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An Inversion of the Obstacle Problem and its Explicit Solution.

HANS LEWY (*)

dedicated to the memory of Guido Stampacchia

The obstacle problem in its simplest form for functions in $\boldsymbol{R}^n$ is this: Given are a smooth domain Ω of $\boldsymbol{R}^n$ and a smooth function $\psi(x)$, $x \in \Omega \cup \partial\Omega$ with $\psi(x) < 0$, $x \in \partial\Omega$. Minimize the Dirichlet integral of a function $u(x)$

$$\int_\Omega (\operatorname{grad} u)^2\, dx = \min\,,$$

among all functions $u(x)$ which vanish on $\partial\Omega$ and which do not exceed $\psi(x)$.

Existence, uniqueness of $u(x)$ and continuity of its first derivatives are well established for the solution.

A more difficult problem is the nature of the set

$$\omega = \{x\colon u(x) = \psi(x)\}\,.$$

For $n = 2$, under the hypothesis of convexity of Ω and analyticity and strong concavity of $\psi(x)$, it was shown in [2], [3] that ω is simply connected and has an analytic boundary. For $n > 2$ it is not known whether or not the same hypotheses imply the same conclusion. However it was proved in [1] that if P is a point of $\partial\omega$ where ω has positive density, then $\partial\omega$ is smooth near P.

Consider together with $\psi(x)$ all obstacles $\psi(x) + c$ with positive constant c, and for which $\max\limits_\Omega \psi(x) = 0$, while, say, $\psi(x) = -\infty$ on $\partial\Omega$. The solutions u of the above problem become functions u_c of the parameter c and so does the set $\omega = \omega_c$. As c varies from 0 to ∞ the set ω_c increases mono-

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tonely. For it is known that the solution $u_c(x)$ is the smallest continuous superharmonic compatible with the imposed inequality and boundary condition. It follows that

$$\psi(x) + c' \leqslant u_{c'}(x) \leqslant u_c(x) + c' - c\,, \qquad \text{if } c < c'$$

because $(u_c + c' - c) \cap u_{c'}$ is also superharmonic and competes with $u_{c'}$. Hence if at x, $\psi(x) + c = u_c(x)$, then also $u_{c'}(x) = \psi(x) + c'$. This observation suggests the problem inverse to that of the determination of ω_c:

Given a smoothly increasing set ω_c, determine the function $\psi(x)$, *i.e.* the obstacle $\psi(x) + c$ which gives rise to the coincidence set ω_c. Contrary to the difficulty of characterization of ω_c, given ψ, this inverse problem admits of an easy explicit solution. That is the first part of this paper.

The second part, motivated by the first, gives examples in which the obstacle problem is solved by the inverse method. In particular we prove that if $\Omega = \mathbf{R}^3$, $\partial\Omega$ the point at ∞, and $\psi(x) = -\sum_1^3 x_j^2/b_j$, $b_j > 0$, then ω_c is a solid ellipsoid $\sum_1^3 x_j^2/a_j \leqslant \text{const.}$, $a_j > 0$.

It is to be expected that the stability of $\partial\omega_c$ under « small » changes of Ω could be established by the method of [4], but we have not carried out its application to the equation (1.2) below.

1. – The inverse problem.

In order to derive the relation for the recovery of ψ from the assignment of ω_c, we assume that $\partial\omega_c$ can be represented by

$$S(x) = c\,, \qquad \operatorname{grad} S \neq 0$$

where $S(x)$ is sufficiently smooth. This is a working hypothesis which restricts the topological behavior of ω_c and which essentially limits the nature of the obstacle and also that of $\partial\Omega$.

Denote by $G(x, y)$ Green's function of the Laplacian for the domain Ω, and consider

$$\int_{\omega_c} -\Delta\psi(y)\, G(x, y)\, dy\,.$$

If $G(x, y)$ is properly normed, this integral represents a function of x which vanishes on $\partial\Omega$, is C^1 in Ω, and which is harmonic in $\Omega - \omega_c$, while in $S < c$ its Laplacian equals that of ψ. These properties characterize a function uniquely and are shared by $u_c(x)$. Thus

$$u_c(x) = \int_{\omega_c} -\Delta\psi(y)\, G(x, y)\, dy\,, \qquad x \in \Omega\,. \tag{1.1}$$

