ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

C. DENSON HILL
GERALDINE TAIANI

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 5, nº 2 (1978), p. 327-380

<http://www.numdam.org/item?id=ASNSP_1978_4_5_2_327_0>

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# Families of Analytic Discs in $\mathbf{C}^n$ with Boundaries on a Prescribed *CR* Submanifold (*).

C. DENSON HILL (**) - GERALDINE TAIANI (**)

*dedicated to Hans Lewy*

## 1. – Introduction.

The inspiration for this work stems from the series of penetrating papers [12], [13], [14], of Hans Lewy. Let $S$ be a sufficiently smooth real hypersurface in $\mathbf{C}^n$ $(n \geqslant 2)$ whose Levi form at the origin does not vanish identically. Then there is an open set $\Omega$ in $\mathbf{C}^n$, lying on one side of $S$, with $S \cap \bar{\Omega}$ a neighborhood of the origin in $S$, such that any sufficiently smooth function $u_0$ on $S \cap \bar{\Omega}$, which satisfies the tangential Cauchy-Riemann equations to $S$ there, has a unique extension to a holomorphic function $u$ in $\Omega$ with $u|_{S \cap \bar{\Omega}} = u_0$. This is the well known theorem of Hans Lewy (presented in [12] for the case $n = 2$; see [22], [10] for a proof when $n > 2$). The same sort of extension phenomenon can also occur, as Lewy demonstrated in [14], when the hypersurface $S$ is replaced by a real submanifold $M$ in $\mathbf{C}^n$ whose codimension is greater than one.

The region $\Omega$ mentioned above is the region swept out by the interiors of an appropriately chosen family of complex one-dimensional analytic discs in $\mathbf{C}^n$, whose boundaries lie on $M$ (or $S$). The holomorphic extension $u$ can be obtained via the Cauchy integral formula by integrating $u_0$ around the boundary of each analytic disc. In the hypersurface case, the requisite family of analytic discs can be obtained simply by an elementary slicing technique, using an appropriate system of local holomorphic coordinates. When $M$ has codimension greater than one, it can be shown that no such elementary slicing technique will work to produce the requisite discs;

(*) Research supported by a National Science Foundation Grant.
(**) SUNY at Stony Brook.
Pervenuto alla Redazione il 1° Giugno 1977.

in fact, in order to find even a single analytic disc with boundary on $M$, it is necessary to solve a certain system of nonlinear singular integral equations: Following the work of Lewy, Bishop [3] introduced a functional equation, involving the Hilbert transform $T$ on the unit circle, which must be solved in order to produce such a disc. Bishop then showed how to solve the functional equation by the method of successive approximations, working in the Sobolev space $H_1(S^1)$ and using the boundedness of $T$ on $L^2(S^1)$. He thereby produced a particular family of analytic discs with boundaries on $M$ which depends on certain parameters involved in the construction. Since then a number of authors [3], [22], [6], [11] have discussed generalizations of Lewy's ideas, in various forms. But they have all relied rather heavily on the use of the discs constructed by Bishop, modifying his argument only by working in a higher Sobolev space $H_s(S^1)$, deriving estimates for higher derivatives, and thereby getting increased smoothness of the discs and their dependence on parameters. This involved a loss of approximately $n/2$ derivatives between the original manifold $M$ and the solution. The best results in that direction are those of Weinstock [21].

In any event experience has shown that the hardest part of the problem in codimension greater than one is involved with the investigation of the analytic discs and their properties; previous work has suffered from an inadequate discussion of these points. See for example [7] where the Lewy extension phenomenon is discussed by merely postulating the existence of a family of discs which sweep out a manifold with certain desired properties.

The purpose of this paper is to give a treatment of these questions about the analytic discs from a more precise and hopefully more useful point of view. Our main results should be thought of as providing local parameterization and lifting theorems for families of analytic discs in $C^n$ with boundaries on a prescribed $CR$ submanifold. Actually we treat these questions in three categories: $C^{k,\alpha}$, real analytic and $C^\infty$. Our viewpoint differs from that of previous authors in that we introduce suitable Banach spaces of parameters, and characterize each analytic disc with boundary on $M$ as the lift of a corresponding parameter disc in the tangent space. In particular, in the $C^{k,\alpha}$ category, we lose only $1+\varepsilon$ derivatives between the original manifold $M$ and the family of discs. To our knowledge these questions have never before been discussed in the real analytic and $C^\infty$ categories. The results alluded to here are summarized in Theorem 5.1, Theorem 6.1, Theorem 7.1, and Theorem 8.1. We have also obtained a useful stability result, Theorem 5.2.

To simplify conceptually the exposition we have derived our results via the implicit function theorem in Banach spaces. But we would like to emphasize that, since the implicit function theorem has the apparatus of the method of successive approximations stored up once and for all in its proof, our method is actually constructive. In order to obtain sharp results we have at certain points used an improved version of the implicit function theorem due to Nijenhuis, which uses the notion of strong differentiability.

We have made an important application of our work in Section 9 to solve a specific problem: Given a suitable manifold $M$ whose Levi form at the origin does not vanish identically, the problem is that of constructing a local manifold-with-boundary, $\tilde{M}$, of real dimension one greater than that of $M$, which is nicely attached along $M$ with $M$ as its boundary. This involves an additional difficulty which has the nature of a « regularity up to the boundary » type of problem. In previous work [3], [22], [21], [6] an $\tilde{M}$ was obtained merely as the image of the set where a certain Jacobian had maximal rank, thereby avoiding singularities which were present, and $M$ was attached to $\tilde{M}$ only in the sense of being in the point-set theoretic closure of $\tilde{M}$. We have achieved rather precise results in Theorem 9.1.

We would like to express our thanks to Michael E. Taylor for a number of useful suggestions, especially the use of the space $B^{\alpha}\{M\}$ in Section 7.

Another application is Theorem 8.2 on the extension of germs of holomorphic functions. As for the original question of extendibility in the sense of Hans Lewy, we have confined ourselves to a brief discussion in Section 10.

## 2. – Bishop's functional equation.

Let $M = M^{n+m} \subset \mathbf{C}^n$ be a real $n+m$ dimensional manifold embedded in $\mathbf{C}^n$. The precise differentiability class of $M$ will be specified later. Let $T_p(M)$ denote the real tangent space to $M$ at a point $p \in M$. The *holomorphic tangent space* at $p$ is the complex vector space $HT_p(M) = T_p(M) \cap \cap \sqrt{-1}\, T_p(M)$, and $\dim_{\mathbf{C}} HT_p(M)$ is the *CR-dimension* of $M$ at $p$. If the $CR$-dimension is minimal; i.e., $\dim_{\mathbf{C}} HT_p(M) = m$, $M$ is said to be *generic* at $p$. $M$ is called an *embedded CR-manifold* if its $CR$-dimension is a constant independent of $p$. Since genericness is an open condition, $M$ is always locally $CR$ near a generic point $p$. In what follows we assume that $M$ is an embedded $CR$-manifold of $CR$ dimension $m$ and of real codimension $l = n - m$.

For any point $p \in M$ we can find an affine complex linear change of coordinates so that $p = 0$ and $T_0(M) = \{y_1 = y_2 = \ldots = y_l = 0\}$ and $HT_0(M) = \{z_1 = z_2 = \ldots = z_l = 0\}$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$. We can express $M$ locally as a graph over its tangent space; so that $M = \{(z_1, \ldots, z_n) | y_j = = h_j(x_1, \ldots, x_l, z_{l+1}, \ldots, z_n), j = 1, \ldots, l\}$ where the $h_j$'s are real valued functions which vanish to 2nd order at the origin.

Consider a map $g: \bar{D} \to \mathbf{C}^n$ which is holomorphic in the *open unit disc* $D \subset \mathbf{C}$ and belongs to some differentiability class on its closure $\bar{D}$. Then $g$, or sometimes the image $g(D)$, will be called an *analytic disc* in $\mathbf{C}^n$. The restriction of $g$ to $S^1 = \partial D$, or sometimes the image $g(S^1)$, will be called the *boundary of the disc*.

Bishop derived a functional equation whose solution leads to the construction of a family of analytic discs whose boundaries lie on $M$. In order to derive this equation, Bishop first noted that if $h = (h_1, \ldots, h_l)$ is identically zero, then $z_1, \ldots, z_l$ are all real on $M$ and must, therefore, be real and constant on any analytic disc with boundary on $M$. Thus, for $h \equiv 0$, each analytic disc in $\mathbf{C}^n$ with boundary on $M$ is uniquely determined by an analytic disc in $\mathbf{C}^m$ in the variables $z_{l+1}, \ldots, z_n$ and by $l$ real constants for the values $z_1, \ldots, z_l$.

In general, suppose $g$ is an analytic disc whose boundary lies on $M$. Then, using an obvious vector notation for $u$ and $w$, we have that

$$g(S^1) = \{(u(e^{i\theta}) + ih(u(e^{i\theta}), w(e^{i\theta})), w(e^{i\theta}))\}$$

where $u_j(e^{i\theta}) + ih_j(u(e^{i\theta}), w(e^{i\theta}))$, $1 \leqslant j \leqslant l$, and $w_k(e^{i\theta})$, $1 \leqslant k \leqslant m$, are boundary values of holomorphic functions in $D$. Consider a $\mathbf{R}^l$ valued harmonic function $U$ in $D$ which belongs to an appropriate differentiability class on $\bar{D}$. Let $V$ be the unique conjugate harmonic function such that $V(0) = 0$. Let $T$ be the operator (acting componentwise) which takes the boundary values of $U$ to the boundary values of $V$. In our case, $h(u, w) - iu$ are the boundary values of a holomorphic function in $D$. Hence $T[h(u, w)]$ and $-u$ must differ by a constant $c = U(0) \in \mathbf{R}^l$, where $U$ denotes the harmonic function with boundary values $u$. Therefore the real part $u$ of the first $l$ components of $g(S^1)$ must satisfy Bishop's functional equation:

$$u(e^{i\theta}) = c - T[h(u(e^{i\theta}), w(e^{i\theta}))] . \tag{2.1}$$

On the other hand, suppose $c \in R^l$ is prescribed and $w: \bar{D} \to \mathbf{C}^m$ denotes an analytic disc in $\mathbf{C}^m$. If $u$ satisfies (2.1) then

$$f(e^{i\theta}) = u(e^{i\theta}) + ih(u(e^{i\theta}), w(e^{i\theta}))$$

are the boundary values of a holomorphic function $f: \bar{D} \to \mathbf{C}^l$ such that $\operatorname{Re} f(0) = c$. It follows that the function $g: \bar{D} \to \mathbf{C}^n$ defined by

$$g(\zeta) = (f(\zeta), w(\zeta)) , \quad \zeta \in \bar{D} \tag{2.2}$$

defines an analytic disc $g(D)$ in $\mathbf{C}^n$ whose boundary $g(S^1)$ lies on $M$. Thus whenever the Bishop functional equation can be solved, the solution provides a lifting:

$$\begin{array}{ccc} & & C^l\times\mathbf{C}^m \cong C^n\\ & \overset{g}{\nearrow} & \Big\downarrow{\scriptstyle (\mathrm{Re}f(0),id)}\\ \bar{D} & \xrightarrow[(c,w)]{} & \mathbf{R}^l\times\mathbf{C}^m \cong T_0(M)\end{array} \tag{2.3}$$

of an arbitrary analytic disc in $T_0(M)$ to an analytic disc in the ambient space with boundary on the graph $M$ over $T_0(M)$.

