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Variations on a theme of Carathéodory

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Variations on a Theme of Carathéodory (*).

EDOARDO VESENTINI (**)

In 1927 N. Kritikos [11] proved that every automorphism of the bounded domain $\{(z^1, z^2) \in \boldsymbol{C}^2 : |z^1| + |z^2| < 1\}$ of $\boldsymbol{C}^2$ leaves the origin fixed. This result—which is one of the first applications of the Carathéodory distance—was reobtained in 1931 by P. Thullen [20], as a by-product of the construction of the groups of automorphisms of bounded Reinhardt domains in $\boldsymbol{C}^2$.

Theorem II of the present paper establishes the following generalization of the theorem of Kritikos to domains in complex Banach spaces. Let M be a measure space, with a positive measure μ, and let B be the open unit ball of the complex Banach space $L^1(M, \mu)$. If $\dim_{\boldsymbol{C}} L^1(M, \mu) > 1$, every (bi-holomorphic) automorphism of B leaves the origin fixed. This result is actually a consequence of an investigation on Kobayashi and Carathéodory distances on domains of locally convex topological complex vector spaces. A result in this area is the fact (Theorem I) that the Carathéodory distance from any given point on such a domain is a continuous logarithmically plurisubharmonic function. This result holds also on any (reduced) connected analytic space X, thus providing a continuous plurisubharmonic function intrinsically associated to X and to any point chosen in X.

In the remaining sections of this paper we compute the Carathéodory and Kobayashi distances on a domain in a complex Banach algebra, and we establish some spectral versions of the Schwarz lemma. This investigation is strictly interwoven with previous results [21] on the logarithmic subharmonicity of the spectral radius. An extension of these results to the *hyperbolic spectral radius*—i.e., to the spectral radius defined in terms of the hyperbolic distance on the unit disc—is also established (Proposition 5.4).

The final part of this paper concentrates on Banach algebras endowed with a hermitian involution, establishing explicit formulas for the Carathé-

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odory and Kobayashi distances in terms of the Pták norm [14, 15]. These formulas yield a generalization of one of the main results in [23] from von Neumann algebras to C^*-algebras with identity (Proposition 6.3).

1. – Preliminaries and plurisubharmonicity.

1. – Let $\Delta = \{\zeta \in \boldsymbol{C}: |\zeta| < 1\}$ be the open unit disc in $\boldsymbol{C}$. The Poincaré-Bergman differential metric $ds^2 = d\zeta\, \overline{d\zeta}/(1 - |\zeta|^2)^2$ defines on Δ a distance

$$\omega(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{1 + |(\zeta_1 - \zeta_2)/(1 - \zeta_1 \bar{\zeta}_2)|}{1 - |(\zeta_1 - \zeta_2)/(1 - \zeta_1 \bar{\zeta}_2)|} \qquad (\zeta_1, \zeta_2 \in \Delta) .$$

Let $\mathcal{E}$ and $\mathcal{E}_1$ be two complex, locally convex, Hausdorff vector spaces, and let A be a domain in $\mathcal{E}$. A holomorphic map $F: A \to \mathcal{E}_1$ is, by definition [13, p. 25], a continuous map F of A into $\mathcal{E}_1$ such that, for every choice of $(x, y) \in A \times (\mathcal{E} \setminus \{0\})$ and every continuous linear form λ_1 on $\mathcal{E}_1$, the scalar-valued function $\zeta \mapsto \lambda_1 \circ F(x + \zeta y)$ is holomorphic on the open set $\{\zeta \in \boldsymbol{C}: x + \zeta y \in A\}$ of $\boldsymbol{C}$. If A_1 is a domain in $\mathcal{E}_1$, we denote by $\operatorname{Hol}(A, A_1)$ the set of all holomorphic maps $F: A \to \mathcal{E}_1$, such that $F(A) \subset A_1$.

The Kobayashi pseudo-distance $d_A(x', x'')$ between two points x', x'' in A is defined as follows. Let $\zeta_1', \zeta_1'', \ldots, \zeta_\nu', \zeta_\nu''$ be ν pairs of points in Δ, and let $f_1, \ldots, f_\nu$ be elements of $\operatorname{Hol}(\Delta, A)$ such that $f_1(\zeta_1') = x'$, $f_j(\zeta_j'') = f_{j+1}(\zeta_{j+1}')$ for $j = 1, \ldots, \nu - 1$, $f_\nu(\zeta_\nu'') = x''$.

The Kobayashi pseudo-distance $d_A(x', x'')$ is, by definition,

$$d_A(x', x'') = \inf \sum_{j=1}^{\nu} \omega(\zeta_j', \zeta_j'') ,$$

where the infimum is taken over all possible choices of $\nu, \zeta_j', \zeta_j'', f_j$ $(j = 1, \ldots, \nu)$.

A simple application of the triangle inequality and of the Schwarz-Pick lemma implies that, for every $f \in \operatorname{Hol}(A, \Delta)$,

$$\omega\big(f(x'), f(x'')\big) \leqslant d_A(x', x'') .$$

Thus, setting

$$c_A(x', x'') = \sup \{\omega\big(f(x'), f(x'')\big) : f \in \operatorname{Hol}(A, \Delta)\},$$

we have [9]

$$c_A(x', x'') \leqslant d_A(x', x'') . \tag{1.1}$$

The function $(x', x'') \mapsto c_A(x', x'')$ is the Carathéodory pseudo-distance on A.

Let A_1 be a domain in $\mathcal{E}_1$, and consider the Kobayashi and Carathéodory pseudo-distances d_{A_1} and c_{A_1}. The above definitions imply that any $F \in \mathrm{Hol}(A, A_1)$ is distance decreasing for both the Kobayashi and Carathéodory pseudo-distances, i.e.,

$$d_{A_1}(F(x'), F(x'')) \leqslant d_A(x', x''), \qquad c_{A_1}(F(x'), F(x'')) \leqslant c_A(x', x'')$$

for all $x', x'' \in A$. In particular: 1) every bi-holomorphic diffeomorphism of A onto A_1, is an isometry for both pseudo distances; 2) if D is a domain in $\mathcal{E}$, such that $D \subset A$, then

$$d_A(x', x'') \leqslant d_D(x', x''), \qquad c_A(x', x'') \leqslant c_D(x', x'') \qquad (x', x'' \in D).$$

Furthermore, the Schwarz-Pick lemma yields [9]

$$c_\Delta = d_\Delta = \omega. \tag{1.2}$$

Let p be a continuous semi-norm on $\mathcal{E}$, and let

$$B_p = \{x \in \mathcal{E}: p(x) < 1\}.$$

LEMMA 1.1. *For every* $x \in B_p$,

$$c_{B_p}(0, x) = d_{B_p}(0, x) = \omega(0, p(x)).$$

PROOF. Let $x \in B_p$, with $p(x) > 0$. The (holomorphic) function $\zeta \mapsto (\zeta/p(x))x$ maps the unit disc into B_p, 0 into 0, and $p(x)$ into x. Thus

$$c_{B_p}(0, x) \leqslant d_{B_p}(0, x) \leqslant \omega(0, p(x)).$$

On the other hand, there exists a continuous linear form λ on $\mathcal{E}$ such that $\lambda(x) = p(x)$ and $|\lambda(y)| \leqslant p(y)$ for all $y \in \mathcal{E}$. Thus $\lambda \in \mathrm{Hol}(B_p, \Delta)$, and therefore

$$\omega(0, p(x)) \leqslant c_{B_p}(0, x).$$

Let $x \neq 0$, but $p(x) = 0$. For any $t > 1$, the holomorphic function $f_t: \zeta \mapsto t\zeta x$ maps Δ into B_p; moreover $f_t(0) = 0$, $f_t(1/t) = x$. Hence $c_{B_p}(0, x) \leqslant$ $\leqslant d_{B_p}(0, x) \leqslant \omega(0, 1/t)$.

Letting $t \to \infty$, we get $c_{B_p}(0, x) = d_{B_p}(0, x) = 0$. The proof of the lemma is complete. Q.E.D.

Let $r > 0$ and let $B_{p,r}$ and Δ_r be the open discs $B_{p,r} = \{x \in \mathcal{E}: p(x) < r\}$, $\Delta_r = \{\zeta \in \boldsymbol{C}: |\zeta| < r\}$. If $x \in \mathcal{E}$, and if $F: \boldsymbol{C} \to \mathcal{E}$ is the holomorphic map $\zeta \mapsto \zeta x$, then $F^{-1}(B_{p,r})$ is the disc Δ_R of radius $R = r/p(x)$, where we set $R = \infty$ and $\Delta_\infty = \boldsymbol{C}$ if $p(x) = 0$. In the latter case, both the Carathéodory and Kobayashi pseudo-distances on Δ_∞ vanish identically. If $0 < R < \infty$, they can be obtained from (1.2) by a homotety: they coincide, and

$$c_{\Delta_R}(0, \zeta) = d_{\Delta_R}(0, \zeta) = \omega\left(0, \frac{\zeta}{R}\right) \qquad (\zeta \in \Delta_R)\,.$$

Let $x \in B_{p,r} \setminus \{0\}$ and let D be a domain in $\mathcal{E}$ such that $F(\Delta_R) \subset D \subset B_{p,r}$.

