are biholomorphic. Then $W|U_a = (H|U_b)^{-1}$ maps U_a biholomorphically onto U_b and $\mathfrak{w}(\square, \xi)|U_a = \pi \circ W|U_a$ maps U_a biholomorphically onto U_c. The map $\mathfrak{w}(\square, \xi): D \to L_\xi$ is locally biholomorphic.

Also we have

$$\begin{aligned}\mathfrak{w}'(x + iy, \xi) &= - i\eta_0(\mathfrak{w}_y(x + iy, \xi)) = - i\eta_0(JF(\mathfrak{w}(x + iy, \xi))) \\ &= f(\mathfrak{w}(x + iy, \xi))\end{aligned}$$

for all $x + iy \in D$; q.e.d.

Below we will see that $\mathfrak{w}(\square, \xi): D \to L(\xi)$ is the universal covering space of $L(\xi)$. Hence $W: D \to N$ is biholomorphic. Take $x_0 \in I_0$ and keep x_0 fixed. According to (3°) the map

$$\chi_{x_0} = \chi(x_0, \square): S \to M\langle e^{x_0}\rangle \tag{5.14}$$

is a diffeomorphism. Since $\mathfrak{w}(x_0 + iy, \xi) \in M\langle e^{x_0}\rangle$ for all $y \in \boldsymbol{R}$ and $\xi \in S$, a map

$$\zeta : \boldsymbol{R} \times S \to S \tag{5.15}$$

of class C^∞ is defined by

$$\zeta(y, \xi) = \chi_{x_0}^{-1}(\mathfrak{w}(x_0 + iy, \xi)) \qquad \forall (y, \xi) \in \boldsymbol{R} \times S . \tag{5.16}$$

LEMMA 5.9. *If $x \in I_0$ and $y \in \boldsymbol{R}$ and $\xi \in S$, then*

$$\mathfrak{w}(x + iy, \xi) = \chi(x, \zeta(y, \xi)) .$$

PROOF. Take $y \in \boldsymbol{R}$ and $\xi \in S$ and keep fixed. For all $x \in I_0$ Lemma 5.8 implies

$$\eta_0(\mathfrak{w}_x(x + iy, \xi)) = \mathfrak{w}'(x + iy, \xi) = f(\mathfrak{w}(x + iy, \xi)) = \eta_0(F(\mathfrak{w}(x + iy, \xi))) .$$

Since η_0 is injective, we obtain

$$\mathfrak{w}_x(x + iy, \xi) = F(\mathfrak{w}(x + iy, \xi))$$
$$\dot{\chi}(x, \zeta(y, \xi)) = F(\chi(x, \zeta(y, \xi)))$$
$$\chi(x_0, \zeta(y, \xi)) = \chi_{x_0}(\zeta(y, \xi)) = \mathfrak{w}(x_0 + iy, \xi) .$$

Therefore $\mathfrak{w}(x + iy, \xi) = \chi(x, \zeta(y, \xi))$; q.e.d.

LEMMA 5.10. *For each $\xi \in S$, the map $\mathfrak{w}(\square, \xi) : D \to L(\xi)$ is the universal covering space of $L(\xi)$.*

PROOF. Take $\xi \in S$ and keep fixed. We distinguish two cases.

1. *Case*: The map $\zeta(\square, \xi) : \boldsymbol{R} \to S$ is injective. We claim that $\mathfrak{w}(\square, \xi)$ is injective. Assume that $x_j \in I_0$ and $y_j \in \boldsymbol{R}$ for $j = 1, 2$ are given with $\mathfrak{w}(x_1 + iy_1, \xi) = \mathfrak{w}(x_2 + iy_2, \xi) = p$. Then $x_1 = \frac{1}{2} \log \tau(p) = x_2$. Also

$$\chi(x_1, \zeta(y_1, \xi)) = p = \chi(x_1, \zeta(y_2, \xi)) .$$

Because χ is injective, we have $\zeta(y_1, \xi) = \zeta(y_2, \xi)$. Hence $y_1 = y_2$; therefore $\mathfrak{w}(\square, \xi)$ is injective. Now Lemma 5.7 and Lemma 5.8 imply that $\mathfrak{w}(\square, \xi) : D \to L(\xi)$ is biholomorphic, which proves the Lemma in the 1. Case.

2. *Case*: The map $\zeta(\square, \xi) : \boldsymbol{R} \to S$ is not injective. We claim that $\mathfrak{w}(\square, \xi)$ is periodic in the direction of the imaginary axis. Define $p_0 =$

$= \chi(x_0, \xi)$. Then $\mathfrak{w}(x_0 + iy) = \sigma(y, p_0, e^{x_0})$. Abbreviate

$$\sigma(y, p) = \sigma(y, p, e^{x_0}) \qquad \forall (y, p) \in \boldsymbol{R} \times M\langle e^{x_0}\rangle .$$

By assumption $y_1 \in \boldsymbol{R}$ and $y_2 \in \boldsymbol{R}$ with $y_1 > y_2$ exist such that $\zeta(y_1, \xi) = \zeta(y_2, \xi)$. Hence $\mathfrak{w}(x_0 + iy_1, \xi) = \mathfrak{w}(x_0 + iy_2, \xi)$ which means $\sigma(y_1, p_0) = \sigma(y_2, p_0)$. Therefore

$$\begin{aligned}\sigma(y_1 - y_2, p_0) &= \sigma(-y_2, \sigma(y_1, p_0)) \\ &= \sigma(-y_2, \sigma(y_2, p_0)) = \sigma(0, p_0) = p_0\end{aligned}$$

with $y_1 - y_2 > 0$. Define $\alpha = \inf\{y \in \boldsymbol{R}^+ | \sigma(y, p_0) = p_0\}$. Since $\dot\sigma(0, p_0) = JF(p_0) \neq 0$, we have $\alpha > 0$. By continuity $\sigma(\alpha, p_0) = p_0$. If $y \in \boldsymbol{R}$, then

$$\begin{aligned}\sigma(y + \alpha, p_0) &= \sigma(y, \sigma(\alpha, p_0)) = \sigma(y, p_0) \\ \zeta(y + \alpha, \xi) &= \chi_{x_0}^{-1}(\sigma(y + \alpha, p_0)) = \chi_{x_0}^{-1}(\sigma(y, p_0)) = \zeta(y, \xi) \\ \mathfrak{w}(x + i(y + \alpha), \xi) &= \chi(x, \zeta(y + \alpha, \xi)) \\ &= \chi(x, \zeta(y, \xi)) = \mathfrak{w}(x + iy, \xi)\end{aligned}$$

for all $x \in I_0$ and $y \in \boldsymbol{R}$. Therefore if $z \in D$ and $n \in \boldsymbol{Z}$, then

$$\mathfrak{w}(z + in\alpha, \xi) = \mathfrak{w}(z, \xi) .$$

Assume that $x_j \in I_0$ and $y_j \in \boldsymbol{R}$ are given for $j = 1, 2$ such that $\mathfrak{w}(x_1 + iy_1, \xi) = \mathfrak{w}(x_2 + iy_2, \xi) = p$. Then $x_1 = \frac{1}{2} \log \tau(p) = x_2$ and

$$\chi(x_1, \zeta(y_1, \xi)) = \chi(x_1, \zeta(y_2, \xi)) .$$

Hence $\zeta(y_1, \xi) = \zeta(y_2, \xi)$ which implies

$$\sigma(y_1, p_0) = \sigma(y_2, p_0) \qquad \text{or} \qquad \sigma(y_1 - y_2, p_0) = p_0 .$$

An integer $n \in \boldsymbol{Z}$ and $r \in \boldsymbol{R}[0, 1)$ exist such that $y_1 - y_2 = n\alpha + r\alpha$. Then $p_0 = \sigma(n\alpha + r\alpha, p_0) = \sigma(r\alpha, p_0)$. The definition of α implies $r = 0$. Hence $x_1 + iy_1 = x_2 + iy_2 + in\alpha$. Let $D_\alpha = D/i\alpha\boldsymbol{Z}$ be the quotient space. The residual map $\alpha: D \to D_\alpha$ is locally biholomorphic and in fact, the universal covering space of D_α. Also $\mathfrak{w}(\square, \xi): D \to L(\xi)$ factors to a biholomorphic map $\mathfrak{w}_0: D_\alpha \to L(\xi)$ such that $\pi \circ \mathfrak{w}_0 = \mathfrak{w}(\square, \xi)$. Hence $\mathfrak{w}(\square, \xi): D \to L(\xi)$ is the universal covering of $L(\xi)$; q.e.d.

LEMMA 5.11. *If* $\xi \in S$, *then* $\zeta(0, \xi) = \xi$.

PROOF. We have $\zeta(0, \xi) = \chi_{x_0}^{-1}(\mathfrak{w}(x_0, \xi)) = \chi_{x_0}^{-1}(\chi_{x_0}(\xi)) = \xi$; q.e.d.

LEMMA 5.12. *If* $\xi \in S$ *and* $\xi_1 \in S$, *then* $L(\xi) = L(\xi_1)$ *if and only if there exists* $y \in \boldsymbol{R}$ *such that* $\xi_1 = \zeta(y, \xi)$.

PROOF. If $L(\xi) = L(\xi_1)$, then $\chi(x_0, \xi_1) \in L(\xi)$. Hence $x \in I_0$ and $y \in \boldsymbol{R}$ exist such that $\chi(x_0, \xi_1) = \mathfrak{w}(x + iy, \xi) = p$. Then $x_0 = \frac{1}{2}\log \tau(p) = x$. Hence (5.14) and (5.16) imply $\xi_1 = \zeta(y, \xi)$. If $\xi_1 = \zeta(y, \xi)$ for some $y \in \boldsymbol{R}$, then $\chi(x_0, \xi_1) = \mathfrak{w}(x_0 + iy, \xi) = q$ with $q \in L(\xi) \cap L(\xi_1)$. Hence $L(\xi) = L(\xi_1)$; q.e.d.

b) The determination of the leaf space.

