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Existence of solutions of nonlinear hyperbolic equations

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Existence of Solutions of Nonlinear Hyperbolic Equations (*).

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1. – Introduction.

In this paper we consider the question of existence of solutions of abstract equations of the form

$$Ex = Nx , \qquad x \in X ,$$

where X is a Hilbert space, E is a linear operator with a possibly infinite dimensional kernel X_0 and such that the partial inverse H of E on the quotient space X/X_0 is bounded, but not necessarily compact. Our theorems therefore apply to quasi-linear hyperbolic partial differential equations and systems, in particular wave equations.

Our purpose here is to point out how much of the recent developments in the theory of nonlinear elliptic partial differential equations can be extended naturally to obtain existence of solutions of nonlinear hyperbolic problems.

In the recent years there has been an extensive literature on the question of existence of solutions to quasilinear elliptic equations of the type $Ex = Nx$ (E being a linear operator with a finite dimensional kernel, the partial inverse H of E being compact, and N nonlinear) and we have already shown [4, 5, 6, 7, 8] that in this situation suitably conceived abstract existence theorems essentially contain most of the results just mentioned for elliptic problems.

It is therefore the purpose of this paper to state and prove analogous abstract existence theorems for the present more general situation (here X_0

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is infinite dimensional and H is bounded but not compact) and to show that known and new specific existence theorems for hyperbolic problems can be naturally derived from the abstract theorems. In particular we derive existence statements for specific problems of the forms $u_{tt} - Au = f(\cdot,\cdot, u)$ or $u_{tt} - Au = f(\cdot,\cdot, u, u_t, u_x)$ will contain as particular cases results which have been proved by W. S. Hall [11, 12] and H. Petzeltova [15] only for $f = \varepsilon g$, ε a small parameter.

In another paper [9] we shall continue this programme showing, as in the elliptic case, that most known specific results for the hyperbolic case, and new simple criteria, can be derived from our abstract theorems.

The basic formulation of our abstract theorems for the hyperbolic case is about the same as in the elliptic case. It appears therefore that some unification has been brought about in a rather large variety of specific situations.

2. – The auxiliary and bifurcation equations.

Let X and Y be real Banach spaces and let $\|x\|_X$, $\|y\|_Y$ denote the norms in X and Y respectively. Let $\mathfrak{D}(E)$ and $\mathfrak{R}(E)$ be the domain and range of the linear operator $E\colon \mathfrak{D}(E) \to Y$, $\mathfrak{D}(E) \subset X$, and let $N\colon X \to Y$ be an operator not necessarily linear. We shall now consider the equation

$$(1) \qquad Ex = Nx\,, \quad x \in \mathfrak{D}(E) \subset X\,.$$

Let $P\colon X \to X$, $Q\colon Y \to Y$ be projection operators (*i.e.*, linear, bounded, and idempotent), with ranges and null spaces given by

$$\mathfrak{R}(P) = PX = X_0\,, \quad \ker P = \mathfrak{R}(I-P) = (I-P)X = X_1\,,$$
$$\mathfrak{R}(Q) = QY = Y_0\,, \quad \ker Q = \mathfrak{R}(I-Q) = (I-Q)\,Y = Y_1\,.$$

We assume that P and Q can be so chosen that $\ker E = PX$, $\mathfrak{R}(E) = Y_1 = (I-Q)\,Y$. This requires that $\ker E$ and $\mathfrak{R}(E)$ be closed in the topologies of X and Y respectively. Then E as a linear operator from $\mathfrak{D}(E) \cap X_1$ into Y_1 is one-to-one and onto, so that the partial inverse H: $Y_1 \to \mathfrak{D}(E) \cap X_1$ exists as a linear operator. We assume that H is a bounded linear operator, not necessarily compact, and that the following axioms hold: (i) $H(I-Q)\,E = I-P$, (ii) $EP = QE$, and (iii) $EH(I-Q) = I-Q$. We have depicted a situation which is rather typical for a large class of differential systems, not necessarily self-adjoint. Let L be a constant such that $\|Hy\|_X \leqslant L\|y\|_Y$ for all $y \in Y$. We have seen in [3] that equation (1) is equivalent to the system of auxiliary and bifurcation equations

$$x = Px + H(I - Q)Nx\,, \tag{2}$$

$$Q(Ex - Nx) = 0\,. \tag{3}$$

If $x^* = Px \in X_0$, and $\ker E = X_0 = PX$, then these equations become

$$x = x^* + H(I - Q)Nx\,, \tag{4}$$

$$QNx = 0\,. \tag{5}$$

Thus, for any $x^* \in X_0$, the auxiliary equation (4) has the form of a fixed point problem $x = Tx$, with $Tx = x^* + H(I - Q)Nx$.

For $X = Y$, a real Hilbert space, Cesari and Kannan [7] have given sufficient conditions for the solvability of equation (1) in terms of monotone operator theory.

Again, for $X = Y$ a real separable Hilbert space, E self-adjoint and N Lipschitzian, Cesari (cf. [3], § 1, ns. 3-5) has shown that it is always possible to choose X and hence P, Y, Q in such a way that the operator T is a contraction map in the norm of X, and thus the auxiliary equation $x = Tx$ has a unique solution in a suitable ball in X by the Banach fixed point theorem. McKenna [14] has extended the result to nonself-adjoint operators E in terms of its dual E^* and the self-adjoint operators EE^* and E^*E. Recently, Cesari and McKenna [10] have indicated the set-theoretic basis for the extension of the basic arguments to rather general situations.

In general for E not necessarily self-adjoint and X, Y real Banach spaces as stated above, Cesari and Kannan in a series of papers have considered the situation where Y is a space of linear operators on X, so that the operation $\langle y, x\rangle$, $Y \times X \to R$ is defined, is linear in both x and y, under the following natural assumptions.

(π_1) $|\langle y, x\rangle| \leqslant K\|y\|_Y\|x\|_X$ for some constant K and all $x \in X$, $y \in Y$.

We can always choose norms in X or Y, or the operation $\langle y, x\rangle$, in such a way that $K = 1$. Furthermore we assume that:

(π_2) for $y \in Y$, we have $y \in \mathfrak{R}(E) = Y_1$, i.e., $Qy = 0$, if and only if $\langle Qy, x^*\rangle = 0$ for all $x^* \in X_0$.

As simple examples of the above situation we have the following. Here G denotes a bounded domain in any t-space R^ν, $t = (t_1, \ldots, t_\nu)$, $\nu \geqslant 1$.

a) $X = Y = L_2(G)$, $|\langle y, x\rangle| = \left|\int_G y(t)x(t)\,dt\right| \leqslant \|y\|\,\|x\|$ with usual norms in L_2.

b) $X = L_2(G)$ with L_2 norm $\|x\|$, $Y = L_\infty(G)$ with norm $\|y\|_\infty$, and then $|\langle y, x\rangle| = \left|(\text{meas } G)^{-\frac{1}{2}}\int_G y(t)x(t)\,dt\right| \leqslant \|y\|_\infty\|x\|$.

c) $X = L_\infty(G)$ with usual norm $\|x\|_\infty$, $Y = L_\infty(G)$ with norm $\|y\|_\infty$, and then again $|\langle y, x\rangle| = \left|(\text{meas } G)^{-1}\int_G y(t)x(t)\,dt\right| \leqslant \|y\|_\infty\|x\|_\infty$.

d) $X = H^m(G)$ with usual Sobolev norm $\|x\|_m$, $Y = L_2(G)$, and then $|\langle y, x\rangle| = \left|\int_G y(t)x(t)\,dt\right| \leqslant \|y\|\,\|x\| \leqslant \|y\|\,\|x\|_m$.

3. – An abstract theorem for the elliptic case.

For the sake of simplicity we limit ourselves here to the case of Hilbert spaces.

Let X, Y be real Hilbert spaces, and let $(\ ,\ )$ denote the inner product in X. Let us consider equation (1) in X with $X_0 = \ker E$ of finite dimension, and let H be compact, and P and Q orthogonal projection operators, hence $\|P\| = 1$, $\|Q\| = 1$, and let $L = \|H\|$.

