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Pseudoconcave Homogeneous Manifolds.

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When we began this project we had the goal of carrying over some of the main techniques in the theory of compact homogeneous manifolds to the case of pseudoconcave homogeneous manifolds. In particular, we were looking toward a theory in the non-compact case which would parallel the Borel-Remmert theory [10]. It turns out that many techniques are rather easily carried over from the compact to the pseudoconcave category. The first three sections of the paper are devoted to this: we give a detailed account of the *normalizer*, *Albanese*, and *meromorphic reduction* fibrations for pseudoconcave complex-homogeneous manifolds. (By complex-homogeneous we mean that there is a *complex* Lie group acting transitively.) The latter is in fact proved for *arbitrary* complex-homogeneous manifolds.

In the second part of the paper we restrict ourselves to non-compact 0-concave complex-homogeneous manifolds. Our main result is a complete description of such manifolds:

MAIN THEOREM. *X is a non-compact 0-concave complex-homegeneous manifold if and only if X is a positive line bundle space over a compact homogeneous rational manifold which can be realized as a linear cone in projective space with its vertex removed.*

Thus, to construct such an X, one starts with an arbitrary compact homogeneous rational manifold Q (alternately called a flag manifold and given as the quotient of a semi-simple complex Lie group and a parabolic subgroup). Then, imbed Q in a hyperplane $\{z_0 = 0\} \cong \boldsymbol{P}^{N-1} \subset \boldsymbol{P}^N$, and connect each point of Q with lines to the point $[1:0:...:0]$. The resulting compact variety is what we call a linear cone. Let X be this cone with the vertex $[1:0:...:0]$ removed. Then X is non-compact and easily seen to be complex-

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homogeneous. An appropriate distance function to the point $[1:0:\ldots:0]$ then gives a 0-concave exhaustion for X. Projection into the hyperplane $\{z_0=0\}$ reveals the line bundle structure of X. An example to keep in mind is the Segre cone $V=\{z\in \boldsymbol{P}^4|z_1z_2=z_3z_4\}$ with the vertex $[1:0:0:0]$ removed, $X=V\setminus\{[1:0:0:0]\}$. Here, X projects to a $\boldsymbol{P}^1\times\boldsymbol{P}^1$ (the 2-dimensional projective quadric) in the hyperplane at infinity.

The remarkable fact is that X is algebraic (it even compactifies to a rational manifold) when, a priori, we do not even know there exist meromorphic functions on X. The proof is based around the special case presented in section 8 where we assume X compactifies to an algebraic variety, V, and the complex Lie group, G, acting transitively on X is a group of colineations of $\boldsymbol{P}^N\supset V$. In this setting we use a flag argument to show that the radical of G has closed, 1-dimensional orbits on X, and in fact these orbits give a homogeneous fibration of X as a line bundle space over a homogeneous compact rational manifold. The general case is reduced to this special case by studying the normalizer fibration (section 9). The base of the normalizer fibration is in the special setting of section 8, so that we need to understand the fiber which is parallelizable (discrete isotropy). The necessary results for this case are presented in sections 4 through 6. There is also a problem if G has no radical, and we eliminate this possibility in section 7. Finally, in section 10 we show that the line bundle structure of X is very ample so that we easily obtain the cone realization mentioned above.

For more information on the characterization of homogeneous cones we refer the reader to the work of Huckleberry and Oeljeklaus [16].

We conclude this introduction with a small prelude to [17]. In [17] we consider non-compact 0-concave manifolds X which are homogeneous under the action of a (*not necessarily complex*) Lie group. This is a bigger class of manifolds than those covered by the present paper. For example, $\boldsymbol{P}^2\setminus\boldsymbol{B}^2$, where $\boldsymbol{B}^2$ is a ball in some $\boldsymbol{C}^2$ in $\boldsymbol{P}^2$, is obviously not holomorphically equivalent to a positive line bundle over a rational manifold. In some sense, such examples as this are not too far from generic. In particular, one can prove that if the group is not complex and $\dim X>2$, then either X is $\boldsymbol{P}^N\setminus\boldsymbol{B}^N$, or X can be realized as the complement of a totally real compact homogeneous manifold of half the dimension in a compact homogeneous rational manifold of a very special type.

0. – Definitions and notation.

In the case of a compact manifold X, it is known that the full group of biholomorphic maps of X onto itself, i.e. the group of *automorphisms* of X, is a complex Lie group [8]. This is not necessarily true when X is non-compact. However, we wish to restrict to this setting. Thus, we define a connected complex manifold X to be *complex-homogeneous* if there exists a complex Lie group with a countable number of components which acts holomorphically and transitively on X. We denote this group by G. If H is the isotropy subgroup of a fixed point $x_0 \in X$, $H := \{g \in G \mid g(x_0) = x_0\}$, then we have the natural identification of X with G/H, the right H-coset space of G. We may always assume that G is connected since the connected component of the identity, G^0, acts transitively on X. In addition, we always assume that the group G acts *effectively* on X, that is, the only element of G which leaves every point of X fixed is the identity. In this way we may consider G to be a subgroup of Aut (X), the group of automorphisms of X, and we will often write $G \subset \mathrm{Aut}(X)$. Note that this last assumption is not restrictive since the closed complex subgroup $L = \{g \in G \mid g(x) = x$ for all $x \in X\}$ is normal in G so that $\tilde{G} := G/L$ is a complex Lie group which acts transitively *and* effectively on X. If K is any closed complex subgroup of G then G/K, the right K-coset space of G, is always a complex manifold. If K contains another closed complex subgroup J, we obtain a natural holomorphic homogeneous fiber bundle. $G/J \to G/K$ with fiber isomorphic to K/J. We say that $X = X(B, \varrho, F)$ is a homogeneous fiber bundle if $X = G/H$, $B = G/K$, $F = K/H$, and $\varrho \colon X \to B$ is given by the natural projection $G/H \to G/K$.

A complex manifold X, $\dim X = n > 1$, is called *p-concave* if there exists a smooth, non-negative function $\varphi \colon X \to \boldsymbol{R}_+$ satisfying:

1) $X_a = \{x \in X \mid \varphi(x) > a\}$ is relatively compact in X for $a > 0$ (1).

2) For some $a_0 > 0$, the Hermitian form

$$\mathfrak{L}(\varphi)_p = \sum \frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} (p) \, dz_i \otimes d\bar{z}_j$$

has at least n-p positive eigenvalues on the complex tangent space $T_{\boldsymbol{C}}(X)_x$, for all $x \in X \setminus \bar{X}_{a_0}$.

Such a φ is called a p-concave exhaustion for X. (See [2] for more details about p-concave manifolds.) In this language, if X is a p-concave manifold, $0 \leq p \leq n-2$, then X is pseudo-concave in the sense of Andreotti [1], and if $p = 0$ then X is strongly pseudo-concave. Every compact manifold is, of course, 0-concave.

(1) Whenever we use the set X_a in this paper we will choose a so that the boundary of X_a is smooth.

We denote the field of meromorphic functions on any connected complex space X by $\varkappa(X)$ and we let $t(X) := \operatorname{transdeg} \varkappa(X)$, the transcendence degree of this field over the constants. A complex space Y will be referred to as *meromorphically separable* if, for any two distinct points $y_1, y_2 \in Y$, there is a meromorphic function m on Y which is not indeterminate at either y_1 or y_2 and such that $m(y_1) \neq m(y_2)$ (i.e. $\varkappa(Y)$ separates the points of Y).

1. – The normalizer fibration.

In this section we present a natural fibration which is well known in the theory of complex homogeneous manifolds [31]. The idea is to look at the adjoint action of G on the tangent space of $X = G/H$ at the point $x_0 = eH$. This amounts to following the orbit of $\mathfrak{h}$ (the Lie algebra of H) under the adjoint action of G in $M_{k,m}$, the Grassmann manifold of k-planes in m-space ($k = \dim \mathfrak{h}$, $m = \dim \mathfrak{g}$). We then obtain a map of X onto this orbit which, in general, could behave wildly in $M_{k,m}$. Of course if the manifold X is compact, Remmert's proper mapping theorem shows that this orbit is a compact analytic set in $M_{k,m}$. In fact, if the orbit is compact it can be shown to be a homogeneous *rational* manifold (see Borel-Remmert [10]). For the more general p-concave manifold X, one must rely on knowledge of the function field $\varkappa(X)$ in order to describe this orbit.

THEOREM 1. *Let X be a connected complex homogeneous manifold, $\dim X = n$. Assume X is p-concave, $0 \leq p \leq n-2$. We write $X = G/H$ and let $N = N_G(H^0)$, the normalizer of H^0 in G. Then the natural projection $\mu\colon G/H \to G/N$ yields a holomorphic homogeneous fiber bundle with G/N Zariski open in an irreducible compact projective algebraic variety and the fiber N/H is group theoretically parallelizable.*

PROOF. The subgroup $N = N_G(H^0)$ is a closed subgroup of G containing H so $\mu\colon G/H \to G/N$ naturally has the structure of a homogeneous fiber bundle. We may write $N/H \cong N'/\Gamma$ where $N' = N/H^0$ is a complex Lie group and $\Gamma = H/H^0$ is discrete. Thus N/H is parallelizable.