We summarize the above discussion in the following:

PROPOSITION 2.1. *An analytic disc $g(D)$ in $\mathbf{C}^n$ with $g(S^1)\subset M$ exists if and only if $u(e^{i\theta})$ is a solution of* (2.1) *corresponding to some choice of the constant $c\in\mathbf{R}^l$ and to some analytic disc $w(D)$ in $\mathbf{C}^m$.*

## 3. – Properties of the Hilbert transform on the circle.

The operator $T$ defined in Section 2 can be expressed in two distinct ways: Since $T$ acts componentwise, there is no loss of generality if we assume that $u$ is a real valued sufficiently regular function defined on the unit circle $S^1$. Then $u(e^{i\theta})$ has a Fourier series

$$u(e^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta),$$

and $Tu$ can be expressed as the conjugate Fourier series of $u$

$$Tu(e^{i\theta}) = \sum_{n=1}^{\infty}(-\,b_n\cos n\theta + a_n\sin n\theta)\,. \tag{3.1}$$

Alternately, $T$ can be written as the limit of a convolution operator with the conjugate Poisson kernel,

$$Q_r(\theta) = \mathrm{Im}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right);$$

i.e.,

$$\begin{aligned} Tu(e^{i\theta}) &= \lim_{r\uparrow 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\varphi})Q_r(\theta-\varphi)\,d\varphi = \\ &= \frac{1}{2\pi}\,\mathrm{p.v.}\int_{-\pi}^{\pi}u\big(e^{i(\theta-\varphi)}\big)\,\mathrm{Im}\left(\frac{1+e^{i\varphi}}{1-e^{i\varphi}}\right)d\varphi \end{aligned} \tag{3.2}$$

(see Hoffman [9] for more details). We shall call $T$ the Hilbert Transform on the unit circle. $T$ is closely related to the classical Hilbert Transform for the line under a transformation of the upper half plane to the unit circle, differing from it by a weight function on the measure. We shall be concerned with the properties of $T$ on the space $C^{k,\alpha}(S^1)$.

For any $K \subset R^n$, $K$ compact and $0 \leqslant \alpha \leqslant 1$, $C^\alpha(K) = C^{0,\alpha}(K)$ is defined by

$$C^\alpha(K) = \left\{ u : K \to \mathbf{R} \,\Big|\, |u|_\alpha \equiv \sup_{x \in K} |u(x)| + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

For $k$ any non-negative integer we define

$$C^{k,\alpha}(K) = \{ u \colon K \to \mathbf{R} \,\big|\, |u|_{k,\alpha} \equiv \sum_{|\beta| \leqslant k} |D^\beta u|_\alpha < \infty \}.$$

(For now our spaces $C^{k,\alpha}(K)$ will consist of real valued functions; in subsequent sections $C^{k,\alpha}(K)$ may sometimes consist of complex valued functions. Exactly which we intend will always be clear from the context, and should cause the reader no confusion.)

$C^{k,\alpha}(K)$ is a Banach algebra under the $|\ |_{k,\alpha}$ norm. We have the following proposition which, in the case $k = 0$, is just the classical theorem of Privaloff.

PROPOSITION 3.1. *Let $g(z) = U(z) + iV(z)$ be holomorphic in $D$ and $0 < \alpha < 1$. If $V(x, y) \in C^{k,\alpha}(S^1) \cap C^k(\bar{D})$ with norm $|V|_{k,\alpha}$ on $S^1$ then $g \in C^{k,\alpha}(\bar{D})$ and $|g|_{k,\alpha} \leqslant c|V|_{k,\alpha}$, where $c$ depends only on $k$ and $\alpha$. Thus $T \colon C^{k,\alpha}(S^1) \to C^{k,\alpha}(S^1)$ is a bounded linear operator. Moreover, $T$ commutes with (tangential) differentiation and $\|T\|_{k,\alpha} \leqslant \|T\|_\alpha$.*

PROOF. This proposition is well-known from several points of view—the reader who is willing to accept it as a fact may skip on to Section 4. However, in order to make this paper as elementary and self-contained as possible, we include a proof of the proposition based on the maximum principle, which is modeled on an elegant proof of Privaloff's theorem due to Bers ([4], p. 401):

Let $g$, $V$ be as above and $k = 0$. For any $\theta$, $\theta'$ we have $|V(e^{i\theta}) - V(e^{i\theta'})| \leqslant |V|_\alpha |e^{i\theta} - e^{i\theta'}|^\alpha$. Fixing $\theta'$, let $\Phi(z)$ denote the single-valued harmonic function in $D$ such that $\Phi(z) = \operatorname{Re}[(1 - ze^{-i\theta'})^\alpha]$ and $\Phi(0) = 1$. We have $(1 - ze^{-i\theta'}) = |1 - ze^{-i\theta'}|(\cos \nu + i \sin \nu)$, where $\nu$ is $\arg(1 - ze^{-i\theta'}) =$ the angle between the straight lines from 1 to the points 0 and $ze^{-i\theta'} =$ the angle between the straight lines from $e^{i\theta'}$ to the points 0 and $z$ (rotating by $e^{i\theta'}$). Thus, $(1 - ze^{-i\theta'})^\alpha = |1 - ze^{-i\theta'}|^\alpha(\cos \alpha\nu + i \sin \alpha\nu) = |z - e^{i\theta'}|^\alpha(\cos \alpha\nu + i \sin \alpha\nu)$. Therefore, $\Phi(z) = |z - e^{i\theta'}|^\alpha \cos \alpha\nu$. For $z$ on the unit circle $0 \leqslant |\nu| \leqslant \pi/2$, implying that

$$|z - e^{i\theta'}|^{\alpha} = \frac{\Phi(z)}{\cos \alpha\nu} \leqslant \frac{\Phi(z)}{\cos \alpha(\pi/2)}.$$

Hence $z \in S^1$ implies

$$\frac{-|V|_{\alpha}\Phi(z)}{\cos \alpha(\pi/2)} \leqslant V(z) - V(e^{i\theta'}) \leqslant \frac{|V|_{\alpha}\Phi(z)}{\cos \alpha(\pi/2)}.$$

Since $V(z) - V(e^{i\theta'})$ is harmonic we have

$$(3.3) \qquad |V(z) - V(e^{i\theta'})| \leqslant \frac{|V|_{\alpha}\Phi(z)}{\cos \alpha(\pi/2)} \quad \text{for all } z \in \bar{D}.$$

Consider the disc $C = \{z \,|\, |z - re^{i\theta'}| < 1 - r\}$. For $z \in C$ we have $|1 - ze^{-i\theta'}| = |e^{i\theta'} - z| \leqslant |z - re^{i\theta'}| + |re^{i\theta'} - e^{i\theta'}| < 1 - r + 1 - r = 2(1 - r)$. Thus for $z \in C$ (3.3) implies

$$|V(z) - V(e^{i\theta'})| \leqslant \frac{|V|_{\alpha}}{\cos \alpha(\pi/2)}[2(1-r)]^{\alpha}.$$

Now $V(z) - V(e^{i\theta'})$ is harmonic in $C \subset D$, so Poisson's formula holds. Thus for $\varrho < 1 - r$ and any $\varphi$,

$$V(\varrho e^{i\varphi} + re^{i\theta'}) - V(e^{i\theta'}) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{[V([1-r]e^{it} + re^{i\theta'}) - V(e^{i\theta'})][(1-r)^2 - \varrho^2]}{(1-r)^2 - 2\varrho(1-r)\cos(\varphi - t) + \varrho^2}\,dt.$$

Calculating the directional derivative in the direction determined by $\varphi$, we have

$$V_{\varrho}(re^{i\theta'}) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{[V([1-r]e^{it} + re^{i\theta'}) - V(e^{i\theta'})](1-r)^2 2(1-r)\cos(\varphi - t)}{(1-r)^4}\,dt,$$

which implies the bound

$$|V_{\varrho}(re^{i\theta'})| \leqslant \frac{|V|_{\alpha}2^{\alpha+1}(1-r)^{\alpha-1}}{\cos \alpha(\pi/2)},$$

independent of $\varphi$. Hence we have a bound on the gradient and it follows that $|V_x(re^{i\theta'})|$ and $|V_y(re^{i\theta'})|$ are both bounded by

$$\frac{|V|_{\alpha}2^{\alpha+1}(1-r)^{\alpha-1}}{\cos \alpha(\pi/2)}.$$

But $g'(z) = V_y(z) + iV_x(z)$ and since $\theta'$ was arbitrary this yields

$$|g'(z)| \leq \frac{8|V|_\alpha}{\cos\alpha(\pi/2)}(1-|z|)^{\alpha-1} \quad \text{for all } z\in\bar{D}\,.$$

Now $|g(z_2)-g(z_1)| = \left|\int_{z_1}^{z_2} g'(\zeta)\,d\zeta\right| = \left|\int_\gamma g'(\zeta)\,d\zeta\right|$ where $\gamma$ is any curve from $z_1$ to $z_2$. Assume $|z_1| \leq |z_2|$. Let $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$. We look at the following three cases:

*a*) $|z_2 - z_1| \leq 1 - |z_1|$;

*b*) $|z_2 - z_1| > 1 - |z_1|$ with $|z_2| \geq \frac{1}{4}$;

*c*) $|z_2 - z_1| > 1 - |z_1|$ with $|z_2| < \frac{1}{4}$.

*Case a*) Since $|z_1| \leq |z_2|$ we have $1-|z_1| \geq 1 - |z_2|$, $|z_1| < 1$, and $|z_2| - |z_1| = r_2 - r_1 \leq |z_2 - z_1|$. Let $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1(t) = r_1e^{it}$, $\theta_1 \leq t \leq \theta_2$ and $\gamma_2(r) = re^{i\theta_2}$, $r_1 \leq r \leq r_2$.

$$|g(z_2)-g(z_1)| = \left|\int_{\gamma_1} g'(\zeta)\,d\zeta + \int_{\gamma_2} g'(\zeta)\,d\zeta\right| \leq$$

$$\leq \left|\int_{\theta_1}^{\theta_2} g'(r_1e^{it})r_1ie^{it}dt\right| + \left|\int_{r_1}^{r_2} g'(re^{i\theta_2})e^{i\theta_2}dr\right| \leq$$

$$\leq C_\alpha|V|_\alpha\left[\int_{\theta_1}^{\theta_2}\frac{r_1}{(1-r_1)^{1-\alpha}}dt + \int_{r_1}^{r_2}\frac{1}{(1-r)^{1-\alpha}}dr\right] \leq$$

$$\leq C_\alpha|V|_\alpha\left[\frac{1}{(1-r_1)^{1-\alpha}}2|z_2-z_1| + \int_{1-r_2}^{1-r_2+|z_2-z_1|}\frac{ds}{s^{1-\alpha}}\right] \leq$$

$$\leq C_\alpha|V|_\alpha\left[\frac{2|z_2-z_1|}{|z_2-z_1|^{1-\alpha}} + \frac{(1-r_2+|z_2-z_1|)^\alpha - (1-r_2)^\alpha}{\alpha}\right] \leq C_\alpha|V|_\alpha|z_2-z_1|^\alpha$$

where $C_\alpha$ denotes a generic constant depending only on $\alpha$.