For $c < c'$, we know

$$u_{c'} = u_c + c' - c \text{ in } \omega_c$$

whence

$$\frac{u_{c'} - u_c}{c' - c} = 1\,, \qquad x \in \omega_c\,.$$

Thus

$$(c' - c)^{-1} \int_{\omega_{c'} - \omega_c} -\Delta\psi(y)\, G(x, y)\, dy = 1\,, \qquad x \in \omega_c\,.$$

Denote by $\delta\nu$ the length of the normal at y on $\partial\omega_c$ as it pierces $\partial\omega_{c'}$. As $c' \to c$, we obtain

$$\int_{\partial\omega_c} -\Delta\psi(y)\, G(x, y) \lim_{c' \to c} \frac{\delta\nu}{c' - c}\, dy = 1\,, \qquad x \in \omega_c\,.$$

Now $\lim\limits_{c' \downarrow c} \big(\delta\nu/(c' - c)\big) = |\text{grad}\, S(y)|^{-1}$ and

$$V_c(x) \overset{\text{def}}{=} \int_{\partial\omega_c} -\Delta\psi(y) |\text{grad}\, S(y)|^{-1} G(x, y)\, dy = 1\,, \qquad x \in \omega_c\,.$$

Evidently $V_c(x)$ is continuous in Ω, harmonic in $\Omega - \omega_c$ and $= 0$ on $\partial\Omega$. Thus $V_c(x)$ is the so-called conductor potential of ω_c relative to Ω. It is well known how to obtain the factor $f(y)$ in a *surface* integral $\int f(y)\, G(x, y)\, dy$ through the jump of the normal derivative. We find, with γ a constant depending on the dimension number n,

$$-\Delta\psi(x) |\text{grad}\, S(x)|^{-1} = \gamma |\text{grad}\, V_c()x|\,, \qquad x \in \partial\omega_c\,.$$

γ is computed from $\psi = -|x|^2$, $\Omega = \mathbf{R}^n$. We find $u_c(x) = k(c)|x|^{2-n}$ for $|x| \geqslant |x_0|$ and $u_c(x) = c - |x|^2$ for $|x| \leqslant |x_0|$. C^1 continuity of $u_c(x)$ gives $(2 - n) k |x_0|^{1-n} = -2|x_0|$ and $c - |x_0|^2 = k|x_0|^{2-n} = 2(n-2)^{-1}|x_0|^2$. Thus $c = n(n-2)^{-1}|x_0|^2$, $|\text{grad}\, S| = \big(2n/(n-2)\big)|x_0|$ and $|\text{grad}\, V_c| = |2 - n||x_0|^{-1}$ result in $\gamma = 1$. Since on $\partial\omega_c$ we have $S(x) = c$ and $V_c(x) = 1$ we can write this also

$$+\Delta\psi(x) = \text{grad}\, S(x)\, \text{grad}\, V_c(x)\,, \qquad S(x) = c\,. \tag{1.2}$$

This relation determines, together with (1.1) or

$$(1.3)\qquad c+\psi(x)=\int_{\omega_c}-\Delta\psi(y)G(x,y)\,dy\,,\qquad x\in\omega_c$$

the obstacle for all $x\in\omega_{c_0}$ if (1.2) holds for all c in $0<c\leqslant c_0$. It is not necessary to know beforehand that $S(x)=\text{const.}$ gives the boundary of the coincidence sets in an obstacle problem; in other words the indeterminacy of a ψ whose Laplacian obeys (1.2) is removed by (1.3) in a way which does not depend on c. To see this put $\psi(x)=\psi_c(x)$,

$$(1.4)\qquad w(x)=|\text{grad}\,V_c(x)|\qquad \text{with } c=S(x)$$

and compute $(\partial/\partial c_0)\psi_{c_0}(x)$ for x in $S(x)<c_0$. We find by differentiating (1.3) at $c=c_0$ and substituting (1.2) under the integral

$$1+\frac{\partial\psi_{c_0}(x)}{\partial c_0}=\int_{\partial\omega_{c_0}}|\text{grad}\,V_{c_0}(y)|G(x,y)\,dy\,,\qquad S(x)<c_0\,.$$

But

$$V_{c_0}(x)=\int_{\partial\omega_{c_0}}|\text{grad}\,V_{c_0}(y)|G(x,y)\,dy$$

as right hand is the evaluation of $\int_{\Omega}-\Delta V_{c_0}(y)G(x,y)\,dy$ with $\Delta V_c(y)$ understood in the sense of distributions. Hence

$$\partial\psi_{c_0}(x)/\partial c_0=0\,,\qquad S(x)<c_0$$

as required.