Let $m = \dim_{\mathbf{C}} G$ and $k = \dim_{\mathbf{C}} H$. We can consider the Lie algebra $\mathfrak{h}$ of H as a point in the Grassmann manifold $M_{k,m}$ consisting of the k-dimensional linear subspaces of the Lie algebra $\mathfrak{g}$ of G. By means of the adjoint map, G acts linearly on $M_{k,m}$ and the isotropy in G of the point $\mathfrak{h} \in M_{k,m}$

is just N. Since $H \subset N$, we can map G/H holomorphically onto B, the G-orbit of $\mathfrak{h}$, which we identify with G/N. Noting that $M_{k,m}$ is a compact projective algebraic variety in some $\boldsymbol{P}_N$, we define $\hat{B}$ to be the smallest compact projective algebraic variety which contains B. We claim that $\dim \hat{B} = \dim B$. If this is true, then G, which acts linearly on $M_{k,m}$ and thus holomorphically on $\hat{B}$, has an open orbit in $\hat{B}$, namely B. The variety $\hat{B}$ is then said to be almost-homogeneous with respect to G. It is not hard to prove (see e.g. Remmert and van de Ven [25]) that the complement in $\hat{B}$ of the open orbit of G is an analytic and thus algebraic set, showing that B is Zariski-open in $\hat{B}$. To prove the claim we first observe that if $\dim B < \dim \hat{B}$ then there are $t = \dim \hat{B}$ algebraically (analytically) independent meromorphic functions $f_1, \ldots, f_t \in \varkappa(\hat{B})$ such that f_t is analytically dependent on $f_1, \ldots, f_{t-1}$ when restricted to B. Then $\mu^* f_t \in \varkappa(X)$ is analytically dependent on the meromorphic functions $\mu^* f_i \in \varkappa(X)$, $1 \leq i \leq t-1$. But Andreotti shows [1] that on p-concave manifolds, analytic dependence implies algebraic dependence of meromorphic functions. Therefore f_t is *algebraically* dependent on $f_1, \ldots, f_{t-1}$ when restricted to B, i.e. there is a polynomial P, such that $P(f_1, \ldots, f_t) = 0$ on B and $P(f_1, \ldots, f_t) \neq 0$ on $\hat{B}$. Writing the f_i's in homogeneous coordinates $[x_0 : \ldots : x_N] \in \boldsymbol{P}_N$ and clearing denominators gives rise to a homogeneous polynomial Q such that $Q[x_0 : \ldots : x_N] = 0$ on B and $Q[x_0 : \ldots : x_N] \neq 0$ on $\hat{B}$. This contradicts the minimality of $\hat{B}$.

To see that the variety $\hat{B}$ is irreducible, let $\hat{B}_1, \ldots, \hat{B}_t$ be the irreducible branches of $\hat{B}$ and let S be the singular set of $\hat{B}$. Then the sets $\hat{B}_1 \setminus S, \ldots, \hat{B}_t \setminus S$ are disjoint, while the set of $B \setminus S$ is connected since S is thin analytic in B for dimension reasons. Therefore $B \setminus S \subset \hat{B}_i \setminus S$ for some i which shows that $B \subset \hat{B}_i$. Therefore, $\hat{B} = \hat{B}_i$, and $\hat{B}$ is irreducible. □

2. – The meromorphic reduction fibration.

In [13], Grauert and Remmert show that for a compact homogeneous manifold X, there exists a meromorphic reduction $\varrho : X \to B$. This means that $\varkappa(X)$ is isomorphic to $\varkappa(B)$, that B is meromorphically separable, and that B is universal with respect to this property. We remark that this reduction exists for arbitrary complex-homogeneous manifolds:

THEOREM 2. *Let X be a connected complex-homogeneous manifold. Then there exists a holomorphic fibration of X, $\varrho : X \to B$, with a group theoretically parallelizable fiber F, having the following properties:*

1) *$X = X(B, \varrho, F)$ is a homogeneous fiber bundle and ϱ is semiproper on closed ϱ-saturated sets* (2).

2) *$\varkappa(B)$ is isomorphic to $\varkappa(X)$, and B is meromorphically separable.*

3) *If $\tau = X \to Y$ is a holomorphic map into any meromorphically separable complex space Y then there exists a unique holomorphic map $\sigma: B \to Y$ such that $\tau = \sigma \circ p$.*

PROOF. For a homogeneous manifold X and $x \in X$ we define

$$F(x) = \cap \{f^{-1}(a) \,|\, (f, a) \in \varkappa(X) \times \boldsymbol{P}^1,\ x \in f^{-1}(a)\} .$$

Clearly, $F(x)$ is an analytic set in X and can be defined by finitely many $f^{-1}(a)$.

LEMMA. *For a homogeneous manifold X the sets $F(x)$, $x \in X$ form a G-invariant analytic partition of X. That is,*

$$g(F(x)) = F(gx) \quad \textit{for all} \quad (g, x) \in G \times X$$
$$\textit{and if} \quad y \in F(x) \quad \textit{then} \quad F(y) = F(x) .$$

PROOF. For any $g \in G$ we have an isomorphism $g^*: \varkappa(X) \to \varkappa(X)$. Then

$$\begin{aligned} F(g(x)) &= \cap \{f^{-1}(a) \,|\, (f, a) \in \varkappa(X) \times \boldsymbol{P}^1,\ g(x) \in f^{-1}(a)\} \\ &= \cap \{g(g^* f)^{-1}(a) \,|\, (g^* f, a) \in \varkappa(X) \times \boldsymbol{P}^1,\ x \in (g^* f)^{-1}(a)\} \\ &= g \cap \{h^{-1}(a) \,|\, (h, a) \in \varkappa(X) \times \boldsymbol{P}^1,\ x \in h^{-1}(a)\} \\ &= gF(x) . \end{aligned}$$

For the second part, if $y \in F(x)$ then $y \in f^{-1}(a)$ for all $(f, a) \in \varkappa(X) \times \boldsymbol{P}^1$ with $x \in f^{-1}(a)$. Therefore $F(y) \subseteq F(x)$. We claim that $x \in F(y)$ which implies the other inclusion, $F(x) \subseteq F(y)$. For suppose $x \notin F(y)$. Then there exists $f \in \varkappa(X)$ and $a \in \boldsymbol{P}^1$ such that $x \notin f^{-1}(a)$ and $y \in f^{-1}(a)$. Let I be the set of indeterminacy of f. It is easy to find a $g \in G$ so that $g(y) \in f^{-1}(a) \setminus I$ and $g(x) \notin$ $\notin f^{-1}(a)$. ($N = \{g \in G \,|\, g(x) \notin X \setminus f^{-1}(a)\}$ is an open neighborhood of the identity in G. So $N(y)$ is an open neighborhood of y and therefore must intersect $f^{-1}(a) \setminus I$). Letting $h = g^* f$, we see that neither x nor y is a point of

(2) A map $\varrho: X \to Y$ is said to be semiproper if, for any compact set K in Y, there exists a compact set $\tilde{K}$ in X such that $\varrho(\tilde{K}) = K \cap \varrho(X)$. A set M in X is said to be ϱ-saturated if $\varrho^{-1}(\varrho(m)) \subset M$ for all $m \in M$.

indeterminacy for h and $h(x) \neq h(y) = a$. This means that $y \notin F(x)$, a contradiction. Therefore $x \in F(y)$, and the Lemma is proved.

The G-invariant partition of X given by $F(x)$, $x \in X$, allows us to put a fiber bundle structure on X in the following canonical way (see Remmert and van de Ven [28]). We fix $x_0 \in X$ and let H be the isotropy subgroup of x_0 in G, so that $X = G/H$. Define J to be the stabilizer of $F(x_0)$ in G, $J = \{g \in G \,|\, gF(x_0) = F(x_0)\}$. Since $F(x_0)$ is an analytic set, J is a closed complex subgroup of G [28]. Note that the above lemma shows that J acts transitively on $F(x_0)$ and $H \subset J$. We define B to be the complex manifold G/J, and let $\varrho\colon X \to B$ be the natural projection $\varrho\colon G/H \to G/J$ with fiber $F := J/H$. Then the G-invariance of $F(x_0)$ implies that $X = X(B, \varrho, F)$ is a homogeneous fiber bundle with $\varrho^{-1}(\varrho(x)) = F(x) \cong F$. To see that ϱ is semiproper on a closed saturated set $V = \varrho^{-1}(\varrho(V))$, we take a compact set $K \subset B$ and look at a finite cover of K, $U_i \subset B$, where on each U_i there is a local trivialization of the bundle $\varphi_i\colon U_i \times F \to \varrho^{-1}(U_i)$. Now

$$\tilde{K} = \cup \overline{\varphi_i(K \cap U_i \times \{x_0\})} \cap V$$

is compact and $\varrho(\tilde{K}) = K \cap \varrho(V)$.

We now show that $\varkappa(B) \cong \varkappa(X)$. We do this by showing the injective morphism $\varrho^*\colon \varkappa(B) \to \varkappa(X)$ is surjective. Let f be any meromorphic function on X and let I be its set of indeterminancy. Then I is empty or has codimension 2. The hypersurfaces $h^{-1}(a)$ for $(h, a) \in \varkappa(X) \times \boldsymbol{P}^1$ are closed ϱ-saturated sets, since $h^{-1}(a) \supset F(x)$ for all $x \in h^{-1}(a)$. Thus $I = f^{-1}(0) \cap f^{-1}(\infty)$ must be a closed ϱ-saturated set. Then ϱ is semiproper on I, and by the semi-proper mapping theorem [19] we know $\varrho(I)$ is analytic. Furthermore $\varrho(I)$ is empty or has codimension 2 because

$$\varrho(I) = \varrho(f^{-1}(0) \cap f^{-1}(\infty)) = \varrho(f^{-1}(0)) \cap \varrho(f^{-1}(\infty)) \quad \text{and} \quad \varrho(f^{-1}(0)),\ \varrho(f^{-1}(\infty))$$

are hypersurfaces in B. (These last statements follow from the ϱ-saturation of the hypersurfaces.) On $X \setminus I$, f is a holomorphic map into $\boldsymbol{P}^1$ which is constant on the fibers of ϱ. Therefore there is a holomorphic map $f'\colon B \setminus \varrho(I) \to \boldsymbol{P}^1$ such that $i^*(\varrho^*(f')) = f$ (where $i\colon X \setminus I \to X$ is the inclusion). Now f' can be extended across $\varrho(I)$ to a meromorphic function $f'\colon B \to \boldsymbol{P}^1$, and so $\varrho^*(f') = f$.