Thus, let $w = (w_1, w_2, \ldots, w_m)$ be an arbitrary orthonormal basis for the finite dimensional space $X_0 = \ker E = PX$, $1 \leqslant m = \dim\ker E < \infty$. For $x^* \in X_0$ we have $x^* = \sum_{i=1}^{m} c_i w_i$, or briefly $x^* = cw$, $c = (c_1, \ldots, c_m) \in R^m$, and then $|c| = \|x^*\|$, where $|c|$ is the Euclidean norm in R^m. The coupled system of equations can now be written in the form $x = cw + H(I-Q)Nx$, and $QNx = 0$. Let $\alpha\colon Y_0 \to X_0$ be a continuous map, not necessarily linear, mapping bounded subsets of Y_0 into bounded subsets of X_0, and such that $\alpha^{-1}(0) = 0$. Thus, $QNx = 0$ if and only if $\alpha QNx = 0$. The following existence theorem holds:

(3.i) Let X, Y *be real Hilbert spaces, and let* E, H, P, Q, N *as in* § 2. *Let* $X_0 = \ker E$ *be nontrivial and finite dimensional, let* H *be linear, bounded and compact, and let* N *be continuous. If there are* r, $R > 0$ *such that* (*a*) *for all* $x^* \in X_0$, $x_1 \in X_1$, $\|x^*\| \leqslant R_0$, $\|x_1\| \leqslant r$, *we have* $\|N(x^* + x_1)\| \leqslant L^{-1}r$; and (*b*) *for all* $\|x^*\| = R_0$, $\|x_1\| \leqslant r$, *we have* $(\alpha QN(x^* + x_1), x^*) \geqslant 0$ [*or* $\leqslant 0$], *then equation* $Ex = Nx$ *has at least a solution* $\|x\|_X \leqslant (R_0^2 + r^2)^{\frac{1}{2}}$.

For a proof of essentially (3.i) and the variant above, we refer to Cesari and Kannan [8, 9], and to Cesari [4, 5, 6] for extensions to Banach spaces and other remarks. A topological proof of essentially theorem (3.i) may be seen in Kannan and McKenna [13].

We present the theorem in the form (3.i) because no requirement is made concerning the behaviour of $N(x^* + x_1)$ outside the set $S = \{(x^*, x_1) \in X, \|x^*\| \leqslant R_0, \|x_1\| \leqslant r\}$, and thus it allows for an arbitrary growth for $N(x)$ as $\|x\| \to \infty$. If (a) $\|Nx\| \leqslant J_0$ for some constant J_0 and all $x \in X$, and (b') for some R_0 inequality (b) in (3.i) holds for all $\|x^*\| \geqslant R_0$ and $\|x_1\| \leqslant LJ_0$, then (a), (b) certainly hold for R_0 as stated in (b') and $r = LJ_0$. In [4, 5, 6] we discuss the determination of suitable constants R, r for various cases of growth of $\|Nx\|$ as $\|x\| \to \infty$.

4. – Preliminary considerations concerning the hyperbolic case.

Let E, N be operators from their domains $\mathfrak{D}(E)$, $\mathfrak{D}(N)$ in $\mathfrak{X}$ to $\mathfrak{Y}$, both $\mathfrak{X}$ and $\mathfrak{Y}$ real Banach or Hilbert spaces, and let us consider the operator equation

$$Ex = Nx$$

as in § 2. Its solutions x in $\mathfrak{X}$ may be expected to be usual solutions, or generalized solutions according to the choice of $\mathfrak{X}$. We shall consider first smaller spaces X and Y, say $X \subset \mathfrak{X}$, $Y \subset \mathfrak{Y}$, both real Hilbert spaces, and we shall assume that the inclusion map $j: X \to \mathfrak{X}$ is compact.

We shall then construct a sequence of elements $\{x_k\}$, $x_k \in X$, which is bounded in X, or $\|x_k\| \leqslant M$. Then, there is a subsequence, say still $\{x_k\}$ for the sake of simplicity, such that $\{jx_k\}$ converges strongly in $\mathfrak{X}$ toward some element ζ. On the other hand, X is Hilbert, hence reflexive, and we can take the subsequence, say still $[k]$, in such a way that $x_k \to x$ weakly in X.

Actually, $\zeta = jx$, that is, ζ is the same element $x \in X$ thought of as an element of $\mathfrak{X}$. In other words:

(4.i) *If $x_k \to x$ weakly in X and $jx_k \to \zeta$ strongly in $\mathfrak{X}$, then $\zeta = jx$.*

Indeed, $j: X \to \mathfrak{X}$ is a linear compact map, hence continuous (see, e.g. [2], p. 285, Th. 17.1). As a consequence, $x_k \to x$ weakly in X implies that $jx_k \to jx$ weakly in $\mathfrak{X}$ (see, e.g. [2], p. 295, pr. no. 12). Since $jx_k \to \zeta$ strongly in $\mathfrak{X}$, we have $\zeta = jx$.

We shall assume that X_1 and X_0 contain finite dimensional subspaces X_{1n}, X_{0n} such that $X_{1n} \subset X_{1,n+1} \subset X_1$, $X_{0n} \subset X_{0,n+1} \subset X_0$, $n = 1, 2, \ldots$, with

$\bigcup_n X_{1n} = X_1$, $\bigcup_n X_{0n} = X_0$, and assume that there are projection operators $R_n\colon X_1 \to X_{1n}$, $S_n\colon X_0 \to X_{0n}$ with $R_n X_1 = X_{1n}$, $S_n X_0 = X_{0n}$ (cf. similar assumptions in Rothe [16]). Since X is a real Hilbert space, we may think of R_n and S_n as orthogonal projections and then $\|R_n x\|_X \leqslant \|x\|_X$, $\|S_n x^*\|_X \leqslant \|x^*\|_X$ for all $x \in X_1$ and $x^* \in X_0$.

Thus, we see that in the process of limit just mentioned, $x_k \to x$ weakly in X, $jx_k \to jx$ strongly in $\mathfrak{X}$, the limit element can still be thought of as belonging to the smaller space X. This situation is well known in the important case $X = W_2^N(G)$, $\mathfrak{X} = W_2^n(G)$, $0 \leqslant n < N$, $X \subset \mathfrak{X}$, G a bounded open set in some R^ν, $\nu \geqslant 1$. Then, the weak convergence $x_k \to x$ in $W_2^N(G)$ implies the strong convergence $jx_k \to jx$ in $W_2^n(G)$, and $\zeta = jx$ is still an element of the smaller space X.

Concerning the subspaces X_{0n} of X_0 it is not restrictive to assume that there is a complete orthonormal system $[v_1, v_2, \ldots, v_n, \ldots]$ in X_0 and that $X_{0n} = \mathrm{sp}\,(v_1, v_2, \ldots, v_n)$, $n = 1, 2, \ldots$ We shall further assume that there is a complete orthonormal system $(\omega_1, \omega_2, \ldots, \omega_n, \ldots)$ in Y, such that $\langle \omega_i, v_j \rangle = 0$ for all $i \neq j$. We shall take $Y_{0n} = \mathrm{sp}\,(\omega_1, \ldots, \omega_n)$ and denote by S'_n the orthogonal projection of Y_0 onto Y_{0n}. Then, $S'_n QNx = 0$ if and only if $\langle QNx, v_j \rangle = 0$, $j = 1, \ldots, n$, and this holds for all $n = 1, 2, \ldots$.

We consider now the coupled system of operator equations

$$x = S_n Px + R_n H(I - Q)Nx\,, \tag{6}$$

$$0 = S'_n QNx\,. \tag{7}$$

We note that we have $S'_n QNx = 0$ if and only if $\langle QNx, x^* \rangle = 0$ for all $x^* \in X_{0n}$.