In section *a*) the parameterization $\mathfrak{w}$ of the foliation associated to τ was constructed and a parameterization of the leaf space Λ by S was provided. The orbit of the curves $\zeta(\square, \xi)\colon \boldsymbol{R} \to S$ determines exactly one leaf. Hence the orbit space is the leaf space. Local consideration at the center, *i.e.* the limit $x \to -\infty$, will enable us to determine these orbits. In order to carry out these asymptotic considerations at the center, we revert to the special coordinates introduced in section 4*c*). In particular we identify $U_{\mathfrak{z}} = U'_{\mathfrak{z}}$ such that $\mathfrak{z}$ is the identity. For the tangent spaces at points of $U_{\mathfrak{z}}$ we use also the identifications explained in (4.8)-(4.16). In particular, the unit sphere S is considered as the unit sphere in $T_0(M) = \mathfrak{T}_0(M) = \boldsymbol{C}^m$. Hence $\xi \in S$ can be considered as a real tangent vector at 0 and as a tangent vector of type (1, 0) at 0 with $\eta_0(\xi) = \xi$ as need may be. Also let (N, ϱ) be the double of M at 0. As in section 4*c*) $(N, \varrho)|U_{\mathfrak{z}}$ is identified with $(\boldsymbol{R} \times S, \mathfrak{r})|U'_{\mathfrak{z}}$ and $(N, \varrho)|B$ with $(\boldsymbol{R}(-t_0, t_0) \times S, \mathfrak{r})$. Recall also the maps ψ of Theorem 4.5 and φ of (4.45). It is important, not to mix up the parameter t in $\psi(t, \xi)$ and $\varphi(t, \xi)$ with the parameter t in $\mathfrak{r}(t, \xi)$. The notation $\boldsymbol{C}^m(t)$ refers to the exhaustion τ_0 of $\boldsymbol{C}^m$ defined by (4.20) where upon $M(t)$ refers to the strictly parabolic exhaustion τ given on M even if $M[t] \subset U_{\mathfrak{z}}$ which is the case for $0 \leqslant t < r_0$ (see (4.19)). Similarly the notations $N[a, b]$ etc. are determined by δ. Observe $\varrho\colon N\langle t\rangle \to M\langle t\rangle$ for all $t \in \boldsymbol{R}[0, \Delta)$. Because of Theorem 4.5 (4), ψ maps $\boldsymbol{R}[0, r_0] \times S$ into $N[0, r_0]$ with $\varrho(N[0, r_0]) = M[r_0] \subset B$. Also we can take $\eta > 0$ so small that $\varrho(\psi(t, \xi)) \in B$ if $-\eta < t < 0$. In view of our identification we have

$$\psi(t, \xi) \in \boldsymbol{R}(-t_0, t_0) \times S \quad \text{if } -\eta < t \leqslant r_0 \tag{5.17}$$

and $\xi \in S$. Therefore these are functions

$$\psi_1 \colon \boldsymbol{R}(-\eta, r_0) \times S \to \boldsymbol{R}(-t_0, t_0) \tag{5.18}$$

$$\psi_2 \colon \boldsymbol{R}(-\eta, r_0) \times S \to S \tag{5.19}$$

of class C^∞ such that $\psi = (\psi_1, \psi_2)$ on $\boldsymbol{R}(-\eta, r_0)$. We have $\psi_1(t, \xi) > 0$ if $0 < t < r_0$ and $\psi_1(t, \xi) < 0$ if $-\eta < t < r_0$ and $\psi_1(0, \xi) = 0$ for all $\xi \in S$. Also the map $\varphi = \varrho \circ \psi$ of (4.45) is given by

$$\varphi(t, \xi) = \psi_1(t, \xi)\psi_2(t, \xi) \tag{5.20}$$

for all $t \in \boldsymbol{R}(-\eta, r_0)$ and $\xi \in S$.

LEMMA 5.13. *There exist functions*

$$\psi_3 \colon \boldsymbol{R}(-\eta, r_0) \times S \to \boldsymbol{R} \quad \text{and} \quad \psi_4 \colon \boldsymbol{R}(-\eta, r_0) \times S \to \boldsymbol{C}^m$$

of class C^∞ such that

$$\psi_1(t, \xi) = t + t^2 \psi_3(t, \xi) \tag{5.21}$$

$$\psi_2(t, \xi) = \xi + t\psi_4(t, \xi) \tag{5.22}$$

for all $t \in \boldsymbol{R}(-\eta, r_0)$ and $\xi \in S$.

PROOF. Recall that we identified $S = S_0 = \{0\} \times S$. By Theorem (4.5) we have $\psi(0, \xi) = \xi$ which reads $\psi(0, \xi) = (0, \xi)$ with $\psi_1(0, \xi) = 0$ and $\psi_2(0, \xi) = \xi$ for all $\xi \in S$. Therefore there are functions $\psi_4 \colon \boldsymbol{R}(-\eta, r_0) \times S \to C^m$ and $\psi_5 \colon \boldsymbol{R}(-\eta, r_0) \times S \to \boldsymbol{R}$ of class C^∞ such that

$$\psi_1(t, \xi) = t\psi_5(t, \xi) \qquad \psi_2(t, \xi) = \xi + t\psi_4(t, \xi)$$

for all $t \in \boldsymbol{R}(-\eta, r_0)$ and $\xi \in S$ with $\dot{\psi}_1(0, \xi) = \psi_5(0, \xi)$. With our identifications we have $\eta_0(\dot{\psi}) = \dot{\psi}$ and

$$\dot{\psi}(t, \xi) = \big(\dot{\psi}_1(t, \xi), \dot{\psi}_2(t, \xi)\big) \in \boldsymbol{R} \times T_p(S) \subseteq \boldsymbol{R} \times \boldsymbol{C}^m$$

where $p = \psi(t, \xi)$. Also

$$\dot{\psi}(t, \xi) = \mathfrak{g}\big(\psi(t, \xi)\big) = \mathfrak{g}\big(\psi_1(t, \xi), \psi_2(t, \xi)\big) \,.$$

By (4.40), this implies

$$\begin{aligned}\dot\psi(0,\xi) &= \mathfrak{g}(\psi_1(0,\xi),\psi_2(0,\xi))\\ &= \mathfrak{g}(0,\xi) = (1, Q_1(\xi)i\xi + Q_3(\xi))\ .\end{aligned}$$

Therefore $\psi_5(0,\xi) = \dot\psi_1(0,\xi) = 1$ for all $\xi \in S$. A function $\psi_3\colon \boldsymbol{R}(-\eta, r_0)\times \times S \to \boldsymbol{R}$ of class C^∞ exists such that

$$\begin{aligned}\psi_5(t,\xi) &= 1 + t\psi_3(t,\xi)\\ \psi_1(t,\xi) &= t + t^2\psi_3(t,\xi)\end{aligned}$$

for all $t \in \boldsymbol{R}(-\eta, r_0)$ and $\xi \in S$; q.e.d.

Now we come to the determination of the orbit of ξ under ζ, which is the fundament of the proof.

THEOREM 5.14. *If $y \in \boldsymbol{R}$ and $\xi \in S$, then $\zeta(y,\xi) = e^{iy}\xi$.*

PROOF. For $t > 0$ abbreviate $\mathfrak{l}(t) = \log t$. Take $\xi \in S$ and $y \in \boldsymbol{R}$ and $t \in \boldsymbol{R}(0, r_0)$. Then $r_0 < \Delta$ and $\mathfrak{l}(t) < \log \Delta = \Delta_0$. Hence

$$\mathfrak{w}(\mathfrak{l}(t) + iy, \xi) = \chi(\mathfrak{l}(t), \zeta(y,\xi)) = \varphi(t, \zeta(y,\xi)) \neq 0\ .$$

Observe that $\varphi(t,\zeta(y,\xi)) \in M\langle t\rangle \subset B$. Hence $\varrho_2^{-1} = \mathfrak{r}_2^{-1}$ can be applied. We have

$$\varrho_2^{-1}(\mathfrak{w}(\mathfrak{l}(t) + iy, \xi)) = \varrho_2^{-1}(\varphi(t,\zeta(y,\xi))) = \psi(t,\zeta(y,\xi))\ .$$

Abbreviate $\gamma(t,y) = \varrho_2^{-1}(\mathfrak{w}(\mathfrak{l}(t) + iy, \xi))$. Then

$$\gamma(t,y) = (t + t^2\psi_3(t,\zeta(y,\xi)), \zeta(y,\xi) + t\psi_4(t,\zeta(y,\xi)))\ .$$

Denote the partial differentiation $\partial/\partial y$ also by affixing the index y. Then

$$\gamma_y(t,y) = \left(t^2\frac{\partial}{\partial y}\psi_3(t,\zeta(y,\xi)),\ \zeta_y(y,\xi) + t\frac{\partial}{\partial y}\psi_4(t,\zeta(y,\xi))\right)$$

for $0 < t < r_0$: The limit of $\gamma_y(t,y)$ for $t \to 0$ exists and is denoted by $\gamma_y(0,y)$ with

(5.23) $$\gamma_y(t,y) = (0, \zeta_y(y,\xi))\ .$$

Considering that η_0 is an identification map we have

$$\mathfrak{w}_y(\mathfrak{l}(t) + iy, \xi) = i\mathfrak{w}'(\mathfrak{l}(t) + iy, \xi) = if(\mathfrak{w}(\mathfrak{l}(t) + iy, \xi)) .$$

Therefore

$$\begin{aligned}\gamma_y(t, y) &= d\mathfrak{x}_2^{-1}(\mathfrak{w}(\mathfrak{l}(t) + iy, \xi), \mathfrak{w}_y(\mathfrak{l}(t) + iy, \xi)) \\ &= d\mathfrak{x}_2^{-1}\big(\varphi(t, \zeta(y, \xi)), f_i(\varphi(t, \zeta(y, \xi)))\big) \\ &= \hat{f}_i(\psi(t, \zeta(y, \xi))) \\ &= \hat{f}_i(\psi_1(t, \zeta(y, \xi)), \psi_2(t, \zeta(y, \xi))) .\end{aligned}$$

The limit $t \to 0$ and (4.35) imply

$$\begin{aligned}\gamma_y(0, y) &= \hat{f}_i(\psi_1(0, \zeta(y, \xi)), \psi_2(0, \zeta(y, \xi))) \\ &= \hat{f}_i(0, \zeta(y, \xi)) \\ &= (0, i\zeta(y, \xi)) .\end{aligned}$$

Comparison implies $\zeta_y(y, \xi) = i\zeta(y, \xi)$ for all $y \in \boldsymbol{R}$. Hence

$$\frac{\partial}{\partial y}\left(e^{-iy}\zeta(y, \xi)\right) = e^{-iy}\big(\zeta_y(y, \xi) - i\zeta(y, \xi)\big) = 0$$

for all $y \in \boldsymbol{R}$. Hence

$$e^{-iy}\zeta(y, \xi) = \zeta(0, \xi) = \xi$$

for all $y \in \boldsymbol{R}$; q.e.d.

Hence ζ determines the Hopf fiberation on S. According to Lemma 5.12, we have $L(\xi) = L(\xi_1)$ if and only if $\xi_1 = e^{iy}\xi$ for some $y \in \boldsymbol{R}$. Let $\boldsymbol{P}(\boldsymbol{C}^m) = \boldsymbol{P}_{m-1}$ be the complex projective space of dimension $m-1$ and let $\boldsymbol{P}\colon \boldsymbol{C}^m - \{0\} \to \boldsymbol{P}_{m-1}$ be the projection. For each $w \in \boldsymbol{P}_{m-1}$, there exists one and only one leaf $L[w] \in \Lambda$ such that $L[w] = L(\xi)$ for each $\xi \in S$ with $\boldsymbol{P}(\xi) = w$. Also $L[w] \neq L[w_1]$ if $w \in \boldsymbol{P}_{m-1}$ and $w_1 \in \boldsymbol{P}_{m-1}$ with $w \neq w_1$: Moreover for each leaf $L \in \Lambda$, there exists one and only one $w \in \boldsymbol{P}_{m-1}$ such that $L = L[w]$. Hence $\boldsymbol{P}_{m-1}$ provides a bijective parameterization of the leaf space Λ.

