Annali della
Scuola Normale Superiore di Pisa
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 6, nº 2 (1979), p. 285-304
<http://www.numdam.org/item?id=ASNSP_1979_4_6_2_285_0>

# Embeddability of Real Analytic Cauchy-Riemann Manifolds (*).

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The local and global embeddability problems for Cauchy-Riemann (C-R) manifolds are as follows:

(I) Which C-R manifolds are locally isomorphic to a generic submanifold of $C^n$?

(II) Which C-R manifolds are globally isomorphic to a generic submanifold of some complex manifold?

It is well-known (see Rossi [10]) that real analytic C-R manifolds are locally embeddable as in (I). In § 2 we prove that real analytic C-R manifolds are also globally embeddable as in (II). This is a generalization of the theorem of Ehresmann-Shutrick (see Shutrick [12]) on the existence of complexifications—a theorem which was also proven independently by Haefliger [8] and by Bruhat and Whitney [4]. It also improves a result proved by Rossi [11]. In § 3 we give a notion of domination of real analytic C-R structures and prove a sort of functorial property concerning their complexifications. In the last section we prove a result about the convexity of the complexification which is a generalization of a theorem of Grauert [6].

## 1. - Preliminaries.

The two best references for the material presented in this section are [1] and Greenfield [7]. We will use the word « manifold » to mean an infinitely

(*) Supported in part by grants of N.S.F. and the Italian Research Council.
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Pervenuto alla Redazione il 27 Gennaio 1978.

differentiable paracompact manifold and the word « submanifold » to mean a subspace of a manifold for which the inclusion map is an embedding.

For each $m$-dimensional manifold $M$, let $T(M)\otimes \boldsymbol{C}$ denote the complexified tangent bundle of $M$ so that

$$T(M)\otimes \boldsymbol{C}_p = \textit{complex span of } \frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_m}(p),$$

if $(x, U)$ is a chart of $M$ with $p \in U$. A *Cauchy-Riemann* (C-R) *structure of type $l$ on $M$* is an $l$-dimensional complex subbundle $A$ of $T(M)\otimes \boldsymbol{C}$ such that

($a$) $A \cap \bar{A} = \{0\}$ (zero section), and

($b$) $A$ is involutive, i.e. $[P, Q]$ is a section of $A$ whenever $P$ and $Q$ are sections of $A$.

Note that the zero section of $T(M)\otimes \boldsymbol{C}$ defines a C-R structure of type 0 on $M$. This trivial C-R structure is called the *totally real structure of $M$.*

Observe from the definition of a C-R structure that $0 \leqslant l \leqslant m/2$ and also that if $A$ is a C-R structure of type $l$ on $M$, then so is $\bar{A}$.

Note also that condition ($a$) above is equivalent to the condition that if $p \in M$ and $P, Q \in A_p$ with $\operatorname{re} P = \operatorname{re} Q$ then $P = Q$. Thus we can define a complex structure on the $2l$-dimensional subbundle $\operatorname{re} A$ of $T(M)$, i.e. a bundle map $J\colon \operatorname{re} A \to \operatorname{re} A$ with $J^2 = -I$, by defining $J$ on $\operatorname{re} A_p$ as follows:

$$JP = Q \quad \textit{if and only if } P + iQ \in A_p . \tag{1.1}$$

A C-R *manifold* is a pair $(M, A)$ where $A$ is a C-R structure on $M$. The C-R manifold $(M, A)$ is said to be *of type* $(m, l)$ if $M$ is an $m$-dimensional manifold and $A$ is a C-R structure of type $l$. A C-R manifold $(M, A)$ is called *real analytic* if $M$ is a real analytic manifold and, for each chart $(x, U)$ of the real analytic atlas of $M$, there exist complex-valued vector fields $P_1, \dots, P_l$ such that

($a$) $A_p =$ *complex span of* $P_1(p), \dots, P_l(p)$ *for each* $p \in U$, *and*

($b$) $P_i = \sum_{j=1}^{m} c_{ij}(\partial/\partial x_j)$ *for* $i = 1, \dots, l$ *with* $c_{ij}\colon U \to \boldsymbol{C}$ *real analytic.*

We will now examine in detail the C-R manifold $(X, HT(X))$, where $X$ is an $n$-dimensional complex manifold and $HT(X)$ is its holomorphic tangent bundle. Note that if $(z, U)$ is a (holomorphic) chart of $X$ and $p \in U$, then

$$T(X)\otimes \boldsymbol{C}_p = \textit{complex span of } \frac{\partial}{\partial z_1}(p), \dots, \frac{\partial}{\partial z_n}(p), \frac{\partial}{\partial \bar{z}_1}(p), \dots, \frac{\partial}{\partial \bar{z}_n}(p),$$

so that each $P \in T(X) \otimes \boldsymbol{C}_p$ can be written uniquely in the form

$$P = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}(p) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial \bar{z}_i}(p) \qquad \textit{with } a_i, b_i \in \boldsymbol{C}.$$

Such a $P$ is called *holomorphic* and an element of $HT(X)_p$ if all the $b_i$ are zero. Similarly, such a $P$ is called *antiholomorphic* and an element of $AT(X)_p$ if all the $a_i$ are zero. Defining the holomorphic and antiholomorphic tangent bundles, $HT(X)$ and $AT(X)$, in the natural way, we see that $HT(X)$ is an $n$-dimensional complex subbundle of $T(X) \otimes \boldsymbol{C}$ with

$$T(X) \otimes \boldsymbol{C} = HT(X) \oplus AT(X) \qquad \textit{and} \qquad \overline{HT(X)} = AT(X).$$

Now $HT(X) \cap AT(X) = \{0\}$ and it is easy to see that $HT(X)$ is involutive. Hence $(X, HT(X))$ is a C-R manifold; in fact, a real analytic C-R manifold.

Henceforth, a complex manifold $X$ will be considered to be a C-R manifold with C-R structure $HT(X)$. When there is no ambiguity we will simply write $X$ for $(X, HT(X))$. We remark, however, that it is customary to take $AT(X)$ as the C-R structure on $X$.

We will now give a detailed description of how a submanifold of a complex manifold can «inherit» a C-R structure. Since the two defining properties of a C-R structure are essentially local, we begin by considering a real submanifold $M$ of $\boldsymbol{C}^n$ of (real) dimension $m$. Fixing $p \in M$ there exists a sufficiently small neighborhood $U$ of $p$ in $\boldsymbol{C}^n$ and $2n - m$ smooth functions

$$f_j \colon U \to \boldsymbol{R} \qquad \textit{for } j = 1, \dots, 2n - m \tag{1.2}$$

such that

(*a*) $M \cap U = \{z \in U \mid f_j(z, \bar{z}) = 0 \ \textit{for } j = 1, \dots, 2n - m\}$,

(*b*) $df_1 \wedge \dots \wedge df_{2n-m} \neq 0 \ \textit{on } U$.