By Lemma 1.1,

$$c_{\Delta_R}(0, F^{-1}(x)) \geqslant c_D(0, x) \geqslant c_{B_{p,r}}(0, x) = \omega\left(0, \frac{p(x)}{r}\right),$$

$$d_{\Delta_R}(0, F^{-1}(x)) \geqslant d_D(0, x) \geqslant d_{B_{p,r}}(0, x) = \omega\left(0, \frac{p(x)}{r}\right).$$

That proves

COROLLARY 1.2. *For all $x \in B_{p,r}$ and for any domain D in $\mathcal{E}$ such that $\Delta_{r/p(x)} \cdot x \subset D \subset B_{p,r}$, we have*

$$c_D(0, x) = d_D(0, x) = \omega\left(0, \frac{p(x)}{r}\right).$$

Now let $p_1, \ldots, p_n$ be continuous seminorms on $\mathcal{E}$, and let D_0 be the domain

$$D_0 = B_{p_1, r_1} \cap \ldots \cap B_{p_n, r_n}$$

for some $r_1 > 0, \ldots, r_n > 0$. Let $x \in D_0$, and suppose that

$$\frac{p_1(x)}{r_1} \geqslant \frac{p_2(x)}{r_2} \geqslant \ldots \geqslant \frac{p_n(x)}{r_n}\,.$$

The function $\zeta \mapsto \zeta x$ maps $\Delta_{r_1/p_1(x)}$ into D_0. Hence Corollary 1.2 yields

$$c_{D_0}(0, x) = d_{D_0}(0, x) = \omega\left(0, \frac{p_1(x)}{r_1}\right),$$

i.e.

$$\begin{aligned} c_{D_0}(0, x) = d_{D_0}(0, x) &= \max\left\{\omega\left(0, \frac{p_j(x)}{r_j}\right): j = 1, \ldots, r\right\} \\ &= \max\left\{c_{B_{p_j, r_j}}(0, x): j = 1, \ldots, n\right\}. \end{aligned}$$

For any $x_0 \in \mathcal{E}$, the domain

$$D_{x_0} = \{x \in \mathcal{E}: p_1(x - x_0) < r_1, \ldots, p_n(x - x_0) < r_n\} \tag{1.3}$$

is the image of D_0 by the translation defined by x_0. Thus

$$c_{D_{x_0}}(x_0, x) = d_{D_{x_0}}(x_0, x) = \max\left\{\omega\left(0, \frac{p_j(x - x_0)}{r_j}\right): j = 1, \ldots, n\right\}. \tag{1.4}$$

Since the open sets (1.3) generate a fundamental system of neighborhoods of x_0, then for any $x_0 \in A$ and any $\varepsilon > 0$, there is a neighborhood U of x_0 in A such that $d_A(x_0, x) < \varepsilon$ for all $x \in U$. Taking into account (1.4) we conclude with

PROPOSITION 1.3. *The functions* $c_A: A \times A \to \boldsymbol{R}$, $d_A: A \times A \to \boldsymbol{R}$ *are continuous.*

2. – In this section we shall show that, for any x_0 in the domain A, the Carathéodory pseudo-distance $c_A(x_0, x)$ is a logarithmically plurisubharmonic function of $x \in A$. We consider first the case $\mathcal{E} = \boldsymbol{C}$, $A = \Delta$.

LEMMA 2.1. *For any* $\zeta_0 \in \Delta$, *the function* $\zeta \mapsto \log \omega(\zeta_0, \zeta)$ *is subharmonic on* Δ.

PROOF. Since the group of holomorphic automorphisms of Δ acts transitively on Δ and isometrically on the Poincaré-Bergman distance, it suffices to prove the lemma when $\zeta_0 = 0$.

The function $\zeta \mapsto \omega(0, \zeta)$ being continuous, we need only show that, for any $a \in \boldsymbol{C}$, the function $\varphi_a: \zeta \mapsto |e^{a\zeta}|\omega(0, \zeta)$ is subharmonic on Δ [16]. Choosing a branch for $\log \zeta$ on $\Delta \setminus \{0\}$, the function $\zeta \mapsto \varphi_a(\zeta)$ is C^∞ on $\Delta \setminus \{0\}$, and

$$\frac{\partial}{\partial \zeta} |\zeta| = \frac{1}{2} (\bar{\zeta}/\zeta)^{\frac{1}{2}}, \qquad \frac{\partial}{\partial \bar{\zeta}} |\zeta| = \frac{1}{2} (\zeta/\bar{\zeta})^{\frac{1}{2}}.$$

Thus, for every $\zeta \in \Delta \setminus \{0\}$,

$$\frac{\partial \varphi_a}{\partial \zeta} = \frac{1}{2}\left(a\varphi_0(\zeta) + \frac{1}{1 - |\zeta|^2} (\bar{\zeta}/\zeta)^{\frac{1}{2}}\right)|e^{a\zeta}|,$$

$$\frac{\partial^2 \varphi_a}{\partial \zeta \partial \bar{\zeta}} = \frac{|e^{a\zeta}|}{2}\left\{\frac{|a|^2}{4} \log \frac{1 + |\zeta|}{1 - |\bar{\zeta}|} + \frac{1}{1 - |\zeta|^2} \operatorname{Re}\left(a\left(\frac{\zeta}{\bar{\zeta}}\right)^{\frac{1}{2}}\right) + \frac{1 + |\zeta|^2}{2|\zeta|(1 - |\zeta|^2)^2}\right\} \geqslant$$

$$\geqslant \frac{|e^{a\zeta}|}{2}\left\{\frac{|a|^2}{4} \log \frac{1 + |\zeta|}{1 - |\zeta|} - \frac{|a|}{1 - |\zeta|^2} + \frac{1 + |\zeta|^2}{2|\zeta|(1 - |\zeta|^2)^2}\right\}.$$

We show now that for $0<t<1$ the trinomial in ϱ

$$\varrho^2 \log\frac{1+t}{1-t} - \frac{2\varrho}{1-t^2} + \frac{1+t^2}{2t(1-t^2)^2} \tag{2.1}$$

is positive. The discriminant is equal to $\big(2/t(1-t^2)^2\big)\sigma(t)$ where

$$\sigma(t) = 2t - (1+t^2)\log\frac{1+t}{1-t}\,.$$

Since

$$\sigma'(t) = \frac{-2t}{1-t^2}\left(2t + (1-t^2)\log\frac{1+t}{1-t}\right) < 0 \qquad \text{for } 0<t<1\,,$$

the function σ is strictly decreasing for $0<t<1$. Being $\sigma(0)=0$, then $\sigma(t)<0$ for $0<t<1$, and the trinomial (2.1) is positive definite. Thus $\partial^2\varphi_a/\partial\zeta\,\partial\bar\zeta>0$ on $\Delta\setminus\{0\}$. Since $\varphi_a(\zeta)>0$ on $\Delta\setminus\{0\}$, then

$$\varphi_a(0) = 0 < \frac{1}{2\pi}\int_0^{2\pi}\varphi_a(re^{i\theta})\,d\theta \qquad \text{for any } 0<r<1\,.$$

Thus φ_a is subharmonic on Δ for all $a\in \boldsymbol{C}$. Q.E.D.

Going back to the general case, let $f\in \operatorname{Hol}(A,\Delta)$, and let $x_0\in A$. Lemma 2.1 implies that the function $x\mapsto \log\omega\big(f(x_0), f(x)\big)$ is a continuous plurisubharmonic function on A [13, theorème 1.2.12, pp. 27-28].

Since the function $x\mapsto \log c_A(x_0, x)$ is a continuous function $A\to$ $\to[-\infty, +\infty)$ (Proposition 1.2), which is by definition the upper envelope of a family of plurisubharmonic functions on A, then we have proved

THEOREM I. *For any x_0 in the domain A, the function $x\mapsto\log c_A(x_0,x)$ is a continuous plurisubharmonic function on A.*

3. – A bounded set $T\subset \boldsymbol{C}$ is a polar set if there exists a subharmonic function $\varphi\not\equiv-\infty$ such that $\varphi=-\infty$ on T. According to a theorem of H. Cartan [2], a bounded subset of $\boldsymbol{C}$ is a polar set if, and only if, its exterior capacity is zero.

Theorem I yields

PROPOSITION 2.2. *Let x_0 be any point in the domain $A\subset\mathcal{E}$, and let $f\colon\Delta\to A$ be a holomorphic map with $x_0\in f(\Delta)$. The set $\{\zeta\in\Delta : c_A\big(x_0, f(\zeta)\big)=0\}$ is either the entire disc Δ, or a polar set.*

In the latter case its exterior capacity is zero. This implies that, *for any* $x_0 \in A$, *the set* $\{x \in A : c_A(x_0, x) = 0\}$ *has no interior points, unless it is* A *itself.*

Although this paper is mainly devoted to the study of invariant metrics on domains in Banach spaces, it is worth noticing that the arguments leading to the proof of Theorem I hold, with no substantial change, in the case where A is a connected (finite dimensional, reduced) complex space. Thus $\log c_A(x_0, \cdot)$ is a continuous plurisubharmonic function on a connected complex space A, for any $x_0 \in A$.

We list a few consequences of this fact.

In [1] A. Andreotti and R. Narasimhan gave a sufficient condition for a complex space A to be a Stein space, bearing on the existence on A of a suitable plurisubharmonic function. In view of this condition and of Theorem I, the following statement holds:

If the connected complex space A *is K-complete and if, for some* $x_0 \in A$, *the sets,*

$$A_k = \{x \in A : c_A(x_0, x) < k\}$$

are relatively compact in A *for all* $k > 0$, *then* A *is a Stein space.*

In particular, if a K-complete connected complex space A is finitely compact for c_A (*i.e.* every bounded closed subset is compact) or, more in particular, if c_A is Cauchy complete, then A is a Stein space.

In [7] H. Horstmann proved that any domain A in $\boldsymbol{C}^n$, for which A_k is relatively compact in A for every $k > 0$, is holomorphically convex. This fact—which was generalized by S. Kobayashi [9] to complex spaces—coupled with K-completeness, yields the above result by a classical theorem of K. Oka. Of course there are Stein spaces A, like for instance $\boldsymbol{C}^n$, on which the Carathéodory distance degenerates completely, or for which the sets A_k are not relatively compact. However, if A is a Stein space and if the sets A_k are relatively compact for all $k > 0$, then by Theorem I and a theorem of R. Narasimhan [12], any A_k is a Stein space which is Runge in A.

2. – « Mittelpunkttreu » automorphisms.

4. – Since holomorphic maps contract the Carathéodory and Kobayashi pseudo-distances, both these pseudo-distances have a built-in Schwarz lemma. In this and the following sections we shall examine explicit forms of this lemma for Banach spaces and Banach algebras, and discuss some applications.

Let $\mathcal{E}$ be a complex Banach space, with norm $\| \ \|$, and let B be the open unit ball in $\mathcal{E}$.

Lemma 1.1 and Theorem I yield

LEMMA 4.1. *The function* $x \mapsto \log\log\big((1+\|x\|)/(1-\|x\|)\big)$ *is plurisubharmonic on* B.

Let B_1 be the open unit ball of a complex Banach space $\mathcal{E}_1$.

If $F\colon B \to B_1$ is any holomorphic map such that $F(0)=0$, then

$$c_{B_1}(0, F(x)) \leqslant c_B(0, x) \qquad \text{for all } x \in B .$$

Since the function $t \mapsto \log\big((1+t)/(1-t)\big)$ is strictly increasing on $[0,1)$, then Lemma 1.1 implies that

$$\|F(x)\| \leqslant \|x\| \qquad \text{for all } x \in B . \tag{4.1}$$

This weak form of the Schwarz lemma can also be obtained by applying the maximum principle to the subharmonic function $\zeta \mapsto \|(1/\zeta)F(\zeta x)\|$ $(\zeta \in \Delta,\ x \in B)$ (cf. [6]). A simple application of the maximum principle along the lines of the classical Schwarz lemma yields part i) of the following lemma. Before stating it, we recall the definition of a *complex extreme point.* Let K be a convex subset of $\mathcal{E}$. A point $x \in K$ is a complex extreme point of K if $y=0$ is the only vector in $\mathcal{E}$ such that the function $\zeta \mapsto x+\zeta y$ maps Δ into K.