2. – Homogeneous S.

Suppose now $\Omega=\boldsymbol{R}^n$, $n>2$, and $S(\lambda x)=\lambda^k S(x)$, $\lambda>0$, $k>0$. With $w(x)$ from (1.4) we find $w(\lambda x)=\lambda^{-1}w(x)$, $(\Delta\psi)(\lambda x)=-|(\text{grad}\,S)(\lambda x)|w(\lambda x)$, whence $(\Delta\psi)(\lambda x)=\lambda^{k-2}\Delta\psi(x)$. Moreover with γ' a constant depending only on n,

$$\begin{aligned}
u_1(x)&=\gamma'\int_{S(y)<1}\Delta\psi(y)|x-y|^{2-n}\,dy\,,\\
u_{\lambda^k}(\lambda x)&=\gamma'\int_{S(y')<\lambda^k}\Delta\psi(y')|\lambda x-y'|^{2-n}\,dy'\\
&=\gamma'\int_{S(y)<1}\lambda^{k-2}\Delta\psi(y)|x-y|^{2-n}\,dy\,\lambda^2=\lambda^k u_1(x)\,,\\
\psi(\lambda x)&=\lambda^k\psi(x)\,.
\end{aligned}$$

3. – We apply the foregoing to the case $n = 3$,

$$S(x) = \sum_1^3 x_i^2/a_i\,, \qquad 0 < a_1 < a_2 < a_3\,. \tag{3.1}$$

Here the conductor potential of $S(x) \leqslant 1$ with respect to $\boldsymbol{R}^3$ is explicitly known in terms of the elliptic coordinates $\lambda_1, \lambda_2, \lambda_3$, solutions of

$$\sum_1^3 x_i^2/(a_i + \lambda) = 1 \tag{3.2}$$

with $-a_1 < \lambda_1 < \infty$, $-a_2 < \lambda_2 < -a_1$, $-a_3 < \lambda_3 < -a_2$. This potential V_1 is a function only of λ_1, $\lambda_1 \geqslant 0$, remaining constant on the surface of each confocal ellipsoid. Its precise value, which incidentally plays no role in the present investigation, is

$$V_1(x) = g(\lambda_1)/g(0)$$

with

$$g(\lambda_1) = \int_{\lambda_1}^{\infty} ((\lambda + a_1)(\lambda + a_2)(\lambda + a_3))^{-\frac{1}{2}}\, d\lambda\,.$$

Accordingly

$$\partial V_1(x)/\partial x_j = -\, g(0)^{-1}(a_1 a_2 a_3)^{-\frac{1}{2}}\, \partial\lambda_1/\partial x_j\,, \qquad \lambda_1 = 0\,.$$

To obtain $\partial\lambda_1/\partial x_j$ at $S(x) = 1$, we put $\lambda = \lambda_1$ in (3.2), take derivatives with respect to x_j and put $\lambda_1 = 0$:

$$2x_j/a_j = \sum_1^3 (x_i^2/a_i^2)\, \partial\lambda_1/\partial x_j\,, \qquad j = 1, 2, 3\,.$$

Thus (1.2) becomes

$$\Delta\psi(x) = -\, 4g(0)^{-1}(a_1 a_2 a_3)^{-\frac{1}{2}}\,, \qquad S(x) = 1\,.$$

From § 2 we gather $\Delta\psi(\mu x) = \Delta\psi(x)$, $\mu > 0$. Hence

$$\Delta\psi(x) = -\, 4g(0)^{-1}(a_1 a_2 a_3)^{-\frac{1}{2}} = -\, K\,, \qquad x \in \boldsymbol{R}^3\,. \tag{3.3}$$

We remark that even if any two successive a_i are equal and of course also if all three a_i are equal, the conductor potential of $S \leqslant 1$ still depends only on λ_1, the largest of the roots of (3.2), and (3.3) holds good also in these cases.

