ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 5, nº 4 (1978), p. 605-631
<http://www.numdam.org/item?id=ASNSP_1978_4_5_4_605_0>

Classical Solution to a Second Order Nonlinear Elliptic System in R_3.

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dedicated to Jean Leray

The paper is concerned with the regularity properties of weak solutions of a Dirichlet boundary value problem for an elliptic nonlinear system of second order. It is well known that there exists a unique solution of the given problem under suitable conditions on coefficients but this solution need be nor bounded nor continuous even if the coefficients are analytic functions (see J. L. Lions [1], Ch. B. Morrey [2], E. Giusti, M. Miranda [3], J. Nečas [4], V. G. Mazja [5]).

In this paper we shall consider coefficients defined only on a subset M of R_{3m}. Similar problem arised in studying the existence of solution in the theory of hyperelasticity (see J. Nečas [4]). The coefficients satisfying the condition of ellipticity non necessarily uniformly will be supposed and the existence of classical solution will be proved in the following (briefly and non-exactly said) sense: if $\tilde{u}$ is a classical solution of the given problem with the right-hand side $\tilde{f}$ and the boundary condition $\tilde{u}_0$ and if we prescribe a continuous path of right-hand sides $\{f(t)\}_{t\in\langle 0,T\rangle}$ and boundary conditions $\{u_0(t)\}_{t\in\langle 0,T\rangle}$ beginning in $\tilde{f}$ and $\tilde{u}_0$ then there are only two possibilities: either there exists a continuous path of regular solutions on the whole $\langle 0, T\rangle$ or there is a critical value $t_0 \in (0, T\rangle$ where the path of solutions ends; in this case the gradient $\nabla u(t)$ tends to the boundary of M or to the point where the ellipticity conditions degenerates or the norms $\|u\|_{3,2} \to \infty$ as $t \to t_0$.

In the first part—§ 2—apriori estimates will be proved. If the given equation is considered as Euler's equation of a functional Φ, it is possible

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Pervenuto alla Redazione il 28 Giugno 1977.

to say that the apriori estimates are based on the local behaviour of the fourth Gâteaux differential of Φ.

In the second part—§ 3— regularity results are proved by means of implicit function theorem.

All the results are formulated for two types of domains—a bounded domain with an infinitely smooth boundary and a half-space. The proof of the apriori estimate for a half space is simpler and shows clearly its basic idea but the second part (the implicit function theorem) brings some complications caused by the nonexistence of compact embeddings. For a bounded domain the proofs in the second part are the standard ones and the proof of the apriori estimate is analogous to that for a half space, but a lot of difficulties is caused by using the suitable partition of unity and complicated test functions.

We shall consider systems of second order with coefficients dependent only on the gradient ∇u.

1. – Notation.

Let Ω be a domain in R_3 with an infinitely smooth boundary and M be a domain in R_{3m} containing the origin. Let us denote by $x = (x_1, x_2, x_3)$ (respectively $\xi = \{\xi_i^r\}$, $i = 1, 2, 3$; $r = 1, \ldots, m$) a generic point of R_3 (respectively R_{3m}) with the norm

$$|x| = \sum_{i=1}^{3} |x_i| \qquad \Big(\text{resp. } |\xi| = \sum_{i,r} |\xi_i^r|\Big).$$

$C_1(\bar{\Omega})$ is the space of all functions with continuous and bounded first derivatives on Ω such that all these first derivatives can be continuously extended on $\bar{\Omega}$. On the Cartesian product $[C_1(\bar{\Omega})]^{3m}$ a pseudonorm

$$\|u\|_{C_1} = \sup\left\{\left|\frac{\partial u_r(x)}{\partial x_i}\right|;\ x \in \bar{\Omega},\ i = 1, 2, 3,\ r = 1, \ldots, m\right\}$$

will be used.

For a bounded domain Ω we shall denote by $[W_2^k(\Omega)]^m$ or $[\mathring{W}_2^k(\Omega)]^m$ the usual Sobolev spaces with the norms

$$\|u\|_{W_2^k} = \|u\|_{k,2} = \Big(\sum_{|\alpha| \leqslant k} \sum_{r=1}^{m} \int_{\Omega} |D^\alpha u_r|^2\Big)^{\frac{1}{2}}$$

and

$$\|u\|_{\mathring{W}_2^k} = \|u\|'_{k,2} = \Big(\sum_{1 \leqslant |\alpha| \leqslant k} \sum_{r=1}^{m} \int_{\Omega} |D^\alpha u_r|^2\Big)^{\frac{1}{2}}.$$

Let us remind that the norms $\|u\|_{k.2}$ and $\|u\|'_{k.2}$ are equivalent on the space $\mathring{W}_2^k$.

For $\Omega = R_3^+ = \{x \in R_3;\, x_3 > 0\}$ we shall denote by $[W_2^k(R_3^+)]^m$ the completion of infinitely smooth functions with compact support in $\overline{R_3^+}$ with respect to the norm

$$\|u\|_{[W_2^k(R_3^+)]^m} = \|u\|_{k.2} = \left(\sum_{|\alpha|\leq k} \sum_{r=1}^{m} \int_{R_3^+} |D^\alpha u_r|^2\right)^{\frac{1}{2}}.$$

By $[W_2^{k'}(R_3^+)]^m$ we shall denote the completion of the space of infinitely smooth functions with compact support in $\overline{R_3^+}$ with respect to the norm

$$\|u\|_{[W_2^{k'}(R_3^+)]^m} = \|u\|'_{k.2} = \left(\sum_{1\leq|\alpha|\leq k} \sum_{r=1}^{m} \int_{R_3^+} |D^\alpha u_r|^2\right)^{\frac{1}{2}}$$

$[\mathring{W}_2^k(R_3^+)]^m$ (respectively $[\mathring{W}_2^{k'}(R_3^+)]^m$) is a completion of the space of all infinitely smooth functions with compact support in R_3^+ with respect to the norm $\|u\|_{k.2}$ (respectively $\|u\|'_{k.2}$).

Let F be a four times continuously differentiable real function defined on M and let us denote

$$a_i^r = \frac{\partial F}{\partial \xi_i^r}, \qquad b_{ij}^{rs} = \frac{\partial^2 F}{\partial \xi_i^r\, \partial \xi_j^s}.$$

Let $a_i^r(0) = 0$. For $u \in [C_1(\overline{\Omega})]^m$ we shall write

$$\nabla u = \left\{\frac{\partial u_r}{\partial x_i}\right\}_{i=1,2,3,\ r=1,\dots,m}$$

and denote the graph of the gradient by

$$\langle \nabla u \rangle = \{\nabla u(x);\, x \in \overline{\Omega}\}.$$

We shall deal with the regularity properties of a weak solution of a Dirichlet problem for a system of the form

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_i^r(\nabla u) - f_i^r\right) = 0 \tag{1.1}$$

on Ω for $r = 1, \dots, m$,

$$u_r = u_{0,r}$$

on $\partial\Omega$ for $r = 1, \dots, m$.

By the weak solution of this problem we shall understand a vector function $u \in [W_2^{1'}(\Omega)]^m$ *such that*

(i) $u - u_0 \in [\mathring{W}_2^{1'}(\Omega)]^m$,

(ii) *the equality*

$$\int_\Omega \sum_{i,r} \frac{\partial \varphi_r}{\partial x_i} (a_i^r(\nabla u) - f_i^r) = 0 \tag{1.2}$$

holds for every $\varphi \in [\mathring{W}_2^{1'}(\Omega)]^m$,

(iii) $\langle \nabla u \rangle \subset M$.

The conditions (i) and (ii) form the usual definition of a weak solution. The condition (iii) is required because of the above described assumption on the domain of F.

2. – Apriori estimates.

A. Estimates on half-space.

In this part of Section 2 the domain $\Omega = R_3^+ = \{x = (x_1, x_2, x_3); x_3 > 0\}$ will be considered.

2.1. LEMMA. *Let* R *be a positive number with the following properties*:

(i) $K_R = \{\xi \in R_{3m}; |\xi| < R\} \subset M$,

(ii) *there exist functions*

$$c\colon (0, R) \to (0, +\infty)\,,$$

$$d\colon (0, R) \to R_1$$

such that, for every $\varrho \in (0, R)$, $\eta \in R_{3m}$, $\xi \in K_\varrho = \{\xi \in R_{3m}, |\xi| < \varrho\}$, *we have*

$$\sum_{i,j,r,s} b_{ij}^{rs}(\xi)\eta_i^r\eta_j^s \geq c(\varrho)\sum_{i,r}(\eta_i^r)^2\,, \qquad \sum_{\substack{i,j,k,l\\ r,s,t,v}} \frac{\partial^2 b_{ij}^{rs}}{\partial\zeta_k^t\,\partial\zeta_l^v}\eta_i^r\eta_j^s\eta_k^t\eta_l^v \leq d(\varrho)\sum_{i,r}(\eta_i^r)^4, \tag{2.1}$$

(iii) $3d(\varrho)\varrho^2 < c(\varrho)$ *for every* $\varrho \in (0, R)$.

Then there exists a continuous function $D\colon (0, R) \to (0, +\infty)$ *such that for every* $u \in [W_2^{3'}(R_3^+) \cap \mathring{W}_2^{1'}(R_3^+)]^m$, *which solves the equation* (1.2) *with* $u_0 = 0$, $f \in [W_2^2(R_3^+)]^{3m}$ *and satisfies the inequality*

(iv) $\|u\|_{C^1(\bar{\Omega})} \leq \varrho$, *the inequality*

$$\|u\|'_{3,2} \leqslant D(\varrho)(\|f\|_{2,2} + 1) \tag{2.2}$$

holds.