If $x \in I_0$ and $y \in \boldsymbol{R}$, if $\xi \in S$ and $n \in \boldsymbol{Z}$, Lemma 5.9 and Theorem 5.14 imply

$$\mathfrak{w}(x + iy, \xi) = \chi(x, e^{iy}\xi) \tag{5.24}$$

$$\mathfrak{w}(x + iy + 2\pi in, \xi) = \mathfrak{w}(x + iy, \xi) . \tag{5.25}$$

LEMMA 5.15. *If* $z_1 \in D$ *and* $z_2 \in D$ *and* ξ *are given such that* $\mathfrak{w}(z_1, \xi) = \mathfrak{w}(z_2, \xi)$, *then* $n \in \boldsymbol{Z}$ *exists such that* $z_2 = z_1 + 2\pi i n$.

PROOF. Let x_j and y_j be the real part and imaginary parts of z_j respectively for $j = 1, 2$. Then

$$\chi(x_1, e^{iy_1}\xi) = \mathfrak{w}(z_1, \xi) = \mathfrak{w}(z_2, \xi) = \chi(x_2, e^{iy_2}\xi) .$$

Because χ is injective, we have $x_1 = x_2$ and $e^{iy_1} = e^{iy_2}$. Therefore an integer n exists such that $y_2 = y_1 + 2\pi n$. Hence $z_2 = z_1 + 2\pi i n$, q.e.d.

Consequently, the 1. Case in the proof Lemma 5.10 is impossible and we have $\alpha = 2\pi$ in the second case.

6. – The parabolic mapping theorem.

Let (M, τ) be a strictly parabolic manifold of dimension m. Then the center $M[0]$ consists of one and only one point denoted by $0 = O_M$. Let Δ be the maximal radius of the exhaustion τ. We continue the same assumptions, conventions and identifications as in section 5. In particular $\boldsymbol{C}^m = \mathfrak{T}_0(M) = T_0(M)$ are identified such that η_0 is the identity. Also $\varkappa_0$ is the hermitian metric on $\boldsymbol{C}^m$. The square length is given by τ_0 in (4.20). As before $\boldsymbol{C}^m(r)$ denotes the open ball of radius r and center 0 in respect to τ_0. Abbreviate $E = \boldsymbol{C}(\Delta)$ and $E_* = E - \{0\}$. Also $S = \boldsymbol{C}^m\langle 1\rangle$ is the unit sphere in $\boldsymbol{C}^m$.

THEOREM 6.1. *For each* $w \in \boldsymbol{P}_{m-1}$, *the leaf* $L[w]$ *is a proper, closed, complex submanifold of complex dimension* 1 *of* M_*. *The closure* $\bar{L}[w] = L[w] \cup \{0\}$ *of* $L[w]$ *in* M *is a proper, closed, complex submanifold of complex dimension* 1 *of* M. *There exists a proper, surjective, differentiable map* $\mathfrak{v}: E \times S \to M$ *satisfying the following conditions.*

(1) *The restriction* $\mathfrak{v}: E_* \times S \to M_*$ *is proper and surjective.*

(2) *If* $\xi \in S$, *then* $\mathfrak{v}(0, \xi) = 0$.

(3) *If* $\xi \in S$ *and* $w = \boldsymbol{P}(\xi)$, *the map* $\mathfrak{v}(\square, \xi): E \to \bar{L}[w]$ *is biholomorphic.*

(4) *If* $z \in E$ *and* $\xi \in S$, *then* $z\mathfrak{v}'(z, \xi) = f(\mathfrak{v}(z, \xi))$.

(5) *If* $\xi \in S \subseteq \mathfrak{T}_0(M)$, *then* $\mathfrak{v}'(0, \xi) = \xi$.

(6) *If* $\xi \in S$ *and* $z \in E$ *and* $\alpha \in \boldsymbol{R}$, *then* $\mathfrak{v}(z, e^{i\alpha}\xi) = \mathfrak{v}(ze^{i\alpha}, \xi)$.

(7) *If* $z \in D$ *and* $\xi \in S$, *then* $\mathfrak{w}(z, \xi) = \mathfrak{v}(e^z, \xi)$.

(8) *If* $z \in E$ *and* $\xi \in S$, *then* $\tau(\mathfrak{v}(z, \xi)) = |z|^2$, *which means* $\mathfrak{v}(z, \xi) \in M\langle |z| \rangle$

(9) *If* $t \in \boldsymbol{R}[0, \Delta)$ *and* $\xi \in S$, *then* $\mathfrak{v}(t, \xi) = \varphi(t, \xi)$.

PROOF. The exponential function maps D onto E_*. By (5.25) the surjective map $\mathfrak{w}: D \times S \to M_*$ of class C^∞ factors to a surjective differentiable map $\mathfrak{v}: E_* \times S \to M_*$ such that (7) holds. If $\xi \in S$ and $w = P(\xi)$, the map $\mathfrak{w}(\square, \xi): D \to L[w]$ is locally biholomorphic and surjective. Hence $\mathfrak{v}(\square, \xi)$: $E_* \to L[w]$ is locally biholomorphic and surjective. By Lemma 5.15, $\mathfrak{v}(\square, \xi)$ is injective. Hence $\mathfrak{v}(\square, \xi): E_* \to L[w]$ is biholomorphic. For each $\xi \in S$, define $\mathfrak{v}(0, \xi) = 0 \in M$. Then (6), (8) and (9) are trivially true for $z = 0$ respectively $t = 0$. If $0 \neq z \in E$ and $\xi \in S$, then $u \in D$ exists such that $z = e^u$. Let x be the real part and y be the imaginary part of u. Then

$$\tau(\mathfrak{v}(z, \xi)) = \tau(\mathfrak{w}(u, \xi)) = e^{2x} = |z|^2 .$$

If $\alpha \in \boldsymbol{R}$, then

$$\begin{aligned} \mathfrak{v}(ze^{i\alpha}, \xi) &= \mathfrak{v}(e^{u+i\alpha}, \xi) = \mathfrak{w}(x + i(y + \alpha)\xi) = \chi(x, e^{i(y+\alpha)}\xi) \\ &= \mathfrak{w}(x + iy, e^{i\alpha}\xi) = \mathfrak{v}(e^u, e^{i\alpha}\xi) = \mathfrak{v}(z, e^{i\alpha}\xi) . \end{aligned}$$

If $t \in \boldsymbol{R}(0, \Delta)$, define $x = \log t$. Then

$$\mathfrak{v}(t, \xi) = \mathfrak{w}(x, \xi) = \chi(x, \xi) = \varphi(e^x, \xi) = \varphi(t, \xi) .$$

Therefore (2), (6), (7), (8) and (9) are proved.

Let K be a compact subset of M_*. Then $K_1 = \mathfrak{v}^{-1}(K)$ is a closed subset of $E_* \times S$. Let $\alpha > 0$ be the minimum of $\sqrt{\tau}$ on K and let $\beta < \Delta$ be the maximum of $\sqrt{\tau}$ on K. If $(z, \xi) \in K_1$, then $\alpha^2 \leqslant \tau(\mathfrak{v}(z, \xi)) = |z|^2 \leqslant \beta^2$. Hence K_1 is contained in the compact subset $(\boldsymbol{C}[\beta] - \boldsymbol{C}(\alpha)) \times S$ of $E \times S$. Therefore K_1 is compact. The map $\mathfrak{v}: E_* \times S \to M_*$ is proper. Hence (1) is proved.

Let U be any open neighborhood of $0 \in M$. Take an open neighborhood V of $0 \in M$ with $0 \in V \subset \overline{V} \subset U$ such that $\overline{V}$ is compact. A number $\varepsilon > 0$ exists such that $\tau(p) > \varepsilon^2$ for all $p \in \partial V$. Since $M(\varepsilon)$ is connected with $0 \in M(\varepsilon)$, we conclude that $M(\varepsilon) \subseteq V$. If $(z, \xi) \in \boldsymbol{C}(\varepsilon) \times S$, then $|z| < \varepsilon$ and $\mathfrak{v}(z, \xi) \in M\langle |z| \rangle \subset M(\varepsilon) \subset U$. Hence $\mathfrak{v}$ is continuous at every point of $\{0\} \times S$. The map $\mathfrak{v}: E \times S \to M$ is continuous. Take a compact subset K of M. Let $\beta < \Delta$ be the maximum of $\sqrt{\tau}$ on K. Then $K_1 = \mathfrak{v}^{-1}(K)$ is a closed subset of $E \times S$. If $(z, \xi) \in K_1$, then $\tau(\mathfrak{v}(z, \xi)) \leqslant \beta^2$. Hence $|z| \leqslant \beta$ and K_1 is contained in the compact subset $\boldsymbol{C}[\beta] \times S$ of $E \times S$. The set K_1 is compact. The map $\mathfrak{v}: E \times S \to M$ is proper.

Take $\xi \in S$ and define $w = P(\xi)$. Then $\mathfrak{v}(\square, \xi): E \to M$ is continuous on E and holomorphic on E_*. Hence $\mathfrak{v}(\square, \xi): E \to M$ is holomorphic. Take $z \in E$. If $z = 0$, then $f(\mathfrak{v}(0, \xi)) = f(0) = 0$. Hence (4) holds. If $z \neq 0$, then

$u \in D$ exists with $e^u = z$. Also

$$\mathfrak{v}'(z, \xi)z = \mathfrak{v}'(e^u, \xi)e^u = \mathfrak{w}'(u, \xi) = f(\mathfrak{w}(u, \xi)) = f(\mathfrak{v}(z, \xi)) .$$

Therefore (4) is proved.

If $0 < t < r_0$ and $\xi \in S$, then Lemma 5.13 and 5.20 apply. Hence

$$\mathfrak{v}(t, \xi) = \varphi(t, \xi) = (t + t^2\psi_3(t, \xi))(\xi + t\psi_4(t, \xi))$$

where $\mathfrak{v}(t, \xi) \in U_3 \subseteq \boldsymbol{C}^m$. Under the identification $\boldsymbol{C}^m = T_0(M) = \mathfrak{T}_0(M)$ we obtain

$$\mathfrak{v}'(0, \xi) = \dot{\varphi}(0, \xi) = \xi$$

which proves (5).

