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L^p regularity for weak solutions of parabolic systems

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L^p Regularity for Weak Solutions of Parabolic Systems.

SERGIO CAMPANATO

Introduction.

Let Ω be a bounded open set of R^n $(n \geqslant 2)$ with a sufficiently regular boundary $\partial\Omega$, for convenience of class C^∞. Let $T > 0$ and $Q = \Omega \times (-T, 0)$; m and N are integers $\geqslant 1$, $(\ |\ )$ and $\|\cdot\|$ are the scalar product and the norm in R^N.

$H^{k,p}(\Omega, R^N)$ and $H_0^{k,p}(\Omega, R^N)$, k integer, are the usual Sobolev spaces of the vectors $u: \Omega \to R^N$; if $p = 2$ we shall write more briefly H^k and H_0^k.

If $u \in H^{k,p}(\Omega, R^N)$ we define

$$
(0.1) \qquad
\begin{aligned}
|u|_{k,p,\Omega} &= \left\{ \int_\Omega \Big(\sum_{|\alpha|=k} \|D^\alpha u\|^2 \Big)^{p/2} dx \right\}^{1/p} \\
\|u\|_{k,p,\Omega} &= \left\{ \int_\Omega \Big(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|^2 \Big)^{p/2} dx \right\}^{1/p} .
\end{aligned}
$$

If $p = 2$ we shall write more briefly $|\cdot|_{k,\Omega}$ and $\|\cdot\|_{k,\Omega}$.

If X is a Banach space and $0 < \theta < 1$, $H^\theta(a, b, X)$ is the space of the vectors $u \in L^2(a, b, X)$ such that

$$
(0.2) \qquad [u]^2_{H^\theta(a,b,X)} = \int_a^b dt \int_a^b \frac{\|u(t) - u(\xi)\|_X^2}{|t - \xi|^{1+2\theta}} d\xi < +\infty
$$

with the usual norm.

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Let $A_{\alpha\beta}(x,t)$, $|\alpha|=m$, $|\beta|=m$, be $N\times N$ matrices defined in Q with

$$(0.3)\qquad A^{hk}_{\alpha\beta}\in L^\infty(Q)\quad\text{and}\quad \Big\{\sum_{hk}\sum_{|\alpha|=m}\sum_{|\beta|=m}(A^{hk}_{\alpha\beta})^2\Big\}^{\frac12}\leqslant M$$

$$(0.4)\qquad \sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}\xi^\beta|\xi^\alpha)\geqslant \nu\sum_{|\alpha|=m}\|\xi^\alpha\|^2$$

for any system $\{\xi^\alpha\}_{|\alpha|=m}$ of vectors of R^N.

Let $f^\alpha(x,t)\in L^2(Q,R^N)$, $|\alpha|\leqslant m$; we consider the Cauchy-Dirichlet problem

$$(0.5)\qquad (-1)^m\sum_{|\alpha|=m}\sum_{|\beta|=m}D^\alpha(A_{\alpha\beta}D^\beta u)+\frac{\partial u}{\partial t}=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha f^\alpha\qquad\text{in } Q$$

$$(0.6)\qquad \begin{aligned}&D^\alpha u=0\quad\text{on } \partial\Omega\times(-T,0),\ |\alpha|\leqslant m-1\\ &u(x,-T)=0\quad\text{for } x\in\Omega\,.\end{aligned}$$

A weak solution of this problem means a vector $u\in L^2(-T,0,H^m(\Omega,R^N))$ such that

$$(0.7)\qquad \int_Q\sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}D^\beta u|D^\alpha\varphi)-\left(u\,\Big|\,\frac{\partial\varphi}{\partial t}\right)dx\,dt=\int_Q\sum_{|\alpha|\leqslant m}(f^\alpha|D^\alpha\varphi)\,dx\,dt$$

$$\forall\varphi\in W_2(Q):\varphi(x,0)=0$$

where

$$(0.8)\qquad W_p(Q)=L^p(-T,0,H^{m,p}_0(\Omega,R^N))\cap H^{1,p}(-T,0,L^p(\Omega,R^N))\,.$$

Further, in general, we write $\delta u=\{D^\alpha u\}_{|\alpha|\leqslant m-1}$ and $Du=\{D^\alpha u\}_{|\alpha|\leqslant m}$ and let

$$f^\alpha=f^\alpha(x,t,\delta u)\qquad\text{for } |\alpha|=m$$

$$f^\alpha=f^\alpha(x,t,Du)\qquad\text{for } |\alpha|\leqslant m-1$$

where f^α are vectors in R^N which are measurable in (x,t), continuous in δu and Du respectively and which satisfy the inequalities [1]

$$(0.9)\qquad \begin{aligned}&\|f^\alpha(x,t,\delta u)\|\leqslant C\Big\{\|g^\alpha\|+\sum_{|\alpha|\leqslant m-1}\|D^\alpha u\|\Big\}\quad\text{for } |\alpha|=m\\ &\|f^\alpha(x,t,Du)\|\leqslant C\Big\{\|g^\alpha\|+\sum_{|\alpha|\leqslant m}\|D^\alpha u\|\Big\}\quad\text{for } |\alpha|\leqslant m-1\end{aligned}$$

(1) We can perhaps impose a more general growth condition, but we shall not deal with this question in this paper.

with $g^\alpha(x, t) \in L^2(Q, R^N)$. We consider solutions $u \in L^2(-T, 0, H_0^m(\Omega, R^N))$ of problem (0.7) with f^α which satisfy (0.9). In this situation have included, in particular, also the linear parabolic systems with the elliptic part not reduced to the principal part alone as in (0.5).

It is well known (see for instance [6], [8]) that if $f^\alpha(x, t) \in L^2(Q, R^N)$ the problem (0.7) has a unique solution and

$$\int_{-T}^{0} \|u\|_{m,\Omega}^2 dt \leqslant C \int_Q \sum_{|\alpha| \leqslant m} \|f^\alpha\|^2 dx\, dt . \tag{0.10}$$

We shall prove in section 1 that $\exists p_0 > 2$ such that if $p \in [2, p_0)$ and

$$\begin{aligned} &f^\alpha(x, t) \in L^p(Q, R^N) && \text{for } |\alpha| = m \\ &f^\alpha(x, t) \in L^p(-T, 0, L^2(\Omega, R^N)) && \text{for } |\alpha| \leqslant m-1 \end{aligned} \tag{0.11}$$

then

$$D^\alpha u \in L^p(Q, R^N) \qquad \text{for } |\alpha| = m \tag{0.12}$$

and the estimate (1.21) holds.

This result is known ([10], [11]) in the particular case when $m = N = 1$ and $f^\alpha = 0$ for $|\alpha| \leqslant m - 1$. This last restriction, however, would not, even in the case $m = N = 1$, permit us to obtain the result (0.12) when f^α satisfy (0.9) and g^α satisfy the conditions (0.11) [see section 3] and hence would not lead us to obtain the L^p local regularity for weak solutions of the system (0.5), namely for vectors $u \in L^2(-T, 0, H^m(\Omega, R^N))$ which satisfy (0.7) $\forall \varphi \in C_0^\infty(Q, R^N)$ [see section 4].