Now suppose Y is a meromorphically separable space and $\tau\colon X \to Y$ is a holomorphic map. Then τ must be constant on the fibers of ϱ. For if $\tau(x) \neq \tau(y)$, there is an $m \in \varkappa(y)$ such that $m(\tau(x)) \neq m(\tau(y))$. But then $\tau^* m \in \varkappa(X)$ and $\tau^* m(x) \neq \tau^* m(y)$ showing $\varrho(x) \neq \varrho(y)$. This allows us to define a continuous map $\sigma\colon B \to Y$ so that $\sigma \circ \varrho = \tau$. Since ϱ is surjective we have that σ is holomorphic [13], and by its definition σ is uniquely determined.

Finally, we show that the fiber F is group theoretically parallelizable. First we observe that the base of the normalizer fibration, G/N, is meromorphically separable, since it is Zariski-open in a projective algebraic variety. By what we have just shown, there exists a map $\sigma\colon G/J \to G/N$ such that $\sigma\circ\varrho = \mu$, where $\mu\colon G/H \to G/N$. Therefore $J \subset N = N_G(G^0)$ and so $F \simeq J/H \simeq (J/H^0)/(H/H^0) = \tilde{J}/\Gamma$, where $\tilde{J}$ is a complex Lie group and Γ is a discrete subgroup. $\square$

3. – The Albanese fibration.

The classical Albanese variety is a universal complex torus $A(X)$ associated with any complex compact Kähler manifold X. That is, there is a holomorphic map $\alpha\colon X \to A(X)$ with $\dim A(X) = \frac{1}{2}b_1(X)$ such that if $\beta\colon X \to T$ is any other holomorphic map of X into a complex torus T, then there exists a holomorphic map $\tau\colon A(X) \to T$ such that $\tau\circ\alpha = \beta$. Blanchard [7] considers the case when X is not Kähler but still compact. He proves the existence of a universal torus, $A(X)$, for which $\dim A(X) \le \frac{1}{2}b_1(X)$. In Theorem 3 we use Blanchard's procedure, after making the appropriate adjustments, for the p-concave case.

THEOREM 3. *Let X be a p-concave manifold,* $0 \le p \le n-2$, $\dim X = n$. *Then there exists a compact complex torus $A(X)$, with* $\dim A(X) \le \frac{1}{2}b_1(X)$, *and a holomorphic map $\alpha\colon X \to A(X)$ such that if $\beta\colon X \to T$ is a holomorphic map of X into any complex torus T, then there is a holomorphic map $\tau\colon A(X) \to T$, unique up to automorphisms of T, such that $\tau\circ\alpha = \beta$. If, in addition, X is complex-homogeneous, then $X = X(A(X), \alpha, F)$ is a homogeneous fiber bundle with connected fiber $F \simeq \alpha^{-1}(0)$.*

REMARK. If we write $X = G/H$, H the isotropy of $x_0 \in X$, then $A(X) \simeq G/K$ where K is the smallest closed normal subgroup of G that contains H. Note that K also contains the commutator subgroup of G.

PROOF. The construction of the Albanese variety follows a general procedure as outlined in Blanchard's paper [7]. Two facts are needed. Let ω be a closed holomorphic 1-form on X.

1) If the real part of ω is zero then ω is zero.

2) If the real parts of the periods of ω are zero then ω is zero.

The first is easy and the second follows from the fact that *there are no non-constant pluriharmonic functions on p-concave manifolds*. In fact if $\psi: X \to \boldsymbol{R}$ is a pluriharmonic function it takes a maximum on $\overline{X}_a$ at a point $p_0 \in \partial \overline{X}_a$. By concavity, we can map a 1-dimensional disk D into X, $i: D \to X$, so that $i(D \setminus \{0\}) \subset X_a$ and $i(0) = p_0$. But then $i^* \psi: D \to \boldsymbol{R}$ is harmonic and takes its maximum at zero. So $i^*\psi$ is constant on D which shows that ψ takes its maximum on $\overline{X}_a$ at some interior point $i(\varepsilon) \in X_a$, and therefore is identically constant by the maximum principle.

It now follows that the set D' of all closed holomorphic 1-forms on X forms a complex vector space whose real dimension is no greater than $b_1(X)$, the first Betti number of X, which is finite for p-concave manifolds, $0 \leq p \leq n-2$ [2]. Let D be the complex dual space of D' (not the antidual space). We define a map $\alpha: H_1(X, \boldsymbol{Z}) \to D$ by $\alpha(\gamma)(\omega) = \int\limits_\gamma \omega$. The map α is a homomorphism and the image of $H_1(X, \boldsymbol{Z})$ under α forms a subgroup of D, which we denote by Δ. It should be noted that Δ generates all of D over the real numbers. For if Δ generated a proper subspace, then there would be a non-zero complex linear form L on D such that $\mathrm{Re}\,(L(\Delta)) = 0$. But then L would define a non-zero closed holomorphic form on X with the real parts of its periods being zero. This contradicts the second remark above. We now let $\overline{\Delta}$ be the smallest closed complex subgroup in D which contains Δ and whose connected component of the identity is a *complex* subspace of D. The quotient $D/\overline{\Delta}$ is then a compact complex torus which we denote by $A(X)$.

The rest of the Albanese construction follows as in the compact case [7]. That is, there is a natural holomorphic map $\alpha: X \to A(X)$ satisfying the above mentioned universal property. We mention here only that the map α is defined by $\alpha(x) = F(x) + \overline{\Delta}$ where $F(x)$ is the linear functional on D given by $F(x)(\omega) = \int\limits_a^x \omega$ (note that F is only well defined up to elements of Δ; see [7] for details).

For the second half of the theorem we must show that $X = X(A(X), \alpha, F)$ is a homogeneous fiber bundle, with connected fiber $F = \alpha^{-1}(0)$ when X is complex-homogeneous. First, we note that for any automorphism $g: X \to X$ we have the holomorphic map $\alpha \circ g: X \to A(X)$ and so, by universality, a holomorphic map $\tau(g): A(X) \to A(X)$ such that $\tau(g) \circ \alpha = \alpha \circ g$. Applying the same argument to g^{-1} we see that $\tau(g)$ has a holomorphic inverse, namely $\tau(g^{-1})$, so $\tau(g)$ is an automorphism of $A(X)$. In addition $\tau(g_1) \circ \tau(g_2) \circ \alpha = \tau(g_1) \circ \alpha \circ g_2 = \alpha \circ g_1 \circ g_2 = \tau(g_1 \circ g_2) \circ \alpha$. Thus we have a homorphism $\tau: G \to T(X)$, (here $T(X)$ denotes the complex group of translations of $A(X)$), defined by the composition of the following holomorphic maps, $p_{x_0}: G \to X$,

$p_{x_0}(g) = g(x_0)$; $\alpha: X \to \alpha(X)$ with $\alpha(x_0) = 0$; and $i: A(X) \to T(X)$, $i(\alpha(x)) =$ = translation by $\alpha(x)$. Furthermore, $\alpha(X)$ can be identified with the complex abelian Lie group $\tau(G)$: $\alpha(X) = \alpha(Gx_0) = \tau(G)\alpha(x_0) \cong \tau(G)$ under i. So we may write $\alpha(X) = \boldsymbol{C}^k/\Gamma$ for some discrete subgroup Γ. Now $\alpha(X)$ must be compact for otherwise we would have non constant pluriharmonic functions on X, e.g. $\alpha^*(\operatorname{Re}(z_i))$ for an appropriate coordinate function z_i in $\boldsymbol{C}^k$. Thus $\alpha(X)$ is a complex torus, and by universality of α we must have $\alpha(X) = A(X)$.

We now show the fibers of α are G-invariant. Let $F(x) = \alpha^{-1}(\alpha(x))$. Then for $g \in G$, $\alpha(gF(x)) = \tau(g) \circ \alpha(F(x)) = \tau(g)\alpha(x) = \alpha(gx)$. Therefore $gF(x) =$ $= F(gx)$ for all $(g, x) \in G \times X$. As in Section 2, once we have such a G-invariant fibration we can put a natural fiber bundle structure on $\alpha: X \to A(X)$. We let $X = G/H$, where H is the isotropy of $K = \{g \in G \mid gF(x_0) = F(x_0)\}$. Then K is a closed subgroup of G containing H, and α can be identified with the natural projection $G/H \to G/K$. The base G/K is biholomorphic to $A(X)$, by the surjectivity of α, and the fiber K/H is biholomorphic to $F(x_0)$.

Since $A(X)$ is an abelian group, K must be a normal subgroup of G containing both H and the commutator subgroup of G. If $\hat{K}$ is any other closed normal subgroup of G containing H, then we have a holomorphic map $G/H \to G/\hat{K}$. Now $G/\hat{K}$ is a complex Lie group which has no non-constant holomorphic functions. (Otherwise G/H has non-constant holomorphic functions, contradicting $\mathfrak{O}(X) \cong \boldsymbol{C}$ for X p-concave.) Therefore $G/\hat{K}$ is abelian [23]. In fact, $G/\hat{K}$ must be a compact torus for otherwise there would be non-constant pluriharmonic functions on $G/\hat{K}$ and therefore on X. Universality of α gives a holomorphic map $\tau: G/K \to G/\hat{K}$ showing that $K \subset \hat{K}$. So K is the smallest closed normal subgroup containing H.

Finally, we show that the fiber F is connected. We let

$$K' = \cup \{K_\lambda \mid K_\lambda \text{ is a connected component of } K, K_\lambda \cap H \neq \emptyset\}$$

and then α can be factored into the holomorphic maps

$$\alpha_1: G/H \to G/K' \quad \text{and} \quad \alpha_2: G/K' \to G/K .$$

The α_1–fiber is now connected and the α_2–fiber is discrete. This shows that G/K' is an abelian Lie group and so, as above, G/K' is a torus. Universality of α gives a holomorphic map $\tau: G/K \to G/K'$ such that $\tau \circ \alpha = \alpha_1$. Therefore $G/K \cong G/K'$ and $F = K/H \cong K'/H$ is connected. $\square$

4. – A fibration lemma.