If $U$ is sufficiently small we can also find *local parametric equations of $M$ on $U$*, i.e. a set of $n$ smooth functions

$$\varphi_i \colon D \to \boldsymbol{C} \qquad \textit{for } i = 1, \dots, n \tag{1.3}$$

defined on an open set $D \subset \boldsymbol{R}^m$ such that

(*a*) $M \cap U = \varphi(D)$, where $\varphi \colon D \to \boldsymbol{C}^n$ is given by the equations

$$\varphi \equiv \{z = \varphi_i(t), \ \textit{for } i = 1, \dots, n\},$$

(*b*) $\displaystyle \operatorname{rk} \frac{\partial(\varphi_1, \dots, \varphi_n, \bar{\varphi}_1, \dots, \bar{\varphi}_n)}{\partial(t_1, \dots, t_m)} = m \qquad \textit{on } D.$

If $M$ is a real analytic submanifold of $\boldsymbol{C}^n$ the functions $f_j$ and $\varphi_i$ described above can be chosen to be real analytic.

Let $F(M)$ denote the sheaf of germs of smooth, complex-valued functions on $\boldsymbol{C}^n$ which vanish on $M$. If $U$ and $f_i$ are as defined in (1.2) then

$$F(M)(U) = \mathcal{S}(U)(f_1, \ldots, f_{2n-m}),$$

where $\mathcal{S}$ is the sheaf of germs of smooth, complex-valued functions on $\boldsymbol{C}^n$. If $U$ and $\varphi_j$ are as defined in (1.3) then

$$F(M)(U) = \{g \in \mathcal{S}(U) | g \circ \varphi = 0 \ \ on \ \ D\}.$$

The *holomorphic tangent space to $M$ at $p \in M$* is now defined by

$$HT(M, \boldsymbol{C}^n)_p = \{P \in HT(\boldsymbol{C}^n)_p | P(f) = 0, \ for \ every \ f \in F(M)_p\}.$$

Note that $HT(M, \boldsymbol{C}^n)_p$ is a complex vector space, and set

$$l(p) = \dim_{\boldsymbol{C}} HT(M, \boldsymbol{C}^n)_p \qquad for \ each \ p \in M.$$

PROPOSITION 1.4. *The function $l(p)$ is an upper semicontinuous function of $p$ along $M$ which satisfies*

$$m - n \leqslant l(p) \leqslant \frac{m}{2}.$$

PROOF. From above $F(M)_p = \mathcal{S}_p(f_1, \ldots, f_{2n-m})$ and hence

$$P = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}(p) \in HT(M, \boldsymbol{C}^n)_p$$

if and only if $P(f_j) = 0$ for $j = 1, \ldots, 2n - m$. Thus $l(p)$ is the dimension of the subspace of vectors $a = (a_1, \ldots, a_n) \in \boldsymbol{C}^n$ such that

$$\sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial z_i}(p) = 0 \qquad for \ j = 1, \ldots, 2n - m.$$

Hence

$$l(p) = n - \operatorname{rk} \frac{\partial(f_1, \ldots, f_{2n-m})}{\partial(z_1, \ldots, z_n)}(p)$$

and we see that $l(p)$ is upper semicontinuous since the rank of a matrix of smooth functions is lower semicontinuous. Moreover, the rank of the above matrix is less than or equal to $2n - m$ and hence $l(p) \geqslant m - n$. Since the functions $f_j$ are real-valued we see that, in self-explanatory notation,

$$\operatorname{rk} \frac{\partial(f)}{\partial(z)} = \operatorname{rk} \frac{\partial(f)}{\partial(\bar{z})} \qquad on\ U .$$

From (1.2) ($b$) it follows that

$$\operatorname{rk} \frac{\partial(f)}{\partial(z, \bar{z})} = 2n - m \qquad on\ U ,$$

so that $\operatorname{rk} \partial(f)/\partial(z) \geqslant \frac{1}{2}(2n - m)$ on $U$ and consequently that $l(p) \leqslant m/2$.

A submanifold $M$ of $C^n$ is said to be *generic at* $p \in M$ if $l(p) = m - n$.

PROPOSITION 1.5. *$M$ is generic at $p \in M$ (and hence in a neighborhood of $p \in M$ by proposition 1.4) if and only if one of the following conditions is satisfied.*

($a$) $\operatorname{rk} \dfrac{\partial(f_1, \ldots, f_{2n-m})}{\partial(z_1, \ldots, z_n)}(p) = 2n - m$ .

($b$) $\operatorname{rk} \dfrac{\partial(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)}(t_0) = n$ , *where* $t_0 \in D$ *with* $\varphi(t_0) = p$ .

PROOF. The first statement follows from the proof of proposition 1.4. For the second statement note that $f_j \circ \varphi = 0$ on $D$ for $j = 1, \ldots, 2n - m$ so that

$$\sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial t_k}(t_0) \frac{\partial f_j}{\partial z_i}(p) + \sum_{i=1}^{n} \frac{\partial \bar{\varphi}_i}{\partial t_k}(t_0) \frac{\partial f_j}{\partial \bar{z}_i}(p) = 0$$

for $j = 1, \ldots, 2n - m$ and $k = 1, \ldots, m$. Thus

$$\varphi_* \frac{\partial}{\partial t_1}(p), \ldots, \varphi_* \frac{\partial}{\partial t_m}(p)$$

defined by

$$\varphi_* \frac{\partial}{\partial t_k}(p) = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial t_k}(t_0) \frac{\partial}{\partial z_i}(p) + \sum_{i=1}^{n} \frac{\partial \bar{\varphi}_i}{\partial t_k}(t_0) \frac{\partial}{\partial \bar{z}_i}(p)$$

are $m$ linearly independent, complex-valued tangent vectors at $p$ which span $T(M) \otimes C_p$. Moreover

$$P = \sum_{k=1}^{m} a_k \varphi_* \frac{\partial}{\partial t_k}(p) \qquad with\ (a_1, \ldots, a_m) \in C^m$$

is an element of $HT(M, \boldsymbol{C}^n)_p$ if and only if

$$\sum_{k=1}^{m} a_k \frac{\partial \bar{\varphi}_i}{\partial t_k}(t_0) = 0 \qquad \text{for } i = 1, \ldots, n .$$

Consequently

$$l(p) = m - \operatorname{rk} \frac{\partial(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)}(t_0)$$

and (*b*) now follows.