In order to obtain the results of sections 3 and 4 stated above, we must first prove some L^p regularity results of this kind:

Let u be a solution of (0.7); if $f^\alpha \in L^2(Q, R^N)$, $|\alpha| \leqslant m$, then, for every α with $0 \leqslant |\alpha| = j \leqslant m - 1$,

$$D^\alpha u \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{with } 2 \leqslant p < \frac{2m}{j} . \tag{0.13}$$

There exists a $p_0 > 2$ such that if $p \in [2, p_0)$ and

$$\begin{aligned} &f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) && \text{if } |\alpha| = m \\ &f^\alpha \in L^2(Q, R^N) && \text{if } |\alpha| \leqslant m-1 \end{aligned} \tag{0.14}$$

then

$$u \in L^p(-T, 0, H_0^m(\Omega, R^N)) \tag{0.15}$$

and

$$(0.16) \qquad D^\alpha u \in L^p(Q, R^N) \quad \text{if } |\alpha| \leqslant m-1\,.$$

These results are proved in section 2 and are also of interest in themselves.

1. – Linear systems. L^p-regularity.

The main result of this section is theorem 1.I which we obtain by proving some lemmas which have an interest in themselves.

LEMMA 1.I. *If* $A_{\alpha\beta}$, $|\alpha| = m$, $|\beta| = m$, *are* $N \times N$ *matrices which satisfy the conditions* (0.3), (0.4) *and if* $K = M(1+\xi)$ *with* $\xi > 0$, *for every system* $\{\lambda^\alpha\}_{|\alpha|=m}$ *of vectors* $\lambda^\alpha \in R^N$ *we have*

$$(1.1) \qquad \left\{\sum_{|\alpha|=m}\left\| K\lambda^\alpha - \sum_{|\beta|=m} A_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12} \leqslant [M(1+\sqrt{1+\xi^2})-\nu]\cdot\left\{\sum_{|\alpha|=m}\|\lambda^\alpha\|^2\right\}^{\frac12}.$$

PROOF. Let $A^*_{\alpha\beta}$ be the adjoint of the matrix $A_{\alpha\beta}$. We denote

$$A^+_{\alpha\beta} = \tfrac12(A_{\alpha\beta} + A^*_{\beta\alpha})\,, \qquad A^-_{\alpha\beta} = \tfrac12(A_{\alpha\beta} - A^*_{\beta\alpha})\,.$$

Then

$$(1.2) \qquad \left\{\sum_{|\alpha|=m}\left\| K\lambda^\alpha - \sum_{|\beta|=m} A_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12} \leqslant$$

$$\leqslant \left\{\sum_{|\alpha|=m}\left\| M\lambda^\alpha - \sum_{|\beta|=m} A^+_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12} + \left\{\sum_{|\alpha|=m}\left\| \xi M\lambda^\alpha - \sum_{|\beta|=m} A^-_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12}.$$

But

$$\left\{\sum_{|\alpha|=m}\left\| M\lambda^\alpha - \sum_{|\beta|=m} A^+_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12} =$$

$$= \sup_\eta\left\{M\sum_{|\alpha|=m}(\lambda^\alpha|\eta^\alpha) - \sum_{|\alpha|=m}\sum_{|\beta|=m}(A^+_{\alpha\beta}\lambda^\beta|\eta^\alpha)\right\} = \sup_\eta \mathcal{A}(\{\lambda^\alpha\},\{\eta^\alpha\})$$

where $\sup_\eta$ is taken over all the vectors $\{\eta^\alpha\}_{|\alpha|=m}$, $\eta^\alpha \in R^N$, such that $\sum_{|\alpha|=m}\|\eta^\alpha\|^2 = 1$.

Since $\mathcal{A}$ is a symmetric and positive bilinear form

$$\mathcal{A}(\{\lambda^\alpha\},\{\eta^\alpha\}) \leqslant \sqrt{\mathcal{A}(\{\lambda^\alpha\},\{\lambda^\alpha\})}\sqrt{\mathcal{A}(\{\eta^\alpha\},\{\eta^\alpha\})} \leqslant$$

$$\leqslant (M-\nu)\left\{\sum_{|\alpha|=m}\|\lambda^\alpha\|^2\right\}^{\frac12}\cdot\left\{\sum_{|\alpha|=m}\|\eta^\alpha\|^2\right\}^{\frac12}.$$

Hence

$$\left\{\sum_{|\alpha|=m}\left\| M\lambda^\alpha - \sum_{|\beta|=m} A^+_{\alpha\beta}\lambda^\beta\right\|^2\right\}^{\frac12} \leqslant (M-\nu)\left\{\sum_{|\alpha|=m}\|\lambda^\alpha\|^2\right\}^{\frac12}. \tag{1.3}$$

Moreover as

$$\sum_{|\alpha|=m}\sum_{|\beta|=m}(A^-_{\alpha\beta}\lambda^\beta|\lambda^\alpha) = 0$$

we have

$$\sum_{|\alpha|=m}\left\|\xi M\lambda^\alpha - \sum_{|\beta|=m} A^-_{\alpha\beta}\lambda^\beta\right\|^2 = M^2\xi^2\sum_{|\alpha|=m}\|\lambda^\alpha\|^2 + \sum_{|\alpha|=m}\left\|\sum_{|\beta|=m}A^-_{\alpha\beta}\lambda^\beta\right\|^2 \leqslant \leqslant M^2(1+\xi^2)\sum_{|\alpha|=m}\|\lambda^\alpha\|^2 \tag{1.4}$$

(1.1) follows from (1.2), (1.3), (1.4).

REMARK 1.I. If $A_{\alpha\beta} = A^*_{\alpha\beta}$, $|\alpha| = m$, $|\beta| = m$, we can assume $K = M$ and it follows from (1.3) that the inequality (1.1) holds with the constant $(M-\nu)$ on the right-hand side.

Let u be the solution of the problem (0.7) with the conditions (0.3), (0.4).

LEMMA 1.II. *There exists a* $p_0 > 2$ *such that if* $p \in [2, p_0)$ *and if*

$$\begin{aligned} &f^\alpha \in L^p(Q, R^N) \qquad &&\text{for } |\alpha| = m, \\ &f^\alpha = 0 &&\text{for } |\alpha| \leqslant m-1 \end{aligned} \tag{1.5}$$

then $u \in L^p(-T, 0, H^{m,p}_0(\Omega, R^N))$ *and*

$$\int_{-T}^{0} |u|^p_{m,p,\Omega}\,dt \leqslant C\int_\Omega\left(\sum_{|\alpha|=m}\|f^\alpha\|^2\right)^{p/2} dx\,dt. \tag{1.6}$$

PROOF. The idea of the proof is standard (see for instance [9]). We obtain the result by proving the following theorem of existence and uniqueness:

There exists a $p_0 > 2$ such that $\forall p \in [2, p_0)$ the problem [2]

$$\begin{aligned} &u \in L^p(-T, 0, H^{m,p}_0(\Omega, R^N)) \\ &\int_Q \sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}D^\beta u|D^\alpha\varphi) - \left(u\,\middle|\,\frac{\partial\varphi}{\partial t}\right) dx\,dt = \int_Q\sum_{|\alpha|=m}(f^\alpha|D^\alpha\varphi)\,dx\,dt \end{aligned} \tag{1.7}$$

has a unique solution and moreover (1.6) holds.

[2] $1/p + 1/p' = 1$.