PROOF. Roughly said the derivatives in the directions of the axis x_1, x_2 will be estimated by choosing suitable test functions, then the normal derivative will be calculated from the equation.

Let us fix $\varrho \in (0, R)$ and deal with the terms with the first and second derivatives at first.

The condition (2.1) (ii) together with $a_i^r(0) = 0$ implies

$$\sum_{i,r} a_i^r(\xi)\, \xi_i^r \geqslant c(\varrho) \sum_{i,r} (\xi_i^r)^2 . \tag{2.3}$$

Putting $\varphi = u$ in (1.2) and using (2.3), we get

$$\|u\|'_{1,2} \leqslant \frac{1}{c(\varrho)} \|f\|_{0,2} . \tag{2.4}$$

Let w' denote the derivative of a function w in the tangential direction, i.e., in the direction of one of the axes x_1, x_2. Let us take $\psi \in [D(R_3^+)]^m$ and put $\varphi = \psi''$ in (1.2). Integrating by parts we get

$$\int_{R_3^+} \sum_{i,j,r,s} b_{ij}^{rs}(\nabla u) \frac{\partial u'_s}{\partial x_j} \frac{\partial \psi'_r}{\partial x_i} = \int_{R_3^+} \sum_{i,r} f_i^{r'} \frac{\partial \psi'_r}{\partial x_i} \tag{2.5}$$

and, putting again $\psi = u$,

$$\|u'\|'_{1,2} \leqslant \frac{1}{c(\varrho)} \|f\|_{1,2} . \tag{2.6}$$

Under the assumption on the function u the equation (1.2) can be written on Ω in the form

$$\sum_{i,j,s} b_{ij}^{rs}(\nabla u) \frac{\partial^2 u_s}{\partial x_i\, \partial x_j} = \sum_i \frac{\partial f_i^r}{\partial x_i} \tag{2.7}$$

for $r = 1, \dots, m$. The matrix (b_{33}^{rs}) is regular with a uniformly bounded inverse and we can estimate $\partial^2 u_r/\partial x_3^2$ by means of (2.6). Thus

$$\|u\|'_{2,2} \leqslant \frac{C}{c(\varrho)} \|f\|_{1,2} . \tag{2.8}$$

Let us now put $\varphi = \psi'''$ for the same function $\psi \in [D(R_3^+)]^m$ in the equation (1.2). Integrating once more by parts, we obtain

$$(2.9)\qquad \int_{R_3^+} \sum_{i,j,r,s} b_{ij}^{rs}(\nabla u) \frac{\partial u_s''}{\partial x_j} \frac{\partial \psi_r''}{\partial x_i} + \int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t}(\nabla u) \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial \psi_r''}{\partial x_i} - \int_{R_3^+} \sum_{i,r} f_i^{r''} \frac{\partial \psi_r''}{\partial x_i} = 0 .$$

Let us put $\psi = u$ and consider the second integral on the left-hand side of (2.9). Integrating by parts for the third time, we obtain

$$(2.10)\qquad I_2 = \sum_{\substack{i,j,k\\ r,s,t}} \int_{R_3^+} \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t}(\nabla u) \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_r''}{\partial x_i} = -\frac{1}{3} \sum_{\substack{i,j,k,l\\ r,s\, t,v}} \int_{R_3^+} \frac{\partial^2 b_{ij}^{rs}}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_r'}{\partial x_i} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l},$$

but the last integral is not less than

$$(2.11)\qquad -\frac{1}{3} d(\varrho) \int_{R_3^+} \sum_{i,r} \left(\frac{\partial u_r'}{\partial x_i}\right)^4 .$$

Using the density of $\overline{D(R_3^+)}$ in the space $W_2^2(R_3^+)$, we can easily prove a special case of L. Nirenberg's interpolation inequality.

2.2. Lemma. *Let $w \in W_2^2(R_N^+)$ be a bounded function on R_N. Then $w' \in L_4(R_N^+)$ and*

$$(2.12)\qquad \|w'\|_{0,4}^4 \leq 9 \|w\|_{0,\infty}^2 \|w''\|_{0,2}^2 .$$

(w' has the same meaning as in the foregoing proof, i.e., the derivative of w in any of the directions $x_1, \ldots, x_{N-1}$).

Now we can estimate the first integral in (2.9) by (ii), the second integral by (2.11), (2.12) (applied to every function $\partial u_r/\partial x_i$) and (iv), the third integral by Hölder's inequality. Thus we have

$$(2.13)\qquad (c(\varrho) - 3d(\varrho)\varrho^2) \|u''\|_{1,2}' \leq \|f\|_{2,2} ,$$

which according to (2.1) (iii) gives an estimate for the tangential derivatives in the form

$$(2.14)\qquad \|u''\|_{1,2}' \leq \frac{1}{c(\varrho) - 3d(\varrho)\varrho^2} \|f\|_{2,2} .$$

The next step is the estimate of the derivatives of the form

$$\frac{\partial^3 u_r}{\partial x_i \partial x_j \partial x_k}$$

where $i, j = 1, 2$, and it immediately follows from

2.3. LEMMA. *There exists a positive constant C such that for every function $w \in W_2^2(R_N^+)$ the inequality*

$$\sum_{i,j=1}^{N-1} \left\| \frac{\partial^2 w}{\partial x_i \, \partial x_j} \right\|_{0,2} \leqslant C \sum_{i=1}^{N-1} \left\| \frac{\partial^2 w}{\partial x_i^2} \right\|_{0,2} \tag{2.15}$$

holds.

The proof of this lemma is an easy consequence of the Plancherel's theorem applied to the function w prolonged on the whole space R_N.

Thus

$$\sum_{i,j=1,2} \left\| \frac{\partial^2 u}{\partial x_i \, \partial x_j} \right\|'_{1,2} \leqslant \frac{C}{c(\varrho) - 3d(\varrho)\varrho^2} \|f\|_{2,2} . \tag{2.16}$$

Let at least one of the indices i, j be less than 3. Then according to Lemma 2.2,

$$\frac{\partial^2 u}{\partial x_i \, \partial x_j} \in L_4(R_3^+) .$$

The L_4-norm of $\partial^2 u / \partial x_3^2$ can be calculated from the system (2.7) as before. Differentiating once more in (2.7) in a tangential direction, we get the equality

$$\sum_{s=1}^{m} b_{33}^{rs}(\nabla u) \frac{\partial^2 u_s'}{\partial x_3^2} = \sum_{i=1}^{3} \frac{\partial f_i^{r'}}{\partial x_i} - \sum_{s=1}^{m} \sum_{\substack{i,j=1 \\ (i,j) \neq (3,3)}}^{3} b_{ij}^{rs}(\nabla u) \frac{\partial^2 u_s'}{\partial x_i \, \partial x_j} - \sum_{\substack{i,j,k \\ s,t}} \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t} (\nabla u) \frac{\partial^2 u_s}{\partial x_i \, \partial x_j} \frac{\partial u_t'}{\partial x_k} . \tag{2.17}$$

All the terms on the right-hand side are L_2-functions and the matrix (b_{33}^{rs}) has a uniformly bounded inverse; therefore

$$\sum_{j,r} \left\| \frac{\partial u_r'}{\partial x_i \, \partial x_j} \right\|_{0,2} \leqslant \frac{C(\varrho)}{c(\varrho) - 3d(\varrho)\varrho^2} \left(1 + \|f\|_{2,2}\right) . \tag{2.18}$$

If now $w' = \partial w / \partial x_3$ in (2.17) we can repeat all the procedures with the (2.18) estimate instead of (2.16) and we get a bound for $\partial^3 u / \partial x_3^3$ in the form

$$\left\| \frac{\partial^3 u}{\partial x_3^3} \right\|_{0,2} \leqslant \frac{A(\varrho)}{c(\varrho) - 3d(\varrho)\varrho^2} \left(1 + \|f\|_{2,2}\right) , \tag{2.19}$$

which together with (2.18), (2.4), and (2.8) accomplishes the proof.

The following lemma deals with the same type of an apriori estimate as Lemma 2.1; we suppose there that u is a solution of a non-homogeneous Dirichlet boundary value problem and that it lies in a neighbourhood of a smooth function $\tilde{u}$.

2.4. LEMMA. *Let* $\tilde{u} \in [W_2^{3'}(R_3^+)]^m$ *and* $\langle \nabla \tilde{u} \rangle \subset M$. *Let* R *be a positive number with the following properties*:

(i) $\tilde{K}_R = \langle \nabla \tilde{u} \rangle + K_R \subset M$,

(ii) *there exist functions*

$$c\colon (0, R) \to (0, \infty)\,,$$

$$d\colon (0, R) \to R_1\,,$$

such that, for every $\varrho \in (0, R)$, $\eta \in R_{3m}$, *and* $\xi \in \tilde{K}_\varrho = \langle \nabla \tilde{u} \rangle + K_\varrho$,

$$(2.20)\qquad \begin{cases} \sum\limits_{i,j,r,s} b_{ij}^{rs}(\xi)\eta_i^r\eta_j^s \geqq c(\varrho) \sum\limits_{i,j} (\eta_i^r)^2\,, \\ \sum\limits_{\substack{i,j,k,l \\ r,s,t,v}} \dfrac{\partial^2 b_{ij}^{rs}(\xi)}{\partial \zeta_k^t\, \partial \zeta_l^v}\, \eta_i^r \eta_j^s \eta_k^t \eta_l^v \leqq d(\varrho) \sum\limits_{i,r} (\eta_i^r)^4\,, \end{cases}$$

(iii) *for every* $\varrho \in (0, R)$, $3d(\varrho)\varrho^2 < c(\varrho)$. *Then there exists a continuous function* $D\colon (0, R) \to (0, \infty)$ *such that, for every* $u \in [W_2^{3'}(R_3^+)]^m$ *which solves the equation* (1.2) *with the boundary value* $u_0 \in [W_2^{3'}(R_3^+)]^m$ *and the right-hand side* $f \in [W_2^2(R_3^+)]^{3m}$ *and satisfies the inequality*

(iv) $\|\tilde{u} - u\|_{C_1} \leqq \varrho$,

the estimate

$$(2.21)\qquad \|u\|'_{3,2} \leqq D(\varrho)\,[1 + \|\tilde{u}\|'^2_{3,2} + \|u_0\|'^2_{3,2} + \|f\|_{2,2}]$$

holds.

