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On the Stability of the Dirichlet Problem for the Vibrating String Equation (*).

GLORIA PAPI FROSALI (**)

1. – Introduction.

The Dirichlet problem for the vibrating string equation:

$$\begin{cases} u_{xx} - u_{tt} = 0\,, & 0 < x < \pi\,, \quad 0 < t < \alpha\pi\,, \\ u(0, t) = u(\pi, t) = 0\,, & \qquad\qquad\quad 0 \leqslant t \leqslant \alpha\pi\,, \\ u(x, 0) = \varphi(x)\,, & 0 \leqslant x \leqslant \pi\,, \\ u(x, \alpha\pi) = \psi(x)\,, & 0 \leqslant x \leqslant \pi\,, \end{cases}$$

where α is a positive constant, is a classical ill-posed problem due to its irregular behaviour. Its solution may neither exist, nor be uniquely determined, nor depend continuously on the data (see [1], [2] and [3]).

Therefore, the above problem cannot be suitably dealt with if φ, ψ and α are known within a certain approximation. Without further information, one cannot answer the following problem:

« When the approximate shape of a vibrating string is given at two different times which are approximately known (e.g. by taking two photographs), find the position of the string at all intermediate times, up to an error whose magnitude can be controlled ».

This leads us to formulate the Dirichlet problem with approximate data and with the additional assumption that the energy of the string is bounded by some absolute constant (see (5) below). This constant can be obviously estimated in any particular case; on the other hand, since the wave equa-

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tion can be applied only to « small » vibrations of the string, such a bound is also implicit in the classical formulation of the problem.

In [2], the Dirichlet problem for the wave equation was studied with the additional assumption of an « a priori » bound for the gradient of the solution.

The present research leads to some problems of diophantine approximation. As far as such problems are concerned, we refer to [4]. For different applications of number-theoretic results to the heat equation see [5].

2. – Main results and comments.

Let $\varphi(x)$ and $\psi(x)$ be functions in $C^2[0,\pi]$ such that $\varphi(0)=\varphi(\pi)= =\psi(0)=\psi(\pi)=0$. Let E, α and δ be positive constants.

We consider solutions u in $C^2([0,\pi]\times[0,+\infty))$ of the following problem:

$$u_{xx}-u_{tt}=0\,,\qquad 0<x<\pi\,,\qquad t>0\,, \tag{1}$$

$$u(0,t)=u(\pi,t)=0\,,\qquad t\geqslant 0\,, \tag{2}$$

$$\|u(x,0)-\varphi(x)\|_{L^2[0,\pi]}\leqslant\delta\pi\sqrt{E}\,, \tag{3}$$

$$\|u(x,\tau\pi)-\psi(x)\|_{L^2[0,\pi]}\leqslant\delta\pi\sqrt{E}\,,\qquad |\tau-\alpha|\leqslant\delta\,, \tag{4}$$

$$\int_0^\pi\big(u_t^2(x,t)+u_x^2(x,t)\big)\,dx\leqslant E\,,\qquad t\geqslant 0\,, \tag{5}$$

for a real number τ depending on u and satisfying $|\tau-\alpha|\leqslant\delta$.

The problem (1)-(5) is the mathematical model describing the physical phenomenon that is formulated in the introduction. The meaning of $|\tau-\alpha|\leqslant\delta$ is that the final time $\alpha\pi$ is known up to a given error. Furthermore, since (3) and (4) yield the position of the string with an error given in the L^2 norm, it is clear that the corresponding constant should be proportional to $\delta\sqrt{E}$, where E is an upper bound for the energy of the string.

We denote by Γ_δ the set of all C^2 solutions of (1)-(5). We note that if $\delta=0$, then the problem (1)-(5) is reduced to the classical Dirichlet problem with the additional assumption (5) that the energy is bounded. It is well known that this problem may have no solutions. Hence it is necessary to assume some compatibility conditions for E, φ and ψ, in order to ensure that Γ_δ is non-empty; however, the study of such conditions goes beyond our purpose. On the other hand, as the system (1)-(5) is supposed to be a good model of a physical phenomenon, we may assume $\Gamma_\delta\neq\varnothing$.