Note that, $\forall p \geqslant 2$, if $F^\alpha \in L^p(Q, R^N)$ and if $K > 0$ then the problem

$$(1.8) \qquad \begin{aligned} & U \in L^p(-T, 0, H_0^{m,p}(\Omega, R^N)) \\ & \int_Q K \sum_{|\alpha|=m} (D^\alpha U | D^\alpha \varphi) - \left(U \,\middle|\, \frac{\partial \varphi}{\partial t}\right) dx\,dt = \int_Q \sum_{|\alpha|=m} (F^\alpha | D^\alpha \varphi)\, dx\,dt \\ & \forall \varphi \in W_{p'}(Q) : \varphi(x, 0) = 0 \end{aligned}$$

has a unique solution and

$$(1.9) \qquad \int_{-T}^{0} |U|^p_{m,p,\Omega}\, dt \leqslant g^p(p, K) \int_\Omega \left(\sum_{|\alpha|=m} \|F^\alpha\|^2 \right)^{p/2} dx\,dt\,.$$

We obtain this result, for instance, using the interpolation theorem of Stampacchia ([2], [12], [1] section 3) between these two situations:

If $p = 2$ there exists only one solution of problem (1.8) and

$$(1.10) \qquad \int_{-T}^{0} |U|^2_{m,\Omega}\, dt \leqslant \frac{1}{K} \int_Q \sum_{|\alpha|=m} \|F^\alpha\|^2\, dx\,dt.$$

If $F^\alpha \in L^\infty(Q, R^N)$ then $D^\alpha U \in \mathcal{E}(Q, R^N)$, $|\alpha| = m$, and

$$(1.11) \qquad \sum_{|\alpha|=m} [D^\alpha U]_{\mathcal{E}(Q,R^N)} \leqslant C \sum_{|\alpha|=m} \|F^\alpha\|_{L^\infty(Q,R^N)}$$

where $\mathcal{E}(Q, R^N)$ is the John-Nirenberg space on Q with respect to the parabolic metric (see [1])

$$(1.12) \qquad d(x, t; y, \tau) = \max\{\|x - y\|, |t - \tau|^{1/2m}\}\,.$$

The estimate (1.11) is obtained as in [1].

In (1.9) we can suppose that

$$(1.13) \qquad \lim_{p \to 2} g(p, K) = g(2, K)$$

and it is known that (see (1.10))

$$(1.14) \qquad g(2, K) \leqslant \frac{1}{K}\,.$$

Let

$$\xi > \frac{M^2 - \nu^2}{2M\nu} \tag{1.15}$$

and let, $\forall u \in L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$ with $p \geqslant 2$, $U = \mathcal{C}_p u$ be the solution of the problem (1.8) with

$$\begin{aligned} K &= M(1+\xi) \\ F^\alpha &= f^\alpha + KD^\alpha u - \sum_{|\beta|=m} A_{\alpha\beta} D^\beta u . \end{aligned}$$

We equip $H_0^{m,p}(\Omega, R^N)$ with the usual norm

$$\|u\|_{H_0^{m,p}(\Omega,\, R^N)} = |u|_{m,p,\Omega} .$$

There exists a $p_0 > 2$ such that, $\forall p \in [2, p_0)$, $\mathcal{C}_p$ is a contraction mapping on $L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$. In fact if $U = \mathcal{C}_p u$ and $V = \mathcal{C}_p v$ then from (1.9) and by lemma 1.I we have

$$\int_{-T}^{Q} |U - V|^p_{p,m,\Omega} dt \leqslant g^p(p, K) \int_Q \left\{ \sum_{|\alpha|=m} \left\| KD^\alpha(u-v) - \sum_{|\beta|=m} A_{\alpha\beta} D^\beta (u-v) \right\|^2 \right\}^{p/2} dx\, dt \leqslant$$

$$\leqslant g^p(p, K) \{ M(1+\sqrt{1+\xi^2}) - \nu \}^p \int_{-T}^{0} |u - v|^p_{m,p,\Omega} dt .$$

But, from (1.13), (1.14), (1.15), we have

$$\lim_{p \to 2} g(p, K) \{ M(1+\sqrt{1+\xi^2}) - \nu \} \leqslant \frac{M(1+\sqrt{1+\xi^2}) - \nu}{M(1+\xi)} < 1 .$$

Then $\exists$ a $p_0 > 2$ such that $\forall p \in [2, p_0)$ $\mathcal{C}_p$ is a contraction mapping of $L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$ into itself, therefore it has a fixed point $u = \mathcal{C}_p u$ which is obviously the solution of (1.7).

The inequality (1.6) follows in a standard way.

Let u be a solution of the problem (0.7) with the hypotheses (0.3), (0.4).

LEMMA 1.III. *If* $p \in [2, p_0 \wedge 2^*)$ (3) *and*

$$\begin{aligned} & f^\alpha = 0 && \textit{for } |\alpha| = m , \\ & f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) && \textit{for } |\alpha| \leqslant m-1 \end{aligned} \tag{1.16}$$

(3) $1/2^* = \frac{1}{2} - 1/n$.

then $u \in L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$ *and*

$$\int_{-T}^{0} |u|^p_{m,p,\Omega}\, dt \leqslant C \int_{-T}^{0} dt \left(\int_Q \sum_{|\alpha| \leqslant m-1} \| f^\alpha \|^2 dx \right)^{p/2}. \tag{1.17}$$

PROOF. For any fixed α with $|\alpha| \leqslant m-1$, let U^α be the solution of the Dirichlet problem

$$U^\alpha \in H_0^{m-|\alpha|} \cap H^{2(m-|\alpha|)}(\Omega, R^N), \qquad (-1)^{m-|\alpha|} \sum_{|\beta| = m-|\alpha|} D^\beta D^\beta U^\alpha = f^\alpha .$$

It is known that

$$\| U^\alpha \|_{2(m-|\alpha|),\Omega} \leqslant C \| f^\alpha \|_{L^2(\Omega, R^N)}$$

and by the imbedding theorems with respect to x

$$\int_Q \left(\sum_{|\beta| = m-|\alpha|} \| D^\beta U^\alpha \|^2 \right)^{p/2} dx \leqslant C \| U^\alpha \|^p_{2(m-|\alpha|),\Omega} \leqslant C \| f^\alpha \|^p_{L^2(\Omega, R^N)} . \tag{1.18}$$

Then, under the hypothesis (1.16), the system (0.7) can be rewritten as follows:

$$\begin{aligned} &\int_Q \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta} D^\beta u | D^\alpha \varphi) - \left(u \,\middle|\, \frac{\partial \varphi}{\partial t} \right) dx\, dt = \\ &= \int_Q \sum_{|\alpha| \leqslant m-1} \sum_{|\beta| = |\alpha| - m} (D^\beta U^\alpha | D^{\alpha+\beta} \varphi)\, dx\, dt \qquad \forall \varphi \in W_2(Q) \colon \varphi(x, 0) = 0 . \end{aligned} \tag{1.19}$$

Since $|\alpha + \beta| = m$, by lemma 1.II and (1.18), we conclude that

$$\int_{-T}^{0} |u|^p_{m,p,\Omega}\, dt \leqslant C \int_Q \sum_{|\alpha| \leqslant m-1} \left(\sum_{|\beta| = m-|\alpha|} \| D^\beta U^\alpha \|^2 \right)^{p/2} dx\, dt \leqslant C \int_{-T}^{0} dt \left(\int_Q \sum_{|\alpha| m \leqslant -1} \| f^\alpha \|^2 dx \right)^{p/2}$$

and this is the required assertion.