We may now define a map $\alpha_n\colon Y_{0n} \to X_{0n}$ by taking

$$\alpha_n y = \sum_1^n \langle y, \omega_i \rangle\, v_i\,.$$

Then, we have $0 = S'_n QNx$ if and only if $0 = \alpha_n S'_n QNx$. We conclude that system (6), (7) is equivalent to system

$$x = S_n Px + R_n H(I - Q)Nx\,, \tag{8}$$

$$0 = \alpha_n S'_n QNx\,. \tag{9}$$

(4.ii) (a lemma) *Under the hypotheses above, let us assume that there are* *constants* R, $r > 0$ *such that*

(*a*) *for all* $x^* \in X_0$, $x_1 \in X_1$, $\|x^*\| \leqslant R$, $\|x_1\| \leqslant r$, *we have*

$$\|N(x^* + x_1)\| \leqslant L^{-1} r ;$$

(*b*) *for all* $\|x^*\| = R_0$, $\|x_1\| \leqslant r$ *we have*

$$\left(\alpha_n S_n' Q N(x^* + x_1), x^*\right) \geqslant 0 \quad [\text{or } \leqslant 0] .$$

Then, for every n, *system* (6), (7) *has at least a solution* $x_n \in \mathfrak{D}(E) \cap (X_{0n} \times X_{1n})$, $x_n = x_{0n}^* + x_{1n}$, $S_n P x_n = x_{0n}^*$, *with* $\|x_n\| \leqslant M = (R_0^2 + r^2)^{\frac{1}{2}}$, M *independent of* n.

PROOF. If we consider the subset C_n of $X_{0n} + X_{1n}$ made up of all $x = x_{0n}^* + x_{1n}$ with $\|x_{0n}^*\| \leqslant R_0$, $\|x_{1n}\| \leqslant r$, we see that

$$\|R_n H(I - Q) N x\| \leqslant r \quad \text{for all } x \in C_n ,$$

$$(\alpha_n S_n' Q N x, x^*) \geqslant 0 \quad [\text{or } \leqslant 0] \text{ for all } x \in C_n \text{ with } x^* = R .$$

In other words, the assumptions actually used in the proofs of (3.i) in [8] and in [13], are satisfied with $L^{-1} r$ replacing J_0 and the same R_0. The proofs of these statements can be repeated verbatim. Now the compactness of the bounded operator $R_n H$ follows from the fact that $R_n H$ has a finite dimensional range, and the finite dimensionality of the kernel of E is now replaced by the fact that the range of $\alpha_n S_n' Q N$ is certainly finite dimensional. The bound $M = (R_0^2 + L^2 J_0^2)^{\frac{1}{2}}$ is now replaced by the bound $M = (R_0^2 + r^2)^{\frac{1}{2}}$, certainly independent of n.

5. – An abstract theorem for the hyperbolic case.

In order to solve the equation $Ex = Nx$ we now adopt a « passage to the limit argument. » We assume that both the Hilbert spaces X and Y are contained in real Banach (or Hilbert) spaces $\mathfrak{X}$ and $\mathfrak{Y}$ with compact injections $j: X \to \mathfrak{X}$, $j': Y \to \mathfrak{Y}$. Actually we can limit ourselves to the consideration of the spaces $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{Y}}$ made up of limit elements from sequences in X and Y respectively as mentioned in § 4. Hence, $\overline{\mathfrak{X}}$ is identical to X and $\overline{\mathfrak{Y}}$ is identical to Y, though they may have different topologies. We shall write $\overline{\mathfrak{X}} = jX$, $\overline{\mathfrak{Y}} = jY$.

Analogously, we take $\mathfrak{X}_0 = jX_0$, $\mathfrak{Y}_0 = j' Y_0$, $\mathfrak{X}_1 = jX_1$, $\mathfrak{Y}_1 = j' Y_1$, and the linear operators $\overline{\mathscr{P}}$: $\overline{\mathfrak{X}} \to \mathfrak{X}_0$, $\mathscr{Q}: \overline{\mathfrak{Y}} \to \mathfrak{Y}_0$ are then defined by $\overline{\mathscr{P}} x = x^*$ in $\overline{\mathfrak{X}}$ if $Px = x_0$ in X; $\mathscr{Q} y = y^*$ in $\mathfrak{Y}$ if $Qy = y^*$ in Y.

We now assume the following:

(C) $x_n \to x$ weakly in X and $jx_n \to jx$ strongly in $\mathfrak{X}$ implies that $Nx_n \to Nx$ strongly in $\mathfrak{Y}$, $S_n Px_n \to Px$ strongly in $\mathfrak{X}$, and $R_n x_n \to x$ strongly in $\mathfrak{X}$.

By (4.ii) there are elements $x_n \in X_n$ such that

$$x_n = S_n P_n x_n + R_n H(I - Q) N x_n \,, \tag{10}$$

$$0 = \alpha_n S'_n Q N x_n \,, \tag{11}$$

where $\|x_n\| \leqslant M$ for all n. Hence, there exists a subsequence, say still $\{x_n\}$, such that $x_n \to x$ weakly in X and $jx_n \to jx$ strongly in $\mathfrak{X}$. Then, by (10) and (11), proceeding to the limit, we have

$$x = Px + H(I - Q)Nx \,, \quad 0 = QNx \,, \quad x \in \overline{\mathfrak{X}} \,.$$

Indeed, as $n \to \infty$, S_n converges to the identity $I\colon Y_0 \to Y_0$ and α_n converges to a homeomorphism $\alpha\colon Y_0 \to Y_0$ in the sense that $S_n y \to y$, $\alpha_n y \to y$ as $n \to \infty$.

We now remark that, in $\overline{\mathfrak{X}}$ the operator E may have no meaning and thus the concept of solution of $Ex = Nx$ has to be properly understood. However, $x \in \overline{\mathfrak{X}}$ and thus, by § 4, x is still an element of X on which E is defined. Further, as a consequence of the hypotheses on P and H, we have $QE = EP = 0$ and $EH(I - Q) = I - Q$. Thus, from the above limit equation we have

$$\begin{aligned} Ex &= EPx + EH(I - Q)Nx \\ &= EPx + (I - Q)Nx + QNx = Nx \,. \end{aligned}$$

We summarize now the hypotheses and the conclusions, concerning the operator equation $Ex = Nx$, we have obtained.

(5.i) THEOREM. *Let $E\colon \mathfrak{D}(E) \to Y$, $\mathfrak{D}(E) \subset X \subset \mathfrak{X}$, E a linear operator, $N\colon X \to Y$ a not necessarily linear operator, X, Y real Hilbert spaces, $\mathfrak{X}$, $\mathfrak{Y}$ real Banach or Hilbert spaces with compact injections $j\colon X \to \mathfrak{X}$, $j'\colon Y \to \mathfrak{Y}$, with projection operators $P\colon X \to X$, $Q\colon Y \to Y$ and decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$, $X_0 = PX = \ker E$, $Y_1 = (I - Q)Y = \operatorname{Range} E$, X_0 infinite dimensional, and let E have the bounded partial inverse $H\colon Y_1 \to X_1$. Let $L = \|H\|$, let $N\colon X \to Y$ be a continuous operator, and let P, Q, H, E, N satisfy* (i), (ii), (iii) *of* § 2. *Let Y be a space of linear operators $\langle y, x\rangle$, or $Y \times X \to$ Reals, satisfying* (π_1), (π_2) *of* § 2. *Let X_{0n}, X_{1n}, Y_{0n} be finite dimensional subspaces of X_0, X_1, Y_0 with orthogonal projection operators $R_n\colon X_1 \to X_1$,*

$S_n: X_0 \to X_0$, $S'_n: Y_0 \to Y_0$ with $R_n X_1 = X_{1n}$, $S_n X_0 = X_{0n}$, $S'_n Y_0 = Y_{0n}$, satisfying (C) *of the present Section. Let $\alpha_n: Y_{0n} \to X_{0n}$ denote the map defined in § 4. If there are constants R, $r > 0$ such that* (*a*) *for all $x^* \in X_0$, $x_1 \in X_1$, $\|x^*\| \leqslant R_0$, $\|x_1\| \leqslant r$, we have $\|N(x^* + x_1))\|_Y \leqslant L^{-1} r$; and* (*b*) *for all $\|x^*\| = R_0$, $\|x_1\| \leqslant r$ we have $(\alpha_n Q N(x^* + x_1), x^*) \geqslant 0$ [or $\leqslant 0$], then the equation $Ex = Nx$ has at least a solution $\|x\|_X \leqslant (R_0^2 + r^2)^{\frac{1}{2}}$.*

In this theorem (5.i) no requirement is made concerning the behavior of $N(x^* + x_1)$ outside the set $S = \{(x^*, x) \in X, \|x^*\| \leqslant R_0, \|x_1\| \leqslant r\}$, and thus it allows for an arbitrary growth for $N(x)$ as $\|x\| \to +\infty$.