REMARK 1.6. From the preceding proof we see that the complex-valued vector fields $\varphi_*(\partial/\partial t_1), \ldots, \varphi_*(\partial/\partial t_m)$ are defined on $\varphi(D) \subset M$ and span $T(M) \otimes \boldsymbol{C}$ at each point of $\varphi(D)$. We also see that

$$l(\varphi(t)) = m - \operatorname{rk} \frac{\partial(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)}(t) \qquad \text{for every } t \in D .$$

Hence if $l(q) = l$ (a constant) for all $q$ in a neighborhood of $p$ in $M$, we can find smooth (or real analytic if the $\varphi_i$'s are real analytic) complex-valued functions $c_{ik}$ for $i = 1, \ldots, l$ and $k = 1, \ldots, m$ which are defined near $p$ in $M$ so that the vector fields

$$P_i = \sum_{k=1}^{m} c_{ik} \varphi_* \frac{\partial}{\partial t_k} \qquad \text{for } i = 1, \ldots, l$$

span $HT(M, \boldsymbol{C}^n)$ at each point of $M$ near $p$.

We now consider $M$ to be an $m$-dimensional submanifold of an $n$-dimensional complex manifold $X$. Let $F(M)$ denote the sheaf of germs of smooth real-valued functions on $X$ which vanish on $M$ and define

$$HT(M, X)_p = \{P \in HT(X)_p | P(f) = 0, \text{ for every } f \in F(M)_p\}$$

for each $p \in M$. Set $l(p) = \dim_{\boldsymbol{C}} HT(M, X)_p$. Note that the complex structure $J$ determined by $HT(X)$ as in (1.1) is defined on $\operatorname{re} HT(X)_p = T(X)_p$ by

$$J\left(\sum_{i=1}^{u} a_i \frac{\partial}{\partial x_i}(p) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial y_i}(p)\right) = -\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}(p) + \sum_{i=1}^{n} a_i \frac{\partial}{\partial y_i}(p)$$

where $(z, U)$ is a chart of $X$ with $p \in U$, $z_j = x_j + iy_j$ and $a_i, b_i \in \boldsymbol{R}$. Now, for each $p \in M$, we see that $\operatorname{re} HT(M, X)_p = T(M)_p \cap JT(M)_p$ and that

$$\operatorname{re} HT(M, X)_p \subset T(M)_p \subset T(M)_p + JT(M)_p \subset T(X)_p . \tag{1.7}$$

Hence $\operatorname{re} HT(M, X)_p$ is the maximal complex subspace of $T(X)_p$ (with complex structure $J$) contained in $T(M)_p$ and $T(M)_p + JT(M)_p$ is the minimal complex subspace of $T(X)_p$ containing $T(M)_p$. Moreover,

$$(1.8) \qquad \dim_{\boldsymbol{C}} \big(T(M)_p + JT(M)_p\big) = m - l(p) .$$

PROPOSITION 1.9. *If $l(p)$ is constant and equal to $l$ on $M$, then $HT(M, X)$ is a* C-R *structure of type $l$ on $M$. Moreover, if $M$ is real analytic, then $\big(M, HT(M, X)\big)$ is a real analytic* C-R *manifold.*

PROOF. From the definition of $HT(M, X)$ we see that $HT(M, X) \cap \cap \overline{HT(M, X)} = \{0\}$ and also that a complex-valued vector field $P$ is a section of $HT(M, X)$ if and only if it is holomorphic and satisfies $PF(M) \subset F(M)$. It thus follows that if $P$ and $Q$ are two such vector fields, then so is $[P, Q]$. From remark (1.6) we see that we can choose a real analytic, local basis for $HT(M, X)$ if $M$ is real analytic.

If $HT(M, X)$ is a C-R structure of type $l$ on $M$ we say that $M$ *inherits a* C-R *structure of type $l$ from $X$*. When there is no ambiguity we will write $HT(M)$ for the inherited C-R structure $HT(M, X)$. Note that $\boldsymbol{R}^n \subset \boldsymbol{C}^n$ inherits the totally real structure from $\boldsymbol{C}^n$ and that every real hypersurface in $\boldsymbol{C}^n$ inherits a C-R structure of type $n-1$ from $\boldsymbol{C}^n$.

We remark again that, classically (as in the study of tangential Cauchy-Riemann equations), one considers $\mathrm{AT(M)} = \overline{HT(M)}$ as the inherited C-R structure on $M$.

An $m$-dimensional submanifold $M$ of an $n$-dimensional complex manifold $X$ is *generic* if it inherits a C-R structure of type $m - n$ from $X$. A generic submanifold of $X$ will always be considered to be a C-R manifold with its inherited C-R structure from $X$.

Two C-R manifolds, $(M, A)$ and $(N, B)$, are said to be *isomorphic* if there exists a diffeomorphism $\psi\colon M \to N$ such that $\psi_* A = B$, where $\psi_*\colon T(M) \otimes \boldsymbol{C} \to T(N) \otimes \boldsymbol{C}$ is the natural map induced by $\psi$. Note that two isomorphic C-R manifolds are necessarily of the same type.

PROPOSITION 1.10. *If $M$ is an $m$-dimensional submanifold of $X$ which inherits a* C-R *structure of type $l$ from $X$, then $\big(M, HT(M)\big)$ is locally isomorphic to a generic submanifold of $\boldsymbol{C}^{m-l}$. If $M$ is real analytic the local isomorphism is also real analytic.*

PROOF. Since the result is local, consider $M$ to be a submanifold of $\boldsymbol{C}^n$ and fix $p \in M$. Recall from (1.7) and (1.8) that $T(M)_p + JT(M)_p$ is the minimal complex subspace of $T(\boldsymbol{C}^n)_p$ containing $T(M)_p$ and that it has dimension $m - l$. Let $\pi_1$ and $\pi_2$ be complex planes of dimensions $m - l$

and $n-m+l$ respectively, for which

(a) $T(\pi_1)_p \oplus T(\pi_2)_p = T(C^n)_p$ (transversal at $p$);

(b) $T(\pi_1)_p = T(M)_p + JT(M)_p$.