We can now state the main theorem

THEOREM 1.I. *If u is a solution of* (0.7) *with the hypotheses* (0.3), (0.4), *there exists a p_0 with $2 < p_0 \leqslant 2^*$ such that if $p \in [2, p_0)$ and if*

$$\begin{aligned} &f^\alpha \in L^p(Q, R^N) && \text{for } |\alpha| = m , \\ &f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) && \text{for } |\alpha| \leqslant m-1 \end{aligned} \tag{1.20}$$

then $u \in L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$ *and*

$$\int_{-T}^{0} |u|^p_{m,p,\Omega}\, dt \leqslant C(Q) \int_Q \Big(\sum_{|\alpha|=m} \|f^\alpha\|^2\Big)^{p/2} dx\, dt + \int_{-T}^{0} dt \Big(\int_\Omega \sum_{|\alpha|\leqslant m-1} \|f^\alpha\|^2 dx\Big)^{p/2} \tag{1.21}$$

2. – Linear systems. Further results on L^p regularity.

We now prove some theorems on L^p regularity which have an interest in themselves and which are used to extend the result of theorem 1.I for solutions of a certain class of non linear parabolic systems and to obtain a similar local result for solutions of linear and non linear systems without boundary conditions.

Let u be the solution of the problem (0.7) with the conditions (0.3), (0.4).

THEOREM 2.1. *If* $0 \leqslant |\alpha| = j \leqslant m-1$ *then we have*

$$D^\alpha u \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{with } 2 \leqslant p < \frac{2m}{j} \tag{2.1}$$

and

$$\int_{-T}^{0} |u|^p_{j,\Omega}\, dt \leqslant C(Q) \Big\{ \int_Q \sum_{|\alpha|\leqslant m} \|f^\alpha\|^2 dx\, dt \Big\}^{p/2}. \tag{2.2}$$

PROOF. If u is the solution of (0.7) it is well known ([6], [8]) that there exists a $U \in L^2(R, H_0^m(\Omega, R^N))$ such that

$$\begin{aligned} &U = u \quad \text{on } Q \\ &\int_R \|U\|^2_{m,\Omega} dt + \int_R |\tau| \|\hat{U}\|^2_{L^2(\Omega,R^N)} d\tau \leqslant \int_Q \sum_{|\alpha|\leqslant m} \|f^\alpha\|^2 dx\, dt \end{aligned} \tag{2.3}$$

where $\frown$ is the Fourier transform in t.

On the other hand if $0 \leqslant j \leqslant m-1$ and $\varepsilon > 0$

$$|\hat{U}|^2_{j,\Omega} \leqslant \varepsilon \|\hat{U}\|^2_{m,\Omega} + c\varepsilon^{j/(j-m)} \|\hat{U}\|^2_{L^2(\Omega,R^N)}$$

where the constant c does not depend on ε. Therefore choosing $\varepsilon = |\tau|^{(j-m)/m}$

$$\int_R |\tau|^{(m-j)/m} |\hat{U}|^2_{j,\Omega} dt \leqslant \int_R \|U\|^2_{m,\Omega} dt + c\int_R |\tau| \|\hat{U}\|^2_{L^2(\Omega,R^N)} d\tau .$$

Then, if $0 \leqslant |\alpha| = j \leqslant m-1$,

$$(2.5)\qquad \begin{aligned} & D^\alpha u \in H^{\theta_j}(-T, 0, L^2(\Omega, R^N)) \qquad \text{with } \theta_j = \frac{1}{2}\frac{m-j}{m} \\ & \text{and} \\ & [D^\alpha u]^2_{H^{\theta_j}(-T,0,L^2(\Omega,R^N))} \leqslant C\int_Q \sum_{|\alpha|\leqslant m} \|f^\alpha\|^2 dx\,dt . \end{aligned}$$

But from the imbedding theorems with respect to t, if $2 \leqslant p < 2m/j$ and $|\alpha| = j$ we have

$$(2.6)\qquad \|D^\alpha u\|_{L^p(-T,0,L^2(\Omega,R^N))} \leqslant C\{\|D^\alpha u\|_{L^2(\Omega,R^N)} + [D^\alpha u]_{H^{\theta_j}(-T,0,L^2(\Omega,R^N))}\} .$$

Then (2.2) follows from the inequalities (2.6), (0.10), (2.5).

The lemma which we shall prove below generalizes theorem 2.III of [3].

LEMMA 2.I. *Let $F \in L^2(-T, 0, H^{-j}(\Omega, R^N))$, $0 \leqslant j \leqslant m$, and let v be the solution of the problem*

$$(2.7)\qquad \begin{aligned} & v \in L^2(-T, 0, H_0^m(\Omega, R^N)) \\ & \int_Q \sum_{|\alpha|=m} (D^\alpha v | D^\alpha \varphi) - \left(v \,\middle|\, \frac{\partial \varphi}{\partial t}\right) dx\,dt = \int_{-T}^{0} \langle F, \varphi\rangle dt \quad (4) \\ & \forall \varphi \in W_2(Q) : \varphi(x, 0) = 0 \end{aligned}$$

then

$$(2.8)\qquad v \in H^{\theta_j}(-T, 0, H_0^m(\Omega, R^N)) \qquad \textit{with } \theta_j = \frac{m-j}{2m}$$

and

$$(2.9)\qquad \|v\|^2_{H^{\theta_j}(-T,0,H_0^m(\Omega,R^N))} \leqslant C\int_{-T}^{0} \|F\|^2_{H^{-j}(\Omega,R^N)} dt .$$

PROOF. It is well known (see (0.10)) that if $j = m$

$$(2.10)\qquad \int_{-T}^{0} |v|^2_{m,\Omega} dt \leqslant C\int_{-T}^{0} \|F\|^2_{H^{-m}(\Omega,R^N)} dt .$$

(4) $\langle\,,\rangle$ denotes the pairing between $H^{-j}(\Omega, R^N)$ and $H_0^j(\Omega, R^N)$.

It is proved in [3] (theorem 2.III) that if $j = 0$

$$v \in H^{\frac{1}{2}}(-T, 0, H_0^m(\Omega, R^N))$$

(2.11)
$$\|v\|^2_{H(-T,0,H_0^m(\Omega,R^N))} \leqslant C\int_{-T}^{0} \|F\|^2_{L^2(\Omega,R^N)} dt\,.$$

Then (2.8), (2.9) follow by interpolation (see [7] prop. 9.4 p. 302).

REMARK 2.I. By the imbedding theorems with respect to t and from (2.8), (2.9) it follows that, $\forall\ 2 \leqslant p < 2m/j$,

(2.12)
$$\|v\|^2_{L^p(-T,0,H_0^m(\Omega,R^N))} \leqslant C\int_{-T}^{0} \|F\|^2_{H^{-j}(\Omega,R^N)} dt\,.$$

Let u be the solution of the problem (0.7) under the assumptions (0.3), (0.4).