However, it is easy to see that, if (*a*) $\|Nx\| \leqslant J_0$ for some constant J_0 and all $x \in X$; and (*b'*) for some R_0 the inequality (*b*) in (5.i) holds for all $\|x^*\| \geqslant R_0$ and $\|x_1\| \leqslant LJ_0$, then (*a*), (*b*) certainly hold for R_0 as stated in (*b'*) and $r = LJ_0$. We shall see in [9] that an analogous determination of R_0 and r can be made in cases of slow growth $\|Nx\| \leqslant J_0 + J_1 \|x\|^\nu$, $0 < \nu < 1$, and even of arbitrary growth $\|Nx\| \leqslant \tilde{\varphi}(\|x\|)$, in particular $\|Nx\| \leqslant J_0 + J_1 \|x\|^\nu$, $\nu \geqslant 1$.

REMARK 1. Also note that the bifurcation equation in the problem, or $\alpha_n S'_n Q N x_n = 0$, can always be replaced by the equation

$$J_n \alpha_n S'_n Q N X_n = 0\,, \tag{12}$$

where $J_n: X_{0n} \to X_{0n}$ is an invertible operator. When this is done, we may require that (*b*) holds with the inequality replaced by

$$(J_n \alpha_n S'_n Q N x, x^*) \geqslant 0 \qquad [\text{or } \leqslant 0]\,. \tag{13}$$

The following corollary of (5.i) is of interest. Again $L = \|H\|$.

(5.ii) *Let $N: X \to Y$ be a continuous map, and there be monotone nondecreasing nonnegative functions $\alpha(R)$, $\beta(R)$, $R \geqslant 0$, and that* (i) *$x \in X$, $\|x\|_X \leqslant R$ implies $\|Nx\| \leqslant \alpha(R)$;* (ii) *$x_1, x_2 \in X$, $\|x_1\|, \|x_2\| \leqslant R$ implies $\|Nx_1 - Nx_2\| \leqslant \beta(R)\|x_1 - x_2\|$. Let us assume further that* (iii) *there are numbers $R_0, r > 0$ such that $L\alpha((R_0^2 + r^2)^{\frac{1}{2}}) < 1$; and $R_0 + L\beta(R_0^2 + r^2)^{\frac{1}{2}} \leqslant r$, and* (iv) *$(\alpha_n S'_n Q N(x^* + x_1), x^*) \geqslant 0$ [or $\leqslant 0$] for all $\|x^*\| \leqslant R_0$ and $\|x_1\| \leqslant r$. Then the equation $Ex = Nx$ has at least a solution $x = x^* + x_1 \in B$, $B = ((x^*, x_1), \|x^*\| \leqslant R_0, \|x_1\| \leqslant r)$.*

PROOF. We proceed as for (5.i) where now we follow ([3], § 1, nos. 3-5). Let $B_n = \{(x^*_{0n}, x_{1n}), x^*_{0n} \in S_n PB, x_{1n} \in R_n(I - B)B\}$. Under the hypotheses of the theorem it can be proved that the truncated auxiliary equation $x = S_n Px + R_n H(I - Q) N_x$ is uniquely solvable for each arbitrary but fixed $S_n Px = x^*_{0n}$ and that the unique solution x_n is of the form $x_n = T(x^*_{0n})$,

$Px_n = x^*_{0n}$, where T is a continuous map from $\{x^* | \|x^*\| \leqslant R_0\}$ into $\{(x^*, x_1) | \|x^*\| \leqslant R_0,\ \|x_1\| \leqslant r\}$. The truncated bifurcation equation is then reduced to $\alpha_n S_n QNT(x^*_{0n}) = 0$. The inequality (iv) is now applied to obtain the existence of a solution x^*_{0n} of this equation. Since $\|x^*_{0n}\| \leqslant R_0$ and $\|x_n\| = \|T(x^*_{0n})\| \leqslant (R_0^2 + r^2)^{\frac{1}{2}}$ and these bounds are independent of n, we can now proceed as for (5.i) to obtain the existence of a solution of the equation $Ex = Nx$.

6. – Applications.

We first consider the existence of solutions $u(t, x)$, periodic in t of period 2π, of the hyperbolic nonlinear problem

$$u_{tt} + u_{xxxx} = f(t, x, u, \ldots)\,, \qquad 0 < x < \pi,\ -\infty < t < +\infty\,, \tag{14}$$

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0\,, \tag{15}$$

$$u(t + 2\pi, x) = u(t, x)\,, \qquad 0 < x < \pi,\ -\infty < t < +\infty\,. \tag{16}$$

Let $I = [0, \pi] \times [0, 2\pi]$, $G = [0, \pi] \times \boldsymbol{R}$. Let D denote the set of all real valued functions $u(t, x)$, 2π periodic in t, of class C^∞ in G and such that $D_x^{2k}\varphi(t, 0) = D_x^{2k}\varphi(t, \pi) = 0$, $k = 0, 1, 2, \ldots$ Let A_m denote the completion of D under the norm

$$\|u\|_m = \Big(\iint_I [(D_t^m u)^2 + (D_x^{2m} u)^2]\, dt\, dx\Big)^{\frac{1}{2}}.$$

Thus, A_m is a real Hilbert space with inner product

$$(u, v)_m = (D_t^m u, D_t^m v) + (D_x^{2m} u, D_x^{2m} v)\,, \qquad u, v \in A_m\,,$$

where $(,)$ denotes the inner product in $L_2(I)$, and thus $A_0 = L_2(I)$. Let E denote the operator defined by $Eu = u_{tt} + u_{xxxx}$. For $g \in A_m$ we shall consider the linear problem $Eu = g$. Then, we say that u is a weak solution of this problem with boundary conditions (15), (16) if $u \in A_m$ and $(u, E\varphi)_m = (g, \varphi)_m$ for all $\varphi \in D$. Then, both equations $Eu = g$ and boundary conditions (15), (16) are understood in the weak sense. A complete orthonormal system in $A_0 = L_2(I)$ is

$$\{e_{kl}^{(0)}\} = \{2^{\frac{1}{2}}\pi^{-1} \sin kt \sin lx,\ 2^{\frac{1}{2}}\pi^{-1} \cos kt \sin lx,\ \pi^{-1} \sin lx\}\,,$$

whose elements can be indexed by $l = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots,$ as usual. A complete orthonormal system in A_m is

$$\{e_{kl}^{(m)}\} = \{\pi^{-1}(k^{2m} + l^{4m})^{-\frac{1}{2}} \sin kt \sin lx,$$
$$\pi^{-1}(k^{2m} + l^{4m})^{-\frac{1}{2}} \cos kt \sin lx, \pi^{-1} l^{-2m} \sin lx\} .$$

It is easy to see that any element $u \in A_m$ has Fourier series $\sum_{kl} a_{kl} e_{kl}^{(m)}$ with $\sum_{kl} a_{kl}^2 < +\infty$, as well as, $\sum_{kl} a_{kl}^2 (k^{2m} + l^{4m}) < +\infty$.