Define $\lambda\colon C^n \to C^{m-l}$ by projecting $C^n$ onto $\pi_1$ parallel to $\pi_2$ and then identifying $\pi_1$ with $C^{m-l}$ by some biholomorphic map. Then $\lambda$ is holomorphic and, by (a) and (b), there is a sufficiently small neighborhood $W$ of $p$ in $M$ on which $\tau = \lambda|_W$ is a diffeomorphism (real analytic if $M$ is real analytic) onto its image. Suppose now that $z=(z_1, \dots, z_n)$ are holomorphic coordinates on $C^n$ and $w=(w_1, \dots, w_{m-l})$ are holomorphic coordinates on $C^{m-l}$, and that

$$w_j = g_j(z_1, \dots, z_n) \qquad \textit{for } j = 1, \dots, m-l$$

are the equations of the holomorphic projection $\lambda\colon C^n \to C^{m-l}$. Also assume that, after possibly choosing a smaller $W$, the smooth functions (real analytic if $M$ is real analytic) $\varphi_i\colon D \to C$ for $i = 1, \dots, n$ give local parametric equations of $M$ on an open set $U$ in $C^n$, as in (1.3), for which $U \cap M = W$. Thus

$$w_j = g_j\big(\varphi_1(t), \dots, \varphi_n(t)\big) \qquad \textit{for } j = 1, \dots, m-l \textit{ and } t \in D$$

gives parametric equations of $\tau W$ on $\tau U \subset C^{m-l}$ and we see by the construction of $\tau$ that

$$\operatorname{rk} \frac{\partial(g)}{\partial(t)} = m - l \qquad \textit{on } D\,.$$

Since this rank is maximal we conclude from proposition 1.5 that $\tau W$ is a generic submanifold of $C^{m-l}$. Since $\lambda$ is holomorphic we have that

$$\lambda_*\left(\frac{\partial}{\partial z_i}(q)\right) = \sum_{j=1}^{m-l} \frac{\partial g_j}{\partial z_i}(q) \frac{\partial}{\partial w_j}(\lambda(q)) \qquad \textit{for every } q \in C^n\,,$$

and hence $\lambda_* HT(C^n) \subset HT(C^{m-l})$. Since $\tau_* = \lambda_{*|T(W)\otimes C}$ we now conclude that

$$\tau_* HT(W, C^n) \subset HT(\tau W, C^{m-l})\,.$$

In fact we have equality in the preceding line since $\tau\colon W \to \tau W$ is a diffeomorphism and $\tau W$ is a generic submanifold of $C^{m-l}$ so both are C-R structures of type $l$ on $\tau W$. Thus $\big(W, HT(W)\big)$ and $\big(\tau W, HT(\tau W)\big)$ are isomorphic and the proof is complete.

The local embeddability problem can be stated as follows.

(I) *Given a* C-R *manifold* $(M, A)$ *can one find, for each* $p \in M$, *a neighborhood* $U$ *of* $p$ *in* $M$ *and an embedding* $\tau\colon U \to \boldsymbol{C}^n$ *such that* $\tau U$ *is a generic submanifold of* $\boldsymbol{C}^n$ *and* $\tau_*(A_{|U}) = HT(\tau U)$?

Note that if $(M, A)$ is of type $(m, l)$ then $n = m - l$.

As we remarked earlier, (I) is solvable for real analytic C-R manifolds. Indeed, one can prove (see [2])

THEOREM 1.11. *If* $(M, A)$ *is a real analytic* C-R *manifold, then* (I) *is solvable with real analytic embeddings* $\tau$.

Note that Nirenberg [9] has given an example of a C-R structure of type one on a neighborhood of the origin in $\boldsymbol{R}^3$ for which (I) is not solvable.

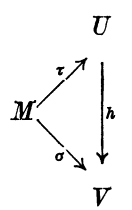
We now turn to the global embeddability problem which is the following:

(II) *Given a* C-R *manifold* $(M, A)$ *for which* (I) *is solvable, can one find a complex manifold* $X$ *and an embedding* $\tau\colon M \to X$ *such that* $\tau M$ *is a generic submanifold of* $X$ *and* $\tau_* A = HT(\tau M)$?

Note again that if $(M, A)$ is of type $(m, l)$, then $X$ is a complex manifold of dimension $m - l$.

We now define a *complexification of the real analytic* C-R *manifold* $(M, A)$ to be a pair $(X, \tau)$ where $X$ is a complex manifold and $\tau\colon M \to X$ is a real analytic, closed embedding such that $\tau M$ is a generic submanifold of $X$ and $\tau_* A = HT(\tau M)$. In the next section we will prove

THEOREM 1.12: (*a*) *Each real analytic* C-R *manifold has a complexification.* (*b*) *If* $(X, \tau)$ *and* $(Y, \sigma)$ *are two complexifications of the real analytic* C-R *manifold* $(M, A)$, *then there exist neighborhoods* $U$ *of* $\tau M$ *in* $X$ *and* $V$ *of* $\sigma M$ *in* $Y$ *and a biholomorphic map* $h\colon U \to V$ *such that the diagram*




*commutes. Moreover,* $h$ *is uniquely defined if* $U$ *is sufficiently small and connected with* $\tau M$.

Thus (II) is solvable for real analytic C-R manifolds and the germ of the complexification of $(M, A)$ along $M$ is unique. The totally real case of theorem 1.12, i.e. the case when $A$ is the totally real structure of $M$, is the theorem of Ehresmann-Shutrick (see Shutrick [12]) which was referred to in the introduction.

## 2. – Global embeddability of real analytic C-R manifolds.

Before proving theorem 1.12 we would like to collect some results concerning real analytic C-R manifolds.

PROPOSITION 2.1. *Suppose that $M = \varphi(D)$ is an m-dimensional, generic, real analytic submanifold of $\boldsymbol{C}^{m-l}$ with real analytic parametric equations*

$$\varphi \equiv \{z_j = \varphi_j(t) \text{ for } j = 1, \ldots, m - l \text{ and } t \in D \subset \boldsymbol{R}^m .$$

*There exist neighborhoods $\tilde{D}$ of $D$ in $\boldsymbol{C}^m$ and $W$ of $M$ in $\boldsymbol{C}^{m-l}$ such that the map $\varphi\colon D \to M$ extends to a holomorphic, open, surjective map $\tilde{\varphi}\colon \tilde{D} \to W$ with the property that*

$$\operatorname{rk} \frac{\partial(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-l})}{\partial(w_1, \ldots, w_m)} = m - l \qquad on\ \tilde{D},$$

*where $w_k = t_k + is_k$ for $k = 1, \ldots, m$ give holomorphic coordinates on $\tilde{D}$.*

PROOF. The functions $\tilde{\varphi}_j(w_1, \ldots, w_m) = \varphi_j(t_1 + is_1, \ldots, t_m + is_m)$ are holomorphic in a neighborhood $\tilde{D}$ of $D$ in $\boldsymbol{C}^m$ and

$$\operatorname{rk} \frac{\partial(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-l})}{\partial(w_1, \ldots, w_m)} = m - l \qquad on\ D,$$

since $M$ is generic and $\tilde{\varphi}_j$ extends $\varphi_j$. Hence the above matrix has maximal rank on a neighborhood of $D$ in $\boldsymbol{C}^m$ which we take to be $\tilde{D}$. The map $\tilde{\varphi}\colon \tilde{D} \to \boldsymbol{C}^{m-l}$ defined by the functions $\tilde{\varphi}_j$ is thus holomorphic and of maximal rank. It is therefore open and we can take $W = \tilde{\varphi}\tilde{D}$ to complete the proof.