*Case b*) $|z_2 - z_1| > 1 - |z_1|$ and $r_2 = |z_2| \geq \frac{1}{4}$. Let $\delta = |z_2 - z_1| - (1 - r_1)$. Let $\gamma = -\gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_1(r) = re^{i\theta_1}$, $r_1 - \delta \leq r \leq r_1$, $\gamma_2(t) = (r_1-\delta)e^{it}$, $\theta_1 \leq t \leq \theta_2$ and $\gamma_3(r) = re^{i\theta_2}$, $r_1 - \delta \leq r \leq r_2$.

$$|g(z_2)-g(z_1)| \leq$$

$$\leq C_\alpha|V|_\alpha\left[\int_{r_1-\delta}^{r_1}\frac{dr}{(1-r)^{1-\alpha}} + \int_{\theta_1}^{\theta_2}\frac{r_1-\delta}{(1-(r_1-\delta))^{1-\alpha}}dt + \int_{r_1-\delta}^{r_2}\frac{dr}{(1-r)^{1-\alpha}}\right] \leq$$

$$\leqslant C_\alpha|V|_\alpha\left[\int_{1-r_1}^{|z_2-z_1|}\frac{ds}{s^{1-\alpha}}+\frac{|r_1-\delta|}{|z_2-z_1|^{1-\alpha}}|\theta_2-\theta_1|+\int_{1-r_2}^{|z_2-z_1|}\frac{ds}{s^{1-\alpha}}\right]\leqslant$$

$$\leqslant C_\alpha|V|_\alpha\left[2\int_0^{|z_2-z_1|}\frac{ds}{s^{1-\alpha}}+\frac{|z_2-z_1|+1}{|z_2-z_1|^{1-\alpha}}8|z_2-z_1|\right]\leqslant$$

$$\leqslant C_\alpha|V|_\alpha\left[\frac{2|z_2-z_1|^\alpha}{\alpha}+24|z_2-z_1|^\alpha\right]\leqslant C_\alpha|V|_\alpha|z_2-z_1|^\alpha$$

where $C_\alpha$ is a generic constant depending only on $\alpha$.

*Case c)* $|z_2-z_1|>1-|z_1|$, $r_1\leqslant r_2<\frac{1}{4}$. Thus

$$|g(z_2)-g(z_1)|\leqslant C_\alpha|V|_\alpha\frac{|z_2-z_1|}{(\frac{3}{4})^{1-\alpha}}\leqslant C_\alpha|V|_\alpha|z_2-z_1|^\alpha$$

again letting $C_\alpha$ be a generic constant.

Therefore, we have shown that $g\in C^\alpha(\bar{D})$.

The linearity of $T$ and the fact that $T$ commutes with differentiation is a consequence of the expression for $T$ in (3.2). The above inequalities imply that $T\colon C^\alpha(S^1)\to C^\alpha(S^1)$ is bounded. Therefore, $T\colon C^{k,\alpha}(S^1)\to C^{k,\alpha}(S^1)$ is bounded with $\|T\|_{k,\alpha}\leqslant\|T\|_\alpha$. For if $u\in C^{k,\alpha}(S^1)$ we have

$$|Tu|_{k,\alpha}=\sum_{j=0}^{k}|D^jTu|_\alpha=\sum_{j=0}^{k}|TD^ju|_\alpha\leqslant\|T\|_\alpha\sum_{j=0}^{k}|D^ju|_\alpha=\|T\|_\alpha|u|_{k,\alpha}.$$

We shall now prove that $V\in C^{k,\alpha}(S^1)\cap C^k(\bar{D})$ implies that $g\in C^{k,\alpha}(\bar{D})$. We have $D^j_\theta V\in C^\alpha(\bar{D})$ for $j\leqslant k$. Since $D^j_\theta V$ is harmonic on $D$ with harmonic conjugate $D^j_\theta U$ we have, from the above that $D^j_\theta g$, $D^j_\theta U\in C^\alpha(\bar{D})$. Since $g$ is analytic on $D$ the Cauchy Riemann equations hold, *i.e.* $D_\theta U(re^{i\theta})=-\tilde{D}_rV(re^{i\theta})$ and $D_\theta V(re^{i\theta})=+\tilde{D}_rU(re^{i\theta})$ for $0<r<1$ where $\tilde{D}_r=rD_r$. Thus for any $0\leqslant l\leqslant j$ we have

$$D^l_\theta\tilde{D}^{j-l}_rU(re^{i\theta})=\begin{cases}\pm D^j_\theta U(re^{i\theta}), & j-l \text{ even}\\ \pm D^j_\theta V(re^{i\theta}), & j-l \text{ odd}\end{cases}$$

and similarly for $D^l_\theta\tilde{D}^{j-l}_rV(re^{i\theta})$, $0<r<1$. This implies that $D^l_\theta\tilde{D}^{j-l}_rV$, $D^l_\theta\tilde{D}^{j-l}_rU\in C^\alpha(\bar{D})$ for all $0\leqslant l\leqslant j\leqslant k$. Since $U$ and $V$ are harmonic on $D$ we have $U$, $V\in C^\infty(D)$. Therefore $D^l_\theta D^{j-l}_rV$, $D^l_\theta D^{j-l}_rU$ and hence $D^l_\theta D^{j-l}_rg$ are $C^\alpha$ continuous on $\bar{D}$ for $0\leqslant l\leqslant j\leqslant k$. Thus $g\in C^{k,\alpha}(\bar{D})$.

All that remains to be proven is that $|g|_{k,\alpha}\leqslant C_{k,\alpha}|V|_{k,\alpha}$, *i.e.* the mapping $G\colon C^{k,\alpha}(S^1)\to C^{k,\alpha}(\bar{D})$ defined by $G(V)=g$ where $g|_{S^1}=-T(V|_{S^1})+iV|_{S^1}$,

$g$ holomorphic in $D$, is continuous. $G$ is a linear map between two Banach Spaces. By appealing to the closed graph theorem, all we need prove is that $G$ has closed graph. Let $(V_j, g_j)$ be a sequence in the graph of $G$. Suppose $V_j \to V$, $g_j \to g$ in $C^{k,\alpha}(S^1)$, $C^{k,\alpha}(\bar{D})$ respectively. In particular, $g_j \to g$ uniformly on $\bar{D}$ and therefore $g$ is holomorphic in $D$. Also

$$g|_{S^1} = \lim_{j\to\infty} (-T(V_{j|_{S^1}}) + iV_{j|_{S^1}}) = -T(V|_{S^1}) + iV|_{S^1}$$

since $T$ is continuous on $C^{k,\alpha}(S^1)$. Thus $g = G(V)$ and we are done.

## 4. – The implicit function theorem for strongly differentiable functions.

In order to solve the Bishop functional equation (2.1) we will make use of the implicit function theorem in Banach spaces. Recall that the standard elementary form of this theorem states that if $f(u, x)$ is a function of class $C^k$ $(k \geqslant 1)$ on a neighborhood of the origin in $E \times X$ into $F$, where $X$, $E$, $F$ are Banach spaces, if $f(0, 0) = 0$ and if the partial differential $D_u f(0, 0)$: $E \to F$ is an isomorphism of $E$ onto $F$, then the equation $f(u, x) = 0$ has a local unique solution $u = \varphi(x)$ of class $C^k$ in some neighborhood of the origin in $X$. Unfortunately this standard theorem does not give the best results when applied to (2.1). To obtain the sharp results we want in section 5 it is necessary to use an improved version of the implicit function theorem due to Nijenhuis [15]. His version uses the notion of strong partial differentiability, which is not standard and is a bit subtle upon first encounter. Therefore we state here its precise definition, list some basic properties, and give the exact statement of the theorem we need. All of this can be found in [15], with proofs.

Consider a function $f(u, x)$ from an open set in $E \times X$ to $F$, where $E$, $F$ are normed vector spaces and $X$ is a topological space. Then $f$ is called *strongly partially differentiable* with respect to $u$ at $(u_0, x_0)$ if:

(i) there is a continuous linear map $L: E \to F$ (also denoted by $D_u f(u_0, x_0)$) such that to every $\varepsilon > 0$ there is a $\delta > 0$ and a neighborhood $N(x_0)$ of $x_0$ in $X$ such that $|u_1 - u_0| < \delta$, $|u_2 - u_0| < \delta$, $x \in N(x_0)$ imply

$$|f(u_2, x) - f(u_1, x) - L(u_2 - u_1)| \leqslant \varepsilon |u_2 - u_1| ,$$

(ii) the map $x \mapsto f(u_0, x)$ of $X$ to $F$ is continuous at $x_0$.

Note that these hypotheses imply that $f$ is continuous at $(u_0, x_0)$.

An $f$ which is independent of $x$ and satisfies (i) above is called simply *strongly differentiable* at $u_0$. Strong differentiability implies differentiability; on the other hand, if $f(u)$ is differentiable in a neighborhood of $u_0$, and the derivative $D_u f(u)$ is continuous in $u$ at $u_0$, then $f$ is strongly differentiable at $u_0$. Thus strong differentiability lies somewhere between differentiability and being of class $C^1$.

If $f(u)$ is strongly differentiable at $u_0$ it follows that $f$ satisfies a Lipschitz condition with respect to $u$ in *a neighborhood* of $u_0$. The usual rules hold for strongly differentiable functions: closure under addition, scalar multiplication, composition of functions, and products (provided the latter are defined).

Whenever $f(u, x)$ is strongly partially differentiable with respect to both $u$ and $x$ at $(u_0, x_0)$, it follows that $f$ is strongly differentiable at $(u_0, x_0)$ with respect to $(u, x)$, and the usual rule holds which relates the total differential to the partial differentials. If $f(u, x)$ is merely partially differentiable with respect to both $u$ and $x$ at $(u_0, x_0)$, but one of the partial derivates is strong, then $f$ is differentiable at $(u_0, x_0)$ with respect to $(u, x)$.

The statement of the theorem of Nijenhuis is as follows:

THEOREM 4.1. *Consider a function $f(u, x, y)$ from an open set in $E \times X \times Y$ into $F$, where $E$, $F$ are Banach spaces, $X$ is a normed vector space, and $Y$ is a topological space. Assume that $f$ is strongly partially differentiable at $(u_0, x_0, y_0)$ with respect to both $u$ and $x$, and that $D_u f(u_0, x_0, y_0)$: $E \to F$ is an isomorphism. Then there exists a neighborhood $N_1$ of $(u_0, x_0, y_0)$ in $E \times X \times Y$ and a neighborhood $N_2$ of $(f(u_0, x_0, y_0), x_0, y_0)$ in $F \times X \times Y$ which are in a one-to-one correspondence under the map*

$$(u, x, y) \mapsto \big(f(u, x, y), x, y\big) = (v, x, y)\,;$$

*the inverse map*

$$(v, x, y) \mapsto \big(\varphi(v, x, y), x, y\big)$$

*is strongly partially differentiable at $(v_0, x_0, y_0) = \big(f(u_0, x_0, y_0), x_0, y_0\big)$ with respect to $F \times X$.*

Note that the hypotheses above tacitly assume that $f(u_0, x_0, y)$ is continuous in $y$ at $y_0$, and the conclusion tacitly implies that the solution $\varphi(v, x, y)$ is continuous in all variables jointly at $(v_0, x_0, y_0)$.