LEMMA 4.2. i) *If equality holds in* (4.1) *at some point* $x_0 \in B\setminus\{0\}$, *then*

$$\|F(\zeta x_0)\| = \|\zeta x_0\| \qquad \textit{for all } \zeta \in \mathbf{C} \textit{ with } |\zeta| < \frac{1}{\|x_0\|} .$$

ii) *Assume that every point with norm one in* $\mathcal{E}_1$ *is a complex extreme point of the closure* $\bar{B}_1$ *of* B_1. *If equality holds in* (4.1) *at some point* $x_0 \in B\setminus\{0\}$, *then*

$$F(\zeta x_0) = \zeta F(x_0) \qquad \textit{for all } \zeta \in \mathbf{C} \textit{ with } |\zeta| < \frac{1}{\|x_0\|} .$$

To prove part, ii) consider the subharmonic function

$$\zeta \mapsto \left\| \frac{1}{\zeta\|x_0\|} F(\zeta x_0) \right\| \qquad \text{for } |\zeta| < \frac{1}{\|x_0\|} ,$$

reaching its maximum, 1, at $\zeta = 1 < 1/\|x_0\|$. Since all points of norm one are complex extreme points of $\bar{B}_1$, then the strong maximum principle [19] implies that the function $(1/\zeta\|x_0\|)F(\zeta x_0)$ is independent of ζ, i.e. there is a vector u of norm one in $\mathcal{E}_1$ such that

$$F(\zeta x_0) = \zeta\|x_0\|u \qquad \text{for all } |\zeta| < \frac{1}{\|x_0\|}.$$

Choosing $\zeta = 1$ we see that $\|x_0\|u = F(x_0)$, and that completes the proof of the lemma. Q.E.D

We shall now apply Lemma 4.2 to the study of a class of non-homogeneous bounded domains.

Let (M, Ξ, μ) be a measure space. Here M is a set, Ξ is a σ-algebra of subsets of M, and μ is a positive measure on Ξ. Let $\mathcal{E} = L^1(M, \mu)$ and let B be the open unit ball

$$B = \left\{x \in \mathcal{E}\colon \|x\| = \int_M |x|\, d\mu < 1\right\}.$$

We will prove the following

THEOREM II. *If* $\dim_{\mathbf{C}} \mathcal{E} > 1$, *every holomorphic automorphism of* B *is* (*the restriction to* B *of*) *a linear isometry of* $\mathcal{E}$.

Let H be a holomorphic automorphism of B. According to a theorem of H. Cartan [4], H is a continuous linear map—and therefore a linear isometry of $\mathcal{E}$—if (and only if) H leaves the origin fixed. Hence all we have to prove is that $H(0) = 0$.

Let $y_0 = H(0)$, and suppose that $y_0 \neq 0$. We shall show that this assumption leads to a contradiction.

Consider the measure $d\psi = y_0(m)\, d\mu(m)$ $(m \in M)$, and let

$$d\psi = h|d\psi|$$

be its polar decomposition; h is a measurable function such that $|h(m)| = 1$ for all $m \in M$. Then

$$|y_0| = \bar{h}y_0 \qquad \text{a.e.}$$

The map $x \mapsto \bar{h}x$ is a linear isometry of $\mathcal{E}$ onto $\mathcal{E}$. Thus, composing H with this isometry, we can assume that $y_0 = H(0)$ is a real positive element of $L^1(M, \mu)$. Since $\dim_{\mathbf{C}} L^1(M, \mu) > 1$, the σ-algebra Ξ contains at least two proper non-empty disjoint subsets on which μ takes finite, positive values. Hence there exists an element $K \in \Xi$, $K \neq M$, such that $\mu(M \setminus K) \in (0, +\infty]$

and that

$$\int_K y_0\, d\mu > 0 .$$

Let $\varphi\colon M \to \boldsymbol{R}$ be the measurable function defined by: $\varphi(m) = -1$ if $m \notin K$, $\varphi(m) = 1$ if $m \in K$. The map $x \mapsto \varphi x$ is a linear isometry of $\mathcal{E}$ onto $\mathcal{E}$, and the map $B \ni x \mapsto \varphi H(x)$ is a (holomorphic) automorphism of B for which

$$H(0) + \varphi H(0) = y_0 - y_0 = 0 \qquad \text{a.e. on } M \setminus K ,$$

$$\int_K \big(H(0)(m) + \varphi(m) H(0)(m)\big)\, d\mu(m) = 2 \int_K y_0(m)\, d\mu(m) > 0 .$$

Let $x_0 = \frac{1}{2}(y_0 + \varphi y_0)$. Then $x_0 = 0$ a.e. on $M \setminus K$, $x_0 \in B$ and

$$\int_K x_0\, d\mu > 0 .$$

Let Aut (B) be the group of all holomorphic automorphisms of B. W. Kaup and H. Upmeier have shown in [8] that there exists a closed complex subspace $\mathcal{F}$ of $\mathcal{E}$ such that the orbit Aut (B) (0) is Aut $(B)(0) = \mathcal{F} \cap B$. Hence there exists an automorphism $F \in \operatorname{Aut}(B)$ such that $F(0) = x_0$.

Let $x \in B$. A subset $\Gamma \subset B$, with $x \in \Gamma$ will be called a *complex geodesic curve at x in B* if there exists a holomorphic map $f\colon \Delta \to B$ such that:

1) $f(\Delta) = \Gamma$, and thus $x = f(\zeta_0)$ for some $\zeta_0 \in \Delta$;
2) $c_B(x, f(\zeta)) = \omega(\zeta_0, \zeta)$ for all $\zeta \in \Delta$.

Note that, by applying first a suitable Moebius transformation of Δ we can always choose $\zeta_0 = 0$.

A result of E. Thorp and R. Whitley enables us to determine all complex geodesic curves at 0. In fact it was shown in [19] that every vector of norm one in $\mathcal{E}$ is a complex extreme point of the closure $\bar{B}$ of B. Thus part ii) of lemma 4.2 shows that all complex geodesic curves at 0 are determined by linear maps $\boldsymbol{C} \to \mathcal{E}$. More precisely, we have

LEMMA 4.3. *For every $x \in B$, $x \neq 0$, the image of Δ by the linear map $\zeta \mapsto (\zeta/\|x\|)x$ is the unique complex geodesic curve at* 0, *containing x.*

We will now construct a family of complex geodesic curves at x_0.

LEMMA 4.4. *Let a and b be two real vectors in $\mathcal{E}$ such that*

$$(4.2) \qquad |a(m)| \leqslant b(m) \quad \text{a.e. on } M , \qquad \int_M b(m)\, d\mu(m) = 1 , \qquad \int_M a(m)\, d\mu(m) = 0$$

and let $f: \Delta \to \mathcal{E}$ *be the holomorphic function on* Δ *defined by*

$$f(\zeta) = \frac{1+\zeta^2}{2} a + \zeta b \qquad (\zeta \in \Delta)\,. \tag{4.3}$$

Then $f(\Delta) \subset B$, *and* $f(\Delta)$ *is a complex geodesic curve at* $f(\zeta)$ *in* B *for all* $\zeta \in \Delta$.

PROOF. For $\zeta = e^{i\theta}$ $(\theta \in \boldsymbol{R})$, $1 + \zeta^2 = 2\cos\theta e^{i\theta}$, and therefore

$$f(e^{i\theta}) = e^{i\theta}(\cos\theta \cdot a + b)\,.$$

Since

$$\cos\theta \cdot a + b \geqslant b - |a| \geqslant 0 \qquad \text{a.e. on } M\,,$$

then

$$\|f(e^{i\theta})\| = \int_M |\cos\theta \cdot a(m) + b(m)|\,d\mu(m) = \int_M \big(\cos\theta \cdot a(m) + b(m)\big)\,d\mu(m) = 1$$

for all $\theta \in \boldsymbol{R}$. Since $f(0) = \frac{1}{2}a$, and $\|\frac{1}{2}a\| \leqslant \frac{1}{2}\|b\| = \frac{1}{2}$, by the maximum principle $f(\zeta) \in B$ for every $\zeta \in \Delta$. Let $\gamma: B \to \Delta$ be the holomorphic map defined by $\gamma(x) = \int_M x(m)\,d\mu(m)$.

For any $\zeta \in \Delta$, $\gamma(f(\zeta)) = \zeta$. Thus, for all ζ_1, ζ_2 in Δ,

$$\omega(\zeta_1, \zeta_2) \geqslant c_B\big(f(\zeta_1), f(\zeta_2)\big) \geqslant \omega\big(\gamma\circ f(\zeta_1), \gamma\circ f(\zeta_2)\big) = \omega(\zeta_1, \zeta_2)\,. \qquad \text{Q.E.D.}$$

To obtain a complex geodesic curve of the above type at x_0 in B we determine now a and b in such a way that $x_0 = f(\zeta_0)$ for some $\zeta_0 \in \Delta$. Since $\|x_0\| = \int_M x_0(m)\,d\mu(m) = \gamma(x_0)$, we must choose $\zeta_0 = \gamma(x_0) = \|x_0\|$, so that the vectors a and b are then related by

$$a = \frac{2}{1 + \|x_0\|^2}\,(x_0 - \|x_0\| b)\,.$$

Thus we choose any real $b \in \mathcal{E}$ such that the first two conditions (4.2) are fulfilled, and these are readily seen to be equivalent to

$$\int_M b(m)\,d\mu(m) = 1\,, \qquad \frac{2}{(1 + \|x_0\|)^2}\,x_0 \leqslant b \qquad \text{a.e. on } M\,. \tag{4.4}$$

The corresponding function expressed by (4.3), which will now be denoted by f_b, is given by

$$f_b(\zeta) = \frac{1+\zeta^2}{1+\|x_0\|^2}(x_0 - \|x_0\| b) + \zeta b .$$

Composing f_b with the Moebius transformation $\zeta \mapsto (\zeta + \|x_0\|)/(1+\|x_0\|\zeta)$, we define the same complex geodesic curve by a new holomorphic function $\Delta \to B$ satisfying conditions 1) and 2) and mapping 0 into x_0. This holomorphic function is expressed in terms of the real vector

$$v = \frac{1-\|x_0\|^2}{1+\|x_0\|^2}\big(2\|x_0\|x_0 + (1-\|x_0\|^2)b\big) . \tag{4.5}$$