PROPOSITION 2.2 (Tomassini [13]). *If $M$ is as in proposition 2.1 and $f\colon M \to \boldsymbol{C}$ is a real analytic function with $f_*\colon HT(M) \to HT(\boldsymbol{C})$, then there exists a holomorphic function $F$ defined in a neighborhood $W$ of $M$ in $\boldsymbol{C}^{m-l}$ which extends $f$. Moreover, if $F_1$ and $F_2$ are holomorphic extensions of $f$ defined on neighborhoods $W_1$ and $W_2$ of $M$ in $\boldsymbol{C}^{m-l}$, then $F_1 = F_2$ on every neighborhood of $M$ in $\boldsymbol{C}^{m-l}$ which is connected with $M$ and contained in $W_1 \cap W_2$.*

PROOF. By proposition 2.1 we can select a neighborhood $\tilde{D}$ of $D$ in $C^m$ so that $\zeta_1 = \tilde{\varphi}_1, \dots, \zeta_{m-l} = \tilde{\varphi}_{m-l}$ are part of a system of holomorphic coordinates $(\zeta_1, \dots, \zeta_m)$ on $\tilde{D}$. Since $f\circ\varphi$ is real analytic we can find a holomorphic function $G$ on $\tilde{D}$ (sufficiently small) such that $G = f\circ\varphi$ on $D$. Since for each $P \in T(M)\otimes C_p$ we have

$$f_* P = Pf\frac{\partial}{\partial z}(f(p)) + P\bar{f}\frac{\partial}{\partial \bar{z}}(f(p)),$$

we see that the condition $f_*: HT(M) \to HT(C)$ is equivalent to the condition

$$P\bar{f} = 0 \qquad \textit{whenever } P \in HT(M)_p \textit{ and } p \in M,$$

or equivalently

$$Qf = 0 \qquad \textit{whenever } Q \in AT(M)_p \textit{ and } p \in M.$$

From the proof of proposition 1.5 the latter condition is in turn equivalent to the condition

$$\left(\sum_{k=1}^m b_k \varphi_* \frac{\partial}{\partial t_k}(p)\right) f = 0$$

*whenever* $p \in M$ *and* $b \in C^m$ *satisfying*

$$\sum_{k=1}^m b_k \frac{\partial \varphi_j}{\partial t_k}(t_0) = 0 \qquad \textit{for } j = 1, \dots, m-l.$$

Thus $d\varphi_1 \wedge \dots \wedge d\varphi_{m-l} \wedge d(f\circ\varphi) = 0$ on $D$, or $d\zeta_1 \wedge \dots \wedge d\zeta_{m-l} \wedge dG = 0$ on $D$. Since

$$dG = \sum_{k=1}^m \frac{\partial G}{\partial \zeta_k} d\zeta_k$$

we see that the holomorphic functions $\partial G/\partial \zeta_k$ for $k = m - l + 1, \dots, m$ vanish on $D$ and hence on $\tilde{D}$ (assuming $\tilde{D}$ is connected with $D$). Thus $G$ is independent of $\zeta_{m-l+1}, \dots, \zeta_m$ and so $G$ factors through $\tilde{\varphi}$, i.e. there exists a holomorphic function $F$ defined on a neighborhood $W$ of $M$ in $C^{m-l}$ such that $G = F\circ\tilde{\varphi}$. Now $F|_M = f$ and the unicity of $F$ follows from the above considerations.

Since the restriction of a holomorphic map is real analytic and maps holomorphic tangent vectors to holomorphic tangent vectors we have

COROLLARY 2.3. *If $F_1$ and $F_2$ are holomorphic functions defined on open sets $W_1$ and $W_2$ in $C^{m-l}$ and $F_1 = F_2$ on a generic submanifold $M$ of $C^{m-l}$,* *then* $F_1 = F_2$ *on every neighborhood of* $M$ *in* $C^{m-l}$ *which is connected with* $M$ *and contained in* $W_1 \cap W_2$.

We now establish theorem 1.12 (*b*), i.e. the unicity of the germ of the complexification of $(M, A)$ along $M$. From corollary 2.3 it suffices to prove this result locally. Fixing $p \in M$ we apply proposition 2.2 to the components of $\sigma\circ\tau^{-1}\colon \tau M \to \sigma M$ near $\tau(p)$ to obtain a holomorphic function $h$ which extends $\sigma\circ\tau^{-1}$ near $\tau(p)$. Similarly, we also obtain a holomorphic function $g$ which extends $\tau\circ\sigma^{-1}$ near $\sigma(p)$. Now $g\circ h$ is a holomorphic function in a sufficiently small neighborhood $U$ of $\tau(p)$ in $X$ and $g\circ h$ is the identity on $U \cap \tau M$. Assuming that $U$ is connected with $\tau M$ we conclude from corollary 2.3 that $g\circ h$ is the identity on $U$. Similarly, $h\circ g$ is the identity on the neighborhood $V = h(U)$ of $\sigma(p)$ in $Y$ which is connected with $\sigma(M)$ and thus $h$ is the desired biholomorphic map. The unicity of $h$ follows from another application of corollary 2.3.

The proof of theorem 1.12 (*a*) is established by a construction similar to that given by Bruhat and Whitney [4] in the case of the totally real structure. (See also [3].)

Let $(M, A)$ be a real analytic C-R manifold of type $(m, l)$ and let $n = m - l$.