## 5. – Solution of the Bishop equation in $C^{k,\alpha}$.

In what follows we will use the following notation: $B$ will denote a compact neighborhood of the origin in $\mathbf{R}^l \times \mathbf{C}^m$. For the function $h$ we introduce the Banach space

$$C_l^{k,\alpha}(B) = C^{k,\alpha}(B) \times \ldots \times C^{k,\alpha}(B) \qquad (l\text{-times}),$$

with norm $|h|_{k,\alpha} \equiv |h|_{k,\alpha}^B$. For the analytic disc $w$ in $\mathbf{C}^m$ we introduce the Banach spaces

$$\mathscr{D}_m^{k,\alpha} = \mathscr{D}^{k,\alpha} \times \ldots \times \mathscr{D}^{k,\alpha} \qquad (m\text{-times})$$

with norm $|w|_{k,\alpha} \equiv |w|_{k,\alpha}^{\bar{D}}$, where $\mathscr{D}^{k,\alpha} = \mathscr{O}(D) \cap C^{k,\alpha}(\bar{D})$ and $\mathscr{O}(D) =$ holomorphic functions in the open unit disc $D$. Actually in this section we shall only use the boundary values

$$w_{|S^1} \in C_m^{k,\alpha}(S^1),$$

and do not need the fact that they fill in to an analytic disc $w \in \mathscr{D}_m^{k,\alpha}$, so that throughout this section $\mathscr{D}_m^{k,\alpha}$ could be replaced by $C_m^{k,\alpha}(S^1)$. But we have introduced the spaces $\mathscr{D}_m^{k,\alpha}$ because we will need them later. If $u$ is an $l$-tuple of bounded functions on $S^1$, or if $w$ is an $m$-tuple of bounded functions on $S^1$ (later on $\bar{D}$), we shall denote the supremum of their Euclidean norms over $S^1$, or $\bar{D}$, by $|u|_\infty$ and $|w|_\infty$, respectively. It will also be convenient to introduce the notation

$$\operatorname{Lip}(h) = \operatorname{Lip}^B(h) \equiv \sup \frac{|h(u_1, w) - h(u_2, w)|}{|u_1 - u_2|}$$

for the partial Lipschitz constant of $h$ with respect to $u$; here the sup is taken over all $(u_1, w)$, $(u_2, w) \in B$ with $u_1 \neq u_2$.

First we consider the question of uniqueness of solutions to the Bishop equation: Let $c$ be a given vector in $\mathbf{R}^l$ and let $w$ be a given $m$-tuple of bounded measurable functions on $S^1$. Corresponding to this choice of the parameters $(c, w)$, consider any bounded measurable solution $u(e^{i\theta})$ of (2.1), considered as an element of $L^1(S^1)$. In order that the composition $h(u, w)$ be well defined, we shall consider only $u$ and $w$ with $|u|_\infty$, $|w|_\infty$ sufficiently small so that $(u, w) \in B$.

PROPOSITION 5.1. *If $h \in C_l^{0,1}(B)$ and $\operatorname{Lip}^B(h) < 1$ then such solutions $u$ of the Bishop equation* (2.1) *are unique.*

PROOF. It follows from the representation (3.1) for $T$, by Pavseval's equality, that $T$ is a bounded operator on $L^2(S^1)$ with $\|T\|_{L^2} = 1$. Let $u_1$, $u_2$ be two solutions of (2.1) as above. Since on $S^1$, $|h(u_1, w) - h(u_2, w)| \leqslant$

$\leqslant \mathrm{Lip}(h)|u_1 - u_2|$, it follows upon subtraction that

$$|u_1 - u_2|_{L^2} \leqslant \|T\|_{L^2} \mathrm{Lip}(h)|u_1 - u_2|_{L^2} < |u_1 - u_2|_{L^2};$$

hence $u_1 = u_2$.

An interesting consequence of the above proposition is the fact that solutions to the Bishop equation are locally unique at a point of strong differentiability of $M$: Namely, consider a function $h(u, w)$ defined for $(u, w)$ in some neighborhood of the origin in $\mathbf{R}^l \times \mathbf{C}^m$ and taking values in $\mathbf{R}^l$; we have

PROPOSITION 5.1$'$. *If $h$ is strongly differentiable at* 0 *and* $dh(0) = 0$, *then sufficiently small bounded solutions $u$ of the Bishop equation* (2.1), *corresponding to sufficiently small paramaters $c$ and $w$, are unique.*

PROOF. It follows from the definition of strong differentiability that in a sufficiently small neighborhood $B$ of the origin, the Lipschitz constant for $h$ is less than one.

REMARK. The above uniqueness results also apply in the case where $u$ and $h$ are complex valued, provided $T$ is extended by linearity to act on the real and imaginary parts of $h$. This remark will be useful in section 6.

Next we state our main theorems concerning the existence and dependence upon parameters of solutions to the Bishop equation in the spaces $C^{k,\alpha}$. It will be convenient to denote the space of parameters $p = (c, w)$ by $P = \mathbf{R}^l \times \mathscr{D}^{k,\alpha}_m$; it is a Banach space with norm $|p|_{k,\alpha} \equiv |(c, w)|_{k,\alpha} = |c| + |w|_{k,\alpha}$, where $|c|$ is the Euclidean norm in $\mathbf{R}^l$.

THEOREM 5.1. *Let $k \geqslant 0$ be an integer and $0 < \alpha < 1$.*

*a) There exists a positive constant $c = c(l)$ such that if $h \in C^{k+s+1}_l(B)$, $s \geqslant 1$, and $\mathrm{Lip}^B(h) < [c\|T\|_\alpha]^{-1}$ then there is a local unique solution $u \in C^{k,\alpha}_l(S^1)$ of* (2.1) *such that $|u|_{k,\alpha}$ is of class $C^s$ in its dependence upon the parameters $p = (c, w)$ measured in the norm $|p|_{k,\alpha}$; i.e., there exists a neighborhood $U = U(k, \alpha)$ of the origin in $P$ such that $u$ is given by a map $u\colon U \to C^{k,\alpha}_l$ of class $C^s$. Moreover, if $h \in C^{k+s+1,1}_l(B)$ then $|u|_{k,\alpha}$ is of class $C^{s,1}$ with respect to $|p|_{k,\alpha}$.*

*b) If $h \in C^{k+1,1}_l(B)$ and $dh(0) = 0$ then there is a local unique solution $u \in C^{k,\alpha}_l(S^1)$ of* (2.1) *such that $|u|_{k,\alpha}$ is Lipschitz continuous in its dependence on the parameters $p = (c, w)$ measured in the norm $|p|_{k,\alpha}$. In fact $|u|_{k,\alpha}$ is strongly differentiable with respect to $|p|_{k,\alpha}$ at the origin in $P$.*

Note that part *b*) above gives a sharper existence result than part *a*); *e.g.*, when $k = 0$ it is a question of assuming that $h \in C^{1,1}$ instead of $h \in C^2$, and in either case we get a solution $u \in C^{\alpha}$. Part *b*) also gives a sharper result about dependence on parameters.

It is also of interest to have a stability theorem which exhibits the dependence of the solution $u$ on the defining function $h$ as well. Let $C_l^{k+1,1}(B, 0)$ denote the Banach subspace of functions $h \in C_l^{k+1}(B)$ such that $h(0) = 0$. We will also introduce a new space of parameters $(p, h) = (c, w, h)$: $\mathscr{P} = P \times C_l^{k+1,1}(B, 0) = \mathbb{R}^l \times \mathscr{O}_m^{k,\alpha} \times C_l^{k+1,1}(B, 0)$. In the following theorem $\mathscr{U}$ will denote a neighborhood of the point $(0, h_0) = (0, 0, h_0)$ in $\mathscr{P}$, and $u_1$, $u_2$ will be the unique solutions of (2.1) which correspond to the parameters $(c_1, w_1, h_1)$, $(c_2, w_2, h_2)$ respectively.

THEOREM 5.2. *Let $k \geq 0$ be an integer and $0 < \alpha < 1$. Assume that $h_0 \in C_l^{k+1,1}(B, 0)$ and $dh_0(0) = 0$. Then there is a neighborhood $\mathscr{U}$ of the point $(0, h_0)$ and there is a constant $C = C(k, \alpha, h_0)$ such that*

$$|u_1 - u_2|_{k,\alpha} \leq C\{|c_1 - c_2| + |w_1 - w_2|^{\bar{D}}_{k,\alpha} + |h_1 - h_2|^{B}_{k+1,1}\}$$

*for all $(c_1, w_1, h_1)$, $(c_2, w_2, h_2) \in \mathscr{U}$. Moreover, $u = u(\cdot, c, w, h)$ is strongly differentiable as a function of $(c, w, h)$ at the point $(0, 0, h_0)$ with respect to the norms indicated above.*

Let $A_k = A_k(\alpha)$ be an open set in

$$C_l^{k,\alpha}(S^1) \times P = C_l^{\alpha,k}(S^1) \times \mathbb{R}^l \times \mathscr{D}_m^{k,\alpha}$$

such that $(u(e^{i\theta}), w(e^{i\theta})) \in B$ for $(u, c, w) \in A_k$. Define the operator $\mathscr{H}$ on $A_k$ by

$$\mathscr{H}(u, p)(e^{i\theta}) = h\big(u(e^{i\theta}), w(e^{i\theta})\big) \tag{5.1}$$

where $p = (c, w)$. For the proofs of the theorems above, we shall need the following

LEMMA 5.1. *Let $k \geq 0$ be an integer and $0 \leq \alpha \leq 1$.*

*a) If $h \in C_l^{k+s+1}(B)$ and $s \geq 0$ then $\mathscr{H}: A_k \to C_l^{k,\alpha}(S^1)$ is of class $C^s$.*

*b) If $h \in C_l^{k+s+1,1}(B)$ and $s \geq 0$ then $\mathscr{H}: A_k \to C_l^{k,\alpha}(S^1)$ is of class $C^{s,1}$.*

*c) If $h \in C_l^{k+1,1}(B)$ and $dh(0) = 0$ then $\mathscr{H}: A_k \to C_l^{k,\alpha}(S^1)$ is strongly differentiable at the origin.*

PROOF OF THEOREM 5.1. For the moment let us assume Lemma 5.1.

*a*) Under the given hypotheses on $h$, let $A^k \subset C_l^{k,\alpha}(S^1)\times P$ and $\mathscr{H}$ be as in Lemma 5.1 and consider the nonlinear mapping $F\colon A_k \to C_l^{k,\alpha}(S^1)$ defined by

$$F(u, c, w) = u - c + T[\mathscr{H}(u, p)].$$

Now we want to find a solution $u$ to the functional equation $F(u, c, w) = 0$. First of all we note that $F(0, 0, 0) = 0$. Both $T$ and the identity $I$ are continuous linear maps from $C_l^{k,\alpha}(S^1)$ to itself; hence they are smooth, and the smoothness of $F$ is precisely that of the smoothness of $\mathscr{H}$. Therefore by the lemma $F$ is of class $C^s$ with $s\geqslant 1$. Since $D_u\mathscr{H}(0, 0)$ corresponds to multiplication by a constant matrix where entries are the first partial derivatives of $h$ evaluated at the origin, it is clear that there is a $c = c(l)$ such that

$$\|T[D_u\mathscr{H}(0, 0)]\|_{k,\alpha}\leqslant \|T\|_{k,\alpha}\|D_u\mathscr{H}(0, 0)\|_{k,\alpha}\leqslant \|T\|_\alpha\, c\,\mathrm{Lip}(h) < 1\,.$$

Hence

$$D_uF(0, 0, 0) = I + T[D_u\mathscr{H}(0,0)]$$

is an isomorphism of $C_l^{k,\alpha}(S^1)$ onto itself. Therefore by the standard implicit function theorem in Banach space, there is a neighborhood of the origin $U = U(k, \alpha) \subset P$ and a map $u\colon U \to C_l^{k,\alpha}(S^1)$ such that $F(u(c, w), c, w) \equiv 0$ for all $(c, w)\in U$. In addition, $u$ is of class $C^s$ with respect to $p = (c, w)$ when both $u$ and $p$ are measured in their appropriate norms. If $h\in C_l^{k+s+1,1}(B)$ one has by Lemma 5.1 that $\mathscr{H}$, and hence $F$, is of class $C^{s,1}$. The usual proof of smoothness of the solution $u$, as in the proof of the standard implicit function theorem, then shows that $u$ is actually of class $C^{s,1}$.