This result (3.3) yields immediately that ψ is analytic at the origin where it must vanish and have a maximum. Since ψ is homogeneous of degree 2 we have thus that $\psi = \sum b_{ij}x_i x_j$. But it is of interest to know that in this way, as the a_j vary, we can obtain $\psi(x) = -\sum_1^3 x_i^2/b_j$ with every triple of positive b_j in order to prove the claim at the end of the introduction.

Observe that (3.3) implies on $S(x) < 1$

$$u_1(x) = 1 + \psi(x) = -K\gamma' \iiint_{S(y)\leqq 1} |x-y|^{-1} dy_1 dy_2 dy_3$$

$$\frac{\partial \psi}{\partial x_i} = -K\gamma' \iiint \frac{\partial}{\partial x_1} |x-y|^{-1} dy_1 dy_2 dy_3 = +K\gamma' \iiint \frac{\partial}{\partial y_1} |x-y|^{-1} dy_1 dy_2 dy_3$$
$$= K\gamma' \iint_{S(y)=1} \frac{dy_2\, dy_3}{|x-y|},$$

$$\frac{\partial \psi}{\partial x_2} = K\gamma' \iint_{S(y)=1} \frac{dy_3\, du_1}{|x-y|}, \quad \frac{\partial \psi}{\partial x_3} = K\gamma' \iint_{S(y)=1} \frac{dy_1\, dy_2}{|x-y|},$$

$$\frac{\partial^2 \psi}{\partial x_1 \partial x_2}(0) = \mathrm{const} \iint_{S(y)=1} \frac{y_1\, dy_2\, dy_3}{|0-y|^3} = 0,$$

since $S(y_1, y_2, y_3) = -S(y_1, -y_2, y_3)$. Similarly

$$\frac{\partial \psi}{\partial x_j}(0) = 0, \qquad \frac{\partial^2 \psi}{\partial x_i \partial x_j}(0) = 0, \qquad i \neq j,$$

$$\frac{\partial^2 \psi}{\partial x_1^2}(0) = -K\gamma' \iint_{S(y)=1} |y|^{-3} y_1\, dy_2\, dy_3, \qquad \frac{\partial^2 \psi}{\partial x_2^2}(0) = -K\gamma' \iint_{S(y)=1} |y|^{-3} y_2\, dy_3\, dy_1,$$

$$\frac{\partial^2 \psi}{\partial x_3^2}(0) = -k\gamma' \iint_{S(y)=1} |y|^{-3} y_3\, dy_1\, dy_2.$$

Put $X = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2$, then

$$\iint_{S(y)=1} |y|^{-3} y_1\, dy_2\, dy_3 = \iint_{|y|=1} X^{-\frac{3}{2}} y_1\, dy_2\, dy_3 (a_1 a_2 a_3)^{\frac{1}{2}}, \ldots$$

and

$$(3.4) \qquad \frac{\partial^2 \psi}{\partial x_1^2}(0) : \frac{\partial^2 \psi}{\partial x_2^2}(0) : \frac{\partial^2 \psi}{\partial x_3^2}(0) = b_1^{-1} : b_2^{-1} : b_3^{-1} = \iint_{|y|=1} X^{-\frac{3}{2}} y_1\, dy_2\, dy_3 :$$
$$: \iint_{|y|=1} X^{-\frac{3}{2}} y_2\, dy_3\, dy_1 : \iint_{|y|=1} X^{-\frac{3}{2}} y_3\, dy_1\, dy_2.$$

We intend to show that given positive $b_1 \leqslant b_2 \leqslant b_3$ there exist positive $a_1 \leqslant \leqslant a_2 \leqslant a_3$ which solve the proportion (3.4). In view of the homogeneity of $S(x)$ in a_1, a_2, a_3 this means that an arbitrary negative definite form $\psi(x) = \sum_1^3 x_i^2/b_i$ leads to an ellipsoidal boundary $\partial\omega_c$ of the coincidence set ω_c.