THEOREM 2.II. *There exists a* p_0, $2 < p_0 \leqslant 2m/(m-1)$, *such that if* $p \in [2, p_0)$ *and if*

(2.13)
$$f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad for\ |\alpha| = m$$

then $u \in L^p(-T, 0, H_0^m(\Omega, R^N))$ *and*

(2.14)
$$\int_{-T}^{0} |u|^p_{m,\Omega}\, dt \leqslant C(Q)\left\{\int_{-T}^{0} dt \left(\int_Q \sum_{|\alpha|=m} \|f^\alpha\|^2 dx\right)^{p/2} + \left[\int_Q \sum_{|\alpha|\leqslant m-1} \|f^\alpha\|^2 dx\, dt\right]^{p/2}\right\}.$$

PROOF. Obviously

$$u = w + \sum_{j=0}^{m-1} v^j = w + v$$

where $v^j \in L^2(-T, 0, H_0^m(\Omega, R^N))$ is the solution of the problem

$$\int_\Omega \sum_{|\alpha|=m} (D^\alpha v^j | D^\alpha \varphi) - \left(v^j \,\middle|\, \frac{\partial\varphi}{\partial t}\right) dx\, dt = \int_\Omega \sum_{|\alpha|=j} (f^\alpha | D^\alpha \varphi)\, dx\, dt$$

$$\forall \varphi \in W_2(Q) : \varphi(x, 0) = 0$$

whereas $w \in L^2(-T, 0, H_0^m(\Omega, R^N))$ is the solution of the problem

$$\int_Q \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta} D^\beta w | D^\alpha \varphi) - \left(w \,\middle|\, \frac{\partial\varphi}{\partial t}\right) dx\, dt = \int_Q \sum_{|\alpha|=m} (F^\alpha | D^\alpha \varphi)\, dx\, dt$$

$$\forall \varphi \in W_2(Q) : \varphi(x, 0) = 0$$

with

$$F^{\alpha} = f^{\alpha} + D^{\alpha} v - \sum_{|\beta|=m} A_{\alpha\beta} D^{\beta} v .$$

By lemma 2.I and in particular from (2.12) we have

$$\int_{-T}^{0} |v^j|^p_{m,\Omega} dt \leqslant C \int_{Q} \sum_{|\alpha|=j} \|f^{\alpha}\|^2 dx\, dt \quad \forall\, 2 \leqslant p < \frac{2m}{j} .$$

Therefore

$$\int_{-T}^{0} |v|^p_{m,\Omega} dt \leqslant C \int_{-T}^{0} \sum_{|\alpha| \leqslant m-1} \|f^{\alpha}\|^2 dx\, dt \qquad \forall\, 2 \leqslant p < \frac{2m}{m-1} \tag{2.15}$$

and hence

$$F^{\alpha} \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{with } 2 \leqslant p < \frac{2m}{m-1} \tag{2.16}$$

Now recalling that it is proved in [4], [5] that there exists a $p_0 > 2$ such that if $p \in [2, p_0)$

$$\int_{-T}^{0} |w|^p_{m,\Omega} dt \leqslant C \int_{-T}^{0} dt \left(\int_{\Omega} \sum_{|\alpha|=m} \|F^{\alpha}\|^2 dx \right)^{p/2} . \tag{2.17}$$

The assertion follows from (2.15), (2.17) and from the fact that $u = v + w$.

Let u be the solution of the problem (0.7) under the assumptions (0.3), (0.4) and let p_0 be as in the previous theorem.

THEOREM 2.III. *If* $p \in [2, p_0 \wedge 2^*)$ *and*

$$f^{\alpha} \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \textit{for } |\alpha| = m$$

then $\forall \alpha \colon 0 \leqslant |\alpha| = j \leqslant m - 1$ *we have*

$$D^{\alpha} u \in L^p(Q, R^N)$$

$$\int_{-T}^{0} |u|^p_{j,p,\Omega} dt \leqslant C(Q) \left\{ \int_{-T}^{0} dt \left(\int_{Q} \sum_{|\alpha|=m} \|f^{\alpha}\|^2 dx \right)^{p/2} + \left[\int_{\Omega} \sum_{|\alpha| \leqslant m-1} \|f^{\alpha}\|^2 dx\, dt \right]^{p/2} \right\} . \tag{2.18}$$

PROOF. From the imbedding theorems with respect to x, $\forall p \leqslant 2^*$ and $\forall\ 0 \leqslant j \leqslant m - 1$ we have

$$|u|^p_{j,p,\Omega} \leqslant C \|u\|^p_{j+1,\Omega}$$

and therefore

$$\int_{-T}^{0} |u|^p_{\delta,p,\Omega}\,dt \leqslant C\int_{-T}^{0} \|u\|^p_{m,\Omega}\,dt\,. \tag{2.19}$$

The assertion follows from this estimate together with theorems 2.I and 2.II.

3. – Quasilinear systems. L^p regularity.

Let us write $\delta u = \{D^\alpha u\}_{|\alpha|\leqslant m-1}$ and $Du = \{D^\alpha u\}_{|\alpha|\leqslant m}$.

Let $u \in L^2(-T, 0, H^m_0(\Omega, R^N))$ be a solution of the quasilinear system

$$\int_Q \sum_{|\alpha|=m}\sum_{|\beta|=m} (A_{\alpha\beta}D^\beta u|D^\alpha\varphi) - \left(u\,\middle|\,\frac{\partial\varphi}{\partial t}\right) dx\,dt = \tag{3.1}$$
$$= \int_Q \sum_{|\alpha|=m} (f^\alpha(x, t, \delta u)|D^\alpha\varphi)\,dx\,dt + \int_Q \sum_{|\alpha|\leqslant m-1} (f^\alpha(x, t, Du)|D^\alpha\varphi)\,dx\,dt$$

$\forall\varphi \in W_2(Q): \varphi(x, 0) = 0$

where

a) $A_{\alpha\beta}$ are $N\times N$ matrices which satisfy the conditions (0.3), (0.4);

b) f^α are vectors in R^N, measurable in (x, t), continuous in δu and Du respectively and which satisfy the inequalities

$$\|f^\alpha(x, t, \delta u)\| \leqslant C\Big\{g^\alpha + \sum_{|\alpha|\leqslant m-1}\|D^\alpha u\|\Big\} \qquad \text{for } |\alpha| = m\,, \tag{3.2}$$
$$\|f^\alpha(x, t, Du)\| \leqslant C\Big\{g^\alpha + \sum_{|\alpha|\leqslant m}\|D^\alpha u\|\Big\} \qquad \text{for } |\alpha| \leqslant m-1$$

where

$$g^\alpha \in L^p(Q) \qquad \text{if } |\alpha| = m \tag{3.3}$$
$$g^\alpha \in L^p(-T, 0, L^2(\Omega)) \qquad \text{if } |\alpha| \leqslant m-1$$

$2 \leqslant p < p_0 \wedge 2^*$, where p_0 is the exponent which appears in theorem 2.II.

In particular also the linear systems not reduced to the principal part alone as in (0.7) are included in this situation.