*b*) Let $A_k$, $\mathscr{H}$ and $F$ be as above. Under the given hypotheses on $h$ we have from Lemma 5.1 that $\mathscr{H}$ is strongly differentiable at the origin. It follows that $F$ is also strongly differentiable at the origin, and since $dh(0) = 0$, $D_uF(0, 0) = I$. Applying Theorem 4.1 we obtain that there exists a neighborhood of the origin $U' = U'(k, \alpha)\subset P$ and a solution map $u\colon U' \to C_l^{k,\alpha}(S^1)$, such that $u$ is strongly differentiable at the origin with respect to $p = (c, w)$. In particular, $|u|_{k,\alpha}$ is Lipschitz continuous with respect to $|p|_{k,\alpha}$, uniformly for $p\in U'$.

PROOF OF LEMMA 5.1. The proof will be done in a number of steps: Note that in general if $g\in C^\alpha(K)$ and $f\in C^{0,1}(g(K))$ then the composition $f\circ g\in C^\alpha(K)$. First we show that if $h\in C_l^{k,\alpha}(B)$ then $\mathscr{H}$ maps $A_k$ into $C_l^{k,\alpha}$. Without loss of generality we may assume that $h$ is scalar valued. Also, for simplicity of notation, we shall often let $v(e^{i\theta}) = (u(e^{i\theta}), w(e^{i\theta}))$ be the vector in $\mathbb{R}^{l+2m}$ consisting of $u$ and the real and imaginary parts of $w$. Then for $0 \leqslant j \leqslant k$ each

$$D_\theta^j h(u(e^{i\theta}), w(e^{i\theta})) \equiv D_\theta^j h(v(e^{i\theta})) = \tag{5.2}$$
$$= \text{a finite sum of terms of the form}$$
$$C \frac{\partial^q h}{\partial v_{i_1} \dots \partial v_{i_q}} (v(e^{i\theta})) D_\theta^{l_1} v_{i_1}(e^{i\theta}) \dots D_\theta^{l_q} v_{i_q}(e^{i\theta}) ,$$

where $C = C(i_1, \dots, i_q)$ is a constant, $0 \leqslant q \leqslant j$, $1 \leqslant i_1 \leqslant \dots \leqslant i_q \leqslant l + 2m$, each $l_r \geqslant 1$ and $l_1 + \dots + l_q = j$. Since each derivative of $h$ of order $\leqslant k$ belongs to $C^{0,1}(B)$ and $D_\theta^j v \in C^\alpha_{l+2m}(S^1)$ for $0 \leqslant j \leqslant k$, and since $C^\alpha$ is a Banach algebra, it follows that

$$|D_\theta^j h(v(e^{i\theta}))|_\alpha < \infty \quad \text{for all } 0 \leqslant j \leqslant k .$$

Hence $|\mathscr{H}(u, p)|_{k,\alpha} < \infty$.

Next we show that if $h \in C_l^{k+1,1}(B)$ then $\mathscr{H}$ is Lipschitz continuous. For $k = 0$ we again assume, without loss of generality, that $h \in C^{1,1}(B)$ is scalar valued. Let $(u, p)$, $(u_0, p_0) \in A_0$ and consider the boundary values $v$, $v_0$ on $S^1$ of the corresponding $(u, w)$, $(u_0, w_0)$. We have

$$|\mathscr{H}(u, p) - \mathscr{H}(u_0, p_0)|_\alpha \leqslant \sup_\theta |h(v(e^{i\theta})) - h(v_0(e^{i\theta}))| +$$
$$+ \sup_{\substack{\theta,\theta' \\ \theta \neq \theta' \bmod 2\pi}} \frac{\pi}{2} \frac{|h(v(e^{i\theta})) - h(v_0(e^{i\theta})) - h(v(e^{i\theta'})) + h(v_0(e^{i\theta'}))|}{|e^{i\theta} - e^{i\theta'}|^\alpha} .$$

Using the normalized form of the Taylor series with integral remainder, and the fact that $h \in C^{1,1}(B)$, we obtain

$$\text{R.H.S.} \leqslant C_1 \sup_\theta |v(e^{i\theta}) - v_0(e^{i\theta})| +$$
$$+ \sup_{\substack{\theta,\theta' \\ \theta \neq \theta' \bmod 2\pi}} \frac{\pi}{2} \frac{1}{|e^{i\theta} - e^{i\theta'}|^\alpha} \left\{ \int_0^1 Dh[v_0(e^{i\theta}) + \tau(v(e^{i\theta}) - v_0(e^{i\theta}))](v(e^{i\theta}) - \right.$$
$$\left. - v_0(e^{i\theta}))\, d\tau - \int_0^1 Dh[v_0(e^{i\theta'}) + \tau(v(e^{i\theta'}) - v_0(e^{i\theta'}))](v(e^{i\theta'}) - v_0(e^{i\theta'}))\, d\tau \right\} .$$

Since the term inside the $\{\}$ can be written as

$$\int_0^1 (Dh[\tau v(e^{i\theta}) + (1 - \tau) v_0(e^{i\theta})] - Dh[\tau v(e^{i\theta'}) + (1 - \tau) v_0(e^{i\theta'})]\, d\tau \cdot [v - v_0](e^{i\theta}) +$$
$$+ \int_0^1 Dh[\tau v(e^{i\theta'}) + (1 - \tau) v_0(e^{i\theta'})]\, d\tau \cdot ([v - v_0](e^{i\theta}) - [v - v_0](e^{i\theta'})) ,$$

it follows that

$$|\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)|_\alpha \leqslant C_1|v-v_0|_\alpha + C_2(|v_0|_\alpha + |v-v_0|_\alpha)|v-v_0|_\alpha + \\ + C_3|v-v_0|_\alpha \leqslant C_4|(u,p)-(u_0,p_0)|_\alpha .$$

Thus $\mathscr{H}\colon A_0 \to C_l^\alpha$ is of class $C^{0,1}$.

Now assume that $h \in C_k^{k+1,1}(B)$. We must estimate

$$|\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)|_{k,\alpha} = \sum_{j=0}^{k} |D_\theta^j[\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)]|_\alpha .$$

But for $0 \leqslant j \leqslant k$ we have that $D_\theta^j[\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)]$ can be expressed as a finite sum of differences of terms of the form (5.2). Putting in all necessary mixed terms, and using the triangle inequality and the given smoothness of $h$, we obtain

$$|D_\theta^j[\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)]|_\alpha \leqslant C|(u,p)-(u_0,p_0)|_{j,\alpha} .$$

Therefore $|\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)|_{k,\alpha} \leqslant C|(u,p)-(u_0,p_0)|_{k,\alpha}$ as desired, where $C$ denotes a generic constant.

Finally we assume that $h \in C_l^{k+2,1}(B)$ and show that $\mathscr{H}\colon A_k \to C_l^{k,\alpha}$ is of class $C^{1,1}$. At a point $(u_0,p_0) \in A_k$ the differential $D\mathscr{H}(u_0,p_0)$ of $\mathscr{H}$ must be the linear transformation from $C_l^{k,\alpha}(S^1)\times P$ to $C_l^{k,\alpha}(S^1)$ which is associated to the $l\times(2l+2m)$ matrix $[h_u(u_0,w_0)|0|h_w(u_0,w_0)]$, where the preceding notation indicates a block decomposition: $[h_u]$ is $l\times l$, $[0]$ is the $l\times l$ zero matrix, and $[h_w]$ is $l\times 2m$. Since all of the first order partial derivatives of $h(u,w)$ are of class $C^{k+1,1}$, it follows from the discussion above that $D\mathscr{H}\colon A_k \to L\{C_l^{k,\alpha}(S^1)\times P, C_l^{k,\alpha}(S^1)\}$, where $L\{X, Y\}$ denotes the space of bounded linear maps from $X$ to $Y$. Using the normalized integral form for the remainder in the Taylor series expansion for $h(v)$, we obtain as above that

$$|\mathscr{H}(u,p)-\mathscr{H}(u_0,p_0)-D\mathscr{H}(u_0,p_0)[(u,p)-(u_0,p_0)]|_{k,\alpha} = \\ = \left|\int_0^1 \sum_{i,j=1}^{l+2m} \frac{\partial^2 h}{\partial v_i\,\partial v_j}(v_0+\tau(v-v_0))(1-\tau)(v_i-v_{0i})(v_j-v_{0j})\,d\tau\right|_{k,\alpha} \leqslant C|v-v_0|^2_{\alpha,k} .$$

Thus $\mathscr{H}$ is differentiable with differential $D\mathscr{H}$.

To show that $\mathscr{H}$ is a $C^{1,1}$ map we need to prove

$$D\mathscr{H}\colon A_k \to L\{C_l^{k,\alpha}\times P, C_l^{k,\alpha}\}$$

is of class $C^{0,1}$; i.e., that for $(u_0, p_0) \in A_k$ we have

$$(5.3)\qquad \sup_{|(u',p')|_{k,\alpha}=1} |[D\mathscr{H}(u, p) - D\mathscr{H}(u_0, p_0)](u', p')|_{k,\alpha} \leqslant C|(u, p) - (u_0, p_0)|_{k,\alpha},$$

with a constant $C$ that is independent of $(u_0, p_0)$. The L.H.S. is bounded by

$$(5.4)\qquad c \max_{i,j} \left|\frac{\partial h_i}{\partial v_j}(u, w) - \frac{\partial h_i}{\partial v_j}(u_0, w_0)\right|_{k,\alpha},$$

where the constant $c = c(l, m)$, since $D\mathscr{H}$ is associated with the Jacobian of $h$. But the first partial derivatives of $h$ belong to $C^{k+1,1}(B)$, so by what was shown above we have that (5.4) is bounded by $C|(u, p) - (u_0, p_0)|_{k,\alpha}$, which is just the desired inequality (5.3).

It follows by induction, continuing this line of argument, that if $h \in C_l^{k+s+1,1}(B)$ then $\mathscr{H}$ is of class $C^{s,1}$ as a mapping from $A_k$ into $C_l^{k,\alpha}$. Thus we have established part *b*) of Lemma 5.1.

To prove part *a*) we shall show that the weaker assumption that $h \in C_l^{k+1}(B)$ actually implies that $\mathscr{H}$ maps $A_k$ into $C_l^{k,\alpha}$ continuously. It is well known that $C^\infty(B)$ is dense in $C^j(B)$ for any integer $j \geqslant 0$ (but $C^\infty$ is not dense in $C^{j,\alpha}$ if $0 < \alpha \leqslant 1$). Thus given $h \in C^{k+1}(B)$ and an $\varepsilon > 0$ there is an $h_1 \in C^\infty(B) \subset C^{k+1,1}(B)$ and an $h_2 \in C^{k+1}(B) \subset C^{k,1}(B)$ such that $h = h_1 + h_2$ and $|h_2|_{k,1} \leqslant |h_2|_{k+1} < \varepsilon$. From part *b*) applied to $h_1$ it follows that for a fixed $(u_0, p_0) \in A_k$,

$$\begin{aligned}|\mathscr{H}(u, p) - \mathscr{H}(u_0, p_0)|_{k,\alpha} &= |h(v) - h(v_0)|_{k,\alpha} \leqslant \\ &\leqslant |h_1(v) - h_1(v_0)|_{k,\alpha} + |h_2(v) - h_2(v_0)|_{k,\alpha} \leqslant \\ &\leqslant C_1|v - v_0|_{k,\alpha} + |h_2(v)|_{k,\alpha} + |h_2(v_0)|_{k,\alpha},\end{aligned}$$

where $C_1 = C_1(h_1)$. In order to handle the last two terms above we observe that for $f \in C^{k,1}(B)$ and $v \in C^{k,\alpha}$ with $|v|_{k,\alpha} \leqslant R$, there is an inequality of the form

$$(5.5)\qquad |f(v)|_{k,\alpha} \leqslant |f(v)|_\infty + C(k, \alpha, R)|f|_{k,1}|v|_{k,\alpha}$$

(Recall that $|\ |_\infty$ denotes the sup norm). The inequality (5.5) follows from (5.2) as in the first paragraph of the proof of part *b*). Thus there is a constant $C_2 = C(k, \alpha, R)$ which is independent of $(u, p)$, $(u_0, p_0) \in A_k$ such that

$$|\mathscr{H}(u, p) - \mathscr{H}(u_0, p_0)|_{k,\alpha} \leqslant C_1|v - v_0|_{k,\alpha} + 2(1 + C_2)\varepsilon \leqslant (3 + 2C_2)\varepsilon$$

if we choose $|(u, p) - (u_0, p_0)|_{k,\alpha} \equiv |v - v_0|_{k,\alpha} < \varepsilon C_1^{-1}$. This shows that $\mathscr{H}$ is continuous.