PART 1. We can find three locally finite open covers of $M$ with the same index set $I$, say $\{V'_i\}$, $\{U'_i\}$ and $\{T'_i\}$ such that

$$V'_i \subset\subset U'_i \subset\subset T'_i \quad \textit{for every } i \in I .$$

For each $i \in I$ we can find local complexifications of $T'_i$ by theorem 1.11, i.e. there exist real analytic isomorphisms $\varphi_i\colon T'_i \to T_i$, where $T_i$ is an $m$-dimensional, generic, real analytic submanifold of $C^n$ and

$$\varphi_{i*}(A_{|T'_i}) = HT(T_i) .$$

We now set

$$U_i = \varphi_i U'_i, \qquad V_i = \varphi_i V'_i$$

$$U_{ij} = \varphi_i(U'_i \cap U'_j) , \qquad V_{ij} = \varphi_i(V'_i \cap V'_j)$$

$$T_{ij} = \varphi_i(T'_i \cap T'_j) .$$

The isomorphism

$$\varphi_j\circ\varphi_i^{-1}\colon T_{ij} \to T_{ji}$$

extends by proposition 2.2 to a biholomorphic map $\psi_{ji}\colon \tilde{T}_{ij} \to \tilde{T}_{ji}$ of a neighborhood $\tilde{T}_{ij}$ of $T_{ij}$ in $C^n$ to a neighborhood $\tilde{T}_{ji}$ of $T_{ji}$ in $C^n$. We can assume that $\tilde{T}_{ij}$ is empty if $T_{ij}$ is empty and that $\psi_{ij} = \psi_{ji}^{-1}$. For every ordered pair $(i, j)$ we can select open sets $\tilde{U}_{ij}$ in $\tilde{T}_{ij}$ such that $\tilde{U}_{ij} \subset\subset \tilde{T}_{ij}$, $\psi_{ji}\tilde{U}_{ij} = \tilde{U}_{ji}$ and

$$(2.4) \qquad \tilde{U}_{ij} \cap T_i = U_{ij} \quad \text{and} \quad \bar{\tilde{U}}_{ij} \cap T_i = \bar{U}_{ij} ,$$

where the bar denotes the closure of the set. Since $\bar{V}_i \cap \psi_{ij}\ (\bar{V}_j \cap \bar{U}_{ji})$ is a compact subset of $U_{ij}$ we can choose open sets $\tilde{W}_{ij} \subset \tilde{T}_{ij}$ such that $\tilde{W}_{ij} \subset\subset \tilde{U}_{ij}$, $\tilde{W}_{ij} = \psi_{ij}(\tilde{W}_{ji})$ and

$$\bar{V}_i \cap \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) \subset \tilde{W}_{ij} .$$

The subsets $\bar{V}_i - \tilde{W}_{ij}$ and $\psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) - \tilde{W}_{ij}$ of $T_i$ are compact and disjoint, and therefore contained in disjoint open sets $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ of $C^n$ respectively, so we have

$$(2.5) \qquad \bar{V}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij} , \qquad \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij} .$$

We now choose $\tilde{A}_i$ open in $C^n$ such that

$$(2.6) \qquad \tilde{A}_i \cap T_i = V_i , \qquad \bar{\tilde{A}}_i \cap T_i = \bar{V}_i$$

and

$$(2.7) \qquad \tilde{A}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij} \quad \textit{for all } j \in I \quad \textit{such that } T_{ij} \textit{ is nonempty.}$$

The last condition can be satisfied since there are only a finite number of $j \in I$ such that $T_{ij}$ is nonempty.

Since $\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}$ is compact and contained in $\tilde{T}_{ij}$ we have

$$\overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \subset \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}) .$$

By (2.4) and (2.6) we have

$$\begin{aligned} \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}) \cap T_j &= \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij} \cap T_i) \\ &= \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij}) \end{aligned}$$

and hence

$$(2.8) \qquad \overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \cap T_j \subset \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij}) .$$

PART 2. For any point $x \in U_i$ there exists an open set $\tilde{U}_i(x)$ in $C^n$ containing $x$ and satisfying the following five conditions:

(1) $\tilde{U}_i(x) \subset \tilde{U}_{ij}$ for every index $j$ such that $x \in U_{ij}$.

(2) $\tilde{U}_i(x) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij}$ for every index $j$ such that $x \in \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$. (Compare (2.5).)

(3) $\tilde{U}_i(x) \cap \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ is empty for every index $j$ such that $\varphi_i^{-1}(x) \notin \overline{V'_j}$.

(4) $\tilde{U}_i(x) \subset \psi_{ij}(\tilde{U}_{ji} \cap \tilde{U}_{jk}) \cap \psi_{ik}(\tilde{U}_{ki} \cap \tilde{U}_{kj})$ for every pair of indices $(j, k)$ such that $x \in U_{ij} \cap U_{ik}$ (i.e. $\varphi^{-1}(x) \in U'_i \cap U'_j \cap U'_k$).

(5) $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{U}_i(x)$ for all $(j, k)$ as in (4).

The conditions (1), (2) and (4) are satisfied because the number of indices involved is finite. Condition (5) is satisfied on $U_{ij} \cap U_{ik}$ and hence in a neighborhood. Condition (3) is trivially satisfied if $\tilde{U}_{ij}$ is empty. Otherwise there are at most finitely many $j$'s such that $\tilde{U}_{ij}$ is nonempty and $\varphi_i^{-1}(x) \notin \overline{V'_j}$ implies $x \notin \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ so $x \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (2.8).

PART 3. Set

$$\tilde{U}_i = \bigcup_{x \in U_i} \tilde{U}_i(x)$$

and let $\tilde{V}_i$ be a neighborhood of $V_i$ in $\boldsymbol{C}^n$ which is contained in $\tilde{A}_i$ and relatively compact in $\tilde{U}_i$. Note from (2.6) that $\tilde{V}_i \cap T_i = V_i$ and $\bar{\tilde{V}}_i \cap T_i = \overline{V}_i$. Setting

$$\tilde{V}_{ij} = \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji}), \qquad \tilde{V}_{ijk} = \tilde{V}_{ij} \cap \tilde{V}_{ik} \tag{2.9}$$

we see that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and $\psi_{ij}\colon \tilde{V}_{ji} \xrightarrow{\sim} \tilde{V}_{ij}$.

Let $y \in \tilde{V}_{ijk}$, so $y \in \tilde{U}_i(x)$ for some $x \in U_i$. Since $\tilde{V}_{ijk}$ intersects $\psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{V}_k \cap \tilde{U}_{ki})$ it also meets $\psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{A}_k \cap \tilde{U}_{ki})$ and hence $x \in \tilde{U}_{ij} \cap \tilde{U}_{ik}$ by (3). Therefore $\psi_{ki}(y) \in \tilde{V}_k \cap \tilde{U}_{kj}$ and $\psi_{jk} \circ \psi_{ki}(y) = \psi_{ji}(y)$ by (2), so

$$\begin{aligned} z = \psi_{ji}(y) &\in \tilde{V}_j \\ &\in \psi_{ji}(\tilde{V}_i \cap \tilde{U}_{ij}) \\ &\in \psi_{jk}(\tilde{V}_k \cap \tilde{U}_{kj}) \\ &\in \tilde{V}_{jik}\,. \end{aligned}$$