PROOF. Repeating the first part of the proof of Lemma 2.1 with the test function $u - u_0$ instead of u, we obtain the inequality

$$(2.22)\qquad \|u\|'_{2,2} \leqq \frac{C}{c(\varrho)}\,[\|f\|_{2,2} + \|u_0\|'_{2,2}]\,.$$

The first difference is caused by the fact that the equality (2.10) does not hold in such a simple form. Integrating by parts in the second integral in (2.9), we obtain

$$(2.23)\quad \int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial \varphi_r''}{\partial x_i} = -\int_{R_3^+} \sum_{\substack{i,j,k,l\\ r,s,t,v}} \frac{\partial^2 b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l} \frac{\partial \varphi_r'}{\partial x_i} - \\ -\int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t} \left[\frac{\partial u_s''}{\partial x_j} \frac{\partial u_t'}{\partial x_k} + \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t''}{\partial x_k}\right] \frac{\partial \varphi_r'}{\partial x_i}.$$

Now $u - u_0$ can be written in the equality instead of φ and

$$(2.24)\quad \int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial (u-u_0)_r''}{\partial x_i} = \\ = -\frac{1}{3}\int_{R_3^+} \sum_{\substack{i,j,k,l\\ r,s,t,v}} \frac{\partial^2 b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_r'}{\partial x_i} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l} + I_2 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{3}\int_{R_3^+} \sum_{\substack{i,j,k,l\\ r,s,t,v}} \frac{\partial^2 b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l} \frac{\partial u_{0r}'}{\partial x_i},$$

$$I_2 = -\frac{2}{3}\int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_{0r}''}{\partial x_i},$$

$$I_3 = \frac{2}{3}\int_{R_3^+} \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}(\nabla u)}{\partial \zeta_k^t} \frac{\partial u_s''}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_{0r}'}{\partial x_i}.$$

Substituting $\psi = u - u_0$ into (2.9) and using (2.24) and (ii), we obtain

$$(2.25)\quad \int_{R_3^+} c(\varrho) \sum_{i,r} \left(\frac{\partial u_r''}{\partial x_i}\right)^2 - \frac{1}{3} d(\varrho) \sum_{i,r} \left(\frac{\partial u_r'}{\partial x_i}\right)^4 \le \int_{R_3^+} \sum_{i,r} f_i^{r''} \frac{\partial (u_r - u_{0r})''}{\partial x_i} + \\ + \sum_{i,j,r,s} b_{ij}^{rs}(\nabla u) \frac{\partial u_s''}{\partial x_j} \frac{\partial u_{0r}''}{\partial x_i} - I_1 - I_2 - I_3.$$

In this case Nirenberg's lemma will be used for the function $v = u - \tilde{u}$. Thus for every positive ε we get

$$\|v'\|_{1,4}'^4 \le 9\varrho^2 \|v''\|_{1,2}'^2,$$

$$\|v''\|_{1,2}'^2 \le (1+\varepsilon)\|u''\|_{1,2}'^2 + C(\varepsilon)\|\tilde{u}''\|_{1,2}'^2,$$

$$\|u'\|_{1,4}'^4 \le (1+\varepsilon) 9\varrho^2 \|u''\|_{1,2}'^2 + C(\varepsilon)\left(\|\tilde{u}''\|_{1,2}'^2 + \|\tilde{u}'\|_{1,4}'^4\right).$$

By means of Nirenberg's lemma and the Fourier transform we get the continuous embedding of the space $W_2^2(R_3^+)$ into $W_4^1(R_3^+)$ and thus

$$\|u'\|'^4_{1,4} \leqq (1+\varepsilon)\, 9\varrho^2 \|u''\|'^2_{1,2} + C(\varepsilon)\|\tilde{u}\|'^2_{3,2}\,. \tag{2.26}$$

Using (2.26) in the inequality (2.25), we have

$$[c(\varrho) - (1+\varepsilon)\, 3d(\varrho)\varrho^2]\, \|u''\|'^2_{1,2} \leqq Z\,, \tag{2.27}$$

where Z contains the terms on the right-hand side of (2.25) and the second term on the right hand side of (2.26). The derivatives of the coefficients b_{ij}^{rs} are uniformly bounded on $\tilde{K}_R$ and we can estimate the right-hand side of (2.25) by

$$C\{\|u_0\|'_{3,2}\cdot\|f\|_{2,2} + \|u''\|'_{1,2}[\|u_0\|'_{3,2} + \|f\|_{2,2}] + \\ + \|u'\|'^3_{1,4}\,\|u_0'\|_{1,4} + \|u'\|^2_{1,4}\,\|u_0\|'_{3,2} + \|u''\|'_{1,2}\,\|u'\|'_{1,4}\,\|u_0'\|_{1,4}\}\,.$$

Further, in virtue of (2.26),

$$Z \leqq C(\varepsilon)\{\|u''\|'^{\frac{3}{2}}_{1,2}[\|f\|_{2,2} + \|u_0\|'_{3,2}] + \\ + \|u''\|'_{1,2}[\|\tilde{u}\|'_{3,2} + \|u_0\|'^2_{3,2} + \|f\|_{2,2} + 1] + \|u_0\|'^2_{3,2} + \|f\|^2_{2,2} + \|\tilde{u}\|'^3_{3,2} + 1\}\,.$$

In order to get a bound for $\|u''\|'_{1,2}$ we shall use a simple algebraic lemma:

2.5. Lemma. *Let $a_1, \ldots, a_n$ be nonnegative real numbers, $x \in R_1$. Then the inequality*

$$x^n \leq \sum_{i=1}^{n} a_i x^{n-i}$$

implies

$$x \leq \sum_{i=1}^{n} a_i^{1/i}\,. \tag{2.28}$$

Choosing a sufficiently small ε, we obtain

$$\|u''\|'_{1,2} \leq C(1 + \|u_0\|'_{3,2} + \|\tilde{u}\|'_{3,2} + \|f\|_{2,2})^2\,. \tag{2.29}$$

The rest of the proof is quite analogous to that of Lemma 2.1.

B. Bounded domain Ω.

In the next lemma, Ω will be a bounded domain with an infinitely smooth boundary in R_3. The required smoothness of the boundary can be described as follows:

The boundary $\partial\Omega$ is described in a neighbourhood of every point by an infinitely differentiable function τ which is defined on a ball

$$B_\sigma = \{x' = (x_1, x_2);\ \|x'\| < \sigma\}$$

in the corresponding coordinate system. The boundary is covered by a finite number P of such system, i.e., for every $x \in \partial\Omega$ there exists a coordinate system in which $x = (x_1, x_2, \tau_i(x_1, x_2))$. Let us suppose

$$(2.30)\qquad \begin{cases} V^i_\sigma = \{x;\ \|x'\| < \sigma,\ \tau_i(x') < x_3 < \tau_i(x') + \sigma\} \subset \Omega\,, \\ U^i_\sigma = \{x;\ \|x'\| < \sigma,\ \tau(x') - \sigma < x_3 < \tau_i(x')\} \subset R_3 - \bar{\Omega}\,, \\ \left|\dfrac{\partial \tau_i}{\partial x_1}(0)\right| + \left|\dfrac{\partial \tau_i}{\partial x_2}(0)\right| = 0\,, \end{cases}$$

for every $i = 1, \ldots, P$ and a suitable $\sigma > 0$. Let us add to $\{V^i_\sigma\}$ a domain V^0_σ with an infinitely smooth boundary and such that

$$(2.31)\qquad \bar{V}^0_\sigma \subset \Omega \qquad \text{and} \qquad \bigcup_{i=0}^{P} V^i_\sigma \supset \Omega\,.$$

Let $\{\gamma^i_\sigma\}_{i=0}^P$ be a partition of unity corresponding to the system $\{V^i_\sigma\}_{i=0}^P$, i.e., $\gamma^0_\sigma \in D(V^0_\sigma)$, the functions γ^i_σ (for $i = 1, \ldots, P$) are infinitely differentiable and supp $\gamma^i_\sigma \subset V^i_\sigma \cup \{x = (x', \tau_i(x'));\ x' \in B^i_\sigma\}$ for $i = 1, \ldots, P$. Moreover, all γ^i_σ are nonnegative and

$$(2.32)\qquad \sum_{i=0}^{P} \gamma^i_\sigma = 1 \qquad \text{on } \Omega\,.$$

The form of the apriori estimate for a bounded domain is quite analogous to that for a half space. We include it for completeness and point out the different part in the proof.