In fact, let $g_v\colon \Delta \to B$ be the function

$$g_v(\zeta) = f_b\left(\frac{\zeta + \|x_0\|}{1+\|x_0\|\zeta}\right) \qquad (\zeta \in \Delta) .$$

Then g_v satisfies conditions 1) and 2), is such that $g_v(0) = x_0$, and has the power series expansion

$$g_v(\zeta) = x_0 + \zeta\Big\{v + \sum_{n=0}^{+\infty}(-1)^n\|x_0\|^n\zeta^{n+1}\big((n+1)(1-\|x_0\|^2)x_0 - (n+2)\|x_0\|v\big)\Big\}, \qquad (\zeta \in \Delta) . \tag{4.6}$$

Let V be the convex set

$$V = \Big\{v \in \mathcal{E}\colon v \text{ real}, \ \int_M v(m)\,d\mu(m) = 1 - \|x_0\|^2, \ \ v(m) \geqslant 2(1-\|x_0\|)x_0(m) \text{ a.e. on } M\Big\}. \tag{4.7}$$

Lemma 4.4 can be rephrased in terms of v as follows:

LEMMA 4.5. *For every $v \in V$ the holomorphic map $g_v\colon \Delta \to B$ defines a complex geodesic curve at $g_v(\zeta)$ in B, for every $\zeta \in \Delta$. Moreover, $g_v(0) = x_0$.*

In order to describe another family of complex geodesic curves at x_0 in B, we shall consider the measure space $(\tilde{M}, \tilde{\Xi}, \tilde{\mu})$, where: $\tilde{M} = M \setminus K$, $\tilde{\Xi}$ is the σ-algebra consisting of the intersections $S \cap \tilde{M}$ $(S \in \Xi)$ and $\tilde{\mu}$ is the restriction of μ to $\tilde{\Xi}$. Let $\tilde{\mathcal{E}} = L^1(\tilde{M}, \tilde{\mu})$, and let $\tilde{B}$ be the open unit ball in $\tilde{\mathcal{E}}$. Denoting by λ the continuous linear form on $\mathcal{E}$,

$$\lambda\colon x \mapsto \int_K x(m)\,d\mu(m) ,$$

let $\alpha\colon B \to \tilde{\mathcal{E}}$ be the holomorphic map defined by

$$\alpha x(\tilde{m}) = \frac{1}{1-\lambda(x)}\, x(\tilde{m}) \qquad (x \in B\,,\ \tilde{m} \in \tilde{M})\,.$$

Denoting by $\|\ \|_{\tilde{\mathcal{E}}}$ the norm in $\tilde{\mathcal{E}}$, for any $x \in B$,

$$\|\alpha x\|_{\tilde{\mathcal{E}}} \leqslant \frac{1}{1-\int\limits_K |x(m)|\, d\mu(m)} \left(\int\limits_M |x(m)|\, d\mu(m) - \int\limits_K |x(m)|\, d\mu(m) \right) < 1\,,$$

i.e. $\alpha(B) \subset \tilde{B}$.

Let $\beta\colon \mathcal{E} \to \tilde{\mathcal{E}}$ be the map defined by:

$$\beta\tilde{x}(m) = (1 - \|x_0\|)\tilde{x}(m) = (1 - \lambda(x_0))\tilde{x}(m) \quad \text{ if } m \in M \setminus K\,,$$
$$\beta\tilde{x}(m) = x_0(m) \qquad \text{ if } m \in K\,.$$

First of all, for any $\tilde{x} \in \tilde{B}$,

$$\begin{aligned}\|\beta\tilde{x}\| &= \int\limits_{M\setminus K} |\beta\tilde{x}(m)|\, d\mu(m) + \int\limits_K |\beta\tilde{x}(m)|\, d\mu(m)\\ &= (1 - \lambda(x_0))\|\tilde{x}\|_{\tilde{\mathcal{E}}} + \lambda(x_0) < 1\end{aligned}$$

i.e.

$$\beta\tilde{B} \subset B\,.$$

Next we prove that β is a holomorphic map. That amounts to showing [6, 13] that for every $\tilde{x} \in \tilde{\mathcal{E}}$, $\tilde{y} \in \tilde{\mathcal{E}} \setminus \{0\}$, $z \in L^\infty(M, \Xi, \mu)$, the scalar valued function on $\boldsymbol{C}$

$$\varphi\colon \zeta \mapsto \int\limits_M \beta(\tilde{x} + \zeta\tilde{y})(m) z(m)\, d\mu(m)$$

is holomorphic. We will prove this fact by applying Morera's theorem, i.e. by showing that, for any closed rectifiable curve l in $\boldsymbol{C}$,

$$\int\limits_l \varphi(\zeta)\, d\zeta = 0\,.$$

Indeed, by Fubini's theorem,

$$\begin{aligned}\int\limits_l \varphi(\zeta)\, d\zeta &= \int\limits_l \Big\{ (1 - \lambda(x_0)) \int\limits_{M\setminus K} (\tilde{x} + \zeta\tilde{y})(m) z(m)\, d\mu(m) + \int\limits_K x_0(m) z(m)\, d\mu(m) \Big\} d\zeta\\ &= (1 - \lambda(x_0)) \int\limits_{M\setminus K} \Big(\int\limits_l (\tilde{x} + \zeta\tilde{y})(m) z(m)\, d\zeta \Big) d\mu(m) + \int\limits_K x_0(m) z(m)\, d\mu(m) \int\limits_l d\zeta\\ &= (1 - \lambda(x_0)) \cdot \int\limits_{M\setminus K} \tilde{y}(m) z(m)\, d\mu(m) \cdot \int\limits_l \zeta\, d\zeta = 0\,.\end{aligned}$$

Thus β is holomorphic. Finally for all $\tilde{x} \in \tilde{B}$, $\tilde{m} \in M$, we have

$$((\alpha \circ \beta)\tilde{x})(\tilde{m}) = \frac{1}{1-\lambda(\beta\tilde{x})}(\beta\tilde{x})(\tilde{m}) = \frac{1-\lambda(x_0)}{1-\lambda(x_0)}\tilde{x}(\tilde{m}) = \tilde{x}(\tilde{m}),$$

i.e.

$$\alpha \circ \beta = \text{identity on } \tilde{B}.$$

Consider now the Carathéodory pseudo-distances c_B and $c_{\tilde{B}}$. For $\tilde{x}_1$, $\tilde{x}_2$ in $\tilde{B}$ we have

$$c_{\tilde{B}}(\tilde{x}_1, \tilde{x}_2) \geqslant c_B(\beta\tilde{x}_1, \beta\tilde{x}_2) \geqslant c_{\tilde{B}}((\alpha\circ\beta)\tilde{x}_1, (\alpha\circ\beta)\tilde{x}_2) = c_{\tilde{B}}(\tilde{x}_1, \tilde{x}_2);$$

hence,

$$c_{\tilde{B}}(\tilde{x}_1, \tilde{x}_2) = c_B(\beta\tilde{x}_1, \beta\tilde{x}_2) \quad \text{for all } \tilde{x}_1, \tilde{x}_2 \in \tilde{B}. \tag{4.8}$$

LEMMA 4.6. *Let $w \in \mathcal{E}$ be such that $w \neq 0$, but $w = 0$ a.e. on K. Then the holomorphic map of Δ into B,*

$$\zeta \mapsto x_0 + \frac{1-\|x_0\|}{\|w\|}\zeta w$$

defines a complex geodesic curve at x_0 in B.

PROOF. If $\tilde{w}$ is the restriction of w to $\tilde{M}$, for every $\zeta \in \Delta$,

$$x_0 + \frac{1-\|x_0\|}{\|w\|}\zeta w = \beta\left(\frac{\zeta}{\|w\|}\tilde{w}\right).$$

The lemma follows then from (4.8) and from Lemma 4.2. Q.E.D.

So far we have constructed two special families of complex geodesic curves at x_0 in B. On the other hand, the existence of the holomorphic automorphism F, mapping 0 into x_0, coupled with Lemma 4.3, yields a complete description of *all* the complex geodesic curves at x_0 in B. In fact, denoting by $dF(0)$ the differential of F at 0, the following statement is a consequence of Lemma 4.3.

LEMMA 4.7. *Let $y \in \mathcal{E}\setminus\{0\}$, and let*

$$u_y = \frac{1}{\|dF(0)^{-1}y\|}dF(0)^{-1}y.$$

For any $\theta \in \boldsymbol{R}$, *the holomorphic map* $h_\theta \colon \Delta \to B$ *expressed by*

$$h_\theta(\zeta) = F(\zeta e^{i\theta} u_y) ,$$

defines a complex geodesic curve at x_0 *in* B. *Moreover, if* $h \colon \Delta \to B$ *is a holomorphic map such that*

$$h(0) = x_0 ;$$

$$h(\Delta) \quad \text{is a complex geodesic curve at } x_0 \text{ in } B ;$$

$$dh(0) = cy \quad \text{for some } 0 \neq c \in \boldsymbol{C} ,$$

then $h = h_\theta$ *for a suitable* $\theta \in \boldsymbol{R}$.

We come now to the proof of Theorem II (*). Among the vectors b satisfying (4.4) we choose two real vectors b' and b'' such that

$$b'(m) = \frac{3x_0(m)}{(1 + \|x_0\|)^2} \quad \text{for } m \in K ,\ b' \geqslant 0 \text{ on } M \setminus K , \quad \int_M b'(m)\, d\mu(m) = 1 ,$$

$$b''(m) = \frac{4x_0(m)}{(1 + \|x_0\|)^2} \quad \text{for } m \in K ,\ b'' \geqslant 0 \text{ on } M \setminus K , \quad \int_M b''(m)\, d\mu(m) = 1 .$$

The vectors

$$v' = \frac{1 - \|x_0\|^2}{1 + \|x_0\|^2} \left(2\|x_0\| x_0 + (1 - \|x_0\|^2) b'\right) ,$$

$$v'' = \frac{1 - \|x_0\|^2}{1 + \|x_0\|^2} \left(2\|x_0\| x_0 + (1 - \|x_0\|^2) b''\right) ,$$

belong to the convex set V defined by (4.7). Since for $m \in K$

$$v'(m) = \frac{1 - \|x_0\|}{1 + \|x_0\|^2} \left(3 - \|x_0\| + 2\|x_0\|^2\right) x_0(m) ,$$

$$v''(m) = 2\,\frac{1 - \|x_0\|}{1 + \|x_0\|^2} \left(2 - \|x_0\| + \|x_0\|^2\right) x_0(m) ,$$

then, for $0 \leqslant t \leqslant 1$,

$$\lambda(tv' + (1-t)v'') = \int_K (tv' + (1-t)v'')(m)\, d\mu(m) =$$

$$= \frac{1 - \|x_0\|}{1 + \|x_0\|^2} \left(4 - t - (2-t)\|x_0\| + 2\|x_0\|^2\right) \|x_0\| .$$

(*) Cf. the Note added in proof at the end of this paper.