The first thing to prove is that the map $a \to b$ given by (3.4) where the a_j are homogeneous coordinates and so are the b_j, can be continuously extended to the boundary of the a set given by $0 < a_1 \leqslant a_2 \leqslant a_3$. This boundary is the triangle of 3 straight segments: from $(0, 0, 1)$ to $(0, 1, 1)$, from $(0, 1, 1)$ to $(1, 1, 1)$, and from $(1, 1, 1)$ to $(0, 0, 1)$. Suppose $a_1 \to 0$ while $0 < a_2 \leqslant a_3$ stay fixed. Evidently each of the integrals on the right of (3.4) increases as a_1 decreases, tending respectively to

$$\iint_{|y|=1} (a_2 y_2^2 + a_3 y_3^2)^{-\frac32} y_1 \, dy_2 \, dy_3 \,,$$

$$\iint_{|y|=1} (a_2 y_2^2 + a_3 y_3^2)^{-\frac32} y_2 \, dy_3 \, dy_1 \,,$$

$$\iint_{|y|=1} (a_2 y_2^2 + a_3 y_3^2)^{-\frac32} y_3 \, dy_1 \, dy_2 \,.$$

Put $y_2 = (1 - y_1^2)^{\frac12} \cos\theta$, $y_3 = (1 - y_1^2)^{\frac12} \sin\theta$. Those limit values are

$$\begin{aligned} b_1^{-1} &= \int_{-1}^{1}\int_{0}^{2\pi} dy_1\, d\theta (1 - y_1^2)^{-\frac32} y_1^2 (a_2 \cos^2\theta + a_3 \sin^2\theta)^{-\frac32} = \infty \,, \\ b_2^{-1} &= \int_{-1}^{1}\int_{0}^{2\pi} dy_1\, d\theta (1 - y_1^2)^{-\frac12} (a_2 \cos^2\theta + a_3 \sin^2\theta)^{-\frac32} \cos^2\theta < \infty \,, \qquad (3.5) \\ b_3^{-1} &= \int_{-1}^{1}\int_{0}^{2\pi} dy_1\, d\theta (1 - y_1^2)^{-\frac12} (a_2 \cos^2\theta + a_3 \sin^2\theta)^{-\frac32} \sin^2\theta < \infty \,, \end{aligned}$$

and thus b_1, b_2, b_3 tend monotonely to limits. The first of these is 0, while the other two are continuous in a_2, a_3. By Dini's theorem the extension of the map (3.4) to $a_1 = 0$ is continuous as long as $0 < a_2 \leqslant a_3$. If now $a_2 \to 0$ while $a_3 = 1$, the limit values of the integrals on $a_1 = 0$ are $b_1^{-1} = \infty$, $b_2^{-1} = \infty$, but

$$\lim b_3^{-1} = \int_{-1}^{1} dy_1 (1 - y_1^2)^{-\frac12} \int_{0}^{2\pi} d\theta |\sin\theta|^{\frac12} < \infty \,.$$

Hence on $a_1 = 0$ we have $b_2 : b_3 \to 0$ as $a_2 : a_3 \to 0$.

Unfortunately this is not enough to imply that the map (3.4) of $a \to b$ can be continuously extended to the boundary of the above triangle. We still must prove that if $a_1 = 1 \leqslant a_2$, $a_3 \to \infty$, $a_2/a_3 \to 0$, then $b_1 : b_2 : b_3 \to 0:0:1$.

We have $a_1 = 1$, $X = Cy_3^2 + B$ with

$$C = a_3 - a_1 \cos^2\theta - a_2 \sin^2\theta\,, \qquad B = a_1 \cos^2\theta + a_2 \sin^2\theta\,,$$

$$y_1 = (1 - y_3^2)^{\frac{1}{2}} \cos\theta\,, \qquad y_2 = (1 - y_3^2)^{\frac{1}{2}} \sin\theta\,,$$

$$y_1\, dy_2\, dy_3 = y_3\, dy_3 \cos^2\theta\, d\theta\,, \quad y_2\, dy_3\, dy_1 = y_3\, dy_3 \sin^2\theta\, d\theta\,, \quad y_3\, dy_1\, dy_2 = y_3^2\, dy_3\, d\theta\,.$$