As in the proof of part $b$) the differentials of $\mathscr{H}$ correspond to multiplication by matrices which involve the various partial derivatives of $h$. Therefore if $h \in C_l^{k+s+1}(B)$ then $\mathscr{H}$ is a map from $A_k$ to $C_l^{k,\alpha}$ of class $C^s$. This completes the proof of part $a$).

For the proof of part $c$) let $\varepsilon > 0$ be given and consider $(u_1, p_1), (u_2, p_2) \in A_k$. Since $dh(0) = 0$ we have

$$|\mathscr{H}(u_2, p_2) - \mathscr{H}(u_1, p_1) - D\mathscr{H}(0)[(u_2, p_2) - (u_1, p_1)]|_{k,\alpha} =$$

$$= |h(v_2) - h(v_1)|_{k,\alpha} =$$

$$= \left| \int_0^1 D\mathscr{H}[v_1 + \tau(v_2 - v_1)](v_2 - v_1)\, d\tau \right|_{k,\alpha} \leqslant$$

$$\leqslant C[|v_1|_{k,\alpha} + |v_2 - v_1|_{k,\alpha}] \cdot |v_2 - v_1|_{k,\alpha},$$

which can be obtained by applying (5.5) to the first partial derivatives of $h$. The last term is bounded by $\varepsilon|v_2 - v_1|_{k,\alpha}$ if we choose $|v_1|_{k,\alpha}$, $|v_2|_{k,\alpha} < \varepsilon(3C)^{-1}$. This shows that $\mathscr{H}$ is strongly differentiable at the origin, and completes the proof of Lemma 5.1.

PROOF OF THEOREM 5.2. To each $h \in C_l^{k+1,1}(B, 0)$ we associate the operator $\mathscr{H}$ defined by (5.1) and define $\mathscr{F} = \mathscr{F}(u, c, w, h)$ by $\mathscr{F} = u - c + T[\mathscr{H}(u, p)]$. Then the nonlinear mapping

$$\mathscr{F}: A_k \times C_l^{k+1,1}(B, 0) \to C_l^{k,\alpha}(S^1)$$

is well-defined. Under the given hypotheses, we claim that $\mathscr{F}$ is strongly differentiable with respect to all of its variables at the point $(0, h_0) = (0, 0, 0, h_0)$. Since $dh_0(0) = 0$ we have then that $D_u\mathscr{F}(0, 0, 0, h_0) = I$, and the desired conclusion of Theorem 5.2 follows from Theorem 4.1 (taking $Y = \emptyset$).

To justify our claim we need only show that $\mathscr{F}$ is strongly partially differentiable with respect to both $(u, p) = (u, c, w)$ and $h$ at the point $(0, h_0)$.

Since $|\mathscr{F}(0, h) - \mathscr{F}(0, h_0)|_{k,\alpha} = |T[h(0) - h_0(0)]|_{k,\alpha} = 0$ it is obvious that $\mathscr{F}(0, h)$ is continuous in $h$ at $(0, h_0)$. Let $D_{u,p}\mathscr{F}$ denote the partial differential of $\mathscr{F}$ with respect to $(u, p) = (u, c, w)$. Since $dh_0(0) = 0$ we have for any $v = (u, c, w) \in C_l^{k,\alpha} \times \mathbf{R}^l \times \mathscr{D}_m^{k,\alpha}$ that $[D_{u,p}\mathscr{F}(0, h_0)](v) = u - c$. Thus if $v_1,\ v_2 \in A_k$ then

$$(5.6)\quad |\mathscr{F}(v_2, h) - \mathscr{F}(v_1, h) - [D_{u,p}\mathscr{F}(0, h_0)](v_2 - v_1)|_{k,\alpha} =$$

$$= |T[\mathscr{H}(v_2) - \mathscr{H}(v_1)]|_{k,\alpha} \leqslant \|T\|_\alpha |h(v_2) - h(v_1)|_{k,\alpha} =$$

$$= \|T\|_\alpha \left|\int_0^1 Dh[v_1 + \tau(v_2 - v_1)](v_2 - v_1)\, d\tau\right|_{k,\alpha} \leqslant$$

$$\leqslant \|T\|_\alpha \Big\{\sup_{\theta,\tau} |Dh[v_1 + \tau(v_2 - v_1)]| + C|h|_{k+1,1}\big(|v_1|_{k,\alpha} + |v_2 - v_1|_{k,\alpha}\big)\Big\}|v_2 - v_1|_{k,\alpha}\,.$$

Given an $\varepsilon > 0$ there exists a $\delta > 0$ such that $|v_1|_{k,\alpha},\ |v_2|_{k,\alpha} < \delta$ imply both

$$C\big[|h_0|_{k+1,1} + 1\big]\big(|v_1|_{k,\alpha} + |v_2 - v_1|_{k,\alpha}\big) < \varepsilon\big(2\|T\|_\alpha\big)^{-1}$$

and

$$\sup_{\theta,\tau,v_1,v_2} |Dh_0[v_1 + \tau(v_2 - v_1)]| < \varepsilon\big(4\|T\|_\alpha\big)^{-1}\,.$$

Choosing $N(h_0) = \{h \,|\, |h - h_0|_{k+1,1} < \min(1, \varepsilon(4\|T\|_\alpha)^{-1})\}$ we have that the L.H.S. of (5.6) is less than $\varepsilon|v_2 - v_1|_{k,\alpha}$. Thus $\mathscr{F}$ is strongly partially differentiable at $(0, h_0)$ with respect to $(u, p)$.

For fixed $h = h_0$ we have already shown in the proof of Theorem 5.1 that $\mathscr{F}(u, c, w, h_0) = F(u, c, w)$ is strongly differentiable, and hence even Lipschitz continuous at the origin $= (0, 0, 0)$. Now let $D_h\mathscr{F}$ denote the partial differential of $\mathscr{F}$ with respect to $h$. We have $[D_h\mathscr{F}(0, h_0)](h) = T[h(0)] = 0$. Therefore, again setting $v = (u, c, w)$,

$$(5.7)\quad |\mathscr{F}(v, h_2) - \mathscr{F}(v, h_1) - [D_h\mathscr{F}(0, h_0)](h_2 - h_1)|_{k,\alpha} =$$

$$= |T[h_2(v) - h_1(v)]|_{k,\alpha} \leqslant \|T\|_\alpha |(h_2 - h_1)(v)|_{k,\alpha} \leqslant$$

$$\leqslant \|T\|_\alpha \Big\{\sup_\theta |(h_2 - h_1)(v)| + C|h_2 - h_1|_{k,1}|v|_{k,\alpha}\Big\}\,.$$

Since $(h_2 - h_1)(0) = 0$ we have that $\sup |(h_2 - h_1)(v)| \leqslant |h_2 - h_1|_{0,1} \sup |v| \leqslant |h_2 - h_1|_{k,1}|v|_{k,\alpha}$. Thus the R.H.S. of (5.7) is less than $(1 + C)\|T\|_\alpha \cdot |h_2 - h_1|_{k,1}|v|_{k,\alpha}$. Given an $\varepsilon > 0$, choose

$$N(0) = \{v \in A_k \,|\, |v|_{k,\alpha} < \varepsilon[(1 + C)\|T\|_\alpha]^{-1}\}\,.$$

Then the L.H.S. is less than $\varepsilon|h_2 - h_1|_{k,1}$, and it follows that $\mathscr{F}$ is strongly partially differentiable at $(0, h_0)$ with respect to $h$. This completes the proof of Theorem 5.2.

## 6. – Solution of the Bishop equation in the real analytic category.

The goal of this section is to prove the theorem below about the existence and dependence upon parameters of real analytic solutions to the Bishop equation for real analytic $h$. Since we have the stability Theorem 5.2 in the $C^{k,\alpha}$ setting, we do not develop here a theorem about the real analytic dependence of the solution $u$ on the defining function $h$. That could easily be done, however; we leave the details to the interested reader.

With $B$ as in section 5 let $\mathfrak{A}(B)$, $\mathfrak{A}(S^1)$ be the space of real valued real analytic functions on $B$, or $S^1$, respectively, and let $\mathfrak{A}_l(B)$, $\mathfrak{A}_l(S^1)$ denote their $l$-fold Cartesian products. $D_\delta$ will denote the open disc about the origin in $\mathbf{C}$ of radius $1+\delta$, and $rS^1$ will be the circle about the origin of radius $r$. $A_\delta$ will stand for a $\delta$-neighborhood of $S^1$ in its complexification, with $S^1$ considered as a real analytic manifold. To be concrete, we will identify $A_\delta$ with the annulus $A_\delta = \{z \in \mathbf{C} | 1-\delta < |z| < 1+\delta\}$. We will also need the Banach spaces

$$\mathscr{A}^\alpha_\delta = \mathcal{O}(A_\delta) \cap C^\alpha(\bar{A}_\delta)\,,$$

as well as its $l$-fold Cartesian product $\mathscr{A}^\alpha_{l,\delta}$. Instead of the norm $|\ |_\alpha$ taken over $\bar{A}_\delta$ we will take the norm in $\mathscr{A}^\alpha_\delta$, or $\mathscr{A}^\alpha_{l,\delta}$, to be

$$\|u\|_{\alpha,\delta} \equiv |u|_\alpha^{(1-\delta)S^1} + |u|_\alpha^{(1+\delta)S^1}\,.$$

The fact that the norms $|\ |_\alpha$ and $\|\ \|_{\alpha,\delta}$ are equivalent on $\mathscr{A}^\alpha_\delta$ follows by the same reasoning as that given in the proof of Proposition 6.1 below (Schauder estimates up to the boundary). The Banach space

$$\mathscr{D}^\alpha(D_\delta) \equiv \mathcal{O}(D_\delta) \cap C^\alpha(\bar{D}_\delta)\,,$$

or its $m$-fold Cartesian product $\mathscr{D}^\alpha_m(D_\delta)$, will however, be taken with its usual norm $|w|_\alpha$ (taken over $\bar{D}_\delta$).

Fix a $\delta_0 > 0$ and an $0 < \alpha < 1$ and consider, in this section only, the new parameter space

$$P = \mathbf{R}^l \times \mathscr{D}^\alpha_m(D_{\delta_0})$$

with norm $\|p\| \equiv \|(c, w)\| = |c| + |w|_\alpha$.