From Sobolev type imbedding theorems (Aubin [1]), as well as by direct estimates on Fourier series and Young-Hausdorff theorems, we know for instance that:

(*a*) If $u \in A_1$, then $u \in C$, $u_x \in L_q$ for any $q < 6$ (at least), u_t, $u_{xx} \in L_2$, and $\|u\|_\infty$, $\|u_x\|_{L_q}$, $\|u_t\|_{L_2}$, $\|u_{xx}\|_{L_2} \leqslant \gamma \|u\|$; for $u \in A_2$, then u, u_t, u_x, $u_{xx} \in C$, u_{tx}, $u_{xxx} \in L_q$ for any $q < 6$, u_{tt}, u_{txx}, u_{xxx}, $u_{xxxx} \in L_2$, and $\|u\|_\infty$, $\|u_t\|_\infty$, $\|u_x\|_\infty$, $\|u_{xx}\|_\infty$, $\|u_{tx}\|_{L_q}$, $\|u_{xxx}\|_{L_q}$, $\|u_{tt}\|_{L_2}$, $\|u_{txx}\|_{L_2}$, $\|u_{xxx}\|_{L_2}$, $\|u_{xxxx}\|_{L_2} \leqslant \gamma \|u\|_2$, where γ is a constant independent of u, and analogous relations hold for any $m \geqslant 1$ (see [9] for details).

As a consequence we have the following:

(*A*) If $f(t, x, u) \in C(G \times R)$ is a given continuous function in $G \times R$, periodic in t, and $u \in A_1$, say $\|u\|_1 \leqslant b$, then u is continuous, $F(t, x) = f(t, x, u(t, x)) = f \circ u$ is continuous in G, 2π-periodic in t, and $\|F\|_\infty \leqslant c$, hence $\|F\|_{L_2} \leqslant d$, where c and d can be made to depend solely on b.

Analogously, if $f(t, x, u, u_x) \in C^2(G \times R^2)$, 2π-periodic in t, and $u \in A_2$, say $\|u\|_2 \leqslant b$, then u, u_x are continuous, $F(t, x) = f(t, x, u(t, x), u_x(t, x)) = f \circ u$ is continuous and $\|F\|_\infty \leqslant C$.

From (*A*) and the expressions of F_t, F_x, F_{xx} it can be shown that F_x is also continuous, that F_t, $F_{xx} \in L_2(I)$, and that $\|F\|_\infty$, $\|F_x\|_\infty \leqslant c$, $\|F_t\|_{L_2}$, $\|F_{xx}\|_{L_2} \leqslant d$, where c and d can be made to depend solely on b and f.

Also, if $f(t, x, u, u_t, u_x) \in C^2(G \times R^2)$, 2π-periodic in t, and $u \in A_2$, say $\|u\|_2 \leqslant b$, then u, u_t, u_x are continuous, $F(t, x) = f \circ u$ is continuous, F_t, F_x, $F_{xx} \in L_2(I)$, (F_{xx} as a distributional derivative), and $\|F\|_\infty \leqslant C$, $\|F_t\|_{L_2}$, $\|F_x\|_{L_2}$, $\|F_{xx}\|_{L_2} \leqslant d$, where again c and d can be made to depend solely on b and f.

In each space A_m we denote by A_{m0} the subspace generated by the elements $e_{kl}^{(m)}$ with $k^2 = l^4$, the kernel of E in A_m, and we decompose then A_m in a direct sum $A_m = A_{m0} + A_{m1}$. Before we proceed we note that if $g \in A_{m1}$ for some m, $(m \geqslant 0)$, then $g = \sum_{kl} a_{kl} e_{kl}^{(m)}$ where $\sum$ ranges over all k with $k^2 \neq l^4$, or ordinary Fourier series

$$g = \sum_{kl} g_{kl} e_{kl}^{(0)} \qquad \text{with } \sum_{kl} g_{kl}^2 (k^m + l^{2m})^2 < +\infty .$$

Let us prove that $u = Hg = \sum_{kl} g_{kl}(k^2 - l^4)^{-1} e_{kl}^{(0)}$ belongs to A_{m+1}. Indeed, for $\varrho_{kl} = (k^2 - l^4)^{-2}(k^2 + l^4)$ we have

$$\sum_{kl} g_{kl}^2 (k^2 - l^4)^{-2}(k^{m+1} + l^{2(m+1)})^2 \leqslant \sum_{kl} g_{kl}^2 \varrho_{kl}(k^m + l^{2m}\varepsilon)^2 ,$$

and all we have to prove is that $|\varrho_{kl}| \leqslant 1$. Indeed, $(l^4 - k^2)^2 = (l^2 - k)^2 \cdot (l^2 + k)^2 \geqslant (l^2 + |k|)^2 \geqslant l^4 + k^2$, and hence $|\varrho_{kl}| \leqslant 1$ for all $k = 0, \pm 1, \pm 2, \ldots$, $l = 1, 2, \ldots, k^2 \neq l^4$. We have proved that $\|u\|_{m+1} = \|Hg\|_{m+1} \leqslant \|g\|_m$ for $g \in A_m$.

(B) If we take $X = A_1$, $Y = A_0 = L_2$, then $X_0 = A_{10}$, $Y_0 = A_{00}$, and we take for $P: X \to X$, $Q: Y \to Y$ the orthogonal projections of X onto X_0, and of Y onto Y_0 respectively. Then $X_1 = (I - P)X = A_{11}$, $Y_1 = (I - P)Y = A_{01}$. Then for the operator H defined above we have $H: Y_1 \to X_1$. If we denote by $\mathfrak{D}(E)$ the range of H in X_1, then $\mathfrak{D}(E) \subset X_1$ and E maps $\mathfrak{D}(E) \cap X_1$ one-one onto Y_1. The hypotheses (i), (ii), (iii) of Section 2 obviously hold, and $\|Hv\|_1 \leqslant \|v\|_0$ for all $v \in Y_1$.

Analogously, we could take $X = A_2$, $Y = A_1$, and assume with Petzeltova [15] that for every $u \in X$ and $F = f \circ u$, we have $F(t, 0) = F(t, \pi) = 0$ for all t. Then, what we have proved in (A) shows that for $u \in X = A_2$ we have $F \in Y = A_1$. Then, any element $v \in Y_1$ is mapped by H into X_1. If we denote by $\mathfrak{R}(E)$ the range of H in X_1 then, as before, E maps $\mathfrak{R}(E) \cap X_1$ one-one onto Y_1, (i), (ii), (iii) of Section 2 hold, and $\|Hv\|_2 \leqslant \|v\|_1$ for all $v \in Y_1$. For cases with $X = A_m$, $m \geqslant 3$, we refer to [9].

To see how artificial is Petzeltova's condition $F(t, 0) = F(t, \pi) = 0$ we consider problem (14)-(16) with $f = 1$, or $f = x$, whose elementary solutions are respectively

$$\begin{aligned} u(t, x) &= 24^{-1}x^4 - 12^{-1}\pi x^3 + 24^{-1}\pi^3 x \\ &= 4\pi^{-1} \sum_{l=1}^{\infty} (2l - 1)^{-5} \sin (2l - 1)x , \end{aligned}$$

and

$$\begin{aligned} u(t, x) &= 120^{-1}x^5 - 36^{-1}\pi^2 x^3 + 7(360)^{-1}\pi^4 x \\ &= 2 \sum_{l=1}^{\infty} (-1)^{l+1} l^{-5} \sin lx , \end{aligned}$$

while

$$F(t, x) = 1 = 4\pi^{-1} \sum_{l=1}^{\infty} (2l - 1)^{-1} \sin (2l - 1)x ,$$

$$F(t, x) = x = 2 \sum_{l=1}^{\infty} (-1)^{l+1} l^{-1} \sin lx$$

are analytic in G but do not belong to A_1 since they do not satisfy Petzeltova's condition. Petzeltova's condition is not needed, however, as we are going to show.