From the hypotheses (3.2), (3.3) it follows that

$$f^\alpha \in L^p(Q, R^N) \qquad \text{for } |\alpha| = m \tag{3.4}$$
$$f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{for } |\alpha| \leqslant m-1\,.$$

In fact since $f^\alpha \in L^2(Q, R^N))$, $|\alpha| \leqslant m$, then by the theorem 2.I

$$D^\alpha u \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{for } |\alpha| \leqslant m-1$$

so that

$$f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{for } |\alpha| = m\,.$$

Then from theorems 2.II and 2.III

$$(3.5) \qquad \begin{aligned} &D^\alpha u \in L^p(-T, 0, L^2(\Omega, R^N)) && \text{for } |\alpha| = m \\ &D^\alpha u \in L^p(Q, R^N) && \text{for } |\alpha| \leqslant m-1 \end{aligned}$$

(3.4) follows from (3.5), (3.2), (3.3). We also obtain the following estimates from theorems 2.I, 2.II and 2.III:

$$(3.6) \qquad \int_{-T}^{0} dt \left(\int_\Omega \sum_{|\alpha| \leqslant m-1} \|f^\alpha\|^2 dx \right)^{p/2} \leqslant$$

$$\leqslant C \left\{ \int_{-T}^{0} dt \left(\int_\Omega \sum_{|\alpha| \leqslant m} |g^\alpha|^2 dx \right)^{p/2} + \left(\int_Q \sum_{|\alpha| \leqslant m} \|D^\alpha u\|^2 dx\, dt \right)^{p/2} \right\}$$

$$(3.7) \qquad \int_Q \sum_{|\alpha| \leqslant m} \|f^\alpha\|^p dx\, dt \leqslant$$

$$\leqslant C \left\{ \int_Q \sum_{|\alpha| = m} |g^\alpha|^p dx\, dt + \left[\int_Q \sum_{|\alpha| \leqslant m-1} |g^\alpha|^2 dx\, dt \right]^{p/2} + \left[\int_Q \sum_{|\alpha| \leqslant m} \|D^\alpha u\|^2 dx\, dt \right]^{p/2} \right\}.$$

Thus the hypotheses of theorem 1.I are satisfied and we can therefore deduce the following

THEOREM 3.I. *If u is a solution of the system* (3.1) *with the hypotheses a*), *b*) *and $2 \leqslant p < p_0 \wedge 2^*$ then $u \in L^p(-T, 0, H_0^{m,p}(\Omega, R^N))$ and*

$$(3.8) \qquad \int_{-T}^{0} |u|^p_{m,p,\Omega}\, dt \leqslant C \left\{ \int_Q \sum_{|\alpha| = m} |g^\alpha|^p dx\, dt + \right.$$

$$\left. + \int_{-T}^{0} dt \left(\int_\Omega \sum_{|\alpha| \leqslant m-1} |g^\alpha|^2 dx \right)^{p/2} + \left[\int_Q \sum_{|\alpha| \leqslant m} \|D^\alpha u\|^2 dx\, dt \right]^{p/2} \right\}.$$

4. – Local L^p regularity.

Let us suppose for convenience that $0 \in \Omega$. Let us denote

$$B(\sigma) = \{x\colon \|x\|_n < \sigma\} \quad \text{and} \quad Q(\sigma) = B(\sigma)\times(-\sigma^{2m}, 0)\,.$$

Let $f^\alpha \in L^2(Q, R^N)$, $|\alpha|\leqslant m$, and let $u \in L^2(-T, 0, H^m(\Omega, R^N))$ be a solution of the system

$$(4.1)\qquad \int\limits_Q \sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}D^\beta u|D^\alpha\varphi) - \left(u\left|\frac{\partial\varphi}{\partial t}\right.\right)dx\,dt = \int\limits_Q \sum_{|\alpha|\leqslant m}(f^\alpha|D^\alpha\varphi)\,dx\,dt$$

$$\forall\varphi \in C_0^\infty(Q, R^N)$$

where $A_{\alpha\beta}$ satisfy the assumptions (0.3), (0.4). In a standard way we can localize the L^p regularity results proved in the sections 1 and 2.

Let us suppose that $Q(3\sigma)\Subset Q$ (5) and $0<\lambda<2$. Let $\theta(x)$ and $\varrho(t)$ be two functions with the properties below:

$$\begin{cases} \theta \in C_0^\infty(R^n), & \theta = 1 \quad \text{for } \|x\|\leqslant\lambda\sigma \\ 0\leqslant\theta\leqslant 1, & \theta = 0 \quad \text{for } \|x\|\geqslant(1+\lambda)\sigma \end{cases}$$

and

$$\begin{cases} \varrho(t) \in C^\infty(R), & \varrho(t) = 1 \quad \text{for } t\geqslant -(\lambda\sigma)^{2m} \\ 0\leqslant\varrho(t)\leqslant 1, & \varrho(t) = 0 \quad \text{for } t\leqslant -[(1+\lambda)\sigma]^{2m}\,. \end{cases}$$

Let c be any vector in R^N. The vector $U = \theta\varrho(u-c)$ belongs to $L^2\big(-[(1+\lambda)\sigma]^{2m}, 0, H_0^m(B([1+\lambda]\sigma), R^N)\big)$ and it is a solution of the Cauchy-Dirichlet problem

$$(4.2)\qquad \int\limits_{Q([1+\lambda]\sigma)} \sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}D^\beta U|D^\alpha\varphi) - \left(U\left|\frac{\partial\varphi}{\partial t}\right.\right)dx\,dt =$$

$$= \int\limits_{Q([1+\lambda]\sigma)} \sum_{|\alpha|\leqslant m}(F^\alpha|D^\alpha\varphi)\,dx\,dt$$

$$\forall\varphi \in W_2(Q([1+\lambda]\sigma))\colon \varphi(x, 0) = 0$$

(5) This means that $B(3\sigma)\Subset\Omega$ and $(3\sigma)^{2m} < T$.

where

$$\tag{4.3} \begin{aligned} \|F^\alpha\| &\leqslant C(\sigma)\Big\{\sum_{|\alpha|=m}\|f^\alpha\| + \sum_{|\alpha|\leqslant m-1}\|D^\alpha(u-c)\|\Big\} \quad \text{for } |\alpha|=m \\ \|F^\alpha\| &\leqslant C(\sigma)\Big\{\sum_{|\alpha|\leqslant m}\|f^\alpha\| + \sum_{|\alpha|\leqslant m}\|D^\alpha(u-c)\|\Big\} \quad \text{for } |\alpha|\leqslant m-1\,. \end{aligned}$$

The next lemma follows from theorems 2.I, 2.II, 2.III.

u is a solution of (4.1).

LEMMA 4.I. *If* $0\leqslant|\alpha|=j\leqslant m-1$ *it follows that*

$$\tag{4.4} D^\alpha u\in L^p(-(2\sigma)^{2m},0,L^2(B(2\sigma),R^N)) \qquad \textit{with } 2\leqslant p<\frac{2m}{j}$$

and

$$\tag{4.5} \begin{aligned} \Big(\int_{-(2\sigma)^{2m}}^{0}|u-c|^p_{j,B(2\sigma)}\,dt\Big)^{1/p} &\leqslant \sum_{h=0}^{m}C_h(\sigma)\Big\{\int_{Q(3\sigma)}\sum_{|\alpha|=h}\|f^\alpha\|^2\,dx\,dt\Big\}^{\frac12}+ \\ &+C(\sigma)\Big\{\int_{Q(3\sigma)}\sum_{|\alpha|=m}\|D^\alpha u\|^2+\sigma^{-2m}\|u-c\|^2\,dx\,dt\Big\}^{\frac12} \end{aligned}$$

where

$$\tag{4.6} \begin{aligned} C_h(\sigma) &= C\sigma^{m-j-h+2m/p} \\ C(\sigma) &= C\sigma^{2m/p-j}\,. \end{aligned}$$