Take $\xi \in S$ and define $w = \boldsymbol{P}(\xi)$. Because $\mathfrak{v}: E_* \times S \rightarrow M_*$ is proper, also $\mathfrak{v}(\square, \xi): E_* \rightarrow M_*$ is proper where $\mathfrak{v}(\square, \xi): E_* \rightarrow L[w]$ is biholomorphic with $\mathfrak{v}'(z, \xi) \neq 0$ for all $z \in E_*$. Hence $L[w]$ carries the induced topology from M_* as the manifold topology and is a closed subset of M_*. Thus $L[w]$ is a proper, closed complex submanifold of dimension 1 of M_* and $\mathfrak{v}(\square, \xi)$: $E_* \rightarrow L[w]$ is biholomorphic. Because $\mathfrak{v}(0, \xi) = 0$ and because $\mathfrak{v}(\square, \xi)$: $E \rightarrow M$ is continuous, $\bar{L}[w] = L[w] \cup \{0\}$ is the closure of $L[w]$ in M. Also $\bar{L}[w]$ continuous $L[w]$ as an analytic set on M where $\mathfrak{v}(\square, \xi): E \rightarrow \bar{L}[w]$ is bijective and holomorphic. Since $\mathfrak{v}'(0, \xi) \neq 0$, the point 0 is a simple point of $\bar{L}[w]$ and $\bar{L}[w]$ is a proper, closed, complex submanifold of dimension 1 of M such that $\mathfrak{v}(\square, \xi): E \rightarrow \bar{L}[w]$ is biholomorphic. Therefore (3) is proved.

By (8) and (4.19) $\mathfrak{v}$ maps $\boldsymbol{C}(r_0) \times S$ into $M(r_0) \subset U_3$. Hence $\mathfrak{v}$ can be considered as a vector function $\mathfrak{v}: \boldsymbol{C}(r_0) \times S \rightarrow \boldsymbol{C}^m$. Take $0 < r < r_0$ and $(z, \xi) \in \boldsymbol{C}(r) \times S$. Then

$$\mathfrak{v}(z, \xi) = \frac{1}{2\pi i} \int_{\boldsymbol{C}\langle r \rangle} \frac{\mathfrak{v}(\zeta, \xi)}{\zeta - z} d\zeta .$$

Therefore $\mathfrak{v}$ is of class C^∞ on $\boldsymbol{C}(r) \times S$. Consequently $\mathfrak{v}: E \times S \rightarrow M$ is a map of class C^∞, q.e.d.

Again identify $\boldsymbol{C}^m = T_0(M) = \mathfrak{T}_0(M)$. Then the ball $\boldsymbol{C}^m(\varDelta)$ is an open subset of the real tangent space of M at 0. Recall the homeomorphism $h: \boldsymbol{C}^m(\varDelta) \rightarrow M$ defined in Theorem 4.6 with $h(t, \xi) = \varphi(t, \xi)$ if $t \in \boldsymbol{R}[0, \varDelta)$ and $\xi \in S$.

Theorem 6.2. (D. Burns) *The homeomorphism $h: \boldsymbol{C}^m(\varDelta) \rightarrow M$ is a diffeomorphism. In fact $h = \exp_0$ is the exponential map at $0 \in M$. If $\varDelta = \infty$, then M is complete in respect to the Kaehler metric.*

PROOF. Take $0 \neq X \in T_0(M)$. Let $\varphi_X: \boldsymbol{R}(\alpha_X, \beta_X) \to M$ be the maximal geodesic through $0 \in M$. Then $\alpha_X < 0 < \beta_X$ and $\varphi_X(0) = 0$ and $\dot{\varphi}_X(0) = X$. Also $\alpha_{-X} = -\beta_X$ and $\beta_{-X} = -\alpha_X$ and $\varphi_{-X}(t) = \varphi_X(-t)$. Define $\xi = X/|X|$ in S and $c = |X| > 0$. According to (7′) section 4d) $\varphi(\square, \xi): \boldsymbol{R}(0, \varDelta) \to M$ is a geodesic. Then $\psi: \boldsymbol{R}[0, \varDelta/c) \to M$ is a geodesic where $\psi(t) = \varphi(ct, \xi)$ with $\psi(0) = \varphi(0, \xi) = 0$ and $\dot{\psi}(0) = c\dot{\varphi}(0, \xi) = c\xi = X$. Hence $\beta_X \geqslant \varDelta/c$ and

$$\varphi_X(t) = \psi(t) = \varphi(ct, \xi) = h(tc\xi) = h(tX)$$

for all $t \in \boldsymbol{R}[0, \varDelta/c)$. Also $\alpha_X = -\beta_{-X} < -\varDelta/c$. If $-\varDelta/c < t < 0$, then $\varphi_X(t) = \varphi_{-X}(-t) = h\big((-t)(-X)\big) = h(tX)$. Hence $\varphi_X(t) = h(tX)$ if $|t| < \varDelta/|X|$. If $0 \neq X \in \boldsymbol{C}^m(\varDelta)$, then $\varDelta/|X| > 1$ and

$$\exp_0(X) = \varphi_X(1) = h(X)\,.$$

Also $\exp_0(0) = 0 = h(0)$. Therefore $h: \boldsymbol{C}^m(\varDelta) \to M$ is the exponential map and as such of class C^∞ and locally diffeomorphic at 0. By Theorem 4.6 the map $h: \boldsymbol{C}^m(\varDelta) \to M$ is a homeomorphism and $h: \boldsymbol{C}^m(\varDelta) - \{0\} \to M_*$ a diffeomorphism. Therefore $h: \boldsymbol{C}^m(\varDelta) \to M$ is a diffeomorphism; q.e.d.

Daniel Burns pointed out to me that h is the exponential map. Using a special coordinate system along $\bar{L}[w]$, he shows that $\bar{L}[w]$ is totally geodesic. Trivially $\varphi(\square, \xi): \boldsymbol{R}[0, \varDelta) \to \bar{L}[w]$ is geodesic for each $\xi \in S$ with $\boldsymbol{P}(\xi) = w$. Since $\bar{L}[w]$ is totally geodesic, $\varphi(\square, \xi)$ is geodesic in M and Theorem 6.2 follows. Subsequently, I was able to prove Lemma 3.7 which yields Theorem 3.8 and the result that $\bar{L}[w]$ is totally geodesic is not explicitly used for the proof of Theorem 6.2 given here. Originally, I failed to establish Lemma 3.7. Hence a term did not cancel in the proof of Theorem 3.8.

The Theorem of Burns will improve the result considerably as can be seen from announcement [17].

LEMMA 6.3. *Take $\xi \in S$. Define $j_\xi: E \to \boldsymbol{C}^m(\varDelta)$ by $j_\xi(z) = z\xi$. Then $h \circ j_\xi = \mathfrak{v}(\square, \xi): E \to M$ is holomorphic.*

PROOF. Take $\xi \in S$ and $z \in E_*$. Define $\zeta = z/|z|$. Then

$$h\big(j_\xi(z)\big) = h(z\xi) = \varphi\big(|z|, \zeta\xi\big) = \mathfrak{v}\big(|z|, \zeta\xi\big) = \mathfrak{v}\big(|z|\zeta, \xi\big) = \mathfrak{v}(z, \xi)\,.$$

If $\xi \in S$, then $h\big(j_\xi(0)\big) = h(0) = 0 = \mathfrak{v}(0, \xi)$. Hence $h \circ j_\xi = \mathfrak{v}(0, \xi)$, q.e.d.

LEMMA 6.4. *Let $\boldsymbol{H}: \boldsymbol{C}^m \to \boldsymbol{C}$ be a function of class C^p with $0 \leqslant p \in \boldsymbol{Z}$ such that $\boldsymbol{H}(w\mathfrak{z}) = w^p \boldsymbol{H}(\mathfrak{z})$ for all $w \in \boldsymbol{C}$ and $\mathfrak{z} \in \boldsymbol{C}^m$. Then $\boldsymbol{H}$ is a homogeneous polynomial of degree p in $\mathfrak{z} = (z_1, \dots, z_m)$ over $\boldsymbol{C}$.*

PROOF. If $p = 0$, then $H(w\mathfrak{z}) = H(\mathfrak{z})$ for all $w \in \boldsymbol{C}$ and $\mathfrak{z} \in \boldsymbol{C}^m$. Take $w = 0$. Then $H(0) = H(\mathfrak{z})$ for all $\mathfrak{z} \in \boldsymbol{C}^m$. Hence H is constant.

Take $p > 0$ and assume that the lemma is established for $p - 1$. Differentiation for z_μ yields $H_{z_\mu}(w\mathfrak{z}) = w^{p-1}H_{z_\mu}(\mathfrak{z})$ for all $w \in \boldsymbol{C}$ and $\mathfrak{z} \in \boldsymbol{C}^m$ where H_{z_μ} has class C^{p-1}. Hence H_{z_μ} is a homogeneous polynomial of degree $p - 1$ in $\mathfrak{z} = (z_1, \ldots, z_m)$ over $\boldsymbol{C}$. Differentiation for w implies

$$\sum_{\mu=1}^{m} H_{z_\mu}(w\mathfrak{z})z_\mu = pw^{p-1}H(\mathfrak{z})$$

since $(\partial/\partial w)(\overline{w}\overline{z}_\mu) = 0$. If we take $w = 1$, we see that H is a homogeneous polynomial of degree p in $\mathfrak{z} = (z_1, \ldots, z_m)$ over $\boldsymbol{C}$; q.e.d.

LEMMA 6.5. *Take $r > 0$. For each $\mathfrak{z} \in \boldsymbol{C}^m$ define $j_\mathfrak{z}\colon \boldsymbol{C} \to \boldsymbol{C}^m$ by $j_\mathfrak{z}(w) = w\mathfrak{z}$. Let $H\colon \boldsymbol{C}^m(r) \to \boldsymbol{C}$ be a function of class C^∞ such that $H \circ j_\xi$ is holomorphic on $\boldsymbol{C}(r)$ for each fixed $\xi \in S$. Then H is holomorphic on $\boldsymbol{C}^m(r)$.*

PROOF. An open neighborhood W of $\{0\} \times \boldsymbol{C}^m$ in $\boldsymbol{C}^{m+1}$ is defined by

$$W = \{(z, \mathfrak{z}) \in \boldsymbol{C} \times \boldsymbol{C}^m \big| \, |z\mathfrak{z}| < r\} .$$

A function $G\colon W \to \boldsymbol{C}$ of class C^∞ is defined by $G(z, \mathfrak{z}) = H(z\mathfrak{z})$. Take $\mathfrak{z} \in \boldsymbol{C}^m$. If $\mathfrak{z} = 0$, then $G(z, 0) = H(0)$ is constant. Hence $G(\square, 0)\colon \boldsymbol{C} \to \boldsymbol{C}$ is holomorphic. Assume that $\mathfrak{z} \neq 0$ and define $\xi = \mathfrak{z}/|\mathfrak{z}|$ in S. Then $G(z, \mathfrak{z}) = H \circ j_\xi(z|\mathfrak{z}|)$ if $|z| < r/|\mathfrak{z}|$. Hence $G(\square, \mathfrak{z})\colon \boldsymbol{C}(r/|\mathfrak{z}|) \to \boldsymbol{C}$ is holomorphic. For each integer $p \geq 0$, define a function $G_p\colon W \to \boldsymbol{C}$ of class C^∞ by

$$G_p(z, \mathfrak{z}) = \frac{1}{p!}\frac{d^p}{dz^p} G(z, \mathfrak{z}) .$$

A function $H_p\colon \boldsymbol{C}^m \to \boldsymbol{C}$ of class C^∞ is defined by $H_p(\mathfrak{z}) = G_p(0, \mathfrak{z})$. We have the Hartogs series development

$$G(z, \mathfrak{z}) = \sum_{p=0}^{\infty} H_p(\mathfrak{z})z^p$$

for all $(z, \mathfrak{z}) \in W$.