To this purpose let $X = A_2$, and take $Y = A_1 + W_1$, with A_1 as above, and W_1 the set of all functions of the form $v_0(t, x) = f_1(t) + xf_2(t)$, $0 \leqslant x \leqslant \pi$, $t \in R$, f_1, f_2 2π-periodic in t and of class C in R. Then, Y is contained in the real Hilbert space $\tilde{Y}$ of all functions $V(t, x)$, $(t, x) \in G$, 2π-periodic in t, $V \in L_2(I)$ and inner product

$$(u, v) = (u, v) + (u_t, v_t) + (u_{xx}, v_{xx}) .$$

Now, if $f(t, x, u, u_t, u_x)$ is of class C^2 in $G \times R^3$, 2π-periodic in t, and $u \in A_2$, then $F(t, x) = f \circ u$ may not be in A_1. However, $f_1(t) = F(t, 0)$, $f_2(t) = \pi^{-1}\big(F(t, \pi) - F(t, 0)\big)$ are 2π-periodic in t and of class C^1 in R, and if we take $F_0(t, x) = f_1(t) + xf_2(t)$, and $F_1(t, x) = F(t, x) - F_0(t, x)$, then $F_1(t, 0) = F_1(t, \pi) = 0$ and we have the decomposition $F = F_0 + F_1$ with $F_1 \in A_1$, $F_0 \in W_1$ and $F \in \tilde{Y}$. Let $P\colon X \to X$, $Q\colon \tilde{Y} \to \tilde{Y}$ denote the orthogonal projections of $X = A_2$ onto A_{20}, and of $\tilde{Y} = A_1 + W_1$ onto A_{10}, and take $X_0 = PX = A_{20}$, $\tilde{Y}_0 = Q\tilde{Y} = A_{10}$, $X_1 = (I - P)x$, $Y_1 = (I - Q)Y$. Note that any function $F = f \circ u = F_0(t, x) + F_1(t, x)$, $F_0 \in W_1$, $F_1 \in A_1$, has Fourier series

$$\begin{aligned} F(t, x) &= f_1(t) + xf_2(t) + F_1(t, x) \\ &= 2^{-1}a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \\ &\quad + \Big[\sum_{l=1}^{\infty} (-1)^{l+1} l^{-1} \sin lx\Big]\Big[2^{-1}c_0 + \sum_{k=1}^{\infty} (c_k \cos kt + d_k \sin kt)\Big] + \sum_k \sum_l a_{kl} e_{kl'}^{(1)} \end{aligned}$$

and we define $u = H(I - P)F$ by taking

$$\begin{aligned} u(t, x) &= (2^{-1}a_0)\Big[4\pi^{-1} \sum_{l=1}^{\infty} (2l-1)^{-5} \sin (2l-1)x\Big] \\ &\quad + 4\pi^{-1} \sum_{k,l}^{\infty} (2l-1)^{-1}[(2l-1)^4 - k^2]^{-1}(a_k \cos kt + b_k \sin kt) \sin (2l-1)x \\ &\quad + 2(2^{-1}c_0)\Big[\sum_{l=1}^{\infty} (-1)^{l+1} l^{-5} \sin lx\Big] + \\ &\quad + 2 \sum_{k,l}^{\infty} (-1)^{l+1} l^{-1}(l^4 - k^2)^{-1}(c_k \cos kt + d_k \sin kt) , \end{aligned}$$

with the convention that the double series ranges only over those k, l with $(2l-1)^4 \neq k^2$, or $l^4 \neq k^2$, respectively.

We have proved in [9] that u belongs to A_2, and thus $H\colon \tilde{Y}_1 \to X = A_2$, and moreover $\|H(I-P)F\| \leqslant \gamma \|F\|_{\tilde{Y}}$. Again we take $\mathfrak{D}(E) \subset X$ as the image of $H\colon \tilde{Y}_1 \to X_1$, and then E is one-one and onto from $\mathfrak{D}(E) \cap X_1$ to $\tilde{Y}_1$.

We shall now consider the elements $y \in Y$ as linear operators on X by taking, as linear operation the inner product in L_2, or $(y, x) = \int_I yx\, dt\, dx$. In each of the cases considered above the operation (y, x) satisfies the assumptions (π_1), (π_2) of no. 2.

For the sake of simplicity, we shall refer to the case $X = A_2$, $m = 2$ and corresponding elements. We shall omit the upper script in the elements $e_k^{(m)}$ when obvious.

We now define the finite dimensional subspaces X_{0n} of X_0 as follows: X_{0n} is the subspace of X_0 in X generated by e_{kl}, $k^2 = l^4$, $l = 1, \ldots, n$. Then let X_{1n} be the subspace of X_1 generated by e_{kl}, $k^2 \neq l^4$, $l = 1, 2, \ldots, n$, $k = 0, \pm 1, \ldots, \pm n$. Let R_n, S_n be the appropriate orthogonal projections given by $R_n\colon X_1 \to X_{1n}$ and $S_n\colon X_0 \to X_{0n}$. Let Y_{0n} be the subspace of Y_0 in Y generated by e_{kl}, $k^2 = l^4$, $l = 1, 2, \ldots, n$. Then S_n' can be defined as the orthogonal projection of Y_0 onto Y_{0n} in Y.

We can now define the map $\alpha_n\colon Y_{0n} \to X_{0n}$ by taking

$$\alpha_n y = \sum_{l=1}^{n} (y, e_{kl}^{(0)})\, e_{kl}^{(0)} .$$

Clearly $S_n' QNu = 0$ if and only if $(QNu, u^*)_{L^2} = 0$ for all $u^* \in X_{0n}$ and further $\alpha_n S_n' QNu = 0$ is equivalent to $S_n' QNu = 0$.

Finally let $J\colon X_0 \to X_0$ be the linear operator defined as follows: for any $u \in X_0$, $u = \sum u_{kl} e_{kl}$, where $\sum$ ranges over all k, l with $k^2 = l^4$, take $J = \sum k^{-1} u_{kl} e_{kl}'$ is obtained by e_{kl} replacing cos kt by sin kt, and sin kt by-cos kt. Then J is an isomorphism. Further $(Ju)_t = u$, and we take $J_n = J$ in (12) and (13). Since any $u \in X_{0n}$ is a solution of the homogeneous problem, also $u_{tt} = -u_{xxxx}$. Hence, for any $u, v \in X_0$,

$$(u, v) = \int_I (u_{tt} v_{tt} + u_{xxxx} v_{xxxx})\, dt\, dx = 2\int_I u_{tt} v_{tt}\, dt\, dx .$$

In order to state our result we shall now note that the results in (B) above can be summarized by saying that there is a monotone nondecreasing positive function $\gamma(R)$, $R \geqslant 0$, such that, $u \in X$, $\|u\| \leqslant R$ implies $F \in Y$, $\|F\| \leqslant \gamma(R)$. We shall, however, take into consideration only the case $m = 2$, $X = A_2$. Moreover, let $L = \|H\|$.

(6.i) *Let $f(t, x, u, u_t, u_x) \in C^2(G \times R^3)$ and let $\gamma(R)$ the corresponding function above. Let us assume that there are constants R_0, r such that* (a) *for* *all* $u^* \in X$, $u_1 \in X_{1n}$, $\|u^*\|_X = R_0$, $\|u_1\|_X \leqslant r$, *we have*

$$\int_I \big(f(t, x, u, u_t, u_x)\big)_t u^*_{tt}\, dt\, dx \leqslant 0 \qquad [\text{or } \geqslant 0]\,,$$

and that (*b*) $L(R) \leqslant r$ *where* $R = (R_0^2 + r^2)^{\frac{1}{2}}$. *Then, the quasi linear hyperbolic problem* (14)-(16) *has at least one solution* $u(t, x) \in A_2$ *with* $\|u\|_2 \leqslant R$.