PROOF. In (4.2) let us take $\lambda=2$. From the estimate (2.2) written for U, for the cylinder $Q(3\sigma)$ and for the vectors F^α, we have

$$\int_{-(2\sigma)^{2m}}^{0}|u-c|^p_{j,B(2\sigma)}\,dt\leqslant C(\sigma)\Big\{\int_{Q(3\sigma)}\sum_{|\alpha|\leqslant m}\|f^\alpha\|^2+\sum_{|\alpha|\leqslant m}\|D^\alpha(u-c)\|^2\,dx\,dt\Big\}^{p/2}.$$

But, $\forall\varepsilon\in(0,1)$ and for $0<j\leqslant m-1$,

$$\tag{4.7} |u|^2_{j,B(3\sigma)}\leqslant\varepsilon|u|^2_{m,B(3\sigma)}+C(\varepsilon,\sigma)\|u-c\|^2_{L^2(B(3\sigma),R^N))}\,.$$

Therefore we have (4.5). Finally (4.6) are proved by the homothetic transformation

$$x=\sigma y\,,\qquad t=\sigma^{2m}\tau\,.$$

LEMMA 4.II. *There exists a* $p_0>2$ *such that* $\forall p\in[2,p_0)$ *if*

$$f^\alpha\in L^p(-T,0,L^2(\Omega,R^N)) \quad \textit{for } |\alpha|=m$$

then $u \in L^p(-\sigma^{2m}, 0, H^m(B(\sigma), R^N))$ *and*

$$
\begin{aligned}
\left\{\int_{-\sigma^{2m}}^{0} |u|^p_{m,B(\sigma)}\,dt\right\}^{1/p} \leqslant\; & C\left\{\int_{-(3\sigma)^{2m}}^{0} dt\left(\int_{B(3\sigma)} \sum_{|\alpha|=m}\|f^\alpha\|^2 dx\right)^{p/2}\right\}^{1/p} + \\
(4.8)\qquad & + \sum_{j=0}^{m-1} C_j(\sigma)\left\{\int_{Q(3\sigma)} \sum_{|\alpha|=j}\|f^\alpha\|^2 dx\,dt\right\}^{\frac12} + \\
& + C(\sigma)\left\{\int_{Q(3\sigma)} \sum_{|\alpha|=m}\|D^\alpha u\|^2 + \sigma^{-2m}\|u-c\|^2 dx\,dt\right\}^{\frac12}
\end{aligned}
$$

where

$$
(4.9)\qquad
\begin{aligned}
C_j(\sigma) &= C\sigma^{2m/p-j}\\
C(\sigma) &= C\sigma^{2m/p-m}\,.
\end{aligned}
$$

PROOF. Let p_0 be the exponent which appears in theorem 2.II. If $p \in [2, p_0)$ we have, by lemma 4.I,

$$
(4.10)\qquad F^\alpha \in L^p(-(2\sigma)^{2m}, 0, L^2(B(2\sigma), R^N)) \quad \text{for } |\alpha| = m\,.
$$

Let us take $\lambda = 1$ in (4.2). The vector U satisfies the estimate (2.4) written for the cylinder $Q(2\sigma)$ and for the vectors F^α. (4.8) follows from this inequality and by lemma 4.I. Finally the (4.9) are proved by the homothetic transformation

$$
x = \sigma y\,, \qquad t = \sigma^{2m}\tau\,.
$$

LEMMA 4.III. *Let p_0 be as in the previous lemma and $p \in [2, p_0 \wedge 2^*)$. If*

$$
f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) \quad \textit{for } |\alpha| = m
$$

and $0 \leqslant |\alpha| = j \leqslant m-1$ *then*

$$
D^\alpha u \in L^p(Q(\sigma), R^N)
$$

and

$$
\begin{aligned}
\left\{\int_{-\sigma^{2m}}^{0} |u-c|^p_{j,p,B(\sigma)}\,dt\right\}^{1/p} \leqslant\; & C_m(\sigma)\left\{\int_{-(3\sigma)^{2m}}^{0} dt\left(\int_{B(3\sigma)} \sum_{|\alpha|=m}\|f^\alpha\|^2 dx\right)^{p/2}\right\}^{1/p} + \\
(4.11)\qquad & + \sum_{h=0}^{m-1} C_h(\sigma)\left\{\int_{Q(3\sigma)} \sum_{|\alpha|=h}\|f^\alpha\|^2 dx\,dt\right\}^{\frac12} + \\
& + C(\sigma)\left\{\int_{Q(3\sigma)} \sum_{|\alpha|=m}\|D^\alpha u\|^2 + \sigma^{-2m}\|u-c\|^2 dx\,dt\right\}^{\frac12}
\end{aligned}
$$

where

$$C_m(\sigma) = C\sigma^{m-j+n/p-n/2}$$
$$(4.12) \qquad C_h(\sigma) = C\sigma^{m-j-h+(2m+n)/p-n/2}$$
$$C(\sigma) = C\sigma^{(n+2m)/p-n/2-j}\,.$$

PROOF. (4.10) holds by lemma 4.I. In (4.2) we take $\lambda = 1$. The vector U satisfies the inequality (2.18) written for the cylinder $Q(2\sigma)$ and for vectors F^α. From this inequality and by lemma 4.I, (4.11) follows. Finally (4.12) are proved by the homothetic transformation

$$x = \sigma y\,, \qquad t = \sigma^{2m}\tau\,.$$

We can now give a local version of theorem 1.I.

THEOREM 4.I. *Let u be a solution of the system* (4.1) *with the assumptions* (0.3), (0.4). *Let $p \in [2, p_0 \wedge 2^*)$ and we suppose that*

$$f^\alpha \in L^p(Q, R^N) \qquad \text{for } |\alpha| = m$$
$$f^\alpha \in L^p(-T, 0, L^2(\Omega, R^N)) \qquad \text{for } |\alpha| \leqslant m-1\,.$$

Then $u \in L^p(-\sigma^{2m}, 0, H_0^{m,p}(B(\sigma), R^N))$ *and*

$$(4.13) \qquad \left\{\int_{-\sigma^{2m}}^{0} |u|^p_{m,p,B(\sigma)}\,dt\right\}^{1/p} \leqslant C\left\{\int_{Q(3\sigma)}\left(\sum_{|\alpha|=m}\|f^\alpha\|^2\right)^{p/2} dx\,dt\right\}^{1/p} + \sum_{j=0}^{m-1} C_j(\sigma)\left\{\int_{-(3\sigma)^{2m}}^{0} dt\left(\int_{B(3\sigma)}\sum_{|\alpha|=j}\|f^\alpha\|^2 dx\right)^{p/2}\right\}^{1/p} + C(\sigma)\left\{\int_{Q(3\sigma)}\sum_{|\alpha|=m}\|D^\alpha u\|^2 + \sigma^{-2m}\|u-c\|^2\,dx\,dt\right\}^{\frac12}$$

where c is any vector in R^N and

$$C_j(\sigma) = C\sigma^{m-j+n(1/p-\frac12)}$$
$$(4.14) \qquad C(\sigma) = C\sigma^{(n+2m)(1/p-\frac12)}$$