In this section we will study certain exceptional cases which arise in later proofs. These cases involve fibrations of a 0-concave homogeneous manifolds which have 1-dimensional fibers. We show that the fiber can never be isomorphic to $\boldsymbol{C}^*$, and if the base is a complex torus, it can also never be isomorphic to $\boldsymbol{C}$. We begin with a discussion of affine bundles and the obstructions to such bundles reducing to line bundles.

Let $X = X(Y, \pi, \boldsymbol{C})$ be an affine bundle over a complex manifold Y. Such a bundle has a local trivialization with respect to some Leray cover $\{U_i\}$ of Y. The transition functions $f_{ij}\colon U_i \cap U_j \to \operatorname{Aut}(\boldsymbol{C})$ are given by $f_{ij}(z_j) = a_{ij} z_j + b_{ij}$, where z_j is a fiber coordinate over U_j. Associated to X is a $\boldsymbol{P}^1$-bundle, $P = P(Y, \pi_1, \boldsymbol{P}^1)$ with transition functions $\left\{\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}\right\}$, and a line bundle $L = L(Y, \pi_2, \boldsymbol{C})$ with transition functions $\{a_{ij}\}$. If we let $[z_0^i, z_1^i]$ be the homogeneous fiber coordinate for P over U_i, then we obtain a bundle preserving holomorphic injection $X \to P$, given locally by $(u, z_i) \to (u, [z_i, 1])$. We will call $E = P \setminus X$ the infinity section of P, which is biholomorphic to Y via the section $s\colon Y \to P$, $s(y) = (y, [1, 0])$, and which is locally defined by the equation $w_i := z_1^i / z_0^i = 0$ on $\pi_1^{-1}(U_i)$. These functions determine transition functions $g_{ij} = w_i / w_j$ on $\pi_1^{-1}(U_i) \cap \pi_1^{-1}(U_j)$ which, when restricted to E, define $N(E)$, the normal bundle of E in P. Since $g_{ij} = w_i/w_j = z_j/z_i = z_j/(a_{ij} z_j + b_{ij}) = 1/(a_{ij} + b_{ij} w_j)$, we have $g_{ij} = 1/a_{ij}$ when restricted to E. Thus, making the identification of E with Y, we have that $N(E)$ is isomorphic to L^*, the dual bundle of L.

Let $\mathfrak{A}$ denote the sheaf of germs of holomorphic maps into $\operatorname{Aut}(\boldsymbol{C}) = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \boldsymbol{C}^*, b \in \boldsymbol{C}\right\}$. Two affine bundles $X = X(Y, \pi, \boldsymbol{C})$ and $X' = X'(Y, \pi', \boldsymbol{C})$ with transition functions $\{f_{ij}\}$ and $\{f'_{ij}\}$ respectively are equivalent if there exist $g_i \in H^0(U_i, \mathfrak{A})$ such that $g_i f_{ij} g_j^{-1} = f'_{ij}$. For then we have a biholomorphic map $g\colon X \to X'$ given locally by $g(u, z_i) = (u, g_i z_i)$, $(u, z_i) \in U_i \times \boldsymbol{C}$, such that $\pi' \circ g = \pi$. Thus to say that the affine structure of X reduces to the line bundle structure of L means that there exist $g_i = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ in $H^0(U_i, \mathfrak{A})$ such that $g_i f_{ij} g_j^{-1} = \begin{pmatrix} a_{ij} & 0 \\ 0 & 1 \end{pmatrix}$, or:

$$(*) \qquad a_i = a_j = \text{constant and } b_{ij} = b_i - a_{ij} b_j .$$

The obstruction to finding the functions b_i satisfying (*) lies in $H^1(Y, \mathfrak{L})$ where $\mathfrak{L}$ is the sheaf of germs of local holomorphic sections of L. To see this, define $c_{ij}=b_{ij}e_j$ on $U_i\cap U_j$, where e_i is a frame for the bundle L over U_i. Using the cocycle conditions on the affine bundle,

$$a_{ij}a_{jk}b_{ki}+a_{ij}b_{jk}+b_{ij}=0$$

and

$$a_{ij}a_{jk}a_{ki}=1\,,\qquad a_{ij}=a_{ji}^{-1}$$

we see that

$$\begin{aligned}c_{ij}+c_{jk}+c_{ki}&=(b_{ij}e_i+b_{jk}e_j+b_{ki}e_k)\\&=(b_{ij}+a_{ij}b_{jk}+a_{ij}a_{jk}b_{ki})\,e_i\\&=0\,.\end{aligned}$$

Thus, $\{c_{ij}\}$ represents a cohomology class in $H^1(Y,\mathfrak{L})$. If $\{c_{ij}\}$ is cohomologous to zero, then $c_{ij}=c_i-c_j$ where $c_i\in H^0(U_i,\mathfrak{L})$. Writing $c_i=b_ie_i$ for the appropriate holomorphic functions b_i on U_i, we have $b_{ij}e_i=b_ie_i-b_je_j=(b_i-a_{ij}b_j)e_i$, and so $b_{ij}=b_i-a_{ij}b_j$ which is condition (*).

We would now like to comment on holomorphic fiber bundles with $\boldsymbol{C}^*$ fibers. These bundles need not be principal bundles in general, but we show here that they are covered in a 2-to-1 way by principal $\boldsymbol{C}^*$-bundles. Let $X=X(Y,\pi,\boldsymbol{C}^*)$ be a holomorphic fiber bundle and consider the principal bundle $\tilde{X}=\tilde{X}(Y,\tilde{\pi},\operatorname{Aut}(\boldsymbol{C}^*))$ associated to X. Since $\operatorname{Aut}(\boldsymbol{C}^*)\cong\boldsymbol{C}^*\times\boldsymbol{Z}_2$, it is clear that the natural map $\beta\colon\tilde{X}\to X$, with $\pi\circ\beta=\tilde{\pi}$, is 2-to-1. Let $\tilde{\pi}_1\colon\tilde{X}\to\tilde{Y}$ and $\tilde{\pi}_2\colon\tilde{Y}\to Y$ be the Stein factorization of the map $\tilde{\pi}$, so that the $\tilde{\pi}_1$-fibers are isomorphic to $\boldsymbol{C}^*$ and the $\tilde{\pi}_2$-fibers are isomorphic to $\boldsymbol{Z}_2$. Since the fibrations $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are locally trivial and Aut $(\boldsymbol{C}^*)$-equivariant, we see that the group of the bundle $\tilde{X}=\tilde{X}(\tilde{Y},\tilde{\pi}_1,\boldsymbol{C}^*)$ is contained in the stabalizer in $\operatorname{Aut}(\boldsymbol{C}^*)$ of the fiber $\boldsymbol{C}^*$. In this case, the fiber $\boldsymbol{C}^*$ is a component of $\operatorname{Aut}(\boldsymbol{C}^*)$ and we know that the $\boldsymbol{Z}_2$ action on $\operatorname{Aut}(\boldsymbol{C}^*)$ interchanges components. Therefore the stabalizer of this $\boldsymbol{C}^*$ fiber is just the identity component, $(\operatorname{Aut}(\boldsymbol{C}^*))^0=\boldsymbol{C}^*$. Thus, $\tilde{X}$ is a principal $\boldsymbol{C}^*$-bundle, and with respect to the map $\tilde{\pi}_2\colon\tilde{Y}\to Y$ of the bases, β is a 2-to-1 bundle map of $\tilde{X}$ onto X.

With these notions in mind, we prove:

LEMMA 4. *Let X be a non-compact, connected, 0-concave manifold. If $X=X(Y,\pi,F)$ is any holomorphic fiber bundle over a compact complex manifold Y with connected fiber, then F cannot be isomorphic to $\boldsymbol{C}^*$. If, in addition, Y is a complex torus and the group of bundle preserving automorphism of X acts transitively on Y, then the fiber cannot be isomorphic to either $\boldsymbol{C}$ or $\boldsymbol{C}^*$.*

PROOF. If the fiber is isomorphic to $\boldsymbol{C}^*$, we consider the principal $\boldsymbol{C}^*$-bundle $\tilde{X}=\tilde{X}(\tilde{Y},\pi_1,\boldsymbol{C}^*)$ given above, along with the 2-to-1 bundle map $\beta: \tilde{X} \to X$. Since the map β is finite, we obtain a 0-concave exhaustion $\varphi \circ \beta$ of $\tilde{X}$, where φ is the 0-concave exhaustion for X. Now, the bundle $\tilde{X}$ is defined by transition functions $f_{ij}: U_i \cap U_j \to \boldsymbol{C}^*$, where $\{U_i\}$ is some Leray cover of $\tilde{Y}$. Let L be the line bundle over $\tilde{Y}$ defined by f_{ij}. Then we obtain a bundle preserving holomorphic injection of $\tilde{X}$ into L (given locally by the injection $U_i \times \boldsymbol{C}^* \to U_i \times \boldsymbol{C}$) so that we may identify $\tilde{X}$ with the complement of the zero section in L. In this way we obtain from the 0-concave exhaustion for $\tilde{X}$, a strongly pseudo-convex neighborhood of the zero-section in L. A theorem of Grauert [12] then shows that some power L^k imbeds $\tilde{Y}$ into projective space, $\boldsymbol{P}^N$. In particular, there is a non-constant section s in $H^0(\tilde{Y}, L^k)$. Letting z be a fiber coordinate in L, we have that sz^{-k} is a non-constant holomorphic function on $\tilde{X}$. This contradicts $\mathfrak{O}(\tilde{X}) \cong \boldsymbol{C}$ for pseudo-concave manifolds.

For the second half of the lemma we assume that Y is a complex torus, and that for any automorphism $\bar{g} \in (\mathrm{Aut}(Y))^0$ there is an automorphism $g \in \mathrm{Aut}(X)$ such that $\pi \circ g = \bar{g} \circ \pi$. By the first half of the lemma we may assume that, if the fiber is 1-dimensional, then it is isomorphic to $\boldsymbol{C}$. With respect to some Leray cover $\{U_i\}$ of Y, we then obtain transition functions $f_{ij}: U_i \cap U_j \to \mathrm{Aut}(C)$ which exhibit X as an affine bundle. The main step in the proof is to reduce the group of this bundle from $\mathrm{Aut}(\boldsymbol{C})$ to $\boldsymbol{C}$ or $\boldsymbol{C}^*$, which we now proceed to do.