2.6. LEMMA. *Let $\tilde{u} \in [W^3_2(\Omega)]^m$ and $\langle \nabla \tilde{u} \rangle \subset M$. Let R be a positive number with the following properties:*

(i) $\tilde{K}_R = \langle \nabla \tilde{u} \rangle + K_R \subset M$,

(ii) *there exist functions*

$$c\colon (0, R) \to (0, +\infty)\,,$$

$$d\colon (0, R) \to R_1\,,$$

such that, for every $\eta \in R_{3m}$, $\varrho \in (0, R)$ *and* $\xi \in \tilde{K}_\varrho = \langle \nabla \tilde{u} \rangle + K_\varrho$,

$$(2.33)\qquad \begin{cases} \sum\limits_{i,j,r,s} b_{ij}^{rs}(\xi)\eta_i^r\eta_j^s \geq c(\varrho) \sum\limits_{i,r} (\eta_i^r)^2 , \\ \sum\limits_{\substack{i,j,k,l\\ r,s,t,v}} \dfrac{\partial^2 b_{ij}^{rs}}{\partial \zeta_k^t \partial \zeta_l^v} \eta_i^r \eta_j^s \eta_k^t \eta_l^v \leq d(\varrho) \sum\limits_{i,r} (\eta_i^r)^4 , \end{cases}$$

(iii) $3d(\varrho)\varrho^2 < c(\varrho)$ *for every* $\varrho \in (0, R)$.

Then there exists a continuous function $D\colon (0, R) \to (0, +\infty)$ *such that for every* $u \in [W_2^3(\Omega)]^m$, *which solves the equation* (1.2) *with the boundary value* $u_0 \in [W_2^3(\Omega)]^m$ *and the right-hand side* $f \in [W_2^2(\Omega)]^{3m}$ *and satisfies the condition*

(iv) $\|u - \tilde{u}\|_{C^1(\Omega)} \leq \varrho$,

the estimate

$$(2.34)\qquad \|u\|_{3,2} \leq D(\varrho)\big(1 + \|\tilde{u}\|_{3,2} + \|u_0\|_{3,2} + \|f\|_{2,2}\big)^2$$

holds.

PROOF. Apriori estimate on V_σ^0 will be obtained by choosing a suitable test function of the form $(\gamma_\sigma^0)^{2K} \cdot u$ for a sufficiently large K. Further we can transform the considered part of V_σ^i into a half-space and use the methods of previous lemma.

With respect to the equivalence of the norms in the space $\mathring{W}_2^3(\Omega)$, it is sufficient to bound the highest derivatives of u. The index σ will be fixed throughout the proof and will be omitted.

PART I. *Estimates on* V^0.

Substituting the test function $\varphi = (\gamma^0)^{2K} \cdot u$ into (2.9), we can divide the left hand side into two parts, one part containing the highest derivatives of u and the second one with the derivatives of non-zero order of $(\gamma^0)^{2K}$. The symbol w' denotes here the first derivative of the function w in any direction. Thus

$$(2.35)\qquad \int\limits_\Omega \sum_{i,j,r,s} b_{ij}^{rs} \frac{\partial u_r''}{\partial x_i} \frac{\partial u_s''}{\partial x_j} (\gamma^0)^{2K} + \sum_{\substack{i,j,k\\ r,s,t}} \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t} \frac{\partial u_r''}{\partial x_i} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} (\gamma^0)^{2K} = Z_1 + Z_2 + Z_3 ,$$

where

$$Z_1 = \int\limits_\Omega \sum_{i,r} f_i^{r\prime} \frac{\partial}{\partial x_i} \big(u \cdot (\gamma^0)^{2K}\big)_r'' ,$$

Z_2 consists of the terms of the form

$$\int\limits_\Omega b_{ij}^{rs} \frac{\partial u_s''}{\partial x_j} D^\alpha u_r D^\beta (\gamma^0)^{2K} ,$$

and Z_3 contains the expressions

$$\int_\Omega \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t} \frac{\partial u_r'}{\partial x_i} \frac{\partial u_s'}{\partial x_j} D^\alpha u_t D^\beta (\gamma^0)^{2K}$$

with $|\beta| \neq 0$, $|\alpha + \beta| = 3$, $D^{\alpha+\beta} w = \partial w''/\partial x_k$.

Using the equality (2.23) for this special case of $\varphi = (\gamma^0)^{2K} \cdot u$, we have that the second integral in (2.35) equals

$$\frac{1}{3} \int_\Omega \sum_{\substack{i,j,k,l \\ r,s,t,v}} \frac{\partial^2 b_{ij}^{rs}}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_r'}{\partial x_i} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l} (\gamma^0)^{2K} - Z_3 + Z_4 + Z_5 , \tag{2.36}$$

where Z_4 contains the expressions

$$\int_\Omega \frac{\partial^2 b_{ij}^{rs}}{\partial \zeta_k^t \partial \zeta_l^v} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_l} D^\alpha u_r D^\beta (\gamma^0)^{2K}$$

and Z_5 the expressions

$$\int_\Omega \frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t} \frac{\partial u_s''}{\partial x_j} \frac{\partial u_t'}{\partial x_k} D^\alpha u_r D^\beta (\gamma^0)^{2K} ,$$

where $|\alpha| \leq 1$ and $D^{\alpha+\beta} w = \partial w'/\partial x_i$. According to (2.33) (ii), (2.35), and (2.36), we obtain

$$\int_\Omega \sum_{i,r} (\gamma^0)^{2K} \left[c(\varrho) \left(\frac{\partial u_r''}{\partial x_i} \right)^2 - \frac{1}{3} d(\varrho) \left(\frac{\partial u_r'}{\partial x_i} \right)^4 \right] \leq \leq c(Z_1 + Z_2 + Z_3 + Z_4 + Z_5) = Z . \tag{2.37}$$

To follow the method of the proof of Lemma 2.1 we have to overcome two technical difficulties. In the first step we must replace u by $v = \tilde{u} - u$ in order to get sharp estimates, but Nirenberg's lemma cannot be used directly for the function $(\partial v_r/\partial x_i)(\gamma^0)^{K/2}$ because the powers of γ^0 on the left-hand side of (2.37) would not agree. We shall use the following modification:

2.7. LEMMA. *Let w be a bounded measurable function on a domain $\Omega \subset R_N$, j be a positive integer less than or equal to N. Suppose that the derivatives*

$$\frac{\partial w}{\partial x_j}, \qquad \frac{\partial^2 w}{\partial x_j^2}$$

exist in a weak sense and are square integrable. Let $\psi \in D(\Omega)$ *be a nonnegative function. Then* $\partial w/\partial x_j \cdot \psi \in L_4(\Omega)$ *and*

$$\int_\Omega \left(\frac{\partial}{\partial x_j}(w\cdot\psi)\right)^4 \leq 9\cdot\sup_{x\in\Omega}|w(x)|^2\cdot\int_\Omega\left(\frac{\partial^2}{\partial x_j^2}(w\cdot\psi)\right)^2\cdot\psi^2 .$$

The lemma can be proved by means of the simple integration by parts and the Hölder's inequality.

If we apply Lemma 2.7 to the functions $w_{ir} = (\partial v_r/\partial x_i)(\gamma^0)^{K/2}$ simultaneously, we shall obtain

$$(2.38)\qquad (c(\varrho) - 3d(\varrho)\varrho^2)\int_\Omega(\gamma^0)^{2K}\sum_{i,r}\left(\frac{\partial v_r''}{\partial x_i}\right)^2 \leq Z + Z_6 + Z_7 + Z_8 .$$

Here Z_6 contains the terms

$$\int_\Omega (D^\alpha v_r\cdot D_\beta(\gamma^0)^{K/2})^4$$

with $|\beta|\neq 0$, $D^{\alpha+\beta}w = \partial w'/\partial x_i$, Z_7 contains the expressions

$$\int_\Omega\left(\frac{\partial v_r'}{\partial x_i}[(\gamma^0)^{K/2}]'\right)^2\cdot(\gamma^0)^K ,$$

and Z_8 the difference between the left-hand side of (2.37) and the same expressions with v instead of u. Returning to the solution u on the left-hand side of (2.38), we get another additional term which can be included into Z_8.

We shall now estimate Z_i by means of

$$I = \left(\int_\Omega(\gamma^0)^{2K}\sum_{i,r}\left(\frac{\partial u_r''}{\partial x_i}\right)^2\right)^{\frac12}$$

to the power less than 2 and the norms of f, $\tilde u$, u_0. Let us recall that

$$\|\nabla v\|_{C^1}\leq\varrho$$

and

$$\|v\|_{1,2}\leq\|(v+\tilde u-u_0)\|_{1,2}+\|\tilde u-u_0\|_{1,2} .$$

The function $v+\tilde u+u_0\in[\mathring{W}_2^1(\Omega)]^m$ and the equivalence of the norms in this space implies that there exists a constant C so that

$$
(2.39)\qquad \begin{cases} \|v\|_{12} \leqslant C\vartheta\,, \\ \|u\|_{12} \leqslant C\vartheta\,, \\ \max\left\{\sum_{i,r}\left[|u_r(x)| + \left|\frac{\partial u_r}{\partial x_i}(x)\right|\right];\ x \in \bar{\Omega}\right\} \leqslant C\vartheta\,, \end{cases}
$$

with $\vartheta = 1 + \|\tilde{u}\|_{3.2} + \|u_0\|_{3.2}$.

If we write

$$
J_n = \left(\int_\Omega \sum_{i,r}\left(\frac{\partial u'_r}{\partial x_i}(\gamma^0)^n\right)^2\right)^{\frac{1}{2}}
$$

and

$$
L = \left(\int_\Omega \sum_{i,r}\left(\frac{\partial u'_r}{\partial x_i}(\gamma^0)^{K/2}\right)^4\right)^{\frac{1}{4}}
$$

we get for $K \geqslant 4$

$$
(2.40)\qquad \begin{cases} Z_1 \leqslant C\|f\|_{2.2}(I + \vartheta + J_{2K-1})\,, \\ Z_2 \leqslant C\cdot I\cdot(\vartheta + J_{K-1})\,, \\ Z_3 \leqslant C\cdot L\cdot(\vartheta + J_{K-1})\,, \\ Z_4 \leqslant C\cdot L^3\cdot\vartheta\,, \\ Z_5 \leqslant C\cdot I\cdot J_{K-2}\cdot\vartheta\,, \\ Z_6 \leqslant C\vartheta\,, \\ Z_7 \leqslant C(J_{K-1}^2 + \vartheta)\,, \\ Z_8 \leqslant C\big(I[1 + \|\tilde{u}\|_{3.2}^2] + L^3\|\tilde{u}\|_{3.2} + \|\tilde{u}\|_{3.2}^4 + 1\big)\,. \end{cases}
$$

To bound the integrals I and J_n we shall use the following inequalities (which can be proved integrating by parts) and Lemma 2.5.