We shall only show that theorem (5.i) applies. Thus, we have to verify hypotheses (*a*) and (*b*) of (5.i). Actually, by Remark 1 of § 5, it is enough to verify that

$$(J\alpha_n S_n' QNu, u^*)_X \leqslant 0$$

for all $u^* \in X_{0n}$, $\|u^*\| = R_0$, and $u = u_1 + u^*$, $u_1 \in X_{1n}$, $\|u_1\| \leqslant r$. Note that here

$$(J\alpha_n S_n' QNu, u^*)_X = 2\big((\alpha_n S_n' QNu)_t, u_{tt}\big)_{L^2}$$

since $J\alpha_n S_n' QNu$ and u^* belong to X_0. Now we have

$$Nu = \sum (Nu, e_{kl})_{L^2} e_{kl}\,,$$

$$QNu = \sum_{k^2 = l^4} (Nu, e_{kl})_{L_2} e_{kl}\,,$$

$$\alpha_n S_n' QNu = \sum{}^* (Nu, e_{kl})_{L_2} e_{kl}\,,$$

where $\sum^*$ denotes any sum extended to all k, l with $1 \leqslant l \leqslant n$, $k^2 = l^4$. Thus

$$(\alpha_n S_n' QN_u)_t = \sum{}^* (Nu, e_{kl})_{L_2} (e_{kl})_t = (Nu)_t\,,$$

and in conclusion

$$(J\alpha_n S_n' QNu, u^*)_X = 2\big((\alpha_n S_n' QNu)_t, u_{tt}\big)_{L_2} = 2\big((QNu)_t, u_{tt}\big) \leqslant 0 \quad [\text{or } \geqslant 0]\,,$$

for all $u \in X_{0n}$, $u_1 \in X_{1n}$, $u = u^* + u_1$, $\|u^*\|_X = R_0$, $\|u_1\|_X \leqslant r$, hence $\|u\| \leqslant$ $\leqslant R = (R_0^2 + r^2)^{\frac{1}{2}}$.

The two remarks above, together with theorem (5.i), show that the truncated coupled system of equations have a solution $u_n = u_n^* + u_{1n}$, $u_n^* \in X_{0n}$, $u_{1n} \in X_{1n}$, and further $\|u_n\|_X$ is bounded independently of n.

Proceeding as in Section 5, we now introduce the space $\mathfrak{X}$. For $\mathfrak{X}$ we choose C, the space of continuous functions on $G = [0, \pi] \times R$, 2π-periodic in t. Then $\|u_n\|_X$ are bounded and since $|u|$, $|u_t|$, $|u_x| \leqslant R$, any bounded sequence $\{u_n\}$ in X generates sequences $\{u_n\}$, $\{(u_n)_t\}$, $\{(u_n)_x\}$ which are equibounded and thus the sequence u_n is also equi-Lipschitzian. By applying Arzela-Ascoli's theorem we obtain that any weak limit element u of $\{u_n\}$ in X is a strong limit in $\mathfrak{X}$. Proceeding to the limit in the coupled system of equations we obtain that $u \in X$ is a solution, in the weak sense, of the original problem.

Here the solution is Lipschitzian, u_t, u_x exist in the strong sense (a.e.) and are bounded; the other derivatives u_{tt}, u_{tx}, u_{xx}, u_{xxx}, u_{xxxx} exist in L_2 in the distributional sense and they satisfy the original equation in the weak sense (a.e.) (in fact they satisfy pointwise a.e.).

We conclude this section with the following remarks. If the nonlinearity involves a small parameter, *i.e.*, f is of the form $\varepsilon g(t, x, u, u_x, u_t)$ then $\gamma(R)$ is replaced by $\varepsilon\gamma(R)$ and Condition (6) of (6.i) can always be satisfied by taking ε sufficiently small.

Let us prove now that the conditions of (6.i) hold under the hypotheses considered in [15], namely $f = \varepsilon g(t, x, u, u_t)$, $\varepsilon > 0$, g of class C^2, and there are constants α, β, μ such that

$$g_{u_t}(t, x, u, v) \geqslant \mu > 2^{-1}(\tau^+ - \tau^-) \quad \text{for } (t, x, u, v) \in G_0 ,$$

$$0 < \beta < \alpha = \mu - 2^{-1}(\tau^+ - \tau^-) , \quad |g_t(t, x, u, v)| \leqslant \beta R_0 \quad \text{for } (t, x, u, v) \in G_0 ,$$

$$\tau^+ = \mathrm{Sup}_{G_0}\, g_u(t, x, u, v) , \quad \tau^- = \mathrm{Inf}_{G_0}\, g_u(t, x, u, v) ,$$

$$G_0 = \{(t, x, u, v) | (t, x) \in G,\ |u| \leqslant R,\ |v| \leqslant R\} , \quad R = (R_0^2 + r^2)^{\frac{1}{2}} .$$

Indeed, for $u_n = u_n^* + u_{1n}$ we have

$$\text{(18)} \quad \big((Nu_n)_t, u_{ntt}^*\big)_{L_2} = \varepsilon(g_t + g_u u_t + g_{u_t} u_{tt}, u_{ntt}^*)_{L_2} \geqslant$$
$$\geqslant \varepsilon\big(g_t + g_u(u_{nt}^* + u_{1nt}) + g_{u_t}(u_{ntt}^* + u_{1ntt}), u_{ntt}^*\big)_{L_2} .$$

On the other hand, for $\|u_n^*\| = R_0$, $\|u_{1n}\|_X \leqslant r$ and $\tau = \mathrm{Sup}_G |g_{u_t}| = \max\{|\tau^+|, |\tau^-|\}$, $\Lambda = \mathrm{Sup}_G |g_u|$, we have

$$\text{(19)} \quad \begin{cases} |(g_t, u_{ntt}^*)| \leqslant \beta R^2 , & |(g_u u_{1nt}, u_{ntt}^*| \leqslant \Lambda r R_0 , \\ |(g_{u_t} u_{1ntt}, u_{ntt}^*)| \leqslant \tau r R_0 , & (g_{u_t} u_{ntt}^*, u_{ntt}^*) \geqslant \mu R_0^2 \end{cases}$$

and we prove below that

$$\text{(20)} \quad |(g_u u_{nt}^*, u_{ntt}^*)| \leqslant 2^{-1}(\tau^+ - \tau^-) \|u_{nt}\|^2 = 2^{-1}(\tau^+ - \tau^-) R_0^2 .$$

This last relation is a consequence of the following simple remark concerning L-integrable functions: if $\varphi \in L_1(I)$ and $\iint_I \varphi\, dt\, dx = 0$ then, for every

$\Psi \in L_1(D)$, $m \leqslant \Psi \leqslant M$, we have

$$(21)\qquad -2^{-1}(M-m)\iint |\varphi|\,dt\,dx \leqslant \iint_I \Psi\varphi\,dt\,dx \leqslant 2^{-1}(M-m)\iint_I |\varphi|\,dt\,dx\,.$$

Indeed, if $I^+ = \{(t, x) \in I|\varphi \geqslant 0\}$ and $I^- = I - I^+$, then

$$\iint_{I^+} \varphi\,dt\,dx = -\iint_{I^-} \varphi\,dt\,dx = 2^{-1}\iint_I |\varphi|\,dt\,dx\,,$$

and

$$\iint_I \Psi\varphi\,dt\,dx = \Big(\iint_{I^+} + \iint_{I^-}\Big) \leqslant M\iint_{I^+} \varphi\,dt\,dx + m\iint_{I^-} \varphi\,dt\,dx = 2^{-1}(M-m)\iint_I |\varphi|\,dt\,dx\,.$$

This is the second inequality (21). Analogously we can prove the first inequality (21). Now by force of inequalities (19) and (20), we see that (18) becomes

$$\big((Nu_n)_t, u^*_{ntt}\big)_{L_2} \geqslant \varepsilon[\mu R_0^2 - \beta R_0^2 - \Lambda r R_0 - \tau r R_0 - r R_0 - 2^{-1}(\tau^+ - \tau^-) R_0^2] =$$
$$= \varepsilon[(\alpha - \beta) R_0^2 - \Lambda r R_0 - \tau r R_0]$$

and by taking $r > 0$ sufficiently small we have $\big((Nu_n)_t, u^*_{ntt}\big)_{L_2} \geqslant \varepsilon(\alpha - \beta) R_0^2/2$.