THEOREM 6.1. *Let $h \in \mathfrak{A}_l(B)$ and $0 < \delta < \delta_0 \leqslant 1$. There exists a positive constant $c = c(l, \alpha, \delta)$ such that if $\mathrm{Lip}^B(h) < [C\|T\|_\alpha]^{-1}$ then there is a local unique solution $u \in \mathfrak{A}_l(S^1)$ of (2.1) such that $u$ is real analytic in its depend-*

*ence upon the parameters* $p = (c, w)$ *measured in the norm* $\|p\|$; *i.e. there exists a neighborhood* $U = U(\alpha, \delta)$ *of the origin in* $P$ *such that* $u$ *is given by the values on* $S^1$ *of a real analytic map* $u: U \to \mathscr{A}^{\alpha}_{l,\delta}$.

For the proof of Theorem 6.1 we will need two propositions. The first is an extension of Proposition 3.1 and the second is the analytic analogue of Lemma 5.1.

We extend the operator $T$, defined on the unit circle, to an operator $\tilde{T}$ defined on the annulus $A_\delta$ by:

$$\tilde{T}f(re^{i\theta}) = \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} f(re^{i(\theta-\varphi)})\,\mathrm{Im}\left(\frac{1+e^{i\varphi}}{1-e^{i\varphi}}\right)d\varphi = \tag{6.1}$$

$$= \frac{1}{2\pi}\lim_{\varrho\uparrow 1}\int_{-\pi}^{\pi} f(re^{i(\theta-\varphi)})\,\mathrm{Im}\left(\frac{1+\varrho e^{i\varphi}}{1-\varrho e^{i\varphi}}\right)d\varphi \equiv \lim_{\varrho\uparrow 1}\tilde{T}_\varrho f(re^{i\theta})\,,$$

where $f \in \mathscr{A}^{\alpha}_{\delta}$ and $1 - \delta \leqslant r \leqslant 1 + \delta$.

PROPOSITION 6.1. *For* $0 < \alpha < 1$ *and* $0 < \delta < 1$, $\tilde{T}$ *is a bounded complex linear operator from* $\mathscr{A}^{\alpha}_{\delta}$ *to itself.*

PROOF. $\tilde{T}$ is obviously complex linear. For each $r \in [1-\delta, 1+\delta]$ define $f_r$ on $S^1$ by $f_r(e^{i\theta}) = f(re^{i\theta})$. Since $f \in \mathscr{A}^{\alpha}_{\delta}$ we have $f_r \in C^{\alpha}(S^1)$; moreover $\tilde{T}f(re^{i\theta}) = Tf_r(e^{i\theta})$. So Proposition 3.1 applied to the real and imaginary parts of $f_r$ yields $\tilde{T}f|_{rS^1} \in C^{\alpha}(rS^1)$ for each $r \in [1-\delta, 1+\delta]$.

Differentiation under the integral sign shows that $\tilde{T}_\varrho f \in \mathcal{O}(A_\delta)$ for $\varrho < 1$, since $f \in \mathscr{A}^{\alpha}_{\delta}$. Consider now a function $f \in C^1(S^1)$ and set

$$g(\theta, \varphi) = \frac{f(e^{i(\theta+\varphi)}) - f(e^{i(\theta-\varphi)})}{2\tan\frac{1}{2}\varphi}\,,$$

$$K_\varrho(\varphi) = \frac{(1-\varrho)^2}{1 - 2\varrho\cos\varphi + \varrho^2}\,.$$

Then as $\varrho \uparrow 1$ we have that $T_\varrho f$ converges uniformly on $S^1$ to

$$Tf(e^{i\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta, \varphi)\,d\varphi\,. \tag{6.2}$$

Namely, we have that

$$T_\varrho f(e^{i\theta}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta, \varphi)\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta, \varphi)K_\varrho(\varphi)\,d\varphi\,,$$

and the R.H.S. of the latter tends uniformly to zero if $f \in C^1(S^1)$ because then there is a uniform bound $|g(\theta, \varphi)| \leqslant M$, and $0 < K_\varrho(\varphi) < 1$, $K_\varrho(\varphi) \to 0$ except at $\varphi = 0$. Returning to our $f \in \mathscr{A}_\delta^\alpha$ we have in particular that $f_r \in C^1(S^1)$ for each $r \in (1-\delta, 1+\delta)$. Thus for such $r$ it follows that $\tilde{T}_\varrho f|_{rS^1} \to \tilde{T}f|_{rS^1}$ uniformly on $rS^1$. By the maximum modulus principle $\tilde{T}_\varrho f \to \tilde{T}f$ uniformly on compact subannuli of $A_\delta$. Hence $\tilde{T}f \in \mathcal{O}(A_\delta)$.

To show that $\tilde{T}f \in C^\alpha(\overline{A}_\delta)$ we observe that the real and imaginary parts of $\tilde{T}f$ are harmonic in $A_\delta$ and, as we have seen, have boundary values on the inner and outer rims $r = 1-\delta$, $1+\delta$ that are in $C^\alpha$. Consider, for example, the outer rim $r = 1+\delta$. We have, regarding $\tilde{T}f(re^{i\theta})$ as an $L^2(S^1)$ valued function of $r$, that

$$\begin{aligned}|\tilde{T}f((1+\delta)e^{i\theta}) - \tilde{T}f(re^{i\theta})|_{L^2} &= |T[f_{1+\delta}(e^{i\theta}) - f_r(e^{i\theta})]|_{L^2} \leqslant \\ &\leqslant \|T\|_{L^2}|f_{1+\delta} - f_r|_{L^2} \leqslant \\ &\leqslant \sqrt{2\pi}|f_{1+\delta} - f_r|_\alpha^{S^1}\end{aligned}$$

which tends to zero because $f \in C^\alpha(\overline{A}_\delta)$. Thus the $C^\alpha$ boundary values are assumed in an $L^2$ sense. It then follows from straightforward estimation of Poisson integrals, as in the proof of the classical Schauder estimates for elliptic equations, that $\tilde{T}f \in C^\alpha(\overline{A}_\delta)$. One could also concoct a proof that $\tilde{T}f \in C^\alpha(\overline{A}_\delta)$ based on the maximum principle, by techniques similar to those used in the proof of Proposition 3.1.

Finally we note that

$$\begin{aligned}\|\tilde{T}f\|_{\alpha,\delta} &= |\tilde{T}f|_\alpha^{(1-\delta)S^1} + |\tilde{T}f|_\alpha^{(1+\delta)S^1} \leqslant \\ &\leqslant |T|_\alpha[(1-\delta)^{-\alpha}|f|_\alpha^{(1-\delta)S^1} + (1+\delta)^\alpha|f|_\alpha^{(1+\delta)S^1}] \leqslant \\ &\leqslant |T|_\alpha \max[(1-\delta)^{-\alpha}, (1+\delta)^\alpha]\|f\|_{\alpha,\delta}\,.\end{aligned}$$

Thus $\tilde{T}$ is bounded on $\mathscr{A}_\delta^\alpha$ and the proof is complete.

By letting $\tilde{T}$ act componentwise we have that $\tilde{T}\colon \mathscr{A}_{l,\delta}^\alpha \to \mathscr{A}_{l,\delta}^\alpha$ is a bounded complex linear operator, whose norm will be denoted by $\|\tilde{T}\|_{\alpha,\delta}$.

Since $h\colon B \to \mathbb{R}^l$ is real analytic there is a compact neighborhood $\tilde{B}$ of the origin in the complexification $\mathbb{C}(\mathbb{R}^l \times \mathbb{C}^m)$ on which $h$ has a holomorphic extension $\tilde{h}\colon \tilde{B} \to \mathbb{C}^l$. It will be convenient to use the following notation: $\mathbb{C}(\mathbb{R}^l) \cong \mathbb{C}^l$, $\mathbb{C}(\mathbb{C}^m) = \mathbb{C}(\mathbb{R}^{2m}) \cong \mathbb{C}^{2m}$ (« forgetting » the complex structure on $\mathbb{C}^m$), $\mathbb{C}(\mathbb{R}^l \times \mathbb{C}^m) \cong \mathbb{C}^l \times \mathbb{C}^{2m} = \mathbb{C}^{l+2m}$; for the real analytic function

$$h = h(u, w) = h(u, w, \overline{w})\,, \qquad (u, w) \in \mathbb{R}^l \times \mathbb{C}^m$$

we express its holomorphic extension by

$$\tilde{h} = \tilde{h}(u, w, w^*), \qquad (u, w, w^*) \in \mathbf{C}^l \times \mathbf{C}^{2m}$$

where $w = w_1 + iw_2$, $w^* = w_1 - iw_2$, and $w_1$, $w_2$ are the complex extensions of the real and imaginary parts of the original $w$. Thus $u =$ real corresponds to the real domain in $\mathbf{C}(\mathbf{R}^l)$ and $w^* = \overline{w}$ corresponds to $\mathbf{C}^m$ which is the real domain in $\mathbf{C}(\mathbf{C}^m)$.

Let $\tilde{A} = \tilde{A}(\alpha, \delta)$ be an open set in $\mathscr{A}^\alpha_{l,\delta} \times \mathbf{C}^l \times \mathscr{A}^\alpha_{2m,\delta}$ such that $(u(re^{i\theta}), w(re^{i\theta}), w^*(re^{i\theta})) \in \tilde{B}$ for all $(u, c, w, w^*) \in \tilde{A}$ and all $re^{i\theta} \in \overline{A}_\delta$. Define the operator $\tilde{\mathscr{H}}$ on $\tilde{A}$ by

$$\tilde{\mathscr{H}}(u, c, w, w^*)(re^{i\theta}) = \tilde{h}(u(re^{i\theta}), w(re^{i\theta}), w^*(re^{i\theta})) . \tag{6.3}$$

PROPOSITION 6.2. *$\tilde{\mathscr{H}}$ maps $\tilde{A}$ into $\mathscr{A}^\alpha_{l,\delta}$ holomorphically.*

PROOF. The composition $\tilde{\mathscr{H}}(u, c, w, w^*) \in \mathcal{O}_l(A_\delta)$ for $(u, c, w, w^*) \in \tilde{A}$ because it is well defined and $(u, c, w, w^*) \in \mathcal{O}_{2l+2m}(A_\delta)$. Since $\tilde{h} \in \mathcal{O}_l(\tilde{B})$ we have that $\tilde{h} \in C^{s,1}_l(\tilde{B})$ for all $s \in \mathbf{N}$, and it follows from a trivial modification of Lemma 5.1 (replacing $S^1$ by $\overline{A}_\delta$) that $\tilde{\mathscr{H}}(u, c, w, w^*) \in C^\alpha_l(\overline{A}_\delta)$ and the mapping is of class $C^\infty$. To conclude that $\tilde{\mathscr{H}}$ maps into $\mathscr{A}^\alpha_{l,\delta}$ holomorphically, we need only observe that the analyticity of $\tilde{h}(u, w, w^*)$ implies that the differential of $\tilde{\mathscr{H}}$ is complex linear.