2.8. LEMMA. *Let $h \in W_2^2(\Omega)$ be bounded by a constant M and $\gamma \in D(\Omega)$. Then there exists a constant C depending only on γ and Ω so that*

$$
(2.41)\qquad \left\|\frac{\partial h}{\partial x_i}\gamma^n\right\|_{0.2} \leqslant C\left(M + M^{\frac{1}{2}}\left\|\frac{\partial^2 h}{\partial x_i^2}\gamma^{2n}\right\|_{0.2}^{\frac{1}{2}}\right)
$$

and

$$
\left\|\frac{\partial h}{\partial x_i}\gamma^{K/2}\right\|_{0.4} \leqslant CM + (3M)^{\frac{1}{2}}\left\|\frac{\partial^2 h}{\partial x_i^2}\gamma^{2K}\right\|_{0.2}^{\frac{1}{2}}.
$$

Putting $h = \partial u_r/\partial x_i$, we obtain

$$\left\{\begin{aligned} J_n &\leq C(\vartheta + \vartheta^{\frac12} I^{\frac12}), \\ L &\leq C\vartheta + (3\vartheta I)^{\frac12}, \end{aligned}\right. \tag{2.42}$$

and with respect to (2.38), (2.40) and Lemma 2.5 we get finally

$$I \leq \frac{C}{c(\varrho) - 3\varrho^2 d(\varrho)} (1 + \vartheta^2 + \|f\|^2_{2,2}), \tag{2.43}$$

and

$$\left\|\frac{\partial^2 u_r}{\partial x_i^2} (\gamma^0)^\varkappa\right\|_{1,2} \leq D(\varrho)(1 + \|\tilde{u}\|^2_{3,2} + \|u_0\|^2_{3,2} + \|f\|^2_{2,2}). \tag{2.44}$$

PART 2. *Estimates on* V^i.

Let us fix an index $i = 1, \ldots, P$ and consider the corresponding coordinate system. Let $\bar{x} = \Phi(x)$ be the coordinates of a point $x \in R_3$ in this system. We shall transform the set V^i into a part of a half-space. Let ψ be a mapping

$$\psi\colon \bar{x} \in \Phi(V^i) \to (\bar{x}_1, \bar{x}_2, \bar{x}_3 - \tau_i(\bar{x}_1, \bar{x}_2)) \in R_3$$

and $\chi = \psi \circ \Phi\colon V^i \to R_3^+$. Then the absolute value of Jacobi's determinant of χ is identically equal to 1. Let us write

$$\alpha_{ij} = \frac{\partial \chi_i}{\partial x_j}.$$

Let us define

$$\tilde{\Omega} = \chi(V^i),$$

$$\bar{f}^r_k\colon \bar{x} \in \tilde{\Omega} \to \sum_{i=1}^{3} \alpha_{ki}(\chi^{-1}(\bar{x}))\, f^r_i(\chi^{-1}(\bar{x})),$$

$$\bar{a}^r_k\colon (\bar{x}, \zeta) \in \tilde{\Omega} \times R_{3m} \to \sum_{i=1}^{3} \alpha_{ki}(\chi^{-1}(\bar{x}))\, a^r_i\Big(\sum_{p=1}^{3} \zeta^s_p \alpha_{pj}(\chi^{-1}(\bar{x}))\Big).$$

If we consider the test function $\varphi \in [\mathring{W}^1_2(V^i)]^m$ in the equality (1.2) and write $\bar{u} = u \circ \chi^{-1}$, we can conclude that $\bar{u}$ solves the equation of the form analogous to (1.2) on $\tilde{\Omega}$ and the interesting part of the boundary $\partial\tilde{\Omega}$ is a part of a plane. Especially we get that for every $\bar{\varphi} \in [\mathring{W}^1_2(\tilde{\Omega})]^m$,

$$\int_{\tilde{\Omega}} \sum_{k,r} \frac{\partial \bar{\varphi}_r}{\partial \bar{x}_k}(\bar{x})[\bar{a}^r_k(\bar{x}, \nabla\bar{u}(\bar{x})) - \bar{f}^r_k(\bar{x})]\, d\bar{x} = 0. \tag{2.45}$$

Now the coefficients $\bar{a}_k^r$ satisfy the conditions (2.33) (ii) with the same constants $c(\varrho)$ and $d(\varrho)$ on the equally transformed balls $\tilde{K}_\varrho$. If we write $\bar{\gamma}^i = \gamma^i \circ \chi^{-1}$ and $\bar{u}_0 = u_0 \circ \chi^{-1}$ we shall repeat the proof of Lemma 2.4 choosing the test function $\bar{\varphi} = (\bar{\gamma}^i)^{2K}(\bar{u} - \bar{u}_0)$. Combining the estimates of Lemma 2.4 and 2.6, we get

$$\int_{\tilde{\Omega}} (\bar{\gamma}^i)^{2K} \sum_{j=1}^{2} \sum_{k=1}^{3} \sum_{r=1}^{m} \left(\frac{\partial^3 \bar{u}_r}{\partial \bar{x}_k\, \partial \bar{x}_j^2}\right)^2 \leqslant C(1 + \|\bar{f}\|_{2,2}^2 + \|\tilde{u}\|_{3,2}^2 + \|\bar{u}_0\|_{3,2}^2) ,$$

and using the equivalence of the norms, we have

$$(2.46) \qquad \int_{\tilde{\Omega}} (\bar{\gamma}^i)^{2K} \sum_{p,j=1}^{2} \sum_{k,r} \left(\frac{\partial^3 \bar{u}_r}{\partial \bar{x}_p\, \partial \bar{x}_j\, \partial \bar{x}_k}\right)^2 \leqslant C(1 + \|\bar{f}\|_{2,2} + \|\tilde{u}\|_{3,2} + \|\bar{u}_0\|_{3,2})^2 .$$

An additional term with the derivatives $\partial \bar{a}/\partial x_i$ does not cause any difficulty as α_{kj} are infinitely smooth functions. The functions

$$\frac{\partial^2 \bar{u}_r}{\partial \bar{x}_k\, \partial \bar{x}_j} (\bar{\gamma}^i)^{K/2}$$

belong to $L_4(\tilde{\Omega})$ according to Nirenberg's lemma for all the indices k, j such that $k + j \leqslant 5$. The second derivative of $\bar{u}$ with respect to $\bar{x}_3$ can be calculated directly from the equation as in the proof of Lemma 2.4. By the same method the last third derivatives can be estimated. Thus we get the inequality

$$(2.47) \qquad \left\|(\bar{\gamma}^i)^K \sum_{j,k,p,r} \frac{\partial^3 \bar{u}_r}{\partial \bar{x}_j\, \partial \bar{x}_k\, \partial \bar{x}_p}\right\|_{0,2} \leqslant C(1 + \|\bar{f}\|_{2,2}^2 + \|\tilde{u}\|_{3,2}^2 + \|\bar{u}_0\|_{3,2}^2) .$$

Returning to the original coordinate system, we obtain the estimates for the solution u on the set V^i and summing these estimates for $i = 0, 1, \ldots, P$, we get

$$\|u\|_{3,2} \leqslant D(\varrho)(1 + \|\tilde{u}\|_{3,2}^2 + \|u_0\|_{3,2}^2 + \|f\|_{2,2}^2) .$$

3. – Main theorem.

The main theorem will be proved by means of the implicit function theorem and the apriori estimates of the last paragraph.

In this part the symbol Ω denotes the half-space R_3^+ or a bounded domain with a smooth boundary, simultaneously.

3.1. THEOREM. *Let* $\tilde{u} \in [W_2^{3'}(\Omega)]^m$ *be a solution of the equation* (1.2) *corresponding to the right-hand side* $\tilde{f} \in [W_2^2(\Omega)]^{3m}$ *and the boundary condition* $\tilde{u}_0 \in [W_2^{3'}(\Omega)]^m$. *Let us suppose that there exists a positive number* R *satisfying the conditions* (2.33) *of Lemma* 2.6.