Let τ be the continuous linear form on $\mathcal{E}$

$$\tau(x) = \int\limits_{M\setminus K} x(m)\,d\mu(m) = \int\limits_{M} x(m)\,d\mu(m) - \lambda(x)\,.$$

Since $tv' + (1-t)v'' \in V$ for $0 \leqslant t \leqslant 1$, then

$$\tau\big(tv' + (1-t)v''\big) = 1 - \|x_0\|^2 - \frac{1-\|x_0\|}{1+\|x_0\|^2}\big(4 - t - (2-t)\|x_0\| + 2\|x_0\|^2\big)\|x_0\| =$$
$$= \frac{1-\|x_0\|}{1+\|x_0\|^2}\big(1 + (t-3)\|x_0\| + (3-t)\|x_0\|^2 - \|x_0\|^3\big)\,.$$

Let $\mathcal{K}$ be the two dimensional complex subspace of $\mathcal{E}$ spanned by v' and v''. Since the restrictions of v' and v'' to K are linearly dependent, while v' and v'' are not, $\mathcal{K}$ contains a vector $w \neq 0$ such that $w = 0$ a.e. on K.

Consider the holomorphic map $B \cap \big(dF(0)^{-1}\mathcal{K}\big) \to \boldsymbol{C}^2$ defined by

$$x \mapsto \big(\lambda\circ F(x),\, \tau\circ F(x)\big) \qquad \big(x \in B \cap \big(dF(0)^{-1}\mathcal{K}\big)\big)\,.$$

For $u \in dF(0)^{-1}\mathcal{K}$, $\|u\| = 1$, consider the power series expansions

$$\lambda\circ F(\zeta u) = \|x_0\| + p_1(u)\zeta + p_2(u)\zeta^2 + \ldots\,,$$
$$\tau\circ F(\zeta u) = \qquad q_1(u)\zeta + q_2(u)\zeta^2 + \ldots\,,$$

where p_ν and q_ν are homogeneous polynomials of degree $\nu = 1, 2, \ldots$, on the two dimensional complex space $dF(0)^{-1}\mathcal{K}$ and $\zeta \in \Delta$. Taking $dF(0)u = v'$ or v'' and comparing with (4.6) we see that $p_1 \neq 0$. Let $u_0 = 1/\|dF(0)^{-1}w\| \cdot$ $\cdot dF(0)^{-1}w$. By Lemma 4.6 and 4.7,

$$p_\nu(u_0) = 0 \quad \text{for } \nu \geqslant 1\,, \qquad q_\nu(u_0) = 0 \quad \text{for } \nu \geqslant 2\,.$$

Hence there exist homogeneous polynomials r_ν and s_ν of degree $\nu = 1, 2, \ldots$, on $dF(0)^{-1}\mathcal{K}$, such that

$$p_\nu = p_1 r_{\nu-1}\,, \qquad q_\nu = p_1 s_{\nu-1} \qquad \text{for } \nu = 2, \ldots\,.$$

Choose now any $v \in \mathcal{K} \cap V$ and let $u = dF(0)^{-1}v$. Then for $\zeta \in \Delta$,

$$\lambda\circ F(\zeta u) = \|x_0\| + \zeta\lambda(v) + \sum_{n=0}^{+\infty}(-1)^n\zeta^{n+2}\|x_0\|^n\big((n+1)(1-\|x_0\|^2)\|x_0\| -$$
$$- (n+2)\|x_0\|\lambda(v)\big)\,,$$

$$\tau\circ F(\zeta u) = \qquad \zeta\tau(v) + \sum_{n=0}^{+\infty}(-1)^{n+1}\|x_0\|^{n+1}\zeta^{n+2}(n+2)\tau(v)\,.$$

Thus we must have $p_1(u) = \lambda(v)$, $q_1(u) = \tau(v)$, $q_2(u) = -2\|x_0\|\tau(v)$, and therefore

$$\lambda(v)s_1(u) = -2\|x_0\|\tau(v)\,.$$

Thus $\tau(v)/\lambda(v)$ should depend linearly on t, for $v = tv' + (1-t)v''$ $(0 \leqslant t \leqslant 1)$, i.e. we should have

$$\frac{1 + (t-3)\|x_0\| + (3-t)\|x_0\|^2 - \|x_0\|^3}{4 - t + (t-2)\|x_0\| + 2\|x_0\|^2} =$$
$$= t\,\frac{1 - 2\|x_0\| + 2\|x_0\|^2 - \|x_0\|^3}{3 - \|x_0\| + 2\|x_0\|^2} + (1-t)\,\frac{1 - 3\|x_0 + 3\|x_0\|^2 - \|x_0\|^3}{4 - 2\|x_0\| + 2\|x_0\|^2}\,.$$

But this is absurd, and this contradiction proves the theorem.

EXAMPLES. 1) Let G be a locally compact topological group (containing more than one element), let μ be a left-invariant Haar measure on G, and let B be the open unit ball of $L^1(G, \mu)$. By Theorem II, every holomorphic automorphism F of B is a linear isometry. A theorem of J. G. Wendel [25] supplies a complete description of the isometric isomorphisms of $L^1(G, \mu)$. According to this theorem, for every isometric isomorphism F of $L^1(G, \mu)$ onto itself, there exists a complex constant β with $|\beta| = 1$, a bi-continuous automorphism γ of G and a continuous character χ of G such that,

$$F(x)(\gamma g) = \beta\chi(g)x(g)$$

for all $g \in G$ and all $x \in L^1(G, \mu)$.

2) Suppose that M consists of two points, m_1, m_2, and let $\mu(m_1) = \mu(m_2) = 1$. Then $\mathcal{E} = L^1(M, \mu)$ can be identified with $\boldsymbol{C}^2$, and the unit ball of $\mathcal{E}$ is

$$B = \{(\zeta^1, \zeta^2) \in \boldsymbol{C}^2 : |\zeta^1| + |\zeta^2| < 1\}\,.$$

In this case Theorem II was proved by N. Kritikos in [11], as one oft he first applications of the notion of Carathéodory's distance (*). His proof —which inspired ours—consisted in examining the Carathéodory metric

(*) A different proof was given by Kritikos in [10], without appealing to the Carathéodory distance, but under the additional hypothesis that any automorphism of B could be extended to a holomorphic map of a neighborhood of $\overline{B}$ into $\boldsymbol{C}^2$. The proof consisted then in examining the behavior of this extension on the boundary of B.

Recent results by W. Kaup and H. Upmeier [8] show that every automorphism of B can be so extended, so that the additional hypothesis turns out to be automatically satisfied.

in neighborhoods of different points. However, the lack of a strong maximum principle prevented Kritikos from proving the uniqueness part of Lemma 4.3. Instead, the burden of the proof lay in a complicated analysis of the 2×2 matrix representing $dF(0)$. This result of Kritikos was re-obtained and generalized by P. Thullen in his classical article [20], in which he gives a complete classification of bounded Reinhardt domains in $\boldsymbol{C}^2$, containing the origin, in terms of their group of automorphisms. (For higher dimensional generalizations of some of Thullen's results and for the relevant bibliographical references cf. [18].)

3. – Spectral versions of the Schwarz lemma.

5. We shall now discuss some spectral versions of the classical Schwarz lemma. Let $\mathcal{A}$ and $\mathcal{A}'$ be complex Banach algebras; let ϱ and ϱ' be their spectral radii, and let

$$C = \{x \in \mathcal{A}: \varrho(x) < 1\}, \qquad C' = \{x' \in \mathcal{A}': \varrho'(x') < 1\}.$$

By the upper semi-continuity of the spectrum [17, p. 37], C and C' are open in $\mathcal{A}$ and $\mathcal{A}'$.

For every $x \in \mathcal{A}$ (or in $\mathcal{A}'$) we denote by $\operatorname{Sp} x$ the spectrum of x, and by $P(x)$ the peripheral spectrum of x: $P(x) = \{\zeta \in \operatorname{Sp} x: |\zeta| = \varrho(x)\}$.

PROPOSITION 5.1. *Let $f: C \to \mathcal{A}'$ be a holomorphic map such that $f(C) \subset \overline{C'}$ (the closure of C') and $f(0) = 0$. Then*

$$\varrho'(f(x)) \leqslant \varrho(x) \quad \textit{for all } x \in C. \tag{5.1}$$

If equality holds at some point $x \in C$, $x \neq 0$, then

$$\varrho'(f(\zeta x)) = \varrho(\zeta x) \quad \textit{for all } \zeta \in \boldsymbol{C} \textit{ with } |\zeta| < \frac{1}{\varrho(x)}; \tag{5.2}$$

moreover the peripheral spectrum $P(f(\zeta x))$ of $f(\zeta x)$ is

$$P(f(\zeta x)) = |\zeta| P(f(x)) \quad \textit{for all } \zeta \in \boldsymbol{C} \textit{ with } |\zeta| < \frac{1}{\varrho(x)}. \tag{5.3}$$

PROOF. Let $y \in \mathcal{A}$ with $0 < \varrho(y) < 1$. The function $\varphi_y: \zeta \mapsto (1/\zeta\varrho(y)) \cdot f(\zeta y)$ is a holomorphic map of the disc $\Delta_{1/\varrho(y)}$ of radius $1/\varrho(y)$ in $\boldsymbol{C}$ into $\mathcal{A}'$. Thus, by Theorem 1 of [21], the function $\varrho' \circ \varphi_y: \zeta \mapsto \varrho'(\varphi_y(\zeta))$ is subharmonic on $\Delta_{1/\varrho(y)}$. Choosing $0 < r < 1/\varrho(y)$, for $|\zeta| = r$, we have

$$\varrho'(\varphi_y(\zeta)) \leqslant \frac{1}{r\varrho(y)}.$$

By the maximum principle this inequality holds for $|\zeta| \leqslant r$. Letting $r \nearrow 1/\varrho(y)$, we obtain

$$(5.4) \qquad \varrho'(\varphi_y(\zeta)) \leqslant 1 \qquad \text{for all } |\zeta| < \frac{1}{\varrho(y)}.$$

Let $x \in C$. If $\varrho(x) > 0$, we choose a real $t > 1$ such that, for $y = tx$, $\varrho(y) = t\varrho(x) < 1$. Being $1/t < 1 < 1/\varrho(y)$, for $\zeta = 1/t$ (5.4) yields (5.1). If $\varrho(x) = 0$, then $\zeta x \in C$ for all $\zeta \in \boldsymbol{C}$. The subharmonic function $\zeta \mapsto \varrho'(f(\zeta x))$ is bounded by 1 on $\boldsymbol{C}$, and therefore [22, Corollary 2.14] is constant. Being $\varrho'(f(0)) = 0$, then $\varrho'(f(\zeta x)) = 0$ for all $\zeta \in \boldsymbol{C}$. This completes the proof of (5.1).