We therefore conclude that $\psi_{ji}(\tilde{V}_{ijk}) \subset \tilde{V}_{jik}$ and by symmetry $\psi_{ij}(\tilde{V}_{jik}) \subset \tilde{V}_{ijk}$, so

$$\psi_{ji}\colon \tilde{V}_{ijk} \xrightarrow{\sim} \tilde{V}_{jik}\,.$$

We also have $\psi_{ij} = \psi_{ji}^{-1}$ and $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{V}_{ijk}$ so $\{\tilde{V}_i, \psi_{ij}\}$ is an amalgamation system and we can construct the amalgamated sum

$$\{\tilde{\Omega}, \psi_i\} = \lim \{\tilde{V}_i, \psi_{ij}\} .$$

This sum is obtained from the disjoint union $\overset{\circ}{\bigcup} \tilde{V}_i$ by dividing out by the equivalence relation

$$x \in \tilde{V}_i \sim y \in \tilde{V}_j \quad \textit{if } x \in \tilde{V}_{ij},\ y \in \tilde{V}_{ji} \textit{ and } y = \psi_{ji}(x) .$$

The space $\tilde{\Omega}$ is a complex manifold if it is Hausdorff. The natural maps $\psi_i \colon \tilde{V}_i \to \tilde{\Omega}$ are given by the canonical open embeddings

$$\tilde{V}_i \to \overset{\circ}{\bigcup} \tilde{V}_i \to \tilde{\Omega} ,$$

and the isomorphisms $\varphi_i$ merge into an isomorphism $\varphi$ of $M$ onto a generically embedded, real analytic, closed submanifold of $\tilde{\Omega}$.

PART 4. The proof will be complete if we show that $\tilde{\Omega}$ has a Hausdorff topology.

We will first show that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and more precisely that $\tilde{V}_{ij} \subset \tilde{W}_{ij}$ (when $T_{ij}$ is nonempty). If $y \in \tilde{V}_{ij}$ then there exists $x \in U_i$ such that $y \in \tilde{U}_i(x)$ since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{U}_i$.

If $x \notin \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ then $\varphi_i^{-1}(x) \notin V_j'$ and hence $y \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (3). Since this contradicts the fact that $y \in \tilde{V}_{ij}$ in (2.9), we have that $x \in \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ and hence by (2) that $y \in \tilde{B}_{ij} \cup \tilde{W}_{ij}$. Since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{A}_i$ we see from (2.7) that $y \in \tilde{A}_{ij} \cup \tilde{W}_{ij}$ and hence that $y \in \tilde{W}_{ij}$ as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ are disjoint. In conclusion we have

$$\tilde{V}_{ij} \subset \tilde{W}_{ij} \subset \tilde{U}_{ij} .$$

Suppose now that $x'$ and $y'$ are two points of $\tilde{\Omega}$ with $x' \neq y'$. Let $x \in \tilde{V}_i$ and $y \in \tilde{V}_j$ such that $\psi_i(x) = x'$ and $\psi_j(y) = y'$. It suffices to show that there exist neighborhoods $A$ of $x$ in $\tilde{V}_i$ and $B$ of $y$ in $\tilde{V}_j$ such that no point of $A$ is equivalent to a point of $B$.

If this was not the case we could find sequences $\{x_k\}$ and $\{y_k\}$ in $\boldsymbol{C}^n$ converging to $x$ and $y$ respectively with $x_k \in \tilde{V}_{ij}$, $y_k \in \tilde{V}_{ji}$ and $x_k = \psi_{ij}(y_k)$ for every $k$. Since $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ we see that $x \in \tilde{U}_{ij}$ and by symmetry $y \in \tilde{U}_{ji}$, so

$x = \psi_{ij}(y)$ by continuity. Thus

$$y \in \tilde{V}_j \cap \tilde{U}_{ji} \subset \tilde{V}_{ji},$$
$$x \in \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji}) = \tilde{V}_{ij}$$

and $y = \psi_{ji}(x)$, so $x$ is equivalent to $y$ which is contrary to our assumption. This completes the proof of theorem 1.12.

## 3. – Domination of C-R structures.

Let $M$ be a real analytic, $m$-dimensional manifold and let $A$ and $B$ be two real analytic C-R structures of types $k$ and $l$ on $M$ respectively. We say that $B$ *dominates* $A$ if $A_p \subset B_p$ for every $p \in M$ (and thus $l \geqslant k$). Note that every real analytic C-R structure on $M$ dominates the totally real structure of $M$.

THEOREM 3.1. *If $B$ dominates $A$ as above and $(X, \tau)$ and $(Y, \sigma)$ are complexifications of $(M, A)$ and $(M, B)$ respectively, then there exist open neighborhoods $U$ of $\tau M$ in $X$ and $V$ of $\sigma M$ in $Y$ and a holomorphic surjective map $h\colon U \to V$ of maximal rank such that the diagram*

$$\begin{array}{ccc} & & U \subset X \\ & \overset{\tau}{\nearrow} & \\ M & & \Big\downarrow h \\ & \underset{\sigma}{\searrow} & \\ & & V \subset Y \end{array}$$

*commutes. Moreover, $h$ is uniquely defined if $U$ is sufficiently small and connected with $\tau M$.*

PROOF. Note that the unicity of $h$ and the sufficiency to prove the result locally both follow from corollary 2.3. We also have

$$\dim_{\mathbb{C}} X = m - k \geqslant m - l = \dim_{\mathbb{C}} Y.$$

PART 1. As a consequence of proposition 2.1 the result holds if $A$ is the totally real structure of $M$.

PART 2. Since $(X, \tau)$ and $(Y, \sigma)$ are complexifications of $(M, A)$ and $(M, B)$ respectively, and $B$ dominates $A$, we have

$$\sigma_* \circ \tau_*^{-1}\colon HT(\tau M, X) \to HT(\sigma M, Y). \tag{3.2}$$

Assume, from the remark at the beginning of this proof, that $M$ is an open set $D \subset \boldsymbol{R}^m$ which is sufficiently small so that $\tau D$ and $\sigma D$ are contained in local holomorphic coordinate systems $z = (z_1, \dots, z_{m-k})$ of $X$ and $w = (w_1, \dots, w_{m-l})$ of $Y$ respectively. The equations

$$\tau \equiv \{z_1 = \tau_i(t) \ \textit{for}\ i = 1, \dots, m-k\ ,$$

*and*

$$\sigma \equiv \{w_j = \sigma_j(t) \ \textit{for}\ j = 1, \dots, m-l$$

with $t \in D$ are real analytic, local parametric equations (see (1.3)) of $\tau M$ and $\sigma M$ respectively, and thus, from (3.2) and the proof of proposition 1.5, we have

$$\sum_{r=1}^{m} a_r \frac{\partial \bar{\sigma}_j}{\partial t_r}(t) = 0 \qquad \textit{for}\ j = 1, \dots, m-l$$