Then for every $\varrho \in (0, R)$ *there exist positive numbers* ε, δ *so that, for each*

$$f \in U_\delta(\tilde{f}) = \{f \in [W_2^2(\Omega)]^{3m};\ \|f - \tilde{f}\|_{2,2} < \delta\}$$

and

$$u_0 \in U_\delta(\tilde{u}_0) = \{u \in [W_2^{3'}(\Omega)]^m,\ \|u - \tilde{u}_0\|_{3,2} < \delta\},$$

there exists a unique solution

$$u \in U_\varepsilon(\tilde{u}) = \\ = \{u \in [W_2^{3'}(\Omega)]^m,\ \|u - \tilde{u}\|'_{3,2} < \varepsilon,\ u - u_0 \in [\mathring{W}_2^{1'}(\Omega)]^m,\ \langle \nabla u \rangle \subset K_\varrho + \langle \nabla \tilde{u} \rangle\}$$

of the equation (1.2) *corresponding to the right hand side* f *and the boundary condition* u_0. *If we denote by* G *the mapping of* $U_\delta(\tilde{f}) \times U_\delta(\tilde{u}_0)$ *into* $U_\varepsilon(\tilde{u})$ *such that* $G(f, u_0) = u$, *then* G *is a continuous mapping from*

$$[W_2^2(\Omega)]^{3m} \times [W_2^{3'}(\Omega)]^m \to [W_2^{3'}(\Omega)]^m .$$

PROOF. Let us put

$$N = \{[u_0, u];\ u, u_0 \in [W_2^{3'}(\Omega)]^m,\ u \in [\mathring{W}_2^{1'}(\Omega)]^m,\ \langle \nabla(u + u_0) \rangle \subset (\bar{K}_\varrho)^0\},$$

$$\mathfrak{G} = [W_2^2(\Omega)]^{3m} \times N,$$

$$X = [W_2^{3'}(\Omega) \cap \mathring{W}_2^{1'}(\Omega)]^m \qquad \text{with the topology of } [W_2^{3'}(\Omega)]^m,$$

$$Y = [W_2^2(\Omega)]^{3m} \times [W_2^{3'}(\Omega)]^m,$$

$$Z = [W_2^1(\Omega)]^m,$$

and let us define the mapping $\Phi: \mathfrak{G} \to Z$ in the following manner:

$$\Phi: (f, u_0, u) \in \mathfrak{G} \to g = \{g_r\}_{r=1}^m \in Z,$$

where

$$g_r = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \{a_i^r(\nabla(u + u_0)) - f_i^r\} .$$

The assertion of the Theorem 3.1 is an immediate consequence of the implicit function theorem applied to the mapping Φ. To verify the assumption of the implicit function theorem we shall need an embedding theorem of this type:

$$u \in W_2^{3'}(\Omega) \Rightarrow \frac{\partial u}{\partial x_i} \in C(\bar{\Omega}) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} \in L_4(\Omega)\,,$$

and, moreover,

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{C(\bar{\Omega})} \leqslant C \|u\|'_{3,2}\,, \tag{3.1}$$

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{0,4} \leqslant C \|u\|'_{3,2} \qquad \text{for } i, j = 1, 2, 3\,. \tag{3.2}$$

These theorems are well known (see J. Nečas [6]) for a bounded domain Ω with a smooth boundary. For the case $\Omega = R_3^+$ the first inequality can be proved by means of Fourier transform and the second one is a consequence of (3.1) and Lemma 2.2.

Thus $\mathfrak{G}$ is an open subset of the Cartesian product $Y \times X$ and

$$\|g\|_{1,2} < C(1 + \|u\|'_{3,2} + \|u_0\|'_{3,2} + \|f\|_{2,2})\,.$$

The convergence of $\{u_n\}$ and $\{u_{0n}\}$ in $[W_2^{3'}(\Omega)]^m$ implies the uniform convergence of their gradients and together with the continuity of b_{ij}^{rs} it gives the continuity of the mapping Φ on $\mathfrak{G}$.

Let us prove that the partial differential Φ_2' exists on $\mathfrak{G}$ and it is a continuous mapping of $\mathfrak{G}$ into the space of all continuous linear mappings of X into Z. The above mentioned embedding theorem implies that the expression

$$\Phi_2'(f^*, u_0^*, u^*)(u) = \left\{ -\sum_{i,j,s} \frac{\partial}{\partial x_i} \left(b_{ij}^{rs}(\nabla(u^* + u_0^*))\right) \cdot \frac{\partial u_s}{\partial x_j} \right\}_{r=1}^m \tag{3.3}$$

belongs to the space Z for every $[u_0^*, u^*] \in N$. Let us write, for u sufficiently near to u^*,

$$\begin{aligned} H = \Phi(f^*, u_0^*, u^* + u) - \Phi(f^*, u_0^*, u^*) - \Phi_2'(f^*, u_0^*, u^*)(u) = \\ = \left\{ -\sum_i \frac{\partial}{\partial x_i} \left\{ a_i^r(\nabla(u_0^* + u^* + u)) - a_i^r(\nabla(u_0^* + u^*)) - \right.\right. \\ \left.\left. - \sum_{j,s} \left[b_{ij}^{rs}(\nabla(u_0^* + u^*)) \frac{\partial u_s}{\partial x_j} \right] \right\} \right\}_{r=1}^m \end{aligned} \tag{3.4}$$

Every component of H is a sum of the expressions of the type

$$\{b_{ij}^{rs}(\nabla(u_0^*+u^*+u))-b_{ij}^{rs}(\nabla(u_0^*+u^*))\}\frac{\partial^2(u_0^*+u^*+u)_s}{\partial x_i\,\partial x_j}- \\ -\sum_{k,t}\frac{\partial b_{ij}^{rs}}{\partial\zeta_k^t}(\nabla(u_0^*+u^*))\frac{\partial u_s}{\partial x_j}\frac{\partial^2(u_0^*+u^*)_t}{\partial x_i\,\partial x_k}\,. \tag{3.5}$$

After easy calculations (the mean value theorem) we get

$$\|H\|_{0,2}\leqslant C\|u\|'_{3,2}\Bigg\{\|u\|'_{3,2}+\|u^*+u_0^*\|'_{3,2}\cdot \\ \cdot\sup\left\{\left|\frac{\partial b_{ij}^{rs}}{\partial\zeta_k^t}(\nabla(u_0^*+u^*+\tau u))-\frac{\partial b_{ij}^{rs}}{\partial\zeta_k^t}(\nabla(u_0^*+u^*))\right|,\qquad x\in\bar{\Omega},\,i,\,j\right\}\Bigg\}$$

where τ is a measurable vector function on Ω with $|\tau|\leqslant 1$. Now the term in the brackets is of the order $o(\|u\|'_{3,2})$. The norm $\|H\|_{1,2}$ can be estimated quite analogously.

Let us prove the continuity of Φ_2'. Let $\{(f_n, u_{0n}, u_n)\}_{n=1}^\infty$ be a sequence in $\mathfrak{G}$ tending to $(f, u_0, u)\in\mathfrak{G}$ and put

$$A_n=[\Phi_2'(f_n, u_{0n}, u_n)-\Phi_2'(f, u_0, u)](v)= \\ =\left\{-\sum_{i,j,s}\frac{\partial}{\partial x_i}\left([b_{ij}^{rs}(\nabla(u_n+u_{0n}))-b_{ij}^{rs}(\nabla(u+u_0))]\cdot\frac{\partial v_s}{\partial x_j}\right)\right\}_{r=1}^m$$

and

$$\eta_n=\sup\{|b_{ij}^{rs}(\nabla(u_n+u_{0n})(x))-b_{ij}^{rs}(\nabla(u+u_0)(x))|,\,x\in\bar{\Omega},\,i,\,j,\,r,\,s\}\,.$$

We get immediately

$$\|A_n\|_{0.2}\leqslant\|v\|'_{2.2}\eta_n+C\|v\|'_{1.2}(\|u_n-u\|'_{2.2}+\|u_{0n}-u_0\|'_{2.2}+\|u+u_0\|'_{2.2}\eta_n)\,.$$

But the convergence of $\{(u_{0n}, u_n)\}_{n=1}^\infty$ to (u_0, u) implies that $\eta_n\to 0$ and thus

$$\|A_n\|_{0.2}\leqslant\|v\|'_{2.2}\varepsilon_n\qquad\text{and}\qquad\varepsilon_n\to 0\,.$$

Analogously we get the inequality

$$\|A_n\|_{1.2}\leqslant\|v\|'_{3.2}\varepsilon_n\qquad\text{and}\qquad\varepsilon_n\to 0\,,$$

and it gives the required result.

The most important part of the proof is that $\Phi_2'(\tilde{f}, \tilde{u}_0, \tilde{u})$ is a continuous one-to-one mapping of X on Z. Clearly Φ_2' is one-to-one: if $\Phi_2'(\tilde{f}, \tilde{u}_0, \tilde{u})\cdot$ $\cdot(v) = 0$, then

$$-\sum_{i,j,s} \frac{\partial}{\partial x_i}\left[b_{ij}^{rs}(\nabla(\tilde{u} + \tilde{u}_0))\frac{\partial v_s}{\partial x_j}\right] = 0$$

on Ω for $r = 1, \dots, m$. Multiplying every equation by v_r, integrating by parts over Ω, and using (2.33) (ii), we get

$$c(\varrho)\|v\|_{1,2}'^2 \leq \int_\Omega \sum_{i,j,r,s} b_{ij}^{rs}(\nabla(\tilde{u} + \tilde{u}_0))\frac{\partial v_s}{\partial x_j}\frac{\partial v_r}{\partial x_i} = 0$$

and it implies that $v = 0$.