Suppose that equality holds in (5.1) at some $x \in C$ with $\varrho(x) > 0$. Choosing as above a real $t > 1$ such that $y = tx \in C$, the function $\varrho' \circ \varphi_y$ attains its maximum, 1, at the point $1/t \in \Delta_{1/\varrho(y)}$. By the maximum principle, equality holds in (5.4) on $\Delta_{1/\varrho(y)}$. That proves (5.2).

According to [21, Proposition 2], if a holomorphic map φ of a domain $D \subset \boldsymbol{C}$ into $\mathcal{A}'$ is such that $\varrho' \circ \varphi$ is constant on D, then the peripheral spectrum $P(\varphi(\zeta))$ is independent of $\zeta \in D$. Hence, if equality holds in (5.1) at some point $x \in C$, $x \neq 0$, there exists a non-empty, compact subset K of the unit circle, such that

$$P(\varphi_y(\zeta)) = K \qquad \text{for all } \zeta \in \boldsymbol{C}, \text{ with } |\zeta| < \frac{1}{\varrho(y)},$$

where $y = tx$ and $t > 1$, are chosen as above. Hence

$$P(f(\zeta y)) = \varrho(\zeta y)K \qquad \left(|\zeta| < \frac{1}{\varrho(y)}\right).$$

For $\zeta = 1/t$, $P(f(x)) = \varrho(x)K$, and (5.3) follows. Q.E.D.

For any $x \in C$, let

$$\tau(x) = \sup\{\omega(0, \zeta) : \zeta \in \operatorname{Sp} x\}.$$

We call $\tau(x)$ the *hyperbolic spectral radius* of x. Since the geodesic line, for the Poincaré-Bergman metric, from 0 to $\zeta \in \Delta$ is the line-segment joining these two points, whose hyperbolic length is

$$\omega(0, \zeta) = \frac{1}{2} \log \frac{1 + |\zeta|}{1 - |\zeta|},$$

then

$$\tau(x) = \frac{1}{2}\log\frac{1+\varrho(x)}{1-\varrho(x)} = \omega\big(0, \varrho(x)\big) \qquad (x \in C)\,. \tag{5.5}$$

The function $t \mapsto \log\big((1+t)/(1-t)\big)$ being strictly increasing on $[0, 1)$ then we obtain from Proposition 5.1, the following statement, where τ' denotes the hyperbolic spectral radius on C'.

PROPOSITION 5.2. *If f is as in Proposition 5.1, then*

$$\tau'\big(f(x)\big) \leqslant \tau(x) \qquad \textit{for all } x \in C\,.$$

If equality holds at some $x \in C$, $x \neq 0$, then

$$\tau'\big(f(\zeta x)\big) = \tau(\zeta x) \qquad \textit{for all } \zeta \in \mathbf{C} \textit{ for which } |\zeta| < \frac{1}{\varrho(x)};$$

moreover (5.2) *and* (5.3) *hold.*

Since C and C' are not necessarily homogeneous, condition $f(0) = 0$ cannot be relased, in general. However, if $\mathcal{A}' = \mathbf{C}$, a similar argument to the classical proof of the Schwarz-Pick lemma implies the following

LEMMA 5.3. *Let $f\colon C \to \Delta$ be a holomorphic map. Then*

$$\omega\big(f(x), f(0)\big) \leqslant \tau(x)$$

for all $x \in C$. If equality holds at some $x \in C$, $x \neq 0$, then

$$\omega\big(f(\zeta x), f(0)\big) = \tau(\zeta x) \qquad \textit{for all } \zeta \in \mathbf{C} \textit{ for which } |\zeta| < \frac{1}{\varrho(x)}\,.$$

Let d_C and c_C be the Kobayashi and Carathéodory pseudodistances on C. For $x \in C$, with $\varrho(x) > 0$, consider the map $f\colon \Delta \to C$ defined by $f(\zeta) = \big(\zeta/\varrho(x)\big)x$. Since $f(0) = 0$, $f\big(\varrho(x)\big) = x$, then, by (5.5),

$$d_C(0, x) \leqslant \omega\big(0, \varrho(x)\big) = \tau(x)\,. \tag{5.6}$$

If $x \neq 0$, but $\varrho(x) = 0$, then for every $a \in \Delta\setminus\{0\}$ the function $f\colon \zeta \mapsto (\zeta/a)x$ maps Δ into C; moreover $f(0) = 0$, and $f(a) = x$. Hence

$$d_C(0, x) \leqslant \omega(0, a)$$

and letting $a \to 0$, we obtain $d_C(0, x)$. Thus (5.6) holds for every $x \in C$, and therefore

$$c_C(0, x) \leqslant d_C(0, x) \leqslant \tau(x) \quad \text{for all } x \in C . \tag{5.7}$$

Thus, if $\mathcal{A}$ contains non-trivial topologically nilpotent elements, both d_C and c_C are (pseudo-distances but) not distances on C.

Since the function $x \mapsto \varrho(x)$ is not always continuous on C (cf. e.g. [17, pp. 282-283]), while d_C is continuous, then (5.7) is not always an equality. However, this is the case if $\mathcal{A}$ is commutative.

LEMMA 5.3. *If $\mathcal{A}$ is a commutative Banach algebra, then*

$$c_C(0, x) = d_C(0, x) = \tau(x) \quad \textit{for every } x \in C .$$

PROOF. Since $\mathcal{A}$ is commutative, ϱ is a continuous semi-norm on $\mathcal{A}$. By the Hahn-Banach theorem, for any $x \in C$ there is a continuous linear form λ on $\mathcal{A}$ such that

$$\lambda(x) = \varrho(x) , \qquad |\lambda(y)| \leqslant \varrho(y) \qquad \text{for all } y \in \mathcal{A} .$$

Hence λ is a holomorphic map of C into Δ, and therefore $\tau(x) = \omega(0, \varrho(x)) = \omega(0, \lambda(x)) \leqslant c_C(0, x)$. Comparison with (5.7) yields the conclusion. Q.E.D.

Let D be a domain in $\boldsymbol{C}$, and let f be a holomorphic mapping of D into C. By Theorem I and Lemma 5.3, if $\mathcal{A}$ is commutative the function $\log\circ\tau\circ f$ is subharmonic on D. We will now prove this fact for every Banach algebra $\mathcal{A}$, thereby extending to the hyperbolic spectral radius Theorem 1′ of [21].

PROPOSITION 5.4. *The function $\zeta \mapsto \log \tau(f(\zeta))$ is subharmonic on D.*

PROOF: Since $\varrho\circ f$ is upper semi-continuous on D, we need only show that, for every $a \in \boldsymbol{C}$, the function

$$\varphi_a \colon \zeta \mapsto |e^{a\zeta}|\tau(f(\zeta)) \qquad (\zeta \in D)$$

is subharmonic on D [16]. Since $\tau\circ f$ is upper semi-continuous, φ_a is upper semi-continuous too. Moreover, by (5.5) φ_a has a power series expansion, converging at every $\zeta \in D$,

$$\varphi_a(\zeta) = |e^{a\zeta}|\sum_{n=0}^{+\infty}\frac{\big(\varrho(f(\zeta))\big)^{2n+1}}{2n+1} = \sum_{n=0}^{+\infty}\frac{1}{2n+1}\big(\varrho\big(e^{a\zeta/(2n+1)}\cdot f(\zeta)\big)\big)^{2n+1} .$$

Since $\zeta \mapsto e^{a\zeta/(2n+1)}\cdot f(\zeta)$ is a holomorphic map of D into $\mathcal{A}$, then $\varrho\big(e^{a\zeta/(2n+1)}f(\zeta)\big)$ is a subharmonic function of $\zeta \in D$ for $n = 1, 2, \ldots,$ [21], and therefore also the function

$$\zeta \mapsto \left(\varrho\left(e^{a\zeta/(2n+1)}f(\zeta)\right)\right)^{2n+1}$$

is subharmonic on D. Hence φ_a is the pointwise limit of an increasing sequence of subharmonic functions. Since φ_a is upper semi-continuous and $\varphi_a(\zeta) < +\infty$ at every $\zeta \in D$, then φ_a is subharmonic. Q.E.D.

6. – Let $\mathcal{A}$ be a complex Banach algebra with an identity e, endowed with an involution $*$. Let $\mathcal{H}(\mathcal{A})$ be the real linear subvariety consisting of all hermitian elements of $\mathcal{A}$. We shall assume throughout the following that the involution is hermitian (*i.e.* that the spectrum of any hermitian element belongs to $\boldsymbol{R}$). No further hypothesis will be made on the involution. In particular we will not require $*$ to be continuous, or equivalently, we will not require $\mathcal{H}(\mathcal{A})$ to be closed in $\mathcal{A}$.

Let $p\colon \mathcal{A} \to \boldsymbol{R}_+$ be the function defined by

$$p(x) = \varrho(x^* x)^{\frac{1}{2}}.$$

We collect now a few known facts, that will be useful in the following.

I) p is a seminorm on $\mathcal{A}$ which is submultiplicative, i.e. $p(xy) \leqslant p(x)p(y)$ for all $x, y \in \mathcal{A}$ [14; 15; 5];

II) $\varrho(x) \leqslant p(x)$ for all $x \in \mathcal{A}$ [14; 15; 5];

III) p is continuous, i.e. there is a constant $k > 0$ such that $p(x) \leqslant k\|x\|$ for all $x \in \mathcal{A}$ [15, (8.2), p. 32].

Let Ω_0 be the set of *positive* elements of $\mathcal{H}(\mathcal{A})$, that is

$$\Omega_0 = \{x \in \mathcal{H}(\mathcal{A})\colon \operatorname{Sp} x \in \boldsymbol{R}_+\}.$$

IV) If $x_1, x_2 \in \Omega_0$, then $x_1 + x_2 \in \Omega_0$ [15, (5.6), p. 24].