As above we let $P = P(Y, \pi_1, \boldsymbol{P}^1)$ be the $\boldsymbol{P}^1$-bundle associated to X, $L = L(Y, \pi_2, \boldsymbol{C})$ be the line bundle associated to X, and $E = P \setminus X$ be the infinity section of P biholomorphic to Y via $s: Y \to P$, $s(y) = (y, [1, 0])$. An automorphism of E, $\bar{g} \in (\mathrm{Aut}(E))^0$, can be identified with an automorphism of Y, $\bar{g} \in (\mathrm{Aut}(Y))^0$. By assumption, there exists an automorphism of X, $g \in \mathrm{Aut}(X)$, such that $\pi \circ g = \bar{g} \circ \pi$. Since g preserves the π-fibers, g extends pointwise in the π_1-fibers to an automorphism of P, $g \in \mathrm{Aut}(P)$. Note that g fixes E and that g restricted to E agrees with $\bar{g}$. The original defining functions of E, $w_i = 0$, are translated by g to $w_i \circ g = 0$, but these still define E. Therefore $N(E)$ is isomorphic to $g^* N(E)$, as a line bundle, and we get a bundle preserving automorphism $\hat{g}: N(E) \cong g^* N(E) \to N(E)$. Now we make the identification of $N(E)$ with L^* as outlined above, to obtain a bundle automorphism $\hat{g}: L^* \to L^*$ which yields a bundle automorphism $\hat{g}: L \to L$. Since this can be done for any $\bar{g} \in (\mathrm{Aut}(Y))^0$, we have that the group of bundle preserving automorphisms of L acts transitively on Y. The same condition holds for L_0, the principal $\boldsymbol{C}^*$-bundle associated to L (since $L_0 \subset L$). A theorem due to Matsushima [20] then implies that L_0 has a holomorphic connection and therefore is topologically trivial (see for e.g. [33]). This implies that L is also topologically trivial.

Now we have two cases: *a*) L is holomorphically trivial; or *b*) L is topologically trivial, but holomorphically non-trivial. In case *a*) there exist $a_i \in H^0(U_i, \mathcal{O}^*)$ such that $a_{ij} = a_i a_j^{-1}$. Setting $g_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}$ we have $g_i f_{ij} g_j^- = \begin{pmatrix} 1 & a_i^{-1} b_{ij} \\ 0 & 1 \end{pmatrix}$, which shows that the affine structure group can be reduced to the group $\boldsymbol{C}$, realizing X as a principal $\boldsymbol{C}$-bundle. In case *b*), a lemma due to Nakano [22] shows that $H^1(Y, \mathfrak{L}) = 0$. As outlined above, this implies that the affine structure of X can be reduced to the line bundle structure of L (with structure group $\boldsymbol{C}^*$). So, we have reduced the structure group to either $\boldsymbol{C}$ or $\boldsymbol{C}^*$, and in either case Matsushima ([20], Prop. 3.4) shows that the bundle X comes from a representation of the fundamental group of Y, $\varrho\colon \pi_1(Y) \to \boldsymbol{C}$. This means that the bundle X is defined by the equivalence relation on $\boldsymbol{C}^n \times \boldsymbol{C}$ ($\boldsymbol{C}^n =$ the universal cover of Y) given by $(w, z) \mathrm{X} (w + \gamma, z + \varrho(\gamma))$ for $\gamma \in \Gamma$, where $\Gamma \cong \pi_1(Y)$ is a lattice in $\boldsymbol{C}^n$, such that $Y = \boldsymbol{C}^n/\Gamma$. Therefore, if we let Γ' be the lattice in $\boldsymbol{C}^{n+1}$ defined by $\{(\gamma, \varrho(\gamma)) \,|\, \gamma \in \Gamma\}$, we have $X = \boldsymbol{C}^{n+1}/\Gamma'$. But then, because X is non-compact, there is some coordinate function z_i on $\boldsymbol{C}^{n+1}$ for which $\mathrm{Re}\,(z_i)$ gives a non-constant pluri-harmonic function on X. As in section 3, this contradicts the 0-concavity of X. Therefore the fiber F of $X \to Y$ cannot be isomorphic to $\boldsymbol{C}$. $\square$

5. – The case of solvable groups.

We now present two brief but useful remarks on the special case when $X = G/H$ is p-concave, $0 \le p \le n-2$, and the complex Lie group G is solvable. These remarks rely on the work on Barth and Otte [6]. First, we note that *the Albanese variety of X is always non-trivial*, $A(X) \neq \{0\}$. To see this, we start with the observation that if $\mathcal{O}(G/H) \cong \boldsymbol{C}$ and G is solvable, then $G/H \cong \tilde{G}/\Gamma$ where $\tilde{G}$ is solvable and Γ is a discrete subgroup, [6]. If $\tilde{G}$ is abelian then $\tilde{G}/\Gamma$ is a p-concave Lie group and [3] shows that $\tilde{G}/\Gamma$ is a compact complex torus. So we need only consider the case where $\tilde{G}$ is non-abelian. Barth and Otte prove [6, propositions 2.5 and 3.2] that under these assumptions, there exists a proper closed normal subgroup $L \subset \tilde{G}$ such that $L \cdot \Gamma$ is closed and $0 < \dim L < \dim \tilde{G}$. We then obtain the non-trivial fibration $\tilde{G}/\Gamma \to \tilde{G}/L \cdot \Gamma = G_1/\Gamma_1$ where $G_1 := \tilde{G}/L$ is solvable and $\Gamma_1 := \Gamma/(L \cap \Gamma)$ is discrete. Note that $0 < \dim G_1 < \dim \tilde{G}$. If G_1 is non-abelian we can repeat this construction and obtain another fibration $G_1/\Gamma_1 \to G_2/\Gamma_2$, with G_2 solvable and Γ_2 discrete. Again we have $0 < \dim G_2 < \dim G_1$. This procedure cannot continue indefinitely so for some k, G_k is abelian and we have the holomorphic map $X \to G_k/\Gamma_k$ given by the composition $X \to G_1/\Gamma_1 \to \dots \to G_k/\Gamma_k$. Since G_k/Γ_k is an abelian Lie Group, it must be compact, for otherwise there are non-constant pluriharmonic functions on G_k/Γ_k which lifts to X, contradicting p-concavity, $0 \leqq p \leqq n-2$ (see Section 2). Therefore G_k/Γ_k is a compact complex torus and $\dim G_k/\Gamma_k > 0$. Since $A(X)$ maps onto G_k/Γ_k by universality, it follows that $A(X)$ is non-trivial.

The second remark we can make is the following:

If $X = G/H$ is a 0-concave manifold and G is a solvable complex Lie group, then X is compact. To prove this, we use an induction argument on the dimension of X. As before, we have $X = G/\Gamma$ with G solvable and Γ discrete. We consider the Albanese fibration $X \to A(X)$, with $A(X) = G/K$ non-trivial by the preceding remark, and with fiber $F = K/\Gamma$ where K is solvable. Since $A(X)$ is non-trivial, $\dim F < \dim X$. Of course, if $\dim F = 0$, then $X = A(X)$ because F is connected and so X is compact. If $\dim F = 1$ then Lemma 4 shows that again X is compact. In particular, if $\dim X = 2$, then X must be compact. If $\dim X > 2$, we may assume $\dim F > 1$, so that F is 0-concave by restricting the exhaustion for X. The induction hypothesis then applies that F is compact. Therefore, X is compact.

6. – The case of discrete isotropy.

Now let $X = G/H$ be an arbitrary non-compact 0-concave complex-homogeneous manifold. We want to show that H cannot be discrete, i.e. that X *is not group theoretically parallelizable.* First we note that the two dimensional case is handled in [11]. Assuming $\dim X > 2$, we can apply a theorem of Andreotti and Siu [4] to obtain a minimal compactification V of X, where V has at most a finite number of normal isolated singularities in $V \setminus X$. We now extend the automorphisms of X to automorphisms of V in the following lemma:

LEMMA 6. *Let $X = G/H$ be a 0-concave homogeneous manifold, where G is an arbitrary connected Lie group acting effectively on X, and let V be a minimal compact complex space such that $X \subset V$. Then $G \subset \mathrm{Aut}(V)$.*

PROOF. Let $Y_{a_0} = V \setminus X_{a_0}$, where $X_{a_0} = \{x \in X \,|\, \varphi(x) > a_0\}$. Since Y_{a_0} has a strongly pseudo-convex exhaustion given by φ, and since V is minimal, it is easy to see that Y_{a_0} is Stein. Choosing another a_0 if necessary, we can imbed Y_{a_0} into a bounded subspace of $\boldsymbol{C}^N$, $j\colon Y_{a_0} \to \boldsymbol{C}^N$. Now consider the « collar » $X_a^b := \{x \in X \,|\, a < \varphi(x) < b\}$ where $a < b < a_0$. Since G is generated by any open neighborhood of the identity we need only extend an arbitrary element $g \in U := \{g \in G \,|\, g(\bar{X}_a^b) \subset Y_{a_0}\}$ (U is an open set in the c-o topology).

Let $\gamma = g \circ j \colon X_a^b \to C^N$. Then γ has components $\gamma_i \colon X_a^b \to C$. By a general Hartogs theorem [29], each γ_i extends as a holomorphic function to $Y_a = V \setminus \bar{X}_a$. Thus, we get a holomorphic map $\gamma \colon Y_a \to C^N$, and it is clear that $\gamma(Y_a) \subset j(Y_{a_0})$. Define the holomorphic map $\tilde{g} \colon V \to V$ to be $j^{-1} \circ \gamma$ on Y_a and g on X_a. Doing the same for g^{-1} (which we can arrange to be an element of U), we get another holomorphic map $\tilde{g}^{-1} \colon V \to V$. Since $\tilde{g} \circ \tilde{g}^{-1} = id$ on the open set X_a, we see that $\tilde{g} \circ \tilde{g}^{-1} = id$ on V. Thus $\tilde{g}$ is invertible and therefore is the desired automorphism of V extending g. This completes the proof of Lemma 6.