REMARK. If we take $J_n = J^4$ in (12) and (13), then we obtain a statement analogous to (6.i) with the main inequality replaced by

$$\int_I f(t, x, u, u_t, u_x)\, u^*\,dt\,dx \leqslant 0 \qquad [\text{or } \geqslant 0]\,.$$

Indeed we have here

$$\begin{aligned}(J_n\alpha_n S_n' QNu, u^*)_X &= 2\big((J^4\alpha_n S_n' QNu)_{tt}, u^*_{tt}\big)\\ &= 2\big((J^2\alpha_n S_n' QNu), u^*_{tt}\big)\\ &= 2(\alpha_n S_n' QNu, u^*)\\ &= 2(QNu, u^*)\,.\end{aligned}$$

Further choices of J_n are considered in [9] leading to a number of simple criteria for existence.

7. – Another application.

We consider here the problem of the periodic solutions $u(t, x)$ of the hyperbolic equation

$$(22)\qquad \begin{cases} u_{tt} - (-1)^p D_x^{2p} u = f(t, x, u)\,, \\ u(t + 2\pi, x) = u(t, x) = u(t, x + 2\pi)\,, \end{cases}$$

where $p > 1$ is any integer, f is a given function in R^3 periodic of period 2π in t and x. Let X be the real Hilbert space of all functions $u(t, x)$ periodic in t and x of period 2π with distributional derivatives u_t, u_{tt}, $D_x^{2p}u$ all in $L_2(I)$, $I = [0, 2\pi]^2$, and inner product $(u, v)_X = (u, v) + (u_{tt}, v_{tt}) + (D_x^{2p}u, D_x^{2p}v)$ where $(,)$ is the usual inner product $L_2(I)$.

A complete orthonormal system in $L_2(I)$ is generated by the functions $e_{kl}(t, x) = \exp(ikt)\exp(ilx)$. From Sobolev imbedding theorems we know that there is a constant γ such that $\|u\|_\infty \leqslant \gamma \|u\|_X$, and also $\|u\|_2$, $\|u_t\|_2$, $\|u_x\|_2$, $\|u_{xx}\|_2 \leqslant \gamma \|u\|_X$. The kernel X of E in X is now the subspace of X generated by the elements e_{kl} with $k^2 = l^{2p}$.

Let X_0, X_1, $Y = L_2(I)$, Y_0, Y_1 be defined as in no. 6, let $\langle y, x\rangle$ denote the inner product in $L_2(I)$, and let X_{0n}, X_{1n}, Y_{0n}, Y_{1n}, S_n, R_n, $S_n'\alpha_n$ be defined as in no. 6. Finally, let $J\colon X_0 \to X_0$ be the linear operator defined as follows: for any $u \in X_0$, $u = \sum_{k^2 = l^{2p}} u_{kl} e_{kl}$, let

$$(23)\qquad Ju = \sum_{k^2 = l^{2p}} (-k^{-2})\, u_{kl} e_{kl}\,.$$

Then, J is an isomorphism. Further $(Ju)_{tt} = u$, and since any $u \in X_{0n}$ is a solution of the homogeneous problem, we also have $u_{tt} = D_x^{2p}u$. For any u, $v \in X_0$, then

$$(u, v)_X = \int_I uv\, dt\, dx + \int_I (u_{tt}v_{tt} + D_x^{2p}uD_x^{2p}v)\, dt\, dx = \int_I uv\, dt\, dx + 2\int_I u_{tt}v_{tt}\, dt\, dx\,.$$

Proceeding as in no. 6 we obtain the following result.

(7.i) *Let f be of class C^1, and there are positive constants R_0, r, C such that, for $(t, x) \in D$ and $|u| \leqslant \gamma R$ we have $|f| \leqslant C$ with $CL \leqslant r$ and $R = (R_0^2 + r^2)^{\frac{1}{2}}$. Let us assume further that, for all $u^* \in X_{0n}$, $u_1 \in X_{1n}$, $\|u^*\|_X = R_0$ and $\|u_1\|_X \leqslant r$, we have*

$$\int_I f(t, x, u)\, u_{tt}^*\, dt\, dx \geqslant 0 \qquad [\text{or } \leqslant 0]\,,$$

and this holds for every n. Then, the nonlinear hyperbolic problem (22) *has at least a solution $u(t, x)$ such that $\|u(t, x)\|_X \leqslant R$.*

We shall only show that theorem (5.i) applies. Thus, we have to verify hypotheses (*a*) and (*b*) of (5.i). In particular we have to prove that $(J\alpha_n S_n' QNu, u^*)_X \geqslant 0$ for all $u^* \in X_{0n}$, $u^* = R_0$, and $u = u_1 + u^*$, $u_1 \in X_{1n}$, $\|u_1\| \leqslant r$.

Note that here by force of (23) and by integrations by parts we have

$$(J\alpha_n S_n' QNu, u^*)_X = \int_I (J\alpha_n S_n' QNu, u^*)\, dt\, dx + 2\int_I (J\alpha_n S_n' QNu)_{tt}, u^*_{tt})\, dt\, dx = \\ = 3\int_I (\alpha_n S_n' QNu, u^*_{tt})\, dt\, dx\,.$$

Now

$$Nu = \sum (Nu, e_{kl})_{L_2} e_{kl}\,,$$

$$QNu = \sum_{l^2 = k^2 p} (Nu, e_{kl})_{L_2} e_{kl}\,,$$

$$S_n' QNu = \sum_{k^2 = l^2 p} (Nu, e_{kl})_{L_2} e_{kl} = \alpha_n S_n' QNu\,.$$

Thus

$$(\alpha_n S_n' QNu, u^*_{tt}) = (Nu, u^*_{tt})_{L_2}\,.$$

Note that the solutions whose existence are guaranteed in (7.i) are Lipschitzian functions u, periodic in t and x of period 2π, with first order derivatives u_t, u_x almost everywhere in the strong sense and bounded, while the other derivatives u_{tt}, u_{tx}, u_{xx}, $D_x^\tau u$, $\tau \leqslant 2p$, all exist in the distributional sense and are L^2-integrable functions.

It remains now to prove that the conditions of Theorem (7.i) hold under the hypotheses considered in [11, 12], namely $f = \varepsilon g(t, x, u)$, $\varepsilon > 0$, g of class C^1, and there are constants $\alpha > 0$, $\mu > 0$ such that

$$g_u(t, x, u) \geqslant \mu\,, \qquad |g_t(t, x, u)| \leqslant \alpha \quad \text{for } (t, x, u) \in G\,,$$

$$G = \{(t, x, u) | (t, x) \in I,\ |u| \leqslant \gamma R\}\,, \qquad R = (R_0^2 + r^2)^{\frac{1}{2}}\,.$$

Indeed, for $u_n = u_n^* + u_{1n}$, we have

$$-(Nu_n, u^*_{ntt}) = \varepsilon\big((Nu_n)_t, u^*_{nt}\big) = \varepsilon\big(g_t + g_u(u^*_{nt} + u_{1nt}), u^*_{nt}\big)\,.$$

On the other hand, for $\|u_n^*\| = R_0$, $\|u_{1n}\|_X \leqslant r$ and $\Lambda = \mathrm{Sup}_G |g_u|$, $K = \mathrm{Sup}_G |g_t|$, we have

$$|(g_t, u^*_{nt})| \leqslant KR_0\,,$$

$$(g_u u^*_{nt}, u^*_{nt}) \geqslant R_0^2\,,$$

$$|(g_u u_{1nt}, u^*_{nt})| \leqslant \Lambda r R_0\,.$$

Hence,

$$-(Nu_n, u^*_{ntt}) \geqslant \varepsilon(\mu R_0^2 - KR_0 - \Lambda r R_0),$$

and the last expression is certainly $\geqslant 2^{-1}\varepsilon\mu R_0^2$ for R_0 sufficiently large and r sufficiently small.

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