By IV), Ω_0 is a convex cone in $\mathcal{H}(\mathcal{A})$. Let Ω be the interior part of Ω_0 for the topology in $\mathcal{H}(\mathcal{A})$. If $x \in \Omega_0$ and if $0 \in \operatorname{Sp} x$, then $x - (1/\nu)e \notin \Omega_0$ for $\nu = 1, 2, \ldots$. Since $x - (1/\nu)e$ tends to x as $\nu \to +\infty$, then $x \notin \Omega$. Conversely, if $\operatorname{Sp} x \subset \boldsymbol{R}_+^* = \{t \in \boldsymbol{R}\colon t > 0\}$, then, by the upper semi-continuity of the function $x \mapsto \operatorname{Sp} x$ [17, p. 35], there is a neighborhood of x in $\mathcal{H}(\mathcal{A})$ all of whose points have their spectra in $\boldsymbol{R}_+^*$. In conclusion

$$\Omega = \{x \in \mathcal{A}\colon \operatorname{Sp} x \subset \boldsymbol{R}_+^*\}.$$

V) If $x \in \Omega_0$ (Ω), there is an element $v \in \Omega_0$ (Ω) such that v commutes with x, and $v^2 = x$ ([3], [15, (1.5), p. 7]). If $x \in \Omega$, then v is invertible and therefore $v \in \Omega$. We shall call such a v a square root of x, and we shall denote it by $x^{\frac{1}{2}}$.

Every $z \in \mathcal{A}$ can be written in a unique way as

$$z = x + iy ,$$

where $x = \frac{1}{2}(x + x^*)$, $y = (1/2i)(x - x^*)$ both belong to $\mathcal{H}(\mathcal{A})$. Let

$$D(\Omega) = \left\{z \in \mathcal{A}: \frac{1}{2i}(z - z^*) \in \Omega\right\} = \{z = x + iy: x \in \mathcal{H}(\mathcal{A}), y \in \Omega\} .$$

Since Ω is convex, $D(\Omega)$ is convex too, hence connected. We shall prove that $D(\Omega)$ is an open homogeneous domain, biholomorphically equivalent to the open unit ball B_p:

$$B_p = \{w \in \mathcal{A}: p(w) < 1\} .$$

For any $w \in B_p$, $\varrho(w) \leqslant p(w) < 1$, hence $1 \notin \operatorname{Sp} w$. Let $U_1 = \{w \in \mathcal{A}: 1 \notin \operatorname{Sp} w\}$. Then $B_p \subset U_1$. By the upper-semicontinuity of the function $w \mapsto \operatorname{Sp} w$ [17, p. 35], U_1 is open in $\mathcal{A}$. Let $\mathfrak{C}_0: U_1 \to \mathcal{A}$ be the holomorphic map defined by

$$\mathfrak{C}_0(w) = i(e + w)(e - w)^{-1} \quad (w \in U_1) . \tag{6.1}$$

Since $e + w$ and $e - w$ commute,

$$\mathfrak{C}_0(w) = i(e - w)^{-1}(e + w) . \tag{6.2}$$

Let $w \in B_p$. Then

$$\begin{aligned} \mathfrak{C}_0(w) - \mathfrak{C}_0(w)^* &= i[(e + w)(e - w)^{-1} + (e - w^*)^{-1}(e + w^*)] \\ &= i(e - w^*)^{-1}[(e - w^*)(e + w) + (e + w^*)(e - w)](e - w)^{-1} \\ &= 2i(e - w^*)^{-1}(e - w^* w)(e - w)^{-1} . \end{aligned}$$

Being $\varrho(w^* w) < 1$, then $\operatorname{Sp}(e - w^* w) \subset (0, 1]$. Let $v \in \Omega$ be a square root of $e - w^* w$. Then

$$\mathfrak{C}_0(w) - \mathfrak{C}_0(w)^* = 2i\big(v(e - w)^{-1}\big)^*\big(v(e - w)^{-1}\big) .$$

Taking into account the fact that v is invertible, we see that

$$\frac{1}{2i}\left(\mathfrak{C}_0(w) - \mathfrak{C}_0(w)^*\right) \in \Omega\,,$$

i.e.

$$\mathfrak{C}_0(B_p) \subset D(\Omega)\,.$$

For $w \in U_1$, (6.1) and (6.2) yield

$$w(\mathfrak{C}_0(w) + ie) = (\mathfrak{C}_0(w) + ie)w = \mathfrak{C}_0(w) - ie\,.$$

LEMMA 6.1. *If $z \in D(\Omega)$, then z is invertible.*

PROOF. Let $z = x + iy$, with $x \in \mathcal{H}(\mathcal{A})$, $y \in \Omega$; let $y^{\frac{1}{2}} \in \Omega$ be a square root of y, and let $y^{-\frac{1}{2}} = (y^{\frac{1}{2}})^{-1}$. Then $y^{-\frac{1}{2}} \in \Omega$, and z can be represented as

$$z = x + iy = iy^{\frac{1}{2}}(e - iy^{-\frac{1}{2}}xy^{-\frac{1}{2}})y^{\frac{1}{2}}\,. \tag{6.3}$$

Since $y^{-\frac{1}{2}}xy^{-\frac{1}{2}}$ is hermitian, then

$$\mathrm{Sp}\,(e - iy^{-\frac{1}{2}}xy^{-\frac{1}{2}}) \subset \{1 - it : t \in \boldsymbol{R}\}\,,$$

showing that $e - iy^{-\frac{1}{2}}xy^{-\frac{1}{2}}$ is invertible. Q.E.D.

Let $z = x + iy$ with $x \in \mathcal{H}(\mathcal{A})$, $y \in \Omega_0$. Then $z + ie \in D(\Omega)$. By Lemma 6.1, $z + ie$ is invertible, i.e., $-i \notin \mathrm{Sp}\,z$.

Let $U_{-i} = \{z \in \mathcal{A} : -i \notin \mathrm{Sp}\,z\}$. Then $D(\Omega) \subset U_{-i}$. By the upper semicontinuity of the function $z \mapsto \mathrm{Sp}\,z$, U_{-i} is open in $\mathcal{A}$. Let $\mathfrak{C}_1 : U_{-i} \to \mathcal{A}$ be the holomorphic map defined by

$$\mathfrak{C}_1(z) = (z - ie)(z + ie)^{-1} \quad (z \in U_{-i})\,. \tag{6.4}$$

Since $z - ie$ and $z + ie$ commute, then $\mathfrak{C}_1(z)$ can also be written

$$\mathfrak{C}_1(z) = (z + ie)^{-1}(z - ie) \quad (z \in U_{-i})\,. \tag{6.5}$$

We prove now that

$$\mathfrak{C}_1(D(\Omega)) \subset B_p\,. \tag{6.6}$$

In fact, let $z = x + iy \in D(\Omega)$, with $x \in \mathcal{H}(\mathcal{A})$, $y \in \Omega$. Then

$$\begin{aligned} e - \mathfrak{C}_1(z)^*\mathfrak{C}_1(z) &= e - (z^* - ie)^{-1}(z^* + ie)(z - ie)(z + ie)^{-1} \\ &= (z^* - ie)^{-1}[(z^* - ie)(z + ie) - (z^* + ie)(z - ie)](z + ie)^{-1} \\ &= 2i(z^* - ie)^{-1}(z^* - z)(z + ie)^{-1} = 4(z^* - ie)^{-1}y(z + ie)^{-1} \\ &= 4(z^* - ie)^{-1}y^{\frac{1}{2}}y^{\frac{1}{2}}(z + ie)^{-1} \\ &= 4\big(y^{\frac{1}{2}}(z + ie)^{-1}\big)^*\big(y^{\frac{1}{2}}(z + ie)^{-1}\big) . \end{aligned}$$

Thus $e - \mathfrak{C}_1(z)^*\mathfrak{C}_1(z) \in \Omega$, and therefore $\operatorname{Sp}\big(\mathfrak{C}_1(z)^*\mathfrak{C}_1(z)\big) \subset [0, 1)$. In conclusion $p\big(\mathfrak{C}_1(z)\big) = \varrho\big(\mathfrak{C}_1(z)^*\,\mathfrak{C}_1(z)\big)^{\frac{1}{2}} < 1$, i.e. $\mathfrak{C}_1(z) \in B_p$. That proves (6.6).

Comparing (6.1) and (6.5) (or (6.2) and (6.4)) we see that

$$\mathfrak{C}_1 \circ \mathfrak{C}_0 = \text{identity on } B_p ,$$

$$\mathfrak{C}_0 \circ \mathfrak{C}_1 = \text{identity on } D(\Omega) .$$

It is readily checked on (6.4) and (6.5) that $\mathfrak{C}_1$ is injective. By consequence, if $z \in U_{-i}$ is such that $\mathfrak{C}_1(z) \in B_p$, then $z = \mathfrak{C}_0\big(\mathfrak{C}_1(z)\big) \in D(\Omega)$. That proves that $D(\Omega) = \mathfrak{C}_1^{-1}(B_p)$. Since $\mathfrak{C}_1$ is continuous, and B_p is open, then $D(\Omega)$ is open.

Denoting by $\mathfrak{C}$ the restriction of $\mathfrak{C}_1$ to $D(\Omega)$, the restriction of $\mathfrak{C}_0$ to B_p is $\mathfrak{C}^{-1}$. Thus the map $\mathfrak{C}: D(\Omega) \to B_p$ is a bi-holomorphic diffeomorphism of $D(\Omega)$ onto B_p; $\mathfrak{C}$ will be called the *Cayley transform*.

We shall prove now that $D(\Omega)$ is affine-homogeneous. Let $z = x + iy \in$ $\in D(\Omega)$ $\big(x \in \mathcal{H}(\mathcal{A}), y \in \Omega\big)$ and let $F_z: \mathcal{A} \to \mathcal{A}$ be the affine automorphism of the Banach space $\mathcal{A}$ defined by

$$F_z(w) = y^{-\frac{1}{2}}(w - x)y^{-\frac{1}{2}} , \tag{6.7}$$

where $y^{\frac{1}{2}} \in \Omega$ is a square root of y, and $y^{-\frac{1}{2}} = (y^{\frac{1}{2}})^{-1}$. For $w = u + iv$ $\big(u, v \in \mathcal{H}(\mathcal{A})\big)$, then

$$F_z(u + iv) = y^{-\frac{1}{2}}(u - x)y^{-\frac{1}{2}} + iy^{-\frac{1}{2}}vy^{-\frac{1}{2}} ,$$

where both $y^{-\frac{1}{2}}(u - x)y^{-\frac{1}{2}}$ and $y^{-\frac{1}{2}}vy^{-\frac{1}{2}}$ are hermitian elements. If $v \in \Omega$, denoting by $v^{\frac{1}{2}} \in \Omega$ a square root of v, we have

$$y^{-\frac{1}{2}}vy^{-\frac{1}{2}} = y^{-\frac{1}{2}}v^{\frac{1}{2}}v^{\frac{1}{2}}y^{-\frac{1}{2}} = (v^{\frac{1}{2}}y^{-\frac{1}{2}})^*\,(v^{\frac{1}{2}}y^{-\frac{1}{2}}) .$$

Since both $v^{\frac{1}{2}}$ and $y^{-\frac{1}{2}}$ are invertible, then $y^{-\frac{1}{2}}vy^{-\frac{1}{2}} \in \Omega$, i.e. $F_z(u + iv) \in D(\Omega)$.