The complex Lie group G now acts almost transitively on V, i.e. G has an open orbit in V, namely X. Therefore, there is a lower dimensional analytic set $E \subset V$ such that $V \setminus E = X$ (see [25]). Note that $E \subset V \setminus \bar{X}_{a_0}$, and $V \setminus \bar{X}_{a_0}$ is Stein, as above. This shows that $V \setminus X = E$ is, in fact, a finite set of points.

We now sketch an argument which appears in detail in [16]. Let $x_0 \in E$. If $G = RS$ is a Levi-Malcev decomposition of G, then S can be linearized at x_0. That is, one imbeds V (locally in a neighborhood of x_0) in the Zariski tangent space of V at x_0, and the linear representation of S acting on this space is almost-faithful. An application of Lie's Flag Theorem to this action shows that any Borel subgroup B in S must have an orbit in X which is at most 1-dimensional.

If the isotropy group H were discrete, then the above argument shows that a Borel subgroup in S is at most 1-dimensional. Thus $S = \{1\}$, and G would be solvable. However, this is ruled out by the remarks in the preceding section. Hence the isotropy group is never discrete.

7. – The case of semi-simple groups.

The next case we wish to consider is when $X = S/H$ is a non-compact 0-concave manifold and the complex Lie group S is semi-simple. We will show in fact that this case is impossible, i.e. *no semi-simple complex Lie group acts transitively on a non-compact* 0*-concave manifold.*

We first show that there exists a homogeneous fibration of X, $S/H \to S/J$ with fiber $J/H \simeq C$ and compact rational base S/J. We note that S is a linear algebraic group, because S is semi-simple. We define M to be a *proper* algebraic subgroup of S which contains H, and whose dimension is maximal with respect to all such subgroups. To see that at least one proper algebraic subgroup of S contains H, we look at the normalizer fibration of section 1. There we had S acting algebraically on a Grassmann manifold and we were able to identify an orbit of S with the coset space S/N, where $N = N_S(H^0)$. Therefore, N is an algebraic subgroup of S which contains H. If $N = S$, then $X = S/H = N/H = (N/H^0)/(H/H^0)$ is 0-concave and parallelizable, which is impossible by the previous section. Thus, N is a proper subgroup of S. Now, by [21], if M^0 is reductive, then S/M^0 is Stein. But then S/M has non-constant holomorphic functions since the map $S/M \to S/M^0$ has finite fiber M/M^0, and therefore S/H has non-constant holomorphic functions, since $S/H \to S/M$. We conclude that M^0 cannot be reductive, so that the subgroup $R_u(M)$ of all unipotent elements in the radical of M is positive dimensional (see [18]). Let $U = (R_u(M))^0 = (R_u(M^0))^0$, and note that $M \subset N_S(U)$. Now, $N_S(U)$ is an algebraic subgroup of S which is proper, because S is semi-simple (S cannot have a non-trivial normal unipotent subgroup!). So, we redefine M to be $N_S(U)$, and observe that M is now a maximal proper subgroup of S. For, if $M' \supset M$, then $\dim M' = \dim M$ by the definition of M so that $(M')^0 = M^0$. This shows that $U = (R_u(M'))^0$ and so $M \supset M'$. According to a standard theorem of reductive groups ([18], Section 30.3, Cor. *b*), M is a parabolic subgroup of S. Thus, S/M is a non-trivial homogeneous compact rational manifold. Since X is non-compact, the fiber of $S/H \to S/M$ must be non-compact connected and positive dimensional. If $\dim M/H = 1$ then $M/H \simeq \boldsymbol{C}$ by Lemma 4 and we have our assertion. If $\dim M/H > 1$ then M/H inherits the 0-concavity of X. Since $\dim M/H < \dim X$, we can apply the following induction hypothesis to M/H: any non-compact complex-homogeneous 0-concave manifold Y is a homogeneous bundle $Y \to Z$ with fiber $\boldsymbol{C}$ and Z a compact homogeneous rational manifold. (That this is true for $\dim Y = 2$ follows from [11]). Thus we get a homogeneous bundle $M/H \to M/J$ with M/J a compact rational manifold and $J/H \simeq \boldsymbol{C}$. Now M contains a maximal connected solvable subgroup B of S since it is parabolic. This subgroup B is also a maximal connected solvable subgroup of M. Since J is a parabolic subgroup of M (M/J is projective algebraic), it must contain a conjugate of B [18]. Therefore, J is a parabolic subgroup of S and so S/J is a compact rational manifold. Then we have the holomorphic fibration $S/H \to S/J$ with fiber $J/H \simeq \boldsymbol{C}$, as desired.

We now outline a geometric argument that shows this situation actually cannot happen. (An algebraic proof appears in [16].) Let K be a maximal compact subgroup of S. If $p \in X$, then $K(p)$ is clearly either a 1-codimensional or 2-codimensional real submanifold of X because it covers the base. In the latter case, the simple-connectivity of the base implies that it is a section of the affine bundle $S/H \to S/J = Q$. If two distinct such K-orbits were 2-codimensional, then the associated line bundle would be topologically trivial and therefore holomorphically trivial (since $H^1(Q, \mathfrak{O}) = 0$). The affine bundle would then be reducible to a principal $\boldsymbol{C}$-bundle as in section 6. However, this bundle is also holomorphically trivial because $H^1(Q, \mathfrak{O}) = 0$. This contradicts the 0-concavity of X, so that K has at most one 2-codimensional orbit.

Considering a representation ϱ of K acting on a fiber of $S/H \to S/J$, we see that $\varrho(K)$ has exactly one fixed point, and K has exactly one 2-codimensional orbit in X. We will call this orbit the *minimal K-orbit* and prove that it must be a complex manifold.

The K-orbits in the Stein space $V \setminus \overline{X}_{a_0}$ are smooth hypersurfaces and are therefore strongly pseudoconvex. We may assume that the minimal normal compactification V of X is singular, as otherwise we may quote the results of E. Oeljeklaus [26] to show that $X = \boldsymbol{P}^n \setminus \{p\}$, and in this case no semi-simple group can act transitively on X. Now applying a result of H. Rossi [30], a strongly pseudoconvex K-orbit bounds a « tube » neighborhood of the zero-section of a negative K-homogeneous line bundle over a homogeneous rational manifold. Thus, we can compactify X by adding the zero-section, E, of this bundle and identify the strongly pseudoconvex neighborhood U of E with the tube neighborhood T in the Rossi bundle. Call this new compactification V'.

Let W be the vector space generated by n-fold wedge products of the complexifications of K-invariant vector fields coming from the Lie algebra of K. Let $\psi \colon X \to \boldsymbol{P}^m$ be the map associated to W, i.e. for a choice of basis $w_0, w_1, \dots, w_m$ of W, $\psi(p) = [w_0(p) : w_1(p) : \dots : w_m(p)]$. If we assume that the minimal K-orbit is *not* a complex manifold, then ψ has no base points because $w_0, w_1, \dots, w_m$ vanish simultaneously only on this orbit. Now the correspondingly defined map of the Rossi bundle is just projection on the base of the bundle. But, T and U are K-invariantly defined and K-equivariantly identified. Thus, ψ has 1-dimensional fibers in U. Since K acts on W, it follows that ψ is K-invariant and maps X onto the rational base Q' of the Rossi bundle. In addition, the homotopy sequence shows that the ψ-fibers are connected.

We have constructed a « new » fibration of X as an affine bundle over a rational manifold Q'. (It is clear that $S = K^{\boldsymbol{C}}$ acts invariantly on this bundle so that the fiber is complex-homogeneous. Lemma 4 then shows it must be $\boldsymbol{C}$.) We know that the normal bundle of E, $N(E)$, in the Rossi bundle is equivalent to the normal bundle of E in V', and that this normal bundle is negative due to the strongly pseudo-convex neighborhoods of E. Let L be the line bundle associated to the affine bundle $X = X(Q', \psi, \boldsymbol{C})$. As in section 4 we have that L is dual to $N(E)$, so that L is positive. Then, by the Kodaira Vanishing Theorem, we have that $H^1(Q', \mathfrak{L}) = 0$, and therefore the affine bundle structure of X can be reduced to the positive line bundle structure of L (see section 4). Thus, X has a 1-point compactification to a rational cone in some $\boldsymbol{P}^m$, as we prove in section 10. The semi-simple group S acts on this cone, fixing the vertex and a complementary hyperplane, D. Thus, $D \cap X$ must be stabalized by S, contrary to the assumption that S acts transitively on X.

The above contradiction resulted from the assumption that the minimal K-orbit is not a complex manifold. Thus, it must be a complex manifold. But, in this case, S would stabalize this orbit and again not act transitively on X. We conclude that no semi-simple complex Lie group acts transitively on X.

8. – The case of linear groups.