In the next part we shall prove an apriori estimate for a linear partial differential equation with continuous coefficients

$$\Phi_2'(\tilde{f}, \tilde{u}_0, \tilde{u})(v) = g\,,$$

saying in fact that $[\Phi_2']^{-1}$ is a continuous mapping from Z in X. Such estimates can be obtained by the multiple use of regularity theorems for linear differential equations (see [7]). We shall prove it here for completeness by a method analogous to the proof of Lemma 2.4. For a bounded domain Ω we shall use the complete continuity of the embedding of the space $W_2^2(\Omega)$ into $C(\bar{\Omega})$ and $W_2^1(\Omega)$ into $L_4(\Omega)$. For a half-space such an assertion does not hold and we replace it by

3.2. LEMMA. *Let ε be a positive real number. Then there exists a constant $C(\varepsilon)$ so that, for each $h \in W_2^2(R_3^+)$,*

$$\sup\{|h(x)|, x \in R_3^+\} \leq \varepsilon\left(\sum_{i,j=1}^{3}\left\|\frac{\partial^2 h}{\partial x_i \partial x_j}\right\|_{0,2}\right) + C(\varepsilon)\|h\|_{0,2}\,, \tag{3.7}$$

$$\left\|\frac{\partial h}{\partial x_i}\right\|_{0,4} \leq \varepsilon\left(\sum_{i,j=1}^{3}\left\|\frac{\partial^2 h}{\partial x_i \partial x_j}\right\|_{0,2}\right) + C(\varepsilon)\|h\|_{0,2}\,. \tag{3.8}$$

PROOF. Let g be an extension of h on the whole space R_3. If we denote by $\hat{g}$ the Fourier transform of the function g, then

$$\sup\{|h(x), x \in R_3^+\} \leq \|\hat{g}\|_{L_1(R_3)}\,. \tag{3.9}$$

Using the Hölder's inequality we get for each positive η the relation

$$\|\hat{g}\|_{L_1(R_3)} \leq \left(\int_{R_3} |\hat{g}\cdot(1 + \eta|\xi|^2)|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int_{R_3} (1 + \eta|\xi|^2)^{-2}\, d\xi\right)^{\frac{1}{2}}. \tag{3.10}$$

The second integral on the left-hand side equals to $\pi\eta^{-\frac{3}{4}}$. Using Minkowski's inequality for the first integral on the left-hand side of (3.10), we get

$$\|\hat{g}\|_{L_1(R_3)} \leq C\eta^{\frac{1}{4}}\left(\sum_{i,j=1}^{3}\left\|\frac{\partial^2 g}{\partial x_i\,\partial x_j}\right\|_{0,2}\right) + C\eta^{-\frac{3}{4}}\|g\|_{0,2}\,. \tag{3.11}$$

We can choose the function g so that there exists a constant C' so that

$$\sum_{i,j=1}^{3}\left\|\frac{\partial^2 g}{\partial x_i\,\partial x_j}\right\|_{L_2(R_3)} \leq C'\left(\sum_{i,j=1}^{3}\left\|\frac{\partial^2 h}{\partial x_i\,\partial x_j}\right\|_{L_2(R_3^+)}\right)$$

and

$$\|g\|_{L_2(R_3)} \leq C'\|h\|_{L_2(R_3^+)}$$

and it gives, together with (3.9) and (3.11), the inequality (3.7). The inequality (3.8) is an easy consequence of (3.7) and Lemma 2.2.

Let us now return to the proof of the Theorem 3.1. Let $g \in Z$ and

$$g = \Phi_2'(\tilde{f}, \tilde{u}_0, \tilde{u})(v) = \left\{-\sum_{i,j,s}\frac{\partial}{\partial x_i}\left(b_{ij}^{rs}(\nabla(\tilde{u}+\tilde{u}_0))\frac{\partial v_s}{\partial x_j}\right)\right\}_{r=1}^{m} \tag{3.12}$$

and $\varphi \in [D(\Omega)]^m$. Multiplying g by φ and integrating by parts we get

$$B(v, \varphi) = \int_\Omega \sum_r g_r\varphi_r = \int_\Omega \sum_{i,j,r,s} b_{ij}^{rs}\frac{\partial v_s}{\partial x_j}\frac{\partial \varphi_r}{\partial x_i}\,, \tag{3.13}$$

The condition (2.33) (ii) and the Lax-Milgram theorem give

$$\|v\|_{1,2}' \leq C\|g\|_{0,2}'\,. \tag{3.14}$$

We shall repeat the procedure of the proofs of Lemma 2.4, respectively 2.6. Let $\Omega = R_3^+$ and let us take $\varphi = \psi''$ for $\psi \in [D(R_3^+)]^m$. (Here again $\psi' = \partial\psi/\partial x_i$ for $i = 1, 2$.) Integrating by parts in (3.13), we obtain the equality

$$\int_{R_3^+}\sum_{i,j,r,s}\left[b_{ij}^{rs}(\nabla(\tilde{u}+\tilde{u}_0))\frac{\partial v_s'}{\partial x_j} + \frac{\partial v_s}{\partial x_j}\cdot \sum_{k,t}\frac{\partial b_{ij}^{rs}}{\partial \zeta_k^t}(\nabla(\tilde{u}+\tilde{u}_0))\frac{\partial(\tilde{u}+\tilde{u}_0)_t'}{\partial x_k}\right]\frac{\partial \psi_r'}{\partial x_i} = \int_{R_3^+}\sum_r g_r\psi_r''\,. \tag{3.15}$$

Putting $\psi = v$ and using (2.33) (ii) and (3.8), we get

$$\|v'\|_{1,2}' \leq \varepsilon C\|v\|_{3,2}'(\|\tilde{u}\|_{3,2}' + \|\tilde{u}_0\|_{3,2}') + C(\varepsilon)(\|g\|_{0,2} + \|\tilde{u}\|_{3,2}' + \|\tilde{u}_0\|_{3,2}')\,. \tag{3.16}$$

The derivatives $\partial^2 v_r/\partial x_3^2$ can be calculated from the equation (3.12). Let now put $\psi = \chi''$, where $\chi \in [D(R_3^+)]^m$, and let us integrate once more by parts in (3.15). Putting $\chi = v$ and using (2.33) (ii), (3.8) and (3.7), we shall have

$$(3.17) \quad \|v''\|'_{1,2} \leq \varepsilon(\|\tilde{u}\|'_{3,2} + \|\tilde{u}_0\|'_{3,2})^2 \cdot \|v\|'_{3,2} + C(\varepsilon)(\|g\|_{1,2} + \|\tilde{u}\|'^2_{3,2} + \|\tilde{u}_0\|'^2_{3,2}) .$$

By standard means we estimate the last third derivatives and get, for sufficiently small ε,

$$(3.18) \qquad \|v\|'_{3,2} \leq C(\|g\|_{1,2} + \|\tilde{u}\|'^2_{3,2} + \|\tilde{u}_0\|'^2_{3,2}) .$$

If Ω is a bounded domain we have to return to the partition of unity and the test functions as in the proof of Lemma 2.6 but basically the proof remains without changes. Lemma 3.2 will be replaced by J. L. Lion's lemma for spaces with the completely continuous embedding (see [1]).

The equality $\Phi_2'(X) = Z$ will be proved by the homotopy method. Let us define

$$(3.19) \qquad b_{ij}^{rs}(t) = (1-t)\delta_{ij}\delta_{rs} + t \cdot b_{ij}^{rs}(\nabla(\tilde{u} + \tilde{u}_0)) ,$$

$$(3.20) \qquad A_t \colon v \in X \to \left\{ -\sum_{i,j,s} \frac{\partial}{\partial x_i}\left(b_{ij}^{rs}(t) \frac{\partial v_s}{\partial x_j} \right) \right\}_{r=1}^m \in Z$$

for every $t \in \langle 0, 1 \rangle$. A_t is a one-to-one continuous linear mapping of X into Z. It is well known that $A_0(X) = Z$. Let Q be maximum of the constant C of (3.18) and the norm of the mapping A_0^{-1}. Let us put

$$\mathfrak{P} = \{ t \in \langle 0. 1 \rangle ;\ A_t(X) = Z .\ \|A_t^{-1}\| \leq Q \} .$$

Then $0 \in \mathfrak{P}$. $\mathfrak{P}$ is a closed subset of $\langle 0, 1 \rangle$: Let $t_n \in \mathfrak{P}$, $t_n \to t$, $g \in Z$, and $v_n = A_{t_n}^{-1}(g)$. Then

$$(3.21) \qquad \|v_n\|'_{3,2} \leq Q \|g\|_{1,2} ,$$

$\{v_n\}$ is a bounded sequence in the Hilbert space X and thus there exists a subsequence (let us denote it by $\{v_n\}$, too) and an element $v \in X$ so that $\{v_n\}$ converges weakly to v. The embedding inequalities imply that

$$A_{t_n}(v_n) \to A_t(v)$$

in Z and thus $A_t(v) = g$. According to (3.21),

$$\|v\|'_{3,2} \leq Q \|g\|_{1,2}$$

and $t \in \mathfrak{P}$. $\mathfrak{P}$ is an open subset in $\langle 0, 1\rangle$: Let $t_0 \in \mathfrak{P}$, $g \in Z$, $v_0 = A_{t_0}^{-1}(g)$. Let us define a mapping $B: X \to X$ in the following manner:

$$B: v \in X \to v + A_{t_0}^{-1}(g - A_t(v)) .$$

Let $\mathfrak{K} = \{x \in X, \|v - v_0\|'_{3.2} \leq 1\}$. Let us prove that the mapping B has a fixed point in $\mathfrak{K}$ for a sufficiently small $|t - t_0|$. But $\mathfrak{K}$ is weakly compact and B is weakly continuous. Moreover,

$$\|B(v) - v_0\|'_{3.2} \leq \|A_{t_0}^{-1}\| \, \|(A_{t_0} - A_t)(v)\| ,$$

where $\|A_{t_0}^{-1}\| \leq Q$, and

$$\|(A_{t_0} - A_t)(v)\|_{1.2} \leq |t_0 - t| \, \|v\|'_{3.2} \cdot S$$

with S that depends on $\|\tilde{u}\|'_{3.2}$, $\|\tilde{u}_0\|'_{3.2}$, and does not depend on v and t. There exists a positive ε (it does not depend on g) so that, for $|t - t_0| < \varepsilon$, $B(\mathfrak{K}) \subset \mathfrak{K}$ and B has a fixed point $v \in \mathfrak{K}$, but such a function solves the equation $A_t(v) = g$.