We now state our results. Let

$$\|u\|^2 = \max_{t\in[0,\alpha\pi]} \int_0^\pi u^2(x,t)\,dx$$

and define the diameter of the set Γ_δ:

$$\operatorname{Diam}\Gamma_\delta = \sup_{v,w\in\Gamma_\delta} \|v-w\| .$$

DEFINITION. For a fixed α, the Dirichlet problem (1)-(5) is stable if

$$\lim_{\delta\to 0} (\operatorname{Diam}\Gamma_\delta) = 0 ,$$

for every E, φ and ψ.

As the functions in Γ_δ are periodic, we may assume $\alpha\in[0,1]$ in the sequel. We prove (Theorem IV) that:

The Dirichlet problem (1)-(5) *is stable if and only if α is irrational.*

The best estimate for Diam Γ_δ is obtained when α belongs to the set of measure zero (see [6] p. 60) of the « badly approximable » irrational numbers, i.e. those having bounded partial quotients in their expansions as simple continued fractions. This estimate is given in the following theorem:

THEOREM I. *Let α be an irrational number and let the simple continued fraction for α have bounded partial quotients $a_n \leqslant A_\alpha$, $n\geqslant 1$. Then*

$$(\operatorname{Diam}\Gamma_\delta)^2 < \delta\pi(A_\alpha+2)^2 E\,\frac{8}{27}\sqrt{30}\,(5+2\sqrt{3}) \qquad \textit{for } 0<\delta\leqslant \frac{3\sqrt{30}}{20\pi} .$$

If α has type $\Omega<\infty$ (1), the estimate we obtain for Diam Γ_δ is given in the following theorem:

THEOREM II. *Let α be an irrational number of type $\Omega<\infty$. Then, for any fixed θ, $\Omega/(\Omega+1)<\theta<1$, there exists a constant $K=K(\theta,\alpha)>0$*

(1) Let α be a fixed irrational. If Ω is the upper bound of the numbers ω satisfying

$$|\alpha - p/q| < 1/q^{1+\omega}$$

for infinitely many $p/q\in\boldsymbol{Q}$, we say that α has type Ω. It is known that almost all numbers α (i.e. all except a set of measure zero) have type $\Omega=1$ (see [6], p. 69).

such that

$$(\mathrm{Diam}\,\Gamma_\delta)^2 < \frac{80\pi^2 E}{(3-\sqrt{5})^2}(K\delta^{1-\theta}+\delta)^2 + 4E\delta^{2\theta}/K^2, \tag{6}$$

for

$$0<\delta<\left\{K\sqrt[4]{20}\sqrt{\pi/(3-\sqrt{5})}\right\}^{2/(2\theta-1)}. \tag{7}$$

The bounds for Diam Γ_δ in theorems I and II both depend on α. The uniformity with respect to α is obtained when α belongs to a subset of $[0, 1]$ having measure μ satisfying $1-\varepsilon<\mu\leqslant 1$, with $\varepsilon>0$ arbitrarily small (see theorem III).

REMARK. As far as the stability of the Dirichlet problem is concerned, it is superfluous to weaken the regularity conditions for the functions in Γ_δ. Indeed, Γ_δ is dense in the set of the weak solutions.

3. – Proofs.

We shall require the following lemma:

LEMMA I. *Let* $f(x, t)\in C^1([0,\pi]\times[0,T])$, *with* $T>0$. *For any* t_1 *and* t_2 *in* $[0, T]$ *such that* $t_1<t_2$, *there exists* $\bar t$ *satisfying* $t_1<\bar t<t_2$ *and*

$$\|f(\cdot, t_2)-f(\cdot, t_1)\|_{L^2[0,\pi]}\leqslant(t_2-t_1)\|f_t(\cdot,\bar t)\|_{L^2[0,\pi]}\,. \tag{8}$$

PROOF. We have

$$\|f(\cdot, t_2)-f(\cdot, t_1)\|^2_{L^2[0,\pi]}=\int_0^\pi[f(x,t_2)-f(x,t_1)]^2\,dx=\int_0^\pi\left(\int_{t_1}^{t_2}f_t(x,t)\,dt\right)^2dx.$$