We are now ready to prove the classification theorem for $X = G/H$ 0-concave when G is a complex linear group:

THEOREM 8. *Let $X = G/H$ be a connected, non-compact, 0-concave, complex-homogeneous manifold which is Zariski-open in a projective algebraic variety $V \subset \boldsymbol{P}^N$. If $G \subset \operatorname{Aut}(\boldsymbol{P}^N)$, then $X = \hat{G}/I$ where $\hat{G}$ and I are complex linear algebraic groups. In addition, $X = X(Q, v, C)$ is a homogeneous affine bundle over a homogeneous compact rational manifold Q. This bundle is given by the natural projection $\hat{G}/I \to \hat{G}/RI$, where R is the radical of $\hat{G}$, and the structure of this bundle can be reduced to that of a positive line bundle. In particular, X is simply connected.*

Before we begin the proof, we would like to remark that there is a fixed set $X_a := \{x \in X \,|\, \varphi(x) > a\}$ (with $a < a_0$) which intersects any positive dimensional analytic subset A of X. To see this, suppose $A \cap X_a = \emptyset$. Then A is contained in the compact set $\bar{V} \setminus X_a$, where $\bar{V}$ is the minimal compactification of X mentioned in section 6. Since $\bar{V} \setminus X$ is only a finite set of points we may apply Remmert-Stein [27] to extend A to a compact analytic set $\tilde{A}$ in $\bar{V} \setminus X_a$. However, $\tilde{A} \subset \bar{V} \setminus \bar{X}_{a_0}$, and $\bar{V} \setminus \bar{X}_{a_0}$ is Stein. This contradicts the fact that A is positive dimensional. Therefore, $A \cap X_a \neq \emptyset$. This remark shows that the base of any fibration of X with positive dimensional fiber is compact: every fiber must intersect the compact set $\bar{X}_a$.

PROOF. Let E be the compact analytic (algebraic) set $V \setminus X$, and let $G(V)$ and $G(E)$ denote the stabalizers of V and E, respectively, in $\operatorname{Aut}(\boldsymbol{P}^N)$. Define $\hat{G} := G(V) \cap G(E)$. Since $G(V)$ and $G(E)$ are linear algebraic groups, so is $\hat{G}$. Now, $G \subset \hat{G}$ and $\hat{G}$ stabalizes X, so we may write $X = \hat{G}(x_0) = \hat{G}/I$ where I is the isotropy of the point $x_0 \in X$.

Let R be the radical of $\hat{G}$, i.e. the largest connected normal solvable subgroup of G. Then R stabalizes a flag in $\boldsymbol{P}^N$:

$$\{0\} = L_0 \subset L_1 \subset \ldots \subset L_N = \boldsymbol{P}^N.$$

Set $k = \min\{i | L_i \cap X \neq \emptyset\}$. If R fixes a point $x \in L_k \cap X$, then it must fix every point of X, since R is normal. Therefore R is contained in I. Using a theorem of Levi-Malcev [18], we express $\hat{G}$ as a semi-direct product $\hat{G} = RS$, where S is semi-simple. Then $I = R(I \cap S)$, and so $X = \hat{G}/I = (\hat{G}/R)/(I/R) = S/L$, where $L = I \cap S$ is an algebraic subgroup of S since I and S are algebraic. This contradicts section 7, so R must have no fixed points and therefore $\dim L_k \cap X > 0$. If $\dim L_k \cap X > 1$, then $L_k \cap X \subset L_k \setminus L_{k-1} \cong \boldsymbol{C}^k$ is an analytic set in X which evidently has non-constant holomorphic functions defined on its top dimensional components. However, these components inherit the 0-concavity of X, and so they can have no non-constant holomorphic functions. This contradiction implies that $\dim L_k \cap X = 1$. We claim that R acts transitively on every connected component, C, of $L_k \cap X$. This follows from the fact that $R(x)$, the orbit of $x \in C$ under R, must be Zariski-open in C, if R is not to have fixed points. But the complement of $R(x)$ in C must either be empty or contain isolated fixed points ($\dim C = 1$). Therefore $R(x) = C$ for $x \in C$. This shows that $R(x)$ is closed in X for all $x \in X$, and since R is normal in $\hat{G}$, these orbits are $\hat{G}$-invariant. So we have the natural fibration $\nu\colon X = \hat{G}/I \to \hat{G}/RI$. The base $\hat{G}/RI = (\hat{G}/R)/(RI/R) = S/(I/R \cap I) = S/P$ is compact since all ν-fibers must intersect $\overline{X}_a$. The fiber $R(x_0) = RI/I$, being complex-homogeneous, non-compact, connected, and 1-dimensional, must be either $\boldsymbol{C}$ or $\boldsymbol{C}^*$. Since X is 0-concave, Lemma 4 implies that the fiber is $\boldsymbol{C}$, i.e. $X \to S/P$ is a homogeneous affine bundle.

We now show that the base S/P is projective algebraic and therefore a compact homogeneous rational manifold. Consider the orbit $S(x)$ for some $x \in X$. Since S is an algebraic group, $S(x)$ is a submanifold of X and is Zariski-open in its closure $\overline{S(x)}$, a subvariety of V. If $S(x)$ is open in X then either $S(x) = X$ which is impossible by section 7, or there is a lower dimensional analytic set A in X such that $S(x) = X \setminus A$. In this case we redefine x to be a point in A, so that we can say $S(x)$ is not open in X. Now, the fibration ν restricted to $S(x)$ is surjective so that $\dim X - 1 \leq \dim S/P \leq \dim S(x) \leq \dim X - 1$. Thus, $S(x) \cap \nu^{-1}(\nu(x))$ is a discrete set. Now, $\overline{S(x)} \cap \overline{\nu^{-1}(\nu(x))}$ is an algebraic subvariety of V which is also discrete since $S(x)$ is Zariski-open in $\overline{S(x)}$ and $\dim \overline{\nu^{-1}(\nu(x))} = 1$. Therefore $S(x) \cap \nu^{-1}(\nu(x))$ must be a finite set and $\nu_0\colon S(x) \to S/P$ a finite map. Since S/P is compact, we have that $S(x)$ is compact and therefore $S(x) = \overline{S(x)}$ is a projective algebraic manifold. It then follows that S/P is projective algebraic (see e.g. [13]).

We note that, since $Q := S/P$ is now a compact homogeneous rational manifold and in particular simply connected, $\nu_0\colon S(x) \to Q$ is biholomorphic and thus defines a section $s := \nu_0^{-1}\colon Q \to X$ of the affine bundle $\nu\colon X \to Q$. Let $f_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}$ be the transition functions for this bundle so that if $s = s_i$ on U_i, then $s_i = a_{ij}s_j + b_{ij}$ on $U_i \cap U_j$ ($\{U_i\}$ a cover of Q). We define $g_i = \begin{pmatrix} 0 & 1 \\ 1 & -s_i \end{pmatrix}$ on U_i and observe that $g_i f_{ij} g_j^{-1} = \begin{pmatrix} a_{ij} & 0 \\ 0 & 1 \end{pmatrix}$. This shows that the affine structure of X reduces to a line bundle structure (see section 4). The 0-concave exhaustion for X now provides a strongly pseudoconvex neighborhood of the zero-section in the dual of this bundle. A theorem of Grauert [12] then shows that this dual bundle is negative, so that in fact X is a positive line bundle. $\square$

9. – The general case.

In this section we show that the conclusion of Theorem 8 applies to an arbitrary non-compact 0-concave complex-homogeneous manifold $X = G/H$ i.e. that *X is a positive line bundle over a compact homogeneous rational manifold*. We begin with the normalizer fibration of section 1, $\mu\colon X \to B$, where B is Zariski-open in an irreducible compact projective algebraic variety $\hat{B} \subset \boldsymbol{P}^K$ and the fiber F is group theoretically parallelizable. If $\dim F > 1$, then F inherits the 0-concavity of X. Section 6 then implies that F is compact. Since the base of any fibration of X with positive dimensional fiber must be compact (see section 7), we have that X is compact, a contradiction. Therefore, $\dim F \leq 1$.

If $\dim F = 1$, then B is again compact and in fact must be a homogeneous rational manifold, as mentioned in section 1. Since F is non-compact connected and complex-homogeneous, it must be isomorphic to $\boldsymbol{C}$ or $\boldsymbol{C}^*$. Lemma 4 eliminates $\boldsymbol{C}^*$, so $F \cong \boldsymbol{C}$. Let $G = RS$ be a Levi-Malcev decomposition of G, where R is the radical of G and S is semi-simple. We recall from section 1 that $B = G/N$ with $N = N_G(H^0)$. A standard flag argument [10] shows that $R \subset N$, so that $B = G/N = (G/R)/(N/R) = S/P$. Now consider the orbit $S(x_0)$, for some $x_0 \in X$. We claim that we can choose x_0 so that $S(x_0)$ is a section of the bundle $\mu\colon X \to B$. Since μ restricted to $S(x_0)$ is surjective, we have $\operatorname{codim} S(x_0) \leq 1$. If $\operatorname{codim} S(x_0) = 1$, then μ_0: $S(x_0) \to B$ is a covering map, and since B is simply connected μ_0 is biholomorphic. Thus we get a section $s := \mu_0^{-1}: B \to X$ with $s(B) = S(x_0)$. If $\operatorname{codim} S(x_0) = 0$ then $S(x_0)$ is open in X. In this case, either $S(x_0) = X$, which is impossible by section 7, or there exists a positive dimensional analytic set A in X such that $S(x_0) = X \setminus A$. We redefine x_0 to be a point in A so that $S(x_0) \subset A$ and therefore $0 < \operatorname{codim} S(x_0) \leq 1$. Thus, $\operatorname{codim} S(x_0) = 1$ and, as in the above case, $S(x_0)$ is a section of the bundle $\mu: X \to B$. Note that using the section s we can reduce the structure of this bundle to that of a line bundle (see section 7). We now choose $x \in X \setminus S(x_0)$. The orbit $S(x)$ must be open in X, for otherwise $S(x)$ is another section of the line bundle $\mu: X \to B$ (as above) distinct from the section $S(x_0)$. However, this implies that the bundle is holomorphically trivial which is impossible (e.g. X would posses non-constant holomorphic functions). Therefore, $S(x) \cap \cap \mu^{-1}(\mu(x))$ is open in the fiber $\mu^{-1}(\mu(x)) = F$. We recall from section 1 that $F = N/H = N(x)$. Thus, $S \cap N$ has an open orbit in F. Notice that $\{x_1\} = S(x_0) \cap F$ so we must have $S \cap N(x_1) = \{x_1\}$. If we define H to be the isotropy of x_1, then $S \cap N \subset H$, and therefore H (and H^0) must have an open orbit in F. However, for any $y \in F$, there is an $n \in N$ such that $n(x_1) = y$, and since N normalizes H^0,

$$H^0(y) = H^0(n(x_1)) = nH^0n^{-1}(n(x_1)) = nH^0(x_1) = \{n(x_1)\} = \{y\}\,.$$

This contradiction implies that $\dim F \neq 1$.