3.3. THEOREM. *Let $\tilde{u} \in [W_2^{3'}(\Omega)]^m$ be the solution of the equation* (1.2) *corresponding to the right-hand side $\tilde{f} \in [W_2^2(\Omega)]^{3m}$ and the boundary value $\tilde{u} \in [W_2^{3'}(\Omega)]^m$. Let us suppose that there exists a positive number R satisfying the condition* (2.33) *of Lemma* 2.6. *Let F be a continuous mapping of $\langle 0, T\rangle$ into Y such that $F(0) = [\tilde{f}, \tilde{u}_0]$. Then there exists a unique continuous mapping U of $\mathfrak{D} \subset \langle 0, T\rangle$ into X so that*

$$\langle \nabla U(t)\rangle \subset (\tilde{K}_R)^0 , \tag{3.22}$$

(3.23) *$U(t)$ solves* (1.2) *with the right-hand side $f(t)$ and the boundary condition $u_0(t)$ such that $[f(t), u_0(t)] = F(t)$ for every $t \in \mathfrak{D}$.*

Moreover there are only two possibilities of the shape of the domain $\mathfrak{D}$: either $\mathfrak{D} = \langle 0, T\rangle$ or there exists a critical value $t_0 > 0$ so that $\mathfrak{D} \supset \langle 0, t_0)$ and

$$\lim_{t \to t_0} \Big(\inf_{x \in \overline{\Omega}} \Big(\operatorname{dist}\big(\{\nabla U(t)(x)\}, \partial \tilde{K}_R\big)\Big)\Big) = 0 . \tag{3.24}$$

PROOF. Theorem 3.1 implies immediately that the domain $\mathfrak{D}$ of the required « solution path » is open. Clearly $0 \in \mathfrak{D}$. Let us consider the component $\mathfrak{D}_1$ of $\mathfrak{D}$ containing the point 0. If $\mathfrak{D}_1 = \langle 0, T\rangle$ then the continuity of U follows from Theorem 3.1. Let $\mathfrak{D}_1 \neq \langle 0, T\rangle$ and let us put $\mathfrak{D}_1 = \langle 0, t_0)$. It remains to prove the condition (3.24). Let $t_n \in \langle 0, t_0)$,

$t_n \to t_0$, and

$$\lim_{n\to+\infty}\left(\inf_{x\in\bar{\Omega}}\left(\operatorname{dist}\left(\{\nabla U(t_n)(x)\}, \partial\tilde{K}_R\right)\right)\right) \geqslant \eta > 0 .$$

Then $\langle \nabla U(t_n)\rangle \subset \tilde{K}_{R-\eta/2}$ for a sufficiently large n, and using the continuity of F at the point t_0 and Lemma 2.4 or 2.6, we get the bound

$$\|U(t_n)\|'_{3,2} \leqslant C$$

for all sufficiently large n. According to the weak compactness of a ball in X, there exists a subsequence $\{U(t_{n_k})\}$ and a function $u \in X$ so that $\{U(t_{n_k})\}$ converges weakly to u in $[W_2^{3'}(\Omega)]^m$. By the usual limiting process we get that u solves the equation (1.2) with the right-hand side $f(t_0)$ and the boundary condition $u_0(t_0)$ and the uniqueness implies that $u = U(t_0)$. With respect to Theorem 3.1 we get the continuation of U on the interval $\langle t_0, t_0+\varepsilon)$, which is a contradiction with the assumption on the domain $\mathfrak{D}_1$.

3.4. THEOREM. *Let $M = R_{3m}$, let $\tilde{u} \in [W_2^{3'}(\Omega)]^m$ be the solution of the equation* (1.2) *corresponding to the right-hand side $\tilde{f} \in [W_2^2(\Omega)]^{2m}$ and the boundary value $\tilde{u}_0 \in [W_2^{3'}(\Omega)]^m$. Let us suppose that there exists a continuous function $c\colon R_{3m} \to \langle 0, +\infty)$ so that for all $\eta, \xi \in R_{3m}$ the inequality*

$$\sum_{i,j,r,s} b_{ij}^{rs}(\xi)\eta_i^r\eta_j^s \geqslant c(\xi)\sum_{i,r}(\eta_i^r)^2 \tag{3.25}$$

holds.

Let F be a continuous mapping of $\langle 0, T\rangle$ into Y such that $F(0) = [\tilde{f}, \tilde{u}_0]$. Then there exists a continuous mapping U of $\mathfrak{D} \subset \langle 0, T\rangle$ into X so that

(3.26) *$U(t)$ solves the equation* (1.2) *with the right-hand side $f(t)$ and the boundary value $u_0(t)$ such that $[f(t), u_0(t)] = F(t)$ for every $t \in \mathfrak{D}$.*

Moreover, there are only two possibilities of the shape of the domain $\mathfrak{D}$: either $\mathfrak{D} = \langle 0, T\rangle$ or there exists a critical point $t_0 \geqslant 0$ so that $\mathfrak{D} \supset \langle 0, t_0)$ and

$$\begin{aligned}&\sup\left\{\|U(t)\|'_{3,2},\ t\in\langle 0, t_0)\right\} = +\infty \quad \textit{or}\\ &\liminf_{t\to t_0}\left(\inf\left\{c(\nabla U(t)(x)),\ x\in\bar{\Omega}\right\}\right) = 0 .\end{aligned} \tag{3.27}$$

PROOF. Let $\mathfrak{D}_1$ and t_0 be defined as in the proof of Theorem 3.3. Let

$$\sup\left\{\|U(t)\|'_{3,2},\ t\in\langle 0, t_0)\right\} < +\infty .$$

Then we can choose a sequence $t_n \to t_0$ and a $u \in [W_2^3(\Omega)]^m$ so that $U(t_n) \rightharpoonup u$ in $[W_2^{3'}(\Omega)]^m$. According to the embedding theorems, u solves the equation with the right-hand side $f(t_0)$ and the boundary value $u_0(t_0)$ so that we can define $U(t_0) = u$. If $t_0 = T$ or

$$\liminf_{t\to t_0}\left(\inf\left\{c(\nabla U(t)(x)),\, x\in\bar\Omega\right\}\right) = 0$$

the proof is completed. Let $t_0 < T$ and

$$\liminf_{t\to t_0}\left(\inf\left\{c(\nabla U(t)(x)),\, x\in\bar\Omega\right\}\right) \geqslant k > 0\ .$$

Then there exists a neighbourhood $\langle\nabla U(t_0)\rangle + K_R$ of $\langle\nabla U(t_0)\rangle$ so that all the assumptions of Theorem 3.1 are satisfied and U is defined on $\langle 0, t_0 + \varepsilon)$, which is a contradiction with the assumption on $\mathfrak{D}_1$.

3.5. THEOREM. *Let $\tilde u \in [W_2^{3'}(\Omega)]^m$ be the solution of the equation* (1.2) *corresponding to the right-hand side $\tilde f \in [W_2^2(\Omega)]^{3m}$ and the boundary value $\tilde u_0 \in [W_2^{3'}(\Omega)]^m$. Let us suppose that there exists a continuous function $c\colon M \to$ $\to \langle 0, +\infty)$ so that for all $\eta \in R_{3m}$, $\xi \in M$ the inequality*

$$\sum_{i,j,r,s} b_{ij}^{rs}(\xi)\eta_i^r\eta_j^s \geqslant c(\xi)\sum_{i,r}(\eta_i^r)^2 \tag{3.28}$$

holds.

Let F be a continuous mapping of $\langle 0, T\rangle$ into Y such that $F(0) = [\tilde f, u_0]$.

Then there exists a continuous mapping U of $\mathfrak{D} \subset \langle 0, T\rangle$ into X so that $U(t)$ solves the equation (1.2) *with the right-hand side $f(t)$ and the boundary value $u_0(t)$ (where $F(t) = [f(t), u_0(t)]$) for every $t \in \mathfrak{D}$.*

Moreover, there are only two possibilities of the shape of the domain $\mathfrak{D}$: either $\mathfrak{D} = \langle 0, T\rangle$ or there exists a critical value $t_0 \in \langle 0, T\rangle$ so that $\mathfrak{D} \subset \langle 0, t_0)$ and

$$\begin{aligned} &\sup\left\{\|U(t)\|'_{3,2},\, t\in\langle 0, t_0)\right\} = +\infty \quad \text{or}\\ &\lim_{t\to t_0}\left(\inf_{x\in\bar\Omega}\left(\operatorname{dist}\left(\{\nabla u(t), (x)\}, \partial M\right)\right)\right) = 0 \quad \text{or}\\ &\liminf_{t\to t_0}\left(\inf\left\{c(\nabla U(t)(x)),\, x\in\bar\Omega\right\}\right) = 0\ . \end{aligned} \tag{3.29}$$

The proof is quite analogous to the proof of Theorem 3.4.

REFERENCES

[1] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969 (Russian transl.: *Nekotoryje metody reshenija nelinejnych krajevych zadac*, Mir, Moscow, 1972).

[2] CH. B. MORREY Jr., *Multiple integrals in the calculus of variations*, Springer, Berlin, 1966.

[3] E. GIUSTI - M. MIRANDA, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Un. Mat. Ital. (4) **2** (1968), pp. 1-8.

[4] J. NEČAS, *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity*, to appear in Beiträge Anal. (1977).

[5] V. G. MAZJA, *Primery neregularnych reshenij kvazilinejnych ellipticeskich uravnenij s analiticeskimi koeficientami*, Funkcional'nyj Analiz, **2** (1968), vyp. 3.

[6] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[7] S. AGMON - A. DOUGLIS - J. NIRENBERG, *Estimates near boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math., **12** (1959), pp. 623-727; *II*, Comm. Pure Appl. Math., **17** (1964), pp. 35-92.