By the Schwarz inequality

$$\left(\int_{t_1}^{t_2}f_t(x,t)\,dt\right)^2\leqslant(t_2-t_1)\int_{t_1}^{t_2}f_t^2(x,t)\,dt\,,$$

whence

$$\|f(\cdot, t_2)-f(\cdot, t_1)\|^2_{L^2[0,\pi]}\leqslant(t_2-t_1)\int_{t_1}^{t_2}\int_0^\pi f_t^2(x,t)\,dx\,dt\,. \tag{9}$$

By the mean value theorem, there exists $\bar t$ such that $t_1<\bar t<t_2$ and

$$\int_{t_1}^{t_2}\int_0^\pi f_t^2(x,t)\,dx\,dt=(t_2-t_1)\int_0^\pi f_t^2(x,\bar t)\,dx\,. \tag{10}$$

From (9) and (10), we obtain (8). This completes the proof of lemma I.

Let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

be the simple continued fraction for α, where the partial quotients a_n are integers such that $a_n \geqslant 1$.

We consider the set of irrational numbers with bounded partial quotients, i.e. the numbers α for which there exists a constant A_α satisfying $a_n \leqslant A_\alpha$ for all n.

We note that if α is a quadratic irrational, then the expansion of α as a simple continued fraction is ultimately periodic, which implies that α has bounded partial quotients.

We are now in position to prove theorem I.

Let $v_1, v_2 \in \Gamma_\delta$. Then there exist τ_1 and τ_2 such that

$$\|v_i(x, \tau_i \pi) - \psi(x)\|_{L^2[0,\pi]} \leqslant \delta\pi\sqrt{E}\,, \qquad |\tau_i - \alpha| \leqslant \delta\,, \qquad i = 1, 2\,. \tag{11}$$

If we let

$$u(x, t) = v_1(x, t) - v_2(x, t)\,, \qquad (x, t) \in [0, \pi] \times [0, +\infty)\,, \tag{12}$$

then $u \in C^2([0, \pi] \times [0, +\infty))$. Moreover, u satisfies (1), (2) and the following conditions:

$$\|u(x, 0)\|_{L^2[0,\pi]} \leqslant 2\delta\pi\sqrt{E}\,, \tag{13}$$

$$\|u(x, \alpha\pi)\|_{L^2[0,\pi]} \leqslant 4\delta\pi\sqrt{E}\,, \tag{14}$$

$$\int_0^\pi \big(u_x^2(x, t) + u_t^2(x, t)\big)\,dx \leqslant 4E\,, \qquad t \geqslant 0\,. \tag{15}$$

It is easy to verify (1), (2), (13) and (15). To prove (14), we note that

$$\|u(x, \alpha\pi)\|_{L^2[0,\pi]} \leqslant \sum_{i=1}^{2} \|v_i(x, \alpha\pi) - v_i(x, \tau_i\pi)\|_{L^2[0,\pi]} + \\ + \sum_{i=1}^{2} \|v_i(x, \tau_i\pi) - \psi(x)\|_{L^2[0,\pi]}\,. \tag{16}$$

Hence, condition (14) follows directly from (8) and (11). We can write the functions satisfying (1) and (2) in the following form:

$$u(x, t) = \sum_{n\geqslant 1} \sin nx \left\{\frac{A_n \sin n(\alpha\pi - t)}{\sin n\alpha\pi} + B_n \sin nt\right\}.$$

Similarly, we can rewrite (13), (14) and (15) as follows:

$$\sum_{n\geqslant 1} A_n^2 \leqslant 8\delta^2\pi E\,, \tag{17}$$

$$\sum_{n\geqslant 1} B_n^2 \sin^2 n\alpha\pi \leqslant 32\delta^2\pi E\,, \tag{18}$$

$$\|u_t(\cdot, t)\|^2_{L^2[0,\pi]} + \|u_x(\cdot, t)\|^2_{L^2[0,\pi]} \leqslant 4E\,, \quad t\geqslant 0\,. \tag{19}$$

Defining

$$\sigma_n = \sqrt{\pi/2}\left(\frac{A_n \sin n(\alpha\pi - t)}{\sin n\alpha\pi} + B_n \sin nt\right),$$

we obtain from (19)

$$\|u_x(\cdot, t)\|^2_{L^2[0,\pi]} = \sum_{n\geqslant 1} n^2 \sigma_n^2 \leqslant 4E\,,$$

whence

$$\|u(\cdot, t)\|^2_{L^2[0,\pi]} = \sum_{n=1}^{N} \sigma_n^2 + \sum_{n=N+1}^{\infty} \sigma_n^2 < \sum_{n=1}^{N} \sigma_n^2 + 4E/N^2\,.$$