If $\dim F = 0$, then $\mu: X \to B$ is a covering map. Again, if B were compact, then it would be a rational manifold and thus simply connected. This would imply that $X = B$ is compact, contrary to our assumption. Therefore B is non-compact and Zariski-open in a projective algebraic variety $\hat{B} \subset \boldsymbol{P}^K$ (Theorem 1). We recall also from the proof of Theorem 1 that $B = \tilde{G}/H$ with $\tilde{G} = \operatorname{Ad}(G) \subset \operatorname{Aut}(\boldsymbol{P}^K)$. We now show that X has an algebraic compactification V and that μ extends as a birational map to $\hat{\mu}: V \to \hat{B}$.

We observe that there exist $n = \dim \hat{B} = \dim X$ algebraically independent meromorphic functions on $\hat{B}$. Thus, there exist n algebraically independent meromorphic functions on X. Andreotti shows [1] that algebraic independence of meromorphic functions implies analytic independence on pseudo-concave manifolds. Therefore by homogeneity we can assure that for any $x_0 \in X$ there are n meromorphic functions on X which are holomorphic and analytically independent near x_0 (i.e. give local coordinates near x_0). Note that we may assume $n > 2$, since the two dimensional case is proved in [11]. We now cover the compact set $\bar{X}_a := \{x \in X \,|\, \varphi(x) \geq a\}$ with a finite number of open sets U^k where on each U^k we have n meromorphic functions $f_1^k, \ldots, f_n^k$ which give local coordinates on U^k. Each meromorphic function f_i^k can be expressed as a ratio of two sections in some holomorphic line bundle $L_{i,k}$. Then $L := \bigotimes_{(i,k)} L_{i,k}$ is a holomorphic line bundle on X whose sections give local coordinates on $\bar{X}_a$. We apply the imbedding theorem of Andreotti and Siu [4] which states that a 0-concave manifold with such a line bundle L is isomorphic to an open set in a compact irreducible projective algebraic variety V. The exhaustion function for X gives a strongly pseudo-convex exhaustion for $V\setminus\bar{X}_a$. Thus, we can blow down the maximal compact analytic set in $V\setminus\bar{X}_a$ to get a new compactification $\bar{V}$ for X. Now, $\bar{V}$ is a minimal compactification for X and, as shown in section 6, $\bar{V}\setminus X$ is a finite set of points. Therefore any meromorphic function on X extends to $\bar{V}$ and can be lifted to V using the modification map. This shows that the injective morphism $i^*\colon \varkappa(V) \to \varkappa(X)$ is also surjective (where $i\colon X \to V$ is the inclusion). This isomorphism allows us to define an injective morphism $\hat{\mu}^* := i^{*-1} \circ \mu^* \circ j^*\colon \varkappa(\hat{B}) \to \varkappa(V)$ (where $j\colon B \to \hat{B}$ is the inclusion). Let $\hat{\mu}\colon V \to \hat{B}$ be the canonical surjective rational map associated to $\hat{\mu}^*$ (see e.g. [24]). Note that $\hat{\mu}$ agrees with μ when restricted to X, so that $\hat{\mu}$ is indeed an extension of μ. In addition, μ is a finite map, since $\hat{\mu}$ is finite ($\dim V = \dim \hat{B}$).

We can now construct a 0-concave exhaustion for B. Define $\hat{\varphi}\colon B \to \boldsymbol{R}_+$ by $\varphi(p) = \varphi(z_1) + \ldots + \varphi(z_r)$, where $z_i \in \mu^{-1}(p)$. Clearly, $\hat{\varphi}$ is smooth on B and strictly plurisubharmonic on $\{p \in B \,|\, \hat{\varphi}(p) < a_0\}$. Furthermore, if $\varphi(z_1) + \ldots + \varphi(z_r) \geq c$, with $z_i \in \mu^{-1}(p)$, then for some fixed $b \in (0, c)$ there must be at least one z_i such that $\varphi(z_i) > b/r =: a$. This shows that the closed set $\bar{B}_c = \{p \in B \,|\, \hat{\varphi}(p) \geq c\}$ is contained in the compact set $\mu(\bar{X}_a)$, and therefore is compact.

Thus $B = \tilde{G}/\tilde{H} \subset \hat{B} \subset \boldsymbol{P}$ is a 0-concave complex-homogeneous manifold with $\tilde{G} \subset \operatorname{Aut}(\boldsymbol{P}^K)$. Therefore, Theorem 8 applies to B, showing it to be simply connected. Therefore, $\mu\colon X \to B$ is biholomorphic and we can apply Theorem 8 to X.

10. – Projective cones.

Now that we know that a non-compact 0-concave complex-homogeneous manifold X can be realized as a positive line bundle $X = X(Q, \pi, \boldsymbol{C})$ over a compact homogeneous rational manifold Q, there is an easy geometrical way to describe X. This description, in fact, provides a convenient characterization of all such manifolds, and demonstrates how the conclusion of Theorem 8 applies to them.

We first show that the line bundle X is very ample. Let W denote the $(N+1)$-dimensional vector space of holomorphic sections of X, and consider the natural map $s: Q \to \boldsymbol{P}^N$ given by $s(q) = [s_0(q) : \ldots : s_N(q)]$ where $s_0, \ldots, s_N$ is some basis for W. (Here we abuse notation and allow $s_i(q)$ to stand for its fiber coordinate.) For any $q \in Q$, we can find a section $t \in W$ which does not vanish at q by finding a $g \in \operatorname{Aut}(Q)$ such that $g(0(q)) \notin 0(Q)$ and defining t to be $g \circ 0$. (We are letting $0: Q \to X$ denote the zero-section.) Therefore, the map s is well-defined, i.e. it has no base points, and $N \geq 0$.

Since the bundle $X = G/H$ is preserved by the action of G, the vector space W is invariant under the action of G and so the map s is invariant under G. This means that the map $s: Q \to \boldsymbol{P}^N$ is given by a homogeneous fibration $Q = G/J \to G/J' \subset \boldsymbol{P}^N$. Now, $Q = S/P$ where S is a semi-simple subgroup of G and P is parabolic. Therefore, $G/J' = S/P'$ where $P' \supset P$, so that P' is parabolic and G/J' is a compact rational manifold. The homotopy sequence then shows that the s-fibers are connected. If F is one such fiber of s, and $i: F \to Q$ is the inclusion map, then $i^*(X)$ must be a trivial line bundle over F, by the definition of F and s. (We can find a section $t \in W$ which is not zero at a point $q \in F$ and therefore not zero on all of F so that $i^*(t)$ trivializes $i^*(X)$). However, if F is positive dimensional, then $i^*(X)$ inherits the 0-concavity of X and so it must be a *positive* line bundle, not trivial. Therefore, F is 0-dimensional showing that s is in fact an imbedding of Q into $\boldsymbol{P}^N$, i.e. that the bundle X is very ample.

Letting z represent the fiber coordinate of the bundle, we construct a holomorphic map $j: X \to \boldsymbol{P}^{N+1}$ given locally by $j(u, z_i) = [z_i(u) : s_i^0(u) : \ldots : s_i^N(u)]$ on $U_i \times \boldsymbol{C}$ (where $s^k = s_i^k$ and $z = z_i$ on U_i). This is a well-defined map which imbeds the zero-section $\{z = 0\}$ of the line bundle into the hyperplane at infinity and maps each fiber of the bundle $\pi^{-1}(q)$ onto the complex line $E_q \setminus [1:0:\ldots:0]$, where E_q is the $\boldsymbol{P}^1$ in $\boldsymbol{P}^{N+1}$ connecting the origin $[1:0:\ldots:0]$ to the point $[0:s^0(q):\ldots:s^N(q)]$. We let V be the 1-point compactification of X, $V = X \cup [1:0:\ldots:0]$.

Given this imbedding, we can easily realize the results of Theorem 8. Since Q is a compact homogeneous rational manifold, there exists a semi-simple group $S \subset \operatorname{Aut}(\boldsymbol{P}^N)$ (here $\boldsymbol{P}^N$ is the hyperplane in which Q is imbedded), such that $Q = S/P$. Let $\hat{G} = \left\{ \begin{pmatrix} 1 & b \\ 0 & A \end{pmatrix} \middle| b \in \boldsymbol{C}^{N+1}, A \in S \right\} \subset \operatorname{Aut}(\boldsymbol{P}^{N+1})$. It is clear that $\hat{G}$ fixes the point $[1:0:\ldots:0] = V \setminus X$ and acts transitively on X. The radical of $\hat{G}$ is $R = \left\{ \begin{pmatrix} 1 & b \\ 0 & I \end{pmatrix} \middle| b \in \boldsymbol{C}^{N+1} \right\}$ which has orbits $R(x) = E_{\pi(x)} \setminus [1:0:\ldots:0]$. The fibration $X = \hat{G}/I \to \hat{G}/RI$ is then explicitly given by the projection of X to the plane at infinity.

On the other hand, given any compact homogeneous rational manifold $Q = S/P$, we can imbed it in a hyperplane $\boldsymbol{P}^N \subset \boldsymbol{P}^{N+1}$ with $S \subset \operatorname{Aut}(\boldsymbol{P}^N)$ and then construct a homogeneous manifold X by defining $\hat{G}$ as above. The orbit $X = \hat{G}(q)$ for some $q \in Q$ will have the same properties described above. Note that there is a natural 0-concave exhaustion for X given by the distance to the origin $[1:0:\ldots:0]$. Thus, we obtain the convenient characterization of a non-compact 0-concave complex-homogeneous manifold as a projective cone over a compact homogeneous rational manifold (with its vertex removed).

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