PROOF. In (4.2) we take $\lambda = 1$. By lemmas 3.I, 3.II, 3.III we have

$$F^\alpha \in L^p(Q(2\sigma), R^N) \qquad \text{for } |\alpha| = m$$
$$F^\alpha \in L^p(-(2\sigma)^{2m}, 0, L^2(B(2\sigma), R^N)) \qquad \text{for } |\alpha| \leqslant m-1\,.$$

Then, by theorem 1.I, we have $D^\alpha U \in L^p(Q(2\sigma), R^N)$, $|\alpha| = m$, and

$$(4.15)\qquad \int_{-(2\sigma)^{2m}}^{0} |U|^p_{m,p,B(2\sigma)} dt \leqslant C(\sigma) \int_{Q(2\sigma)} \Big(\sum_{|\alpha|=m} \|F^\alpha\|^2\Big)^{p/2} dx\,dt + \\ + C(\sigma)\int_{-(2\sigma)^{2m}}^{0} dt \Big(\int_{B(2\sigma)} \sum_{|\alpha|\leqslant m-1} \|F^\alpha\|^2 dx\Big)^{p/2}.$$

Taking into account (4.3) and the lemmas 4.I, 4.II, 4.III

$$(4.16)\qquad \int_{-\sigma^{2m}}^{0} |u|^p_{m,p,B(\sigma)} dt \leqslant C(\sigma) \int_{Q(3\sigma)} \Big(\sum_{|\alpha|=m} \|f^\alpha\|^2\Big)^{p/2} dx\,dt + \\ + C(\sigma) \int_{-(3\sigma)^{2m}}^{0} dt \Big(\int_{B(3\sigma)} \sum_{|\alpha|\leqslant m-1} \|f^\alpha\|^2 dx\Big)^{p/2} + C(\sigma)\Big\{\int_{Q(3\sigma)} \sum_{|\alpha|\leqslant m} \|D^\alpha(u-c)\|^2 dx\,dt\Big\}^{p/2}.$$

But, by (4.7),

$$(4.17)\qquad \int_{Q(3\sigma)} \sum_{|\alpha|\leqslant m} \|D^\alpha(u-c)\|^2 dx\,dt \leqslant C(\sigma) \int_{Q(3\sigma)} \sum_{|\alpha|=m} \|D^\alpha u\|^2 + \|u-c\|^2 dx\,dt$$

(4.13) follows from (4.16), (4.17) and finally we specify the dependence of various constants on σ by the homothetic transformation

$$x = \sigma y\,, \qquad t = \sigma^{2m}\tau\,.$$

We shall also prove an analogous result of local L^p regularity for solutions of quasilinear systems of the type considered in section 3.

Let $u \in L^2(-T, 0, H^m(\Omega, R^N))$ be a solution in Q of the quasilinear system

$$(4.18)\qquad \int_Q \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta} D^\beta u | D^\alpha \varphi) - \Big(u \,\Big|\, \frac{\partial\varphi}{\partial t}\Big) dx\,dt = \\ = \int_Q \sum_{|\alpha|=m} (f^\alpha(x, t, \delta u)|D^\alpha\varphi)\,dx\,dt + \int_Q \sum_{|\alpha|\leqslant m-1} (f^\alpha(x, t, Du)|D^\alpha\varphi)\,dx\,dt$$

$$\forall\varphi \in C_0^\infty(Q, R^N)$$

where $A_{\alpha\beta}$ and f^α satisfy the assumptions *a*), *b*) of section 3, in particular $p \in [2, p_0 \wedge 2^*)$. Reasoning as in section 3 we can prove that for every cylinder $Q(2\sigma)\Subset Q$ [6] we have

$$
(4.19)\qquad \begin{aligned} &f^\alpha(x,t,\delta u)\in L^p(Q(2\sigma),R^N) && \text{for } |\alpha|=m \text{ and}\\ &f^\alpha(x,y,Du)\in L^p(-(2\sigma)^{2m},0,L^2(B(2\sigma),R^N)) && \text{for } |\alpha|\leqslant m-1\,. \end{aligned}
$$

And by a tedious but elementary computation, for $\sigma\leqslant 1$, we obtain the estimate

$$
(4.20)\qquad \left\{\int_{Q(\sigma)}\Big(\sum_{|\alpha|=m}\|f^\alpha\|^2\Big)^{p/2}dx\,dt\right\}^{1/p}+\sum_{j=0}^{m-1}C_j(\sigma)\left\{\int_{-\sigma^{2m}}^{0}dt\Big(\int_{B(\sigma)}\sum_{|\alpha|=j}\|f^\alpha\|^2dx\Big)^{p/2}\right\}^{1/p}\leqslant
$$
$$
\leqslant C\left\{\int_{Q(3\sigma)}\Big(\sum_{|\alpha|=m}|g^\alpha|^2\Big)^{p/2}dx\,dt\right\}^{1/p}+\sum_{j=0}^{m-1}C_j(\sigma)\left\{\int_{-(3\sigma)^{2m}}^{0}dt\Big(\int_{B(3\sigma)}\sum_{|\alpha|=j}|g^\alpha|^2dx\Big)^{p/2}\right\}^{1/p}+
$$
$$
+\,C(\sigma)\left\{\int_{Q(3\sigma)}\sum_{|\alpha|\leqslant m}\|D^\alpha u\|^2+\sigma^{-2m}\|u-c\|^2dx\,dt\right\}^{\frac12}
$$

where $C_j(\sigma)$ and $C(\sigma)$ are as in (4.14).

The (4.19) are exactly the hypotheses of theorem 4.I, so we can state the following

THEOREM 4.II. *If u is a solution of the system* (4.18) *under the assumptions a), b) of section* 3, *in particular $p\in[2,p_0\wedge 2^*)$ and*

$$
(4.21)\qquad \begin{aligned} &g^\alpha\in L^p(Q) && \textit{for } |\alpha|=m\\ &g^\alpha\in L^p(-T,0,L^2(\Omega)) && \textit{for } |\alpha|\leqslant m-1 \end{aligned}
$$

then $\forall\ Q(3\sigma)\Subset Q$ *and* $\sigma\leqslant 1$

$$
(4.22)\qquad \begin{aligned}\left\{\int_{-\sigma^{2m}}^{0}|u|^p_{m,p,B(\sigma)}dt\right\}^{1/p}\leqslant C&\left\{\int_{Q(3\sigma)}\Big(\sum_{|\alpha|=m}|g^\alpha|^2\Big)^{p/2}dx\,dt\right\}^{1/p}+\\ &+\sum_{j=0}^{m-1}C_j(\sigma)\left\{\int_{-(3\sigma)^{2m}}^{0}dt\Big(\int_{B(3\sigma)}\sum_{|\alpha|=j}|g^\alpha|^2dx\Big)^{p/2}\right\}^{1/p}+\\ &+C(\sigma)\left\{\int_{Q(3\sigma)}\sum_{|\alpha|\leqslant m}\|D^\alpha u\|^2+\sigma^{-2m}\|u-c\|^2dx\,dt\right\}^{\frac12}\end{aligned}
$$

where c is any vector in R^N and $C_j(\sigma)$, $C(\sigma)$ are as in (4.14).

[6] $B(2\sigma)\Subset\Omega$ and $(2\sigma)^{2m}<T$.

We note that, in particular, the estimate (4.22) follows from (4.13) and (4.20).

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