*whenever* $t \in D$ *and* $a \in \boldsymbol{C}^m$ *satisfying*

$$\sum_{r=1}^{m} a_r \frac{\partial \bar{\tau}_i}{\partial t_r}(t) = 0 \qquad \textit{for}\ i = 1, \dots, m-k\ .$$

That is (after conjugating),

$$d\sigma_j \wedge d\tau_1 \wedge \dots \wedge\ \tau_{m-k} = 0 \qquad \textit{on}\ D\ \textit{for}\ j = 1, \dots, m-l\ .$$

Let $\tilde{\tau}$ and $\tilde{\sigma}$ be holomorphic extensions of $\tau$ and $\sigma$ respectively, to a sufficiently small neighborhood $\tilde{D}$ of $D$ in $\boldsymbol{C}^m$ such that $\tilde{D} \cap \boldsymbol{R}^m = D$ and

$$d\tilde{\sigma}_j \wedge d\tilde{\tau}_1 \wedge \dots \wedge d\tilde{\tau}_{m-k} = 0 \qquad \textit{on}\ \tilde{D}\ \textit{for}\ j = 1, \dots, m-l\ .$$

It now follows that the map $\tilde{\sigma}$ factors through the map $\tilde{\tau}$, i.e. there exists a holomorphic function $h\colon \tilde{\tau}\tilde{D} \to \tilde{\sigma}\tilde{D}$ such that $\tilde{\sigma} = h \circ \tilde{\tau}$ on $\tilde{D}$. Since $\tilde{D}$ is a complexification of $M$ with its totally real structure, it follows from part 1 that $\tilde{\tau}$ and $\tilde{\sigma}$ are surjective and of maximal rank if $\tilde{D}$ is sufficiently small. Hence $h$ is also a surjective map of maximal rank, and the proof is complete.

## 4. – Convexity of the complexification.

By a theorem of Grauert [6] we know that there is a complexification $(X, \tau)$ of a real analytic manifold $M$ with its totally real structure in which $X$ is Stein. Since the totally real structure is the C-R structure of type zero and Stein is the same as 0-complete we are lead to

CONJECTURE 4.1. *If* $(M, A)$ *is a real analytic* C-R *manifold of type* $(m, l)$, *then there exists a complexification* $(X, \tau)$ *of* $(M, A)$ *for which* $X$ *is* $l$*-complete* (*).

The following theorem states that the above conjecture holds if $M$ is compact.

THEOREM 4.2. *If* $(M, A)$ *is a compact, real analytic* C-R *manifold of type* $(m, l)$ *and* $(X, \tau)$ *is a complexification of* $(M, A)$, *then there exists a neighborhood* $U$ *of* $\tau M$ *in* $X$ *which is an* $l$*-complete manifold.*

REMARK. Since $(U, \tau)$ is a complexification of $(M, A)$ whenever $U$ is an open neighborhood of $\tau M$ in $X$, we see that $\tau M$ has a fundamental system of neighborhoods in $X$ which are $l$-complete.

PROOF OF (4.2). Let $\{U_i, z^i\}$ be a locally finite covering of $\tau M$ by holomorphic charts of $X$ such that for each $i$ there exist (see (1.2)) real analytic functions $f_j^i \colon U_i \to \boldsymbol{R}$ for which

(a) $\tau M \cap U_i = \{z \in U_i \mid f_j^i(z, \bar{z}) = 0 \ \textit{for} \ j = 1, \ldots, m - 2l\}$, *and*

(b) $\operatorname{rk} \dfrac{\partial(f_1^i, \ldots, f_{m-2l}^i)}{\partial(z_1^i, \ldots, z_{m-l}^i)} = m - 2l \quad \textit{on} \ U_i$.

For each index $i$ define $v^i \colon U_i \to \boldsymbol{R}$ by

$$v^i = \sum_{j=1}^{m-2l} |f_j^i|^2$$

and note from (a) that

$$\partial\bar{\partial} v^i(z) = 2 \sum_{j=1}^{m-2l} |\partial f_j^i(z)|^2 \geqslant 0 \quad \textit{for each } z \in \tau M \cap U_i .$$

It now follows from (b) that $\mathfrak{L}(v^i)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in \tau M \cap U_i$.

(*) An $n$-dimensional complex manifold $X$ is *$l$-complete* if there exists a smooth function $\varphi \colon X \to \boldsymbol{R}$ such that

(1) $\{z \in X | \varphi(z) < c\}$ is relatively compact for every $c \in \boldsymbol{R}$, and

(2) at each point $z_0 \in X$ the Levi form of $\varphi$,

$$\mathfrak{L}(\varphi)_{z_0}(u) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} (z_0) u_i \bar{u}_j ,$$

has at least $n - l$ positive eigenvalues.

Let $\{\varrho_i\colon U_i \to \boldsymbol{R}\}$ be a partition of unity subordinate to the cover $\{U_i\}$ with $\varrho_i \geqslant 0$ and set

$$\theta = \sum_i \varrho_i v_i .$$

There exists a sufficiently small open neighborhood $W$ of $\tau M$ in $X$ such that $\theta$ is smoothly defined in $W$ with $\theta \geqslant 0$, $\theta(z) = 0$ if and only if $z \in \tau M$, and with $\mathfrak{L}(\theta)_z$ (the Levi form of $\theta$ at $z$) having at least $m - 2l$ positive eigenvalues for each $z \in W$.

Since $\tau M$ is a compact subset of $W$ there exists an open neighborhood $U$ of $\tau M$ in $W$ with $U$ relatively compact in $W$. Hence $\partial U = \overline{U} - U$ is a compact subset of $W$ which does not meet $\tau M$ and we set

$$\delta = \min_{z \in \partial U} \theta(z) > 0 .$$

If $V = \{z \in U | \theta(z) < \delta\}$, then $\overline{V} \subset \overline{U}$ and $g(z) = (1/\delta)\theta(z)$ has the following properties on $V$:

(1) $0 \leqslant g(z) < 1$ and $g(z) = 0$ if and only if $z \in \tau M$.

(2) $\mathfrak{L}(g)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in V$.

We now define $\varphi\colon V \to R$ by

$$\varphi(z) = \frac{1}{1 - g(z)} .$$

Thus $\{z \in V | \varphi(z) < c\}$ is relatively compact in $V$ for every $c \in \boldsymbol{R}$ and also $\mathfrak{L}(\varphi)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in V$. Since $V$ is an $(m - l)$-dimensional complex manifold we see that it is $l$-complete and (4.2) is established.

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