Vice versa, let $y^{-\frac{1}{2}}vy^{-\frac{1}{2}} = v' \in \Omega$. If $v'^{\frac{1}{2}} \in \Omega$ is a square root of v' then

$$v = y^{\frac{1}{2}}v'^{\frac{1}{2}}v'^{\frac{1}{2}}y^{\frac{1}{2}} = (v'^{\frac{1}{2}}y^{\frac{1}{2}})^*(v'^{\frac{1}{2}}y^{\frac{1}{2}}) ,$$

and therefore $v \in \Omega$, i.e. $w = u + iv \in D(\Omega)$. In conclusion, $F_z(w) \in D(\Omega)$ if, and only if, $w \in D(\Omega)$. That proves that F_z defines an affine automorphism of $D(\Omega)$. Since, for any $z \in D(\Omega)$, $F_z(z) = ie$, then $D(\Omega)$ is affine homogeneous. Summarizing the above results, we state

PROPOSITION 6.2. *Let $\mathcal{A}$ be a Banach algebra with unit, endowed with a hermitian involution. The Cayley transform maps the convex domain $D(\Omega)$ bi-holomorphically onto the domain B_p. The domain $D(\Omega)$ is affine-homogeneous. Thus $D(\Omega)$ and B_p are homogeneous.*

LEMMA 1.1 implies that the Kobayashi and Carathéodory pseudodistances coincide on B_p, and therefore also on $D(\Omega)$:

$$c_{B_p} = d_{B_p}, \qquad c_{D(\Omega)} = d_{D(\Omega)} .$$

For $z_1,\ z_2 \in D(\Omega)$

$$c_{D(\Omega)}(z_1, z_2) = c_{D(\Omega)}(F_{z_1}(z_1), F_{z_1}(z_2)) = c_{D(\Omega)}(ie, F_{z_1}(z_2)) .$$

Since $\mathfrak{C}(ie) = 0$, Lemma 1.1 yields

$$c_{D(\Omega)}(z_1, z_2) = \omega\Big(0, p\big(\mathfrak{C}(F_{z_1}(z_2))\big)\Big) .$$

Let $z_2 = x_2 + iy_2$, $x_2 \in \mathcal{H}(\mathcal{A})$, $y_2 \in \Omega$. Let $y_2^{\frac{1}{2}} \in \Omega$ be a square root of y_2 and let $y_2^{-\frac{1}{2}} = (y_2^{\frac{1}{2}})^{-1}$. Then, by (6.4) and (6.7),

$$\begin{aligned}\mathfrak{C}(F_{z_1}(z_2)) &= \big(y_1^{-\frac{1}{2}}(z_2 - x_1)y_1^{-\frac{1}{2}} - ie\big)\big(y_1^{-\frac{1}{2}}(z_2 - x_1)y_1^{-\frac{1}{2}} + ie\big)^{-1}\\ &= y_1^{-\frac{1}{2}}(z_2 - x_1 - iy_1)y_1^{-\frac{1}{2}}\big(y_1^{-\frac{1}{2}}(z_2 - x_1 + iy_1)y_1^{-\frac{1}{2}}\big)^{-1}\\ &= y_1^{-\frac{1}{2}}(z_2 - z_1)(z_2 - z_1^*)^{-1}y_1^{\frac{1}{2}} ,\end{aligned}$$

and therefore

$$c_{D(\Omega)}(z_1, z_2) = d_{D(\Omega)}(z_1, z_2) = \omega\Big(0, p\big(y_1^{-\frac{1}{2}}(z_2 - z_1)(z_2 - z_1^*)^{-1}y_1^{\frac{1}{2}}\big)\Big) , \qquad \big(z_1, z_2 \in D(\Omega)\big) . \tag{6.8}$$

In general p is only a semi-norm. If it is a norm and if $\mathcal{A}$ is complete with respect to p, then—B_p being homogeneous—the Carathéodory distance on B_p

is complete [24, théoréme 2, p. 279], and therefore also the Carathéodory distance on $D(\Omega)$ is complete.

For example, if $\mathcal{A}$ is a C^*-algebra with identity, then for $z \in \mathcal{A}$,

$$p(z) = \varrho(z^* z)^{\frac{1}{2}} = \|z^* z\|^{\frac{1}{2}} = \|z\| .$$

Hence B_p is the open unit ball B for the norm $\| \ \|$. $D(\Omega)$ is biholomorphically equivalent to B, and all previous requirements are fulfilled. Thus we have

PROPOSITION 6.3. *If $\mathcal{A}$ is a C^*-algebra with identity, then B and $D(\Omega)$ are complete metric spaces for their Carathéodory (and Kobayashi) distances.*

This proposition extends Theorem IV of [23] from von Neumann algebras to C^*-algebras with identity.

EXAMPLES. 1) Let $\mathcal{A}$ be a *commutative* Banach algebra with identity, endowed with a hermitian involution. In this case ϱ is a submultiplicative norm on $\mathcal{A}$. Hence, by II), we have

$$\varrho(z) \leqslant p(z) = \varrho(z^* z)^{\frac{1}{2}} \leqslant \varrho(z^*)^{\frac{1}{2}} \varrho(z)^{\frac{1}{2}} = \varrho(z) \qquad (z \in \mathcal{A}) ,$$

whence $\varrho(z) = p(z)$. Thus $B_p = C = \{z \in \mathcal{A}: \varrho(z) < 1\}$, and by Proposition 6.2, C is homogeneous. Since $\varrho(z) \leqslant \|z\|$, then $B \subset C$.

2) Let G be a discrete abelian group containing more than one element. Let μ be the counting measure on G, and let $\mathcal{A}$ be the convolution algebra on $L^1(G, \mu)$. Then C is homogeneous, while B is not, by Theorem II. Is there any homogeneous domain D such that $B \subset D \subsetneqq C$?

3) If G consists of two elements, e and g, and $\mu(e) = \mu(g) = 1$, then $L^1(G, \mu) \simeq \boldsymbol{C}^2$,

$$B = \{(\zeta^1, \zeta^2) \in \boldsymbol{C}^2 : |\zeta^1| + |\zeta^2| < 1\} .$$

The convolution in $L^1(G, \mu)$ is defined as follows. For $z' = (\zeta'^1, \zeta'^2)$, $z'' = (\zeta''^1, \zeta''^2)$ in $\boldsymbol{C}^2$

$$(z' * z'')(e) = \zeta'^1 \zeta''^1 + \zeta'^2 \zeta''^2 \qquad (z' * z'')(g) = \zeta'^1 \zeta''^2 + \zeta'^2 \zeta''^1 .$$

The dual group of G is G itself. For any $z = (\zeta^1, \zeta^2)$, the Gelfand transform $\hat{z}$ is defined by

$$\hat{z}(e) = \zeta^1 + \zeta^2 , \qquad \hat{z}(g) = \zeta^1 - \zeta^2 .$$

Thus

$$\varrho(z) = \max\left(|\zeta^1 + \zeta^2|, |\zeta^1 - \zeta^2|\right),$$

and

$$C = \{z = (\zeta^1, \zeta^2) : |\zeta^1 + \zeta^2| < 1\,, \quad |\zeta^1 - \zeta^2| < 1\}\,.$$

Hence C is a polydisc, and there is no bounded homogeneous domain $D \subset \mathbf{C}^2$ such that $B \subset D \subsetneqq C$.

4) Going back to formula (6.8) in the general case, let $z_1 = ie$, $z_2 = iy$, with $y \in \Omega$. Since $y - e$ and $y + e$ commute, then

$$p\big((z_2 - ie)(z_2 + ie)^{-1}\big) = p\big((y - e)(y + e)^{-1}\big) = \varrho\big((y - e)(y + e)^{-1}\big)\,.$$

By the spectral mapping theorem,

$$\varrho\big((y - e)(y + e)^{-1}\big) = \max\left\{\left|\frac{t-1}{t+1}\right| : t \in \operatorname{Sp} y\right\} =$$
$$= \max\left\{\max\left\{\frac{t-1}{t+1} : t \in \operatorname{Sp} y\right\},\ -\min\left\{\frac{t-1}{t+1} : t \in \operatorname{Sp} y\right\}\right\}.$$

Since

$$\min\{t : t \in \operatorname{Sp} y\} = \frac{1}{\varrho(y^{-1})}\,, \qquad \max\{t : t \in \operatorname{Sp} y\} = \varrho(y)\,,$$

then

$$p\big((y - e)(y + e)^{-1}\big) = \max\left\{\frac{\varrho(y) - 1}{\varrho(y) + 1}, \frac{\varrho(y^{-1}) - 1}{\varrho(y^{-1}) + 1}\right\}.$$

A simple discussion shows then that

$$c_{D(\Omega)}(ie, iy) = d_{D(\Omega)}(ie, iy) = \tfrac{1}{2}\max\left(\log \varrho(y), \log \varrho(y^{-1})\right).$$

This formula was obtained in [23, Theorem II and (8.4)] under the additional condition that the involution $*$ be locally continuous.

Note added in proof, October 1978.

The proof of theorem II is considerably simplified by the following result established by T. J. Suffridge (*Starlike and convex maps in Banach spaces,* Pacific J. Math., **46** (1973), pp. 575-589; cf. theorem 8, pp. 584-586).

With the same notations as in theorem II, let $f \in \operatorname{Hol}(B, L^1(M, \mu))$ be such that $f(B)$ is an open convex subset of $L^1(M, \mu)$ and that f is a bi-holomorphic map of B onto $f(B)$. If $\dim_{\mathbf{C}} L^1(M, \mu) > 1$, then the map $x \mapsto f(x) - f(0)$ is the restriction to B of a continuous linear map of $L^1(M, \mu)$ onto itself.

In view of this result and of lemma 4.3, the image by F of any complex geodesic curve at 0 in B belongs to a complex affine line through $x_0 = F(0)$. Hence any complex geodesic curve at x_0 must belong to a complex affine line. If $x_0 \neq 0$, that contradicts lemmas 4.4 and 4.5.

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