Take $w \in \boldsymbol{C}$ and $\mathfrak{z} \in \boldsymbol{C}^m$. If $w = 0$ or $\mathfrak{z} = 0$ or $r = \infty$ define $s = \infty$. If $w \neq 0$ and $\mathfrak{z} \neq 0$ and $r < \infty$ define $s = r/(|w\mathfrak{z}|)$. Take any $z \in \boldsymbol{C}(s)$. Then $(z, w\mathfrak{z}) \in W$ and $(zw, \mathfrak{z}) \in W$. Therefore

$$\sum_{p=0}^{\infty} H_p(w\mathfrak{z})z^p = G(z, w\mathfrak{z}) = H(zw\mathfrak{z}) = G(zw, \mathfrak{z}) = \sum_{p=0}^{\infty} H_p(\mathfrak{z})w^p z^p .$$

Therefore $H_p(w\mathfrak{z}) = H_p(\mathfrak{z})w^p$. By Lemma 6.3 H_p is a polynomial of degree p over $\boldsymbol{C}$ in $\mathfrak{z} = (z_1, \ldots, z_m)$. In particular $H_p\colon \boldsymbol{C}^m \to \boldsymbol{C}$ is holomorphic.

For each $t \in \boldsymbol{R}(0, r)$ define $\mu(r) = \operatorname{Max}\{|H(\mathfrak{z})|\,\big|\,|\mathfrak{z}| \leqslant t\}$. Take $0 < t < s < r$ and set $\theta = t/s < 1$. If $|z| \leqslant 1$ and $|\mathfrak{z}| \leqslant s$, then $(z, \mathfrak{z}) \in W$ and $|G(z, \mathfrak{z})| = |H(z\mathfrak{z})| \leqslant \mu(s)$. Hence $|H_p(\mathfrak{z})| \leqslant \mu(s)$. If $|\mathfrak{z}| \leqslant t$, then $|\theta\mathfrak{z}| \leqslant s$ and $|H_p(\mathfrak{z})| = \theta^{-p}|H_p(\theta\mathfrak{z})| \leqslant \theta^{-p}\mu(s)$. Therefore the series

$$H(\mathfrak{z}) = G(1, \mathfrak{z}) = \sum_{p=0}^{\infty} H_p(\mathfrak{z})$$

converges uniformly for all $\mathfrak{z} \in \boldsymbol{C}^m[t]$ for each $t \in \boldsymbol{R}(0, r)$. The function H is holomorphic on $\boldsymbol{C}^m(r)$; q.e.d.

THEOREM 6.6 (The parabolic mapping theorem). *Let (M, τ) be a strictly parabolic manifold of dimension m. Let $\Delta > 0$ be the maximal radius of the exhaustion τ. On $\boldsymbol{C}^m$, define $\tau_0\colon \boldsymbol{C}^m \to \boldsymbol{R}_+$ by $\tau(z) = |z|^2$. Then there exists a biholomorphic map $h\colon \boldsymbol{C}^m(\Delta) \to M$ such that $\tau \circ h = \tau_0$.*

PROOF. If $\Delta = \infty$, define $\sigma = \tau$. If $\Delta < \infty$, define $\sigma = \tau/(\Delta^2 - \tau)$. Then σ is an exhaustion of M with maximal radius ∞ and with $dd^c\sigma > 0$ on M. Therefore M is a Stein manifold. A proper, injective, smooth, holomorphic map $k\colon M \to \boldsymbol{C}^{2m+1}$ exists such that $N = k(M)$ is a proper, closed, complex submanifold of $\boldsymbol{C}^{2m+1}$. The map $k\colon M \to N$ is biholomorphic. Define $k = (k_1, \ldots, k_{2m+1})$ and $H_\mu = k_\mu \circ h\colon \boldsymbol{C}^m(\Delta) \to \boldsymbol{C}$ for each $\mu = 1, \ldots, 2m+1$. For each $\xi \in S$, define $j_\xi\colon \boldsymbol{C} \to \boldsymbol{C}^m$ by $j_\xi(z) = z\xi$. Then $H_\mu \circ j_\xi\colon \boldsymbol{C}(\Delta) \to \boldsymbol{C}$ is holomorphic for each $\xi \in S$. By Lemma 6.5, H_μ is holomorphic on $\boldsymbol{C}^m(\Delta)$. Hence $H = k \circ h\colon \boldsymbol{C}^m(\Delta) \to N$ is a holomorphic diffeomorphism and consequently $H\colon \boldsymbol{C}^m(\Delta) \to N$ is a biholomorphic map. Hence $h = k^{-1} \circ H\colon \boldsymbol{C}^m(\Delta) \to M$ is biholomorphic with $\tau_0 = \tau \circ h$ by Theorem 4.6, q.e.d.

REMARK 1. The biholomorphic map h is an isometry of exhaustions $\tau_0 = \tau \circ h$ and of Kaehler metrics $h^*(dd^c\tau) = dd^c\tau_0$.

REMARK 2. The map h can be defined a priori. Let $\varkappa$ be the Kaehler metric on M with exterior form $dd^c\tau$. The center $M[0]$ consists of one and only one point 0. Then $\varkappa$ defines a hermitian metric $\varkappa_0$ on the holomorphic tangent space $\mathfrak{T}_0(M)$ at $0 \in M$. Define $\tau_0(z) = \varkappa_0(z, z)$. Consider $\boldsymbol{C}^m(\Delta)$ as the ball in $\mathfrak{T}_0(M)$ defined by $\tau_0(z) < \Delta^2$. Identify the real tangent space $T_0(M)$ with $\mathfrak{T}_0(M)$ such that η_0 is the identity. Then $h = \exp_0\colon \boldsymbol{C}^m(\Delta) \to M$ is the exponential map.

REMARK 3. A weaker version of Theorem 6.6 was announced in [17] and Theorem 6.6 was obtained only for those exhaustions with holomorphic vector field f. D. Burns observed that the leaves $\bar{L}[w]$ are totally geodesic and that therefore h is the exponential map. Once h is recognized to be differentiable at 0 the biholomorphy of h follows easily. As a consequence f is holomorphic. A variation of the proof of Burns was given here. His contribution, which led to a considerable improvement, is gratefully acknowledged here.

Appendix.

a) Proof of Theorem 4.1.

Take $x_0 \in M[0, \Delta)$. Then $\delta(x_0) < \Delta$. Take r with $\delta(x_0) < r < \Delta$. Since $M[0, r]$ is compact, a point $x_1 \in M[0, r)$ exists such that $\delta(x_1) \leqslant \delta(x)$, for all $x \in M[0, r]$. Then $\delta(x_1) \leqslant \delta(x_0) < r$. Hence $x_1 \in M[0, r)$. If $x_1 \in M(0, r)$, then $d\delta(x_1, v) = 0$ for all $v \in T_{x_1}(M)$ which contradicts $d\delta(x_1, Y(x_1)) = 1$. Hence $x_1 \in M\langle 0 \rangle = S$. The set S is not empty.

A number $\eta_0 > 0$ exists such that $d\delta(x, Y(x)) \neq 0$ for all $x \in M(-\eta_0, \Delta)$. Hence δ has no critical point on $M(-\eta_0, \Delta)$ and $M\langle r \rangle$ is a proper, compact differentiable submanifold of M for each $r \in \boldsymbol{R}(-\eta_0, \Delta)$.

1. *Step*. The construction of the flow φ. For each $q \in M$ select a local one parameter group

$$\varphi^q \colon \boldsymbol{R}(-\varepsilon_q, \varepsilon_q) \times U_q \to M$$

of diffeomorphisms of Y such that 1°, 2° and 3° hold and such that $q \in U_q$. Then

$$V_0 = \bigcup_{q \in M} \boldsymbol{R}(-\varepsilon_q, \varepsilon_q) \times U_q$$

is an open neighborhood of $\{0\} \times M$ in $\boldsymbol{R} \times M$. Therefore there exists a positive continuous function ϱ_1 on M such that

$$V_1 = \{(t, p) \in \boldsymbol{R} \times M \,\big|\, |t| < \varrho_1(p)\}$$

is contained in V_0. Here V_1 is open with

$$\{0\} \times M \subset V_1 \subseteq V_0 \subseteq \boldsymbol{R} \times M\,.$$

Take $(t_0, p_0) \in V_1$. Then $q \in M$ exists such that $t_0 \in \boldsymbol{R}(-\varepsilon_q, \varepsilon_q)$ and $p_0 \in U_q$. Define $\varphi(t_0, p_0) = \varphi^q(t_0, p_0)$. If $s \in M$ and $t_0 \in \boldsymbol{R}(-\varepsilon_s, \varepsilon_s)$ and $p_0 \in U_s$.

Then $\varphi^q(\square, p_0)$ and $\varphi^s(\square, p_0)$ are integral curves of Y with $\varphi^q(0, p_0) = p_0 = \varphi^s(0, p_0)$. Hence $\varphi^q(t, p_0) = \varphi^s(t, p_0)$ for all $t \in \mathbf{R}(-\varepsilon_q, \varepsilon_q) \cap \mathbf{R}(-\varepsilon_s, \varepsilon_s)$. In particular, $\varphi^q(t_0, p_0) = \varphi^s(t_0, p_0)$. Therefore $\varphi\colon V_1 \to M$ is well defined and differentiable. We have

(1°) *If* $p \in M$, *then* $\varphi(0, p) = p$.

(2°) *If* $(t, p) \in V_1$, *then* $\dot{\varphi}(t, p) = Y(\varphi(t, p))$.

(3°) *For each* $(t_0, p_0) \in V_1$, *there exists an open neighborhood* $U(t_0, p_0)$ *of* p_0 *in* M *such that*

$$\varphi_{t_0} = \varphi(t_0, \square)\colon U(t_0, p_0) \to \varphi_{t_0}(U(t_0, p_0))$$

is a diffeomorphism onto the open image.

(4°) *If* $(s, p) \in V_1$ *and* $(t, \varphi_s(p)) \in V_1$ *and* $(t + s, p) \in V_1$, *then*

$$\varphi_{t+s}(p) = \varphi_t(\varphi_s(p)) .$$

Proof of (3°). Take $q \in M$ with $(t_0, p_0) \in \mathbf{R}(-\varepsilon_q, \varepsilon_q) \times U_q$. An open neighborhood $U(t_0, p_0)$ of p_0 exists such that $U(t_0, p_0) \subseteq U_q$ and such that $-\varrho_1(p) < t_0 < \varrho_1(p)$ for all $p \in U(t_0, p_0)$. Hence $\varphi(t_0, p) = \varphi^q(t_0, p)$ for all $p \in U(t_0, p_0)$ and

$$\varphi_{t_0} = \varphi^q_{t_0}\colon U(t_0, p_0) \to \varphi_{t_0}(U(t_0, p_0))$$

is a diffeomorphism onto the open image set, which proves (3°).