We now have the following bound:

$$\|u(\cdot, t)\|^2_{L^2[0,\pi]} < (\pi/2) \max_{n=1,N} (\sin n\alpha\pi)^{-2} \sum_{n=1}^{N} \left[A_n^2 \sin^2 n(\alpha\pi - t) + \right.$$
$$\left. + B_n^2 \sin^2 nt \sin^2 n\alpha\pi + 2|A_n||B_n||\sin n(\alpha\pi - t)||\sin nt||\sin n\alpha\pi|\right] + 4E/N^2.$$

Therefore, from (17) and (18) it follows that

$$\max_{t\in[0,\alpha\pi]} \|u(\cdot, t)\|^2_{L^2[0,\pi]} = \|u\|^2 < 40\delta^2\pi^2 E \max_{n=1,N} (\sin n\alpha\pi)^{-2} + 4E/N^2, \qquad N = 1, 2, \ldots\,.$$

Since the partial quotients in the continued fraction for α are bounded by A_α, from the theory of continued fractions (see [6] p. 37) we easily obtain

$$\max_{n=1,N} (\sin n\alpha\pi)^{-2} < \left(\sin \frac{\pi}{(A_\alpha + 2)N}\right)^{-2}, \qquad N = 1, 2, \ldots\,.$$

Since $\sin x \geqslant (3\sqrt{3}/2\pi)x$ for $x \in [0, \pi/3]$, we have for every N

$$\|u\|^2 < (160/27)\,\delta^2\pi^2 E(A_\alpha + 2)^2 N^2 + 4E/N^2 . \tag{20}$$

Now let

$$g(t) = (160/27)\,\delta^2\pi^2 E(A_\alpha + 2)^2 t^2 + 4E/t^2 .$$

The minimum value of g for $t > 0$ is attained at

$$\bar{t} = (27/40)^{1/4}\big(\delta\pi(A_\alpha + 2)\big)^{-1/2} .$$

Since g is an increasing function on the interval $[\bar{t}, +\infty)$, we have

$$g([\bar{t} + 1]) < g(\bar{t} + 1).$$

Hence, assuming $\delta < 3\sqrt{30}/20\pi$, we obtain:

$$\|u\|^2 < \delta\pi(A_\alpha + 2)^2 E\,\frac{8\sqrt{30}(5 + 2\sqrt{3})}{27} ,$$

which proves theorem I.

We now give a proof of theorem II. The proof depends on some results obtained in [4].

By corollary 6 of [4], since α has type $\Omega < \infty$, there exist $K = K(\theta, \alpha) > 0$ and, for any $\delta > 0$, a number $\xi \in \boldsymbol{R}\setminus\boldsymbol{Q}$ such that

$$|\xi - \alpha| < \delta , \tag{21}$$

and

$$\max_{n=1,N} (\sin n\pi\xi)^{-2} \leqslant \big(\sin(\pi(3 - \sqrt{5})/2N)\big)^{-2}, \tag{22}$$

for all $N \geqslant K\delta^{-\theta}$.

From (21) it follows that $|\xi - \tau| \leqslant 2\delta$, for every τ satisfying $|\tau - \alpha| \leqslant \delta$.

If u is defined by (12), we obtain from (16)

$$\|u(x, \xi\pi)\|_{L^2[0,\pi]} \leqslant 6\delta\pi\sqrt{E} . \tag{23}$$

Therefore, u satisfies conditions (1), (2), (13), (15) and (23).