Because of the symmetries of the integrands it suffices to integrate from 0 to 1 in y_3 and from 0 to $\pi/2$ in θ. We find with

$$h(y_3) = \int \frac{y_3\, dy_3}{(Cy_3^2 + B)^{\frac{3}{2}}} = -\, C^{-1}(Cy_3^2 + B)^{-\frac{1}{2}}$$

that

$$b_1^{-1} = \int_0^{\pi/2} \cos^2\theta \left(\frac{1}{CB^{\frac{1}{2}}} - \frac{1}{C(C+B)^{\frac{1}{2}}} \right) d\theta$$

$$b_2^{-1} = \int_0^{\pi/2} \sin^2\theta \left(\frac{1}{CB^{\frac{1}{2}}} - \frac{1}{C(C+B)^{\frac{1}{2}}} \right) d\theta$$

$$b_3^{-1} = \int_0^{\pi/2} d\theta \int_0^1 X^{-\frac{3}{2}} y_3\, dy_3 = \int_0^{\pi/2} d\theta \int_0^1 (h(1) - h(y_3))\, dy_3$$

$$= \int_0^{\pi/2} d\theta \int_0^1 \left(\frac{1}{C(Cy_3^2 + B)^{\frac{1}{2}}} - \frac{1}{C(C+B)^{\frac{1}{2}}} \right) dy_3\,.$$

Now $B/C \to 0$ uniformly in θ as $a_2/a_3 \to 0$, and

$$\int_0^1 \frac{dy_3}{(Cy_3^2 + B)^{\frac{1}{2}}} = \frac{1}{\sqrt{C}} \left(\log \sqrt{C/B} + O(1) \right)$$

uniformly in θ as $B/C \to 0$.

Thus, as $a_2/a_3 \to 0$

$$b_3^{-1} = a_3^{-\frac{3}{2}} \int_0^{\pi/2} \frac{1}{2} \log \frac{C}{B}\, d\theta (1 + o(1))$$

$$b_1^{-1} = a_3^{-\frac{3}{2}} \int_0^{\pi/2} \cos^2\theta \left(\frac{C}{B}\right)^{\frac{1}{2}} d\theta (1 + o(1))$$

$$b_2^{-1} = a_3^{-\frac{3}{2}} \int_0^{\pi/2} \sin^2\theta \left(\frac{C}{B}\right)^{\frac{1}{2}} d\theta (1 + o(1)) .$$

Hence

$$0 \leqslant \lim_{\substack{a_3 \to \infty \\ a_2/a_3 \to 0}} \frac{b_1}{b_3} = \lim_{\cdots} \frac{\frac{1}{2}\int_0^{\pi/2} \log (C/B)\, d\theta}{\int_0^{\pi/2} \cos^2\theta (C/B)^{\frac{1}{2}}\, d\theta} ,$$

$$0 \leqslant \lim_{\cdots} \frac{b_2}{b_3} = \lim_{\cdots} \frac{\frac{1}{2}\int_0^{\pi/2} \log (C/B)\, d\theta}{\int_0^{\pi/2} \sin^2\theta (C/B)^{\frac{1}{2}}\, d\theta} .$$

Both of these limits are 0. This would be immediate if the integrations in numerators and denominators were extended from ε to $\pi/2 - \varepsilon$, where ε is a small positive value, since then $\sin^2\theta$ and $\cos^2\theta$ stay away from 0 and the ratios of the integrands tends (uniformly) to 0 as $a_2/a_3 \to 0$. But integrating from 0 to $\pi/2$ increases the denominators while for small $\varepsilon > 0$

$$\left(\int_0^{\varepsilon} + \int_{\pi/2-\varepsilon}^{\pi/2}\right) \log \frac{C}{B}\, d\theta < \frac{1}{2} \int_0^{\pi/2} \log \frac{C}{B}\, d\theta$$

since as $a_2/a_3 \to 0$,

$$\int_0^{\pi/2} \log \frac{C}{B}\, d\theta > \frac{\pi}{2} \log \frac{a_3}{2a_2}, \qquad \left(\int_0^{\varepsilon} + \int_{\pi/2-\varepsilon}^{\pi/2}\right) \log \frac{C}{B}\, d\theta < 2\varepsilon \log \frac{a_3}{a_2} - \int_0^{\pi/2} \log \sin^2\theta\, d\theta .$$

This completes the proof that the map $a \to b$ has a continuous extension to the closure of our triangle.

The next task is to ascertain that there is b in $0 < b_1 < b_2 < b_3$ such there is an a which is mapped by (3.4) into this b.