Proof of (4°). Fix $p \in M$ and $s \in \mathbf{R}$ with $|s| < \varrho_1(p)$. Then

$$I = \{t \in \mathbf{R} \,\big|\, |t + s| < \varrho_1(p) \text{ and } |t| < \varrho_1(\varphi_s(p))\}$$

is an open interval with $0 \in I$. Define $\lambda\colon I \to M$ and $\mu\colon I \to M$ by $\lambda(t) = \varphi(t + s, p)$ and $\mu(t) = \varphi(t, \varphi_s(p))$ for all $t \in I$. Then λ and μ are integral curves of Y with $\lambda(0) = \varphi(s, p) = \varphi(0, \varphi_s(p)) = \mu(0)$. Therefore $\lambda = \mu$, which proves (4°).

Take $p \in M$. Then there exists one and only one differentiable extension of $\varphi(\square, p)$ again called $\varphi(\square, p)$ to a maximal interval $\mathbf{R}(-\varrho_1(p), \varrho_2(p))$ such that

(1*) *If* $p \in M$, *then* $\varphi(0, p) = p$.

(2*) *If* $p \in M$ *and* $t \in \mathbf{R}(-\varrho_1(p), \varrho_2(p))$, *then* $\dot{\varphi}(t, p) = Y(\varphi(t, p))$.

(3*) *If* $p \in M$, *then* $0 < \varrho_1(p) \leqslant \varrho_2(p) \leqslant +\infty$.

Observe that $-\varrho_1(p)$ was kept fixed. Define

$$V_2 = \{(t, p) \in \boldsymbol{R} \times M \mid -\varrho_1(p) < t < \varrho_2(p)\} .$$

Then $V_1 \subseteq V_2$. As above, the assertion (4*) is easily proved:

(4*) *If* $(s, p) \in V_2$ *and* $(t, \varphi(s, p)) \in V_2$ *and* $(t + s, p) \in V_2$, *then*

$$\varphi(t + s, p) = \varphi(t, \varphi(s, p)) .$$

Let V_3 be the set of all interior points (t, p) of V_2 such that φ is of class C^∞ on a neighborhood of (t, p) and such that φ_t is a local diffeomorphism at p. Then V_3 is an open subset of V_2.

(5*) $V_3 = V_2$.

Proof of (5*). For each $p \in M$ define

$$\varrho_3(p) = \sup \{t \in \boldsymbol{R} \mid \boldsymbol{R}(-\varrho_1(p), t) \times \{p\} \subseteq V_3\} .$$

Then $\varrho_1(p) \leqslant \varrho_3(p) \leqslant \varrho_2(p)$. Take $p_0 \in M$. Assume that $\varrho_3(p_0) < \varrho_2(p_0)$. Define $h\colon V_2 \to \boldsymbol{R}$ by

$$h(t, p) = t + \varrho_1(\varphi(t, p)) .$$

Then $h(t, p) > t$. Take $s \in \boldsymbol{R}$ with $\varrho_3(p_0) < s < h(\varrho_3(p_0), p_0)$. Then $r \in \boldsymbol{R}$ exists such that

$$0 < r < \varrho_3(p_0) < s < h(r, p_0) .$$

Because $\boldsymbol{R}[0, r] \times \{p_0\}$ is a compact subset of V_3, an open neighborhood U_0 of p_0 exists such that $\boldsymbol{R}[0, r] \times U_0 \subseteq V_3$. Then $\boldsymbol{R}(-\varrho_1(p), r] \times \{p\} \subseteq V_3$ for all $p \in U_0$. Hence $\varrho_3(p) > r$ for all $p \in U_0$. Since $h(r, \square)$ is continuous on U_0, an open neighborhood U_1 of p_0 with $U_1 \subseteq U_0$ exists such that $s < h(r, p)$ for all $p \in U_1$. Then

$$W = \{(t, p) \in \boldsymbol{R} \times U_1 \mid -\varrho_1(\varphi(r, p)) + r < t < h(r, p)\}$$

is open in $\boldsymbol{R} \times U_1$. A differentiable map $\chi\colon W \to M$ is defined by

$$\chi(t, r) = \varphi(t - r, \varphi(r, p))$$

because $(t - r, \varphi_r(p)) \in V_1$ if $(t, p) \in W$.

Take $p \in U_1$. Then $\chi(r, p) = \varphi(r, p)$ and

$$\dot{\chi}(t, p) = Y(\chi(t, p)) \quad \text{if } -\varrho_1(\varphi(r, p)) + r < t < h(r, p)$$
$$\dot{\varphi}(t, p) = Y(\varphi(t, p)) \quad \text{if } -\varrho_1(p) < t < \varrho_2(p) .$$

Hence $\varphi(t, p) = \chi(t, p)$ in a neighborhood of r. The maximality of $\varrho_2(p)$ implies $h(r, p) \leqslant \varrho_2(p)$ and $\varphi(t, p) = \chi(t, p)$ if $r < t < h(r, p)$. Because $s < h(r, p)$ for all $p \in U_1$ we have $\varphi = \chi$ on $\boldsymbol{R}(r, s) \times U_1$. Hence $\boldsymbol{R}(r, s) \times U_1 \subseteq \operatorname{Int} V_1$ and φ is differentiable on $\boldsymbol{R}(r, s) \times U_1$ with

$$d\varphi_t(p) = d\varphi_{t-r}(\varphi(r, p)) \circ d\varphi_r(p) .$$

Because $(r, p) \in V_3$, the differential $d\varphi_r(p)\colon T_p(M) \to T_{\varphi(r,p)}(M)$ is a linear isomorphism. Since $(t - r, \varphi_r(p)) \in V_1$, the differential $d\varphi_{t-r}(\varphi(r, p))\colon T_{\varphi(r,p)}(M) \to T_{\varphi_t(p)}(M)$ is a linear isomorphism. Hence $d\varphi_t(p)\colon T_p(M) \to T_{\varphi_t(p)}(M)$ is a linear isomorphism. Therefore $\boldsymbol{R}(r, s) \times U_1 \subseteq V_3$. In particular $\varrho_3(p) \geqslant s$ for all $p \in U_1$ which contradicts $\varrho_3(p_0) < s$. This proves (5*).

(6*) *If* $(t, p) \in V_2$ *with* $t \geqslant 0$ *and* $\delta(p) \geqslant 0$, *then* $\delta(\varphi(t, p)) = t + \delta(p)$.

Proof of (6*). First assume that $\delta(p) > 0$. Since $\varphi(0, p) = p$, a largest number $\varrho_4(p) \in \boldsymbol{R}(0, \varrho_2(p))$ exists such that $\delta(\varphi(t, p)) > 0$ if $0 \leqslant t < \varrho_4(p)$. Then $\varphi(t, p) \in M(0, \Delta)$ for all $t \in \boldsymbol{R}[0, \varrho_4(p))$. Therefore

$$\frac{d}{dt}\,\delta(\varphi(t, p)) = d\delta(\varphi(t, p), \dot{\varphi}(t, p)) = d\delta(\varphi(t, p), Y(\varphi(t, p))) = 1$$

for $0 \leqslant t < \varrho_4(p)$. Therefore

$$\delta(\varphi(t, p)) = t + \delta(\varphi(0, p)) = t + \delta(p) \qquad \forall t \in \boldsymbol{R}[0, \varrho_4(p)) .$$

Now consider the case $\delta(p) = 0$. Because δ has no critical points on $S = M\langle 0\rangle = \delta^{-1}(0)$, the set S has the structure of a proper, compact differentiable submanifold of M bounding $M(0, \Delta)$. Hence a sequence $\{p_\lambda\}_{\lambda \in \boldsymbol{N}}$ with $\delta(p_\lambda) > 0$ converges to p. Take t with $0 < t < \varrho_1(p)$. A number $\lambda_0 \in \boldsymbol{N}$ exists such that $0 < t < \varrho_1(p_\lambda)$ for all $\lambda \geqslant \lambda_0$. Therefore $\delta(\varphi(t, p_\lambda)) = t + \delta(p_\lambda)$. Now $\lambda \to \infty$ implies $\delta(\varphi(t, p)) = t + \delta(p)$ for all $t \in \boldsymbol{R}[0, \varrho_1(p))$. A largest number $\varrho_4(p) \in \boldsymbol{R}(0, \varrho_2(p))$ exists such that $\delta(\varphi(t, p)) > 0$ if $0 < t < \varrho_4(p)$. As before we obtain $\delta(\varphi(t, p)) = t + \delta(p)$ if $0 \leqslant t < \varrho_4(p)$.

Assume that $\varrho_4(p) < \varrho_2(p)$. Then $\delta\big(\varphi(\varrho_4(p), p)\big) = \varrho_4(p) + \delta(p) > 0$. Therefore $s \in \boldsymbol{R}(\varrho_4(p), \varrho_2(p))$ exists such that $\delta(\varphi(t, p)) > 0$ if $t \in \boldsymbol{R}(0, s)$ which implies $s \leqslant \varrho_4(p)$. Contradiction! Therefore $\varrho_4(p) = \varrho_2(p)$ which proves (6*).

(7*) *If* $p \in M[0, \Delta)$, *then* $\varrho_2(p) = \Delta - \delta(p)$.

Proof of (7*). Take $p \in M[0, \Delta)$ and assume that $r = \varrho_2(p) + \delta(p) < \Delta$. Take $t \in \mathbf{R}(0, \varrho_2(p))$. Then $\delta(\varphi(t, p)) = t + \delta(p)$ and $\varphi(t, p) \in M[0, r]$, which is compact. A sequence $\{t_\nu\}_{\nu \in N}$ with $0 < t_\nu < \varrho_2(p)$ exists such that $t_\nu \to \varrho_2(p)$ and $\varphi(t_\nu, p) \to q \in M[0, r]$ for $\nu \to \infty$. Take $\nu \in \mathbf{N}$. Then

$$\chi(t, p) = \varphi(t - t_\nu, \varphi(t_\nu, p))$$

is defined for all $t \in \mathbf{R}$ with

$$-\varrho_1(\varphi(t_\nu, p)) + t_\nu < t < \varrho_1(\varphi(t_\nu, p)) + t_\nu$$

with $\chi(t_\nu, p) = \varphi(t_\nu, p)$ and $\dot{\chi}(t, p) = Y(\chi(t, p))$ and $\dot{\varphi}(t, p) = Y(\varphi(t, p))$. By the maximality of $\varrho_2(p)$ we have

$$\varrho_1(\varphi(t_\nu, p)) + t_\nu \leqslant \varrho_2(p) .$$

The limit $\nu \to \infty$ implies

$$\varrho_2(p) < \varrho_1(q) + \varrho_2(p) \leqslant \varrho_2(p)$$

which is impossible. Therefore $\varrho_2(p) + \delta(p) \geqslant \Delta$. If $0 \leqslant t < \varrho_2(p)$, then $t + \delta(p) = \delta(\varphi(t, p)) \leqslant \Delta$. Hence $\varrho_2(p) + \delta(p) \leqslant \Delta$. Together we obtain $\varrho_2(p) + \delta(p) = \Delta$ which proves [7*).