The solutions of the problem (1), (2), (13), (15) and (23) are the functions $u \in C^2([0, \pi]\times[0, +\infty))$ of the form

$$u(x, t) = \sum_{n\geqslant 1} \sin nx \left\{\frac{A_n \sin n(\xi\pi - t)}{\sin n\xi\pi} + B_n \sin nt\right\} ,$$

which satisfy (17), (19) and

$$\sum_{n\geqslant 1} B_n^2 \sin^2 n\xi\pi < 72\delta^2\pi E .$$

As in the proof of theorem I we obtain

$$\|u\|^2 < 80\delta^2\pi^2 E \max_{n=1,N} (\sin n\xi\pi)^{-2} + 4E/N^2, \qquad N = 1, 2, \dots .$$

Using (22) and $\sin x \geqslant (2/\pi)x$ for $x \in [0, \pi/2]$, we obtain

$$\|u\|^2 < \frac{80\pi^2 E}{(3-\sqrt{5})^2}\, \delta^2 N^2 + 4E/N^2, \qquad N \geqslant K\delta^{-\theta} . \tag{24}$$

Let

$$g(t) = \frac{80\pi^2 E}{(3-\sqrt{5})^2}\, \delta^2 t^2 + 4E/t^2, \qquad t > 0 . \tag{25}$$

The minimum of g for $t > 0$ is attained at

$$\bar{t} = \frac{\sqrt{3-\sqrt{5}}}{\sqrt[4]{20}\,\sqrt{\delta\pi}} .$$

It follows from (7) that $\bar{t} < K\delta^{-\theta}$.

Let $\bar{N}$ be the integer $\geqslant K\delta^{-\theta}$ for which the right side of (24) is minimum. Since g is increasing on the interval $[\bar{t}, +\infty)$, $\bar{N}$ satisfies $K\delta^{-\theta} \leqslant \bar{N} < K\delta^{-\theta} + 1$. Hence

$$\|u\|^2 < g(K\delta^{-\theta} + 1) ,$$

and finally

$$\|u\|^2 < \frac{80\pi^2 E}{(3-\sqrt{5})^2} (K\delta^{1-\theta} + \delta)^2 + 4E\delta^{2\theta}/K^2,$$

which proves theorem II.

The following statement is easily proved by means of theorem II above and theorem 4 in [4].

THEOREM III. *Let θ and ε be fixed, $(1/2) < \theta < 1$, and let*

$$K = 2\big(\theta/(2\theta - 1)\big)^{1-\theta} (1/\varepsilon)^{1-\theta} .$$

For any δ satisfying (7), let $\mathcal{S}_\delta$ be the set of the numbers α for which (6) holds.

Then

$$\mu\{\mathcal{G}_\delta \cap [0, 1]\} > 1 - \varepsilon ,$$

where μ denotes the Lebesgue measure.

We conclude with the proof of the following:

THEOREM IV. *The Dirichlet problem* (1)-(5) *is stable if and only if α is irrational. Moreover, if α is irrational then* $\lim_{\delta\to 0} (\mathrm{Diam}\, \Gamma_\delta) = 0$ *uniformly in φ and ψ.*

Let $\alpha \notin \boldsymbol{Q}$. By corollary 9 of [4], there exist a function $f(\delta)$ such that

$$\lim_{\delta\to 0} f(\delta) = \infty , \qquad \lim_{\delta\to 0} \delta f(\delta) = 0 \tag{26}$$

and, for any sufficiently small δ, a number $\xi \notin \boldsymbol{Q}$ satisfying (21) and (22) for all $N \geqslant f(\delta)$. The same argument given in the proof of theorem II (with $K\delta^{-\theta}$ replaced by $f(\delta)$) shows that

$$\|u\|^2 < g(f(\delta) + 1) ,$$

where g is defined by (25), i.e.

$$\|u\|^2 < \frac{80\pi^2 E}{(3-\sqrt{5})^2}\,(\delta f(\delta) + \delta)^2 + \frac{4E}{f^2(\delta)} .$$

By (26), this yields

$$\lim_{\delta\to 0} (\mathrm{Diam}\, \Gamma_\delta) = 0 .$$

If $\alpha \in \boldsymbol{Q}$, $\alpha = p/q$, $(p, q) = 1$, let $E = 1$, and let φ and ψ be identically zero. Then Γ_δ contains all the eigenfunctions of the form

$$u_n(x, t) = C_n \sin nqx \sin nqt , \qquad n \in \boldsymbol{N} ,$$

with $C_n^2 \leqslant 1/\pi n^2 q^2$. Hence

$$\mathrm{Diam}\, \Gamma_\delta = \sup_{v,w\in\Gamma_\delta} \|v - w\| \geqslant \|u_n\| = \sqrt{\alpha}\pi C_n/2 ,$$

and the Dirichlet problem is not stable.

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