Now we have just observed that $\lim b_1^{-1} = \infty$, $\lim b_2^{-1} < \infty$ as $a_1 \downarrow 0$. Hence for a small a_1/a_2 we have $b_1 < b_2$. Next let $a_2 = 1$, a_3 large and $a_1 \downarrow 0$. By (3.5)

$$\lim_{a_1 \downarrow 0} a_3^{\frac{3}{2}} b_2^{-1} = \int_{-1}^{1} dy_1 (1 - y_1^2)^{-\frac{1}{2}} \int_0^{2\pi} \left(\frac{a_2}{a_3} \cos^2\theta + \sin^2\theta \right)^{-\frac{3}{2}} \cos^2\theta \, d\theta$$

$$\lim_{a_1 \downarrow 0} a_3^{\frac{3}{2}} b_3^{-1} = \int_{-1}^{1} dy_1 (1 - y_1^2)^{-\frac{1}{2}} \int_0^{2\pi} \left(\frac{a_2}{a_3} \cos^2\theta + \sin^2\theta \right)^{-\frac{3}{2}} \sin^2\theta \, d\theta \, .$$

Now if a_2/a_3 is small, the first limit is large, but the second limit remains below $\int_{-1}^{1} (1 - y_1^2)^{-\frac{1}{2}} dy_1 \int_0^{2\pi} |\sin\theta|^{\frac{1}{2}} d\theta$. It follows that if a_1 is small, $a_2 = 1$ and a_3 sufficiently large we have $b_1 < b_2 < b_3$.

Having established the continuous extension of the map $a \to b$ to the closed triangle we note that the vertices are mapped on themselves and so are the sides whose equations are resp. $a_1 = 0$, $a_2 = a_3$, and $a_1 = a_2$. The order of an interior point b of the triangle with respect to its boundary is 1. Take for this b a point which is image of an a in the map. Let b' be another point, image of another interior point a' of the triangle. We claim that b' must lie within the interior of the triangle. Otherwise join a to a' by the straight segment whose image in the b-plane would have to cross the boundary of the triangle. But at the crossing point $\hat{b}$ we know that $\hat{b}$ is image of an $\hat{a}$ which lies on the same side of the triangle as $\hat{b}$. Now given $\hat{b}$ there is at most one ratio $a_1 : a_2 : a_3$ for an ellipsoid $\sum x_i^2/a_i = \text{const}$, boundary of the coincidence set ω_c. Thus there is a contradiction, and b' lies within the triangle. It now follows in familiar fashion that all of the b-triangle is image of the a-triangle, and the map is one-one and continuous both ways.

4. – An example of the method of this paper in which however the condition grad $S \neq 0$ is not strictly observed in this:

With $p > 0$ a constant put

$$w = |x - x_1|^{-1} + p|x - x_2|^{-1}$$

which is harmonic in $\boldsymbol{R}^3 - \{x_1\} - \{x_2\}$ and vanishes at ∞.

We take $\Omega = \boldsymbol{R}^3$ and $\psi = -w^{-2}$. Then $\psi(\infty) = -\infty$, $\max \psi = \psi(x_1) = \psi(x_2) = 0$. (1.2) suggests putting $S = g(w)$ so that if $w = \text{const} = a$,

$c = g(a)$. We then find $V_c = w/a$ for $w(x) \leqslant a$; $\operatorname{grad} S = g'(a) \operatorname{grad} w$ and $\operatorname{grad} V_c = a^{-1} \operatorname{grad} w$ on $w = a$.

(1.2) becomes

$$- 6w^{-4}(\operatorname{grad} w)^2 = g'(a)\, a^{-1}(\operatorname{grad} w)^2 , \qquad w = a ,$$

whence

$$g(a) = 3a^{-2} = c ,$$

the constant of integration being 0 on account of $c = 0$ for $a = \infty$ (*i.e.* for $x = x_1$ or $x = x_2$). On the segment joining x_1 to x_2 there is a point $x' = \lambda x_1 + (1 - \lambda) x_2$ with $0 < \lambda < 1$ from $-(\lambda - 1)^{-2} + p\lambda^{-2} = 0$ for which $\operatorname{grad} w(x') = 0$. We observe that this singularity is harmless for the conclusion about the solution of the obstacle problem with the present $\psi + c$. The value of c for which $\partial\omega_c$ has a singular point is easily obtained as $c = 3|x_1 - x_2|^2 p^{-1}$. For smaller c, $\partial\omega_c$ consists of two, for larger c it consists of one smooth (analytic) surfaces.

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