2. *Step*. The construction of ψ. Because $S \neq \emptyset$ is compact, a number $\eta > 0$ exists such that $\varrho_1(p) > \eta$ and $d\delta(p, Y(p)) \neq 0$ if $p \in M[-\eta, 0]$. Then $Y(p) \neq 0$ if $p \in M[-\eta, 0)$. A differentiable map

$$\psi\colon \mathbf{R}(-\eta, \Delta) \times S \to M$$

is defined by $\psi(t, p) = \varphi(t, p)$ for all $t \in \mathbf{R}(-\eta, \Delta)$ and $p \in S$. Step 1 implies

(1′) *If* $p \in S$, *then* $\psi(0, p) = p$.

(2′) *If* $p \in S$ *and* $t \in \mathbf{R}(-\eta, \Delta)$, *then* $\dot{\psi}(t, p) = Y(\psi(t, p))$.

(3′) *If* $p \in S$ *and* $t \in \mathbf{R}[0, \Delta)$, *then* $\delta(\psi(t, p)) = t$.

Therefore points (1), (2) and (4) of Theorem 4.1 are proved.

(4′) *If* t, s *and* $t + s$ *belong to* $\mathbf{R}[0, \Delta)$ *and if* $p \in S$, *then*

$$\psi(t + s, p) = \varphi(t, \psi(s, p)) .$$

Proof of (4′). We have $-\varrho_1(p) < s < \varrho_2(p) = \Delta$ and $-\varrho_1(p) < t + s < \varrho_2(p) = \Delta$. Also $\delta(\psi(s, p)) = s \geqslant 0$. Hence $\varrho_2(\psi(s, p)) = \Delta - s$. Therefore $-\varrho_1(\varphi(s, p)) < t < \Delta - s = \varrho_2(\psi(s, p))$. Hence $\psi(t + s, p) = \varphi(t + s, p) = \varphi(t, \varphi(s, p)) = \varphi(t, \psi(s, p))$, which proves (4′).

(5′) *The map* $\psi\colon \boldsymbol{R}[0, \Delta) \times S \to M[0, \Delta)$ *is injective.*

Proof of (5′). If $t \in \boldsymbol{R}[0, \Delta)$ and $p \in S$, then $\psi(t, p) \in M[0, \Delta)$ by (3′). Assume that $\psi(t_0, p_0) = \psi(t_1, p_1)$ where (t_0, p_0) and (t_1, p_1) belong to $\boldsymbol{R}[0, \Delta) \times S$. Then $t_0 = \delta(\psi(t_0, p_0)) = \delta(\psi(t_1, p_1)) = t_1$. Because $\psi(\square, p_0)$ and $\psi(\square, p_1)$ are integral curves of Y with $\psi(t_0, p_0) = \psi(t_1, p_1)$ we have $\psi(t, p_0) = \psi(t, p_1)$ for all $t \in \boldsymbol{R}(-\eta, \Delta)$. Hence $p_0 = \psi(0, p_0) = \psi(0, p_1) = p_1$. The map ψ is injective on $\boldsymbol{R}[0, s) \times S$, which proves (5′).

(6′) *The map* ψ *is locally diffeomorphic at every point* $(t_0, p_0) \in \boldsymbol{R}[0, \Delta) \times S$.

Proof of (6′). Define $p_1 = \psi(t_0, p_0) \in M[0, \Delta)$. Then $\delta(p_1) = t_0 > 0$ and $d\delta(p_1) \neq 0$. The map $d\varphi_{t_0}(p_0)\colon T_{p_0}(M) \to T_{p_1}(M)$ is a linear isomorphism. Now $\varphi_{t_0} = \psi_{t_0}\colon S \to M\langle t_0\rangle$ implies that

$$d\psi_{t_0}(p_0)\colon T_{p_0}(S) \to T_{p_1}(M\langle t_0\rangle)\ .$$

This injective linear map is an isomorphism since $T_{p_0}(S)$ and $T_{p_1}(M\langle t_0\rangle)$ have dimension $m - 1$. Identify the tangent space of $\boldsymbol{R}$ at t_0 with $\boldsymbol{R}$ such that $\partial/\partial t = 1$ at t_0. Then

$$\boldsymbol{R} \oplus T_{p_0}(S) = T_{(t_0, p_0)}(\boldsymbol{R} \times S)\ .$$

Also $d\psi(t_0, p_0)|T_{p_0}(S) = d\psi_{t_0}(p_0)|T_{p_0}(S)$. Hence the image of $d\psi(t_0, p_0)$ contains the linear subspace $T_{p_1}(M\langle t_0\rangle)$. Here

$$T_{p_1}(M\langle t_0\rangle) = \{v \in T_{p_1}(M) | d\delta(p_1, v) = 0\}\ .$$

Therefore $Y(p_1) \in T_{p_1}(M) - T_{p_1}(M\langle t_0\rangle)$ where

$$d\psi\left((t_0, p_0), \frac{\partial}{\partial t}(t_0)\right) = \dot{\psi}(t_0, p_0) = Y(\psi(t_0, p_0)) = Y(p_1)\,.$$

Hence $d\psi(t_0, p_0)\colon T_{(t_0, p_0)}(\boldsymbol{R} \times S) \to T_{p_1}(M)$ is surjective and for dimensions reason a linear isomorphism, which proves (6′).

(7′) *The map* $\psi\colon \boldsymbol{R}[0, \Delta) \times S \to M[0, \Delta)$ *is a diffeomorphism.*

Proof of (7′). Let q be an accumulation point of $\psi(\boldsymbol{R}[0, \Delta) \times S)$. A sequence $\{(t_\nu, p_\nu)\}_{\nu \in \boldsymbol{N}}$ of points $(t_\nu, p_\nu) \in \boldsymbol{R}[0, \Delta) \times S$ exists such that $\psi(t_\nu, p_\nu) \to q$ for $\nu \to \infty$. Since S is compact, we may assume that $p_\nu \to p \in S$ for $\nu \to \infty$. Then

$$t_\nu = \delta(\psi(t_\nu, p_\nu)) \to \delta(q) \qquad \text{for } \nu \to \infty$$
$$\psi(\delta(q), p) = \lim_{\nu \to \infty} \psi(t_\nu, p_\nu) = q\,.$$

Hence $\psi(\boldsymbol{R}[0, \Delta) \times S)$ is closed in $M[0, \Delta)$. By (6′), $\psi(\boldsymbol{R}[0, \Delta) \times S)$ is not empty and open in the connected space $M[0, \Delta)$. Hence

$$\psi\colon \boldsymbol{R}[0, \Delta) \times S \to M[0, \Delta)$$

is a surjective map and consequently a diffeomorphism, which proves (7′)

If $t \in \boldsymbol{R}[0, \Delta)$, then $\psi_t\colon S \to M\langle t \rangle$ is an injective, local diffeomorphism. If $q \in M\langle t \rangle$, then $q = \psi(s, p)$ for some $(s, p) \in \boldsymbol{R}[0, \Delta) \times S$. Hence $s = \delta(\psi(s, p)) = \delta(q) = t$. Therefore $\psi_t(p) = q$. The map $\psi_t\colon S \to M\langle t \rangle$ is also surjective and consequently a diffeomorphism; q.e.d.

Obviously, the number $\eta > 0$ can be taken so small that

$$\psi\colon \boldsymbol{R}(-\eta, \Delta) \times S \to M$$

is a diffeomorphism onto the open image set.

b) Proofs of the properties of the double.

Proof of property (7): Define $\hat{U} = \varrho^{-1}(U_{\mathfrak{x}})$ and $\hat{U}_j = \hat{U} \wedge N_j$ for $j = 1, 2$. Define $\tilde{U} = \mathfrak{r}^{-1}(U'_{\mathfrak{x}})$ and $\tilde{U}_1 = (\boldsymbol{R}^- \times S) \cap \tilde{U}$ and $\tilde{U}_2 = (\boldsymbol{R}^+ \times S) \cap \tilde{U}$. Then $S_0 \subseteq \hat{U}$ and $\tilde{S}_0 = \{0\} \times S \subseteq \tilde{U}$. Define $U_* = U_{\mathfrak{x}} - \{0\}$ and $U'_* = U'_{\mathfrak{x}} - \{0\}$. The restrictions

$$\varrho_j = \varrho | \hat{U}_j\colon \hat{U}_j \to U_* \qquad \mathfrak{r}_j = \mathfrak{r} | \tilde{U}_j\colon \tilde{U}_j \to U'_*$$

are diffeomorphisms with

$$\mathfrak{r}_1^{-1}(\mathfrak{y}) = \left(-|\mathfrak{y}|, \frac{-\mathfrak{y}}{|\mathfrak{y}|}\right) \qquad \mathfrak{r}_2^{-1}(\mathfrak{y}) = \left(|\mathfrak{y}|, \frac{\mathfrak{y}}{|\mathfrak{y}|}\right)$$

for all $\mathfrak{y} \in U'_*$. A diffeomorphism $f\colon \hat{U} - S_0 \to \tilde{U} - \tilde{S}_0$ is defined by setting $f = \mathfrak{r}_1^{-1} \circ \mathfrak{x} \circ \varrho_j$ on $\hat{U}_j$: Then $f(\hat{U}_j) = \tilde{U}_j$ and $\mathfrak{r} \circ f = \mathfrak{x} \circ \varrho$.

We have to show that f extends to a diffeomorphism $f\colon \hat{U} \to \tilde{U}$. Take $a \in S_0$. A patch $\mathfrak{z}\colon U_{\mathfrak{z}} \to U'_{\mathfrak{z}}$ at a exists satisfying the properties

(α) *We have* $a \in U_{\mathfrak{z}} \subset \hat{U}$ *and* $\mathfrak{z}(a) = 0 \in U'_{\mathfrak{z}} \subseteq \mathbf{R}^m$.

(β) *Open connected subsets* V *of* $\mathbf{R}$ *with* $0 \in V$ *and* W *of* $\mathbf{R}^{m-1}$ *with* $0 \in W$ *exist such that* $U'_{\mathfrak{z}} = V \times W$.

(γ) Define $V_1 = \mathbf{R}^- \cap V$ and $V_2 = \mathbf{R}^+ \cap V$, then $\mathfrak{z}(\hat{U}_j \cap U_{\mathfrak{z}}) = V_j \times W$ for $j = 1, 2$ and $\mathfrak{z}(S_0 \cap U_{\mathfrak{z}}) = \{0\} \times W$.

Define $h = \mathfrak{x} \circ \varrho \circ \mathfrak{z}^{-1}\colon U'_{\mathfrak{z}} \to U'_{\mathfrak{z}}$. If $y \in W$, then $h(t, y) = 0$. Therefore a map $g\colon U'_{\mathfrak{z}} \to \mathbf{R}^m$ of class C^∞ exists such that $h(t, y) = tg(t, y)$ for all $(t, y) \in V \times \times W = U'_{\mathfrak{z}}$. Then $g = (g^1, \ldots, g^m)$ and $h = (h^1, \ldots, h^m)$. If $y \in W$, write $y = (y_2, \ldots, y_m)$. The Jacobian determinant of ϱ in respect to these coordinates is given by

$$\begin{aligned}\Delta(t, y) &= \det\big(h_t(t, y), h_{y_2}(t, y), \ldots, h_{y_m}(t, y)\big)\\ &= t^{m-1} \det\big(g(t, y), g_{y_2}(t, y), \ldots, g_{y_m}(t, y)\big)\\ &\quad + t^m \det\big(g_t(t, y), g_{y_2}(t, y), \ldots, g_{y_m}(t, y)\big)\,.\end{aligned}$$

Because ϱ branches of order 0, we conclude that

$$\det\big(g(0, y), g_{y_2}(0, y), \ldots, g_{y_m}(0, y)\big) \neq 0 \quad \forall y \in W\,.$$

The vectors $g(0, y)$, $g_{y_2}(0, y), \ldots, g_{y_m}(0, y)$ are linearly independent for each $y \in W$; in particular $g(0, y) \neq 0$ for all $y \in W$.

A map $F\colon U'_{\mathfrak{z}} \to \mathbf{R} \times S$ of class C^∞ is defined by

$$F(t, y) = \left(t|g(t, y)|, \frac{g(t, y)}{|g(t, y)|}\right)$$

for all $(t, y) \in V \times W = U'_{\mathfrak{z}}$. Let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathbf{R}^m$. Then

$$\begin{aligned}F_t(0, y) &= \left(|g(0, y)|, \frac{g_t(0, y)}{|g(0, y)|} - \frac{\langle g_t(0, y), g(0, y)\rangle}{|g(0, y)|^3}\, g(0, y)\right)\\ F_{y_\mu}(0, y) &= \left(\quad 0 \quad, \frac{g_{y_\mu}(0, y)}{|g(0, y)|} - \frac{\langle g_{y_\mu}(0, y), g(0, y)\rangle}{|g(0, y)|^3}\, g(0, y)\right).\end{aligned}$$

Assume that $\lambda_1 F_t(0, y) + \lambda_2 F_{y_2}(0, y) + \ldots + \lambda_m F_{y_m}(0, y) = 0$. Then $\lambda_1 = 0$. Define $\mathfrak{a} = \lambda_2 g_{y_2}(0, y) + \ldots + \lambda_m g_{y_m}(0, y)$. Then

$$\mathfrak{a}|g(0, y)|^2 = \langle \mathfrak{a}, g(0, y)\rangle g(0, y)$$

which implies $\lambda_2 = \ldots = \lambda_m = 0$. The map F is a local diffeomorphism at every point of $\{0\} \times W$. Hence $f = F \circ \mathfrak{z}$ is a local diffeomorphis mat every point of S_0.

If $x \in \hat{U}_1 \cap U_{\mathfrak{z}}$, then $\mathfrak{z}(x) = (t, y)$ with $0 > t \in V$ and $y \in W$. Therefore

$$\hat{f}(x) = F(t, y) = \left(-|tg(t, y)|, \frac{-tg(t, y)}{|tg(t, y)|}\right) = \left(-|h(t, y)|, \frac{-h(t, y)}{|h(t, y)|}\right)$$
$$= \mathfrak{r}_1^{-1}(h(t, y)) = \mathfrak{r}_1^{-1}\circ\mathfrak{x}\circ\varrho_1\circ\mathfrak{z}^{-1}(t, y) = f(x).$$

If $x \in \hat{U}_2 \cap U_{\mathfrak{z}}$, then $\mathfrak{z}(x) = (t, y)$ with $0 < t \in V$ and $y \in W$. Therefore

$$\hat{f}(x) = F(t, y) = \left(|tg(t, y)|, \frac{tg(t, y)}{|tg(t, y)|}\right) = \left(|h(t, y)|, \frac{h(t, y)}{|h(t, y)|}\right)$$
$$= \mathfrak{r}_2^{-1}(h(t, y)) = \mathfrak{r}_2^{-1}\circ\mathfrak{x}\circ\varrho_2\circ\mathfrak{z}^{-1}(t, y) = f(x).$$

Hence $\hat{f}$ extends f onto $U_{\mathfrak{z}}$ with $\hat{f}(U_{\mathfrak{z}} \cap S_0) \subseteq \tilde{S}_0$. Hence f extends to a local diffeomorphism $f\colon \hat{U} \to \tilde{U}$. Because $f\colon \hat{U} - S_0 \to \tilde{U} - \tilde{S}_0$ is injective. Also $f\colon \hat{U} \to \tilde{U}$ is injective. Also $f\colon S_0 \to \tilde{S}_0$ is a local diffeomorphism. Hence $f\colon S_0 \to \tilde{S}_0$ is a diffeomorphism and $f\colon \hat{U} \to \tilde{U}$ is surjective. Therefore $f\colon \hat{U} \to \tilde{U}$ is a diffeomorphism with $\mathfrak{r}\circ f = \mathfrak{x}\circ\varrho$ and $f(\hat{U}_j) = \tilde{U}_j$ for $j = 1, 2$. Consequently, f is an isomorphism of doubles over $\mathfrak{x}$; q.e.d.

Now, property (8) follows trivially.

Proof of Property (9): Take a patch $\mathfrak{x}\colon U_{\mathfrak{x}} \to U'_{\mathfrak{x}}$ of M at 0 with $\mathfrak{x}(0) = 0$. Assume that $U_{\mathfrak{x}}$ is connected. Define

$$M_0 = \mathfrak{r}^{-1}(U'_{\mathfrak{x}}) \qquad A_{10} = (\boldsymbol{R}^- \times S) \cap M_0 \qquad A_{20} = (\boldsymbol{R}^+ \times S) \cap M$$
$$U_* = U_{\mathfrak{x}} - \{0\} \qquad U'_* = U'_{\mathfrak{x}} - \{0\} \qquad M_1 = M_2 = M_*$$

and regard M_1 and M_2 as disjoint copies. Also let A_{0j} equal U_* as a subset of M_j for $j = 1, 2$. Define $A_{12} = \emptyset = A_{21}$ and $A_{jj} = M_j$ for $j = 1, 2$. Then $\mathfrak{r}_j = \mathfrak{r}|A_{j0} \to U'_*$ is a diffeomorphism. Define the diffeomorphisms $\gamma_{j0} = \mathfrak{x}^{-1}\circ\mathfrak{r}_j\colon A_{j0} \to A_{0j}$ and $\gamma_{0j} = \gamma_{j0}^{-1}$ for $j = 1, 2$. Define $A_{jj} = M_j$ and let $\gamma_{jj}\colon A_{jj} \to A_{jj}$ be the identity for $j = 1, 2$. The assumptions of Theorem 17.1 in [1] are satisfied in the differentiable case. Therefore there exists a connected, differentiable manifold N, open subsets N_j and diffeomorphisms $\gamma_j\colon M_j \to N_j$ such that $N = N_0 \cup N_1 \cup N_2$ and such that $\gamma_j(A_{ij}) = \gamma_i(A_{ji}) = N_i \cap N_j$ if $A_{ij} \neq \emptyset$ and such that $\gamma_i^{-1}\circ\gamma_j = \gamma_{ij}\colon A_{ij} \to A_{ji}$ if $A_{ij} = \emptyset$. For $j = 1, 2$, the map

$$\varrho_j = \gamma_j^{-1}\colon N_j \to M_j = M_*$$

is a diffeomorphism. The map

$$\varrho_0 = \mathfrak{x}^{-1}\circ\mathfrak{r}\circ\gamma_0^{-1}\colon N_0 \to U_{\mathfrak{x}}$$

is of class C^∞. Observe $N_1 \cap N_2 = \emptyset$. Take $x \in N_0 \cap N_j$ with $j = 1$ or 2. Then $x = \gamma_j(y)$ with $y \in A_{0j}$ and

$$\gamma_0^{-1}(x) = \gamma_0^{-1}(\gamma_j(y)) = \gamma_{0j}(y) = \mathfrak{r}_j^{-1}(\mathfrak{x}(y)) .$$

Hence $\varrho_0(x) = \mathfrak{x}^{-1}\circ\mathfrak{r}\circ\gamma_0^{-1}(x) = y = \gamma_j^{-1}(x) = \varrho_j(x)$. One and only one map ϱ of class C^∞ is defined such that $\varrho|N_j = \varrho_j$ for $j = 0, 1, 2$. In particular $\mathfrak{x}\circ\varrho_0 = \mathfrak{r}\circ\gamma_0^{-1}$. Hence

$$S_0 = \varrho^{-1}(0) = \varrho_0^{-1}(0) = \gamma_0\circ\mathfrak{r}^{-1}(0) = \gamma_0(\{0\}\times S)$$

is a proper, compact, differentiable submanifold of N such that $N = N_1 \cup \cup N_2 \cup S_0$ is a disjoint union. The restrictions

$$\varrho_j = \varrho|N_j\colon N_j \to M_*$$

are diffeomorphisms. Hence $\varrho(N) = \varrho_0(S_0) \cup \varrho_1(N_1) = \varrho_2(N_2) = M$. The map ϱ is surjective.

Let K be a compact subset of M. Let V be an open neighborhood of $0 \in M$ with $V \subseteq \overline{V} \subset U_\mathfrak{x}$ such that $\overline{V}$ is compact. Then $K_0 = K \cap \overline{V}$ and $K' = K - V$ are compact subsets of N with $K = K_0 \cup K'$. Here $K_0 \subset U_\mathfrak{x}$ and $K' \subset M_*$. Hence $\tilde{K}_j = \varrho_j^{-1}(K')$ are compact. Since $\mathfrak{x}\circ\varrho_0 = \mathfrak{r}\circ\gamma_0^{-1}$, the map $\varrho_0\colon M_0 \to U_\mathfrak{x}$ is proper. Hence $\tilde{K}_0 = \varrho_0^{-1}(K_0)$ is compact. Then $\varrho^{-1}(K) = = \tilde{K}_0 \cup \tilde{K}_1 \cup \tilde{K}_2$ is compact. Therefore ϱ is proper. Because $\mathfrak{x}\circ\varrho_0 = \mathfrak{r}\circ\gamma_0^{-1}$, the map $\varrho_0 = \varrho|N_0$ branches of order 0 on $S_0 \subseteq N_0$. Therefore (N, ϱ) is a double of M at 0; q.e.d.

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