PROOF OF THEOREM 6.1. Define $\tilde{F}: \tilde{A} \to \mathscr{A}^\alpha_{l,\delta}$ by

$$\tilde{F}(u, c, w, w^*) = u - c + \tilde{T}[\tilde{\mathscr{H}}(u, c, w, w^*)] . \tag{6.4}$$

We will find a solution $\tilde{u} \in \mathscr{A}^\alpha_{l,\delta}$ to the functional equation $\tilde{F}(u, c, w, w^*) = 0$. Clearly $\tilde{F}(0, 0, 0, 0) = 0$, and $\tilde{F}$ is holomorphic by Propositions 6.1 and 6.2. As in the proof of Theorem 5.1 there is a constant $c = c(l)$ such that

$$\begin{aligned}\|\tilde{T}[D_u \tilde{\mathscr{H}}(0)]\|_{\alpha,\delta} &\leq \|\tilde{T}\|_{\alpha,\delta} \|D_u \tilde{\mathscr{H}}(0)\|_{\alpha,\delta} \leq \\ &\leq \|\tilde{T}\|_{\alpha,\delta} c \operatorname{Lip}(h) \leq \\ &\leq \|T\|_\alpha \max[(1-\delta)^{-\alpha}, (1+\delta)^\alpha] c \operatorname{Lip}(h) < \\ &< 1 ,\end{aligned}$$

if $\operatorname{Lip}(h) < \{c(l) \max[(1-\delta)^{-\alpha}, (1+\delta)^\alpha] \|T\|_\alpha\}^{-1}$. Here we have used the Cauchy-Riemann equations to estimate at the origin the first partial derivatives of $\tilde{h}$ in pure imaginary directions in terms of those of $h$ in pure real directions—which in turn are dominated by Lip$(h)$. It follows that $D_u\tilde{F}(0, 0, 0, 0)$ is an isomorphism of the complex Banach space $\mathscr{A}^{\alpha}_{l,\delta}$ onto itself; hence we can apply the implicit function theorem for holomorphic maps. Thus there is a neighborhood of the origin $\tilde{U} = \tilde{U}(\alpha, \delta)$ in $\mathbf{C}^l \times \mathscr{A}^{\alpha}_{2m,\delta}$ and a holomorphic map $\tilde{u}\colon \tilde{U} \to \mathscr{A}^{\alpha}_{l,\delta}$ such that

$$\tilde{F}(\tilde{u}(c, w, w^*), c, w, w^*) \equiv 0 \tag{6.5}$$

for all $(c, w, w^*) \in \tilde{U}$.

Now in (6.5), if we choose $c$ to be real, $w^*$ to be equal to $\overline{w}$ on $S^1$, and set $r = 1$ we obtain

$$\tilde{u}(e^{i\theta}) = c - T[\tilde{h}(\tilde{u}(e^{i\theta}), w(e^{i\theta}), \overline{w}(e^{i\theta}))] . \tag{6.6}$$

The constant $c(l, \alpha, \delta)$ can be chosen so that the hypotheses imply that Lip$(h) < 1$. Thus by Proposition 5.1 the solution $\tilde{u}$ of (6.6) must agree with the real valued $C^{\alpha}$ solution $u$ of (2.1) which was constructed in Theorem 5.1, since $u$ also solves (6.6); hence $\tilde{u}$ is real valued on $S^1$.

To finish the proof, we merely have to choose $(w, w^*)$ as follows: Let $w \in \mathscr{D}^{\alpha}_m(D_{\delta_0})$ be given. For any $0 < \delta < \delta_0$, its restriction to $\overline{A}_{\delta}$ belongs to $\mathscr{A}^{\alpha}_{m,\delta}$. Let $w^*$ be the holomorphic extension of the real analytic function $\overline{w}|_{S^1}$. The Cauchy-Schwarz estimates for the derivatives of $w$ can be easily used to show that the power series expansion of $w^*$ about any point on $S^1$ converges on a disc of radius $\delta'$ for any $0 < \delta' < \delta_0$. Therefore the restriction of $w^*$ to $\overline{A}_{\delta}$ exists and belongs to $\mathscr{A}^{\alpha}_{m,\delta}$. The norm $\|w\|_{\alpha,\delta}$ in the space $\mathscr{A}^{\alpha}_{m,\delta}$ can be trivially dominated by the norm $|w|_{\alpha}$ taken over $\overline{D}_{\delta_0}$. The map from $\mathscr{D}^{\alpha}_m(D_{\delta_0})$ to $\mathscr{A}^{\alpha}_{m,\delta}$ defined by $w \mapsto w^*$ as above is linear and is obviously closed. Hence by the closed graph theorem there is a constant $C$ such that $\|w^*\|_{\alpha,\delta} \leqslant C|w|_{\alpha}$. Thus there is a neighborhood of the origin $U = U(\alpha, \delta)$ in $\mathbf{R}^l \times \mathscr{D}^{\alpha}_m(D_{\delta_0})$ for which $(c, w, w^*) \in \tilde{U}$ if $(c, w) \in U$. Therefore we have produced a map $u\colon U \to \mathscr{A}^{\alpha}_{l,\delta}$ whose values on $S^1$ define a solution in $\mathfrak{A}_l(S^1)$ to (2.1). Since $\tilde{u}$ is holomorphic and the linear map from $\mathscr{D}^{\alpha}_m(D_{\delta_0})$ to $\mathscr{A}^{\alpha}_{2m,\delta}$ is bounded, it follows that the composition $u$ is a real analytic map. This completes the proof of the theorem.

## 7. – Solution of the Bishop equation in the $C^{\infty}$ category.

Suppose that $h \in C^{\infty}$. Then for any fixed integer $k \geqslant 0$, the results of section 5 give a solution $u_k$ of (2.1), with parameter domain $U_k$ , that is of class $C^k$. For $k' \leqslant k$ we have by uniqueness that $u_{k'} \equiv u_k$ on their common domain $U_{k'} \cap U_k$. But unfortunately, as $k \uparrow \infty$, the arguments of section 5 do not rule out the possibility that the parameter domains $U_k$ might shrink down; hence one cannot conclude directly that there exists an open set $U$ in some appropriate parameter space on which there is a $C^\infty$ solution $u$ to (2.1). The purpose of this section is to fix up that difficulty so as to obtain a local $C^\infty$ solution.

Let $M = \{M_k\}_{k=0}^\infty$ be any sequence of positive real numbers. We shall need to consider the normed linear space $B^\alpha\{M\}$, where $0 \leqslant \alpha \leqslant 1$, of all real valued (or complex valued-just which is intended will be clear from the context) $C^\infty$ functions $f$ defined on $S^1$ for which the norm

$$\|f\|_M \equiv \sup_{k \geqslant 0} \frac{|D_\theta^k f|_\alpha^{S^1}}{M_k} < \infty .$$

Note that $B^\alpha\{M\}$ is a Banach space: All one has to check is that $B^\alpha\{M\}$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $B^\alpha\{M\}$. In particular $\{f_n\}$ is a Cauchy sequence in $C^\infty(S^1)$, so there is an $f \in C^\infty(S^1)$ such that $D^k f_n \to D^k f$ in $C^\alpha(S^1)$ for each $k \geqslant 0$. Since a Cauchy sequence is bounded, there is an $L$ such that

$$\sup_{k \geqslant 0} \frac{|D^k f_n|_\alpha}{M_k} \leqslant L$$

for all $n$. Taking limits we get that $|D^k f|_\alpha / M_k \leqslant L$ for all $k$; hence $f \in B^\alpha\{M\}$. Given $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that $n, m \geqslant N$ implies

$$\frac{|D^k f_m - D^k f_n|_\alpha}{M_k} < \varepsilon$$

for all $k$. But for each fixed $k$, there is an $N_k$ for which

$$\frac{|D^k f - D^k f_m|_\alpha}{M_k} < \varepsilon ,$$

provided $m \geqslant N_k$. Thus for all $k$ we have that

$$\frac{|D^k f - D^k f_n|_\alpha}{M_k} < 2\varepsilon$$

if $n \geqslant N$, and it follows that $f_n \to f$ in $B^\alpha\{M\}$.

We will let $B_l^\alpha\{M\}$ and $B_m^\alpha\{M\}$ denote the Cartesian products of $B^\alpha\{M\}$ $l$ times and $m$ times, respectively. For simplicity we will denote the norm in either space again by $\| \ \|_M$. As in section 5, $B$ will be a compact neighborhood of the origin in $\mathbb{R}^l \times \mathbb{C}^m$. As in section 6, $D_\delta$ is the open disc of radius $1+\delta$, and $\mathscr{D}^\alpha_m(D_\delta) = \mathscr{O}_m(D_\delta) \cap C^\alpha_m(\bar{D}_\delta)$ with norm $|w|_\alpha$ (taken over $\bar{D}_\delta$).

THEOREM 7.1. *Let $h \in C^\infty_l(B)$ and $0 < \alpha < 1$. Then there exists a positive constant $c = c(l)$, and there exists a sequence $M = \{M_k\}$ such that:*

*a) If $\mathrm{Lip}^B(h) < [C\|T\|_\alpha]^{-1}$ then there is a local unique real valued $C^\infty$ solution $u \in B^\alpha_l\{M\}$ of (2.1); i.e. there is a neighborhood $U = U(\alpha, M)$ of the origin in $\mathbb{R}^l \times B^\alpha_m\{M\}$ such that $u$ is given by a map $u\colon U \to B^\alpha_l\{M\}$ where $\|u\|_M$ depends in a $C^\infty$ way on the parameters $\|p\|_M \equiv |c| + \|w\|_M$ for $p = (c, w) \in U$.*

*b) For any fixed $\delta_0 > 0$ the sequence $M = \{M_k\}$ can be chosen so that $u$ is defined for the parameters in a suitable neighborhood $U_0 = U_0(\alpha, \delta_0, M)$ of the origin in $\mathbb{R}^l \times \mathscr{D}^\alpha_m(D_{\delta_0})$. Moreover $\|u\|_M$ is $C^\infty$ in its dependence on the parameters measured in the norm $\|p\| \equiv |c| + |w|_\alpha$ (taken over $\bar{D}_{\delta_0}$).*

REMARKS. 1) The choice of the sequence $M$ depends on the function $h$, but it can be chosen to work uniformly for all $h$ in any bounded set in $C^\infty_l(B)$.

2) The $w$ in *a*) actually refers to the boundary values on $S^1$; thus we could choose as parameters discs $w \in \mathscr{D}^\alpha_m\{M\}$, where $\mathscr{D}^\alpha_m\{M\}$ denotes the subspace of $\mathscr{O}_m(D) \cap C^\infty_m(\bar{D})$ of functions with boundary values in $B^\alpha_m\{M\}$. But the choice of parameters $w \in \mathscr{D}^\alpha_m(D_{\delta_0})$ in *b*) is simpler and useful for applications.

3) Given any sequence $N = \{N_k\}_{k=0}^\infty$, it is possible to choose $M$ to dominate $N$; i.e. $M_k \geqslant N_k$ for all $k$, as can easily be seen from the method of proof. Thus $\mathscr{D}^\alpha_m\{N\} \subset \mathscr{D}^\alpha_m\{M\}$, so (2.1) can be solved for parameter discs $w \in \mathscr{D}^\alpha_m\{N\}$.

PROOF OF THE THEOREM. Let $v$ be the $l + 2m$ dimensional real vector valued function consisting of $u$ and the real and imaginary parts of $w$. For any multi-index $\gamma$ we set $h_\gamma(v) = (D^\gamma h)(v)$ and let $\mathscr{H}_\gamma$ denote the function defined as in (5.1) which corresponds to $h_\gamma$. $A\{M\}$ will be the open cylinder in $B^\alpha_l\{M\} \times \mathbb{R}^l \times B^\alpha_m\{M\}$ of the form $\{\|v\|_M < R\}$, where $R$ is chosen small enough that $v\colon S^1 \to B$ whenever $(u, c, w) \in A\{M\}$, so that the compositions $H_\gamma$ are defined. The method of proof relies on choosing the sequence $M$ so that i) $B^\alpha\{M\}$ is closed under multiplication and ii) $\mathscr{H}_\gamma$ maps $A\{M\}$ into a bounded set in $B^\alpha_l\{M\}$, for each multi-index $\gamma$.

We will determine the $M_k$ recursively so that, for each $\gamma$,

$$\sup_{k \geqslant |\gamma|} \frac{|D^k_\theta h_\gamma(v)|_\alpha}{M_k}$$

is bounded. Set

$$a_j = \sup_{|\gamma|=j} |h_\gamma|^B_{0,1}$$

and choose $M_0 = M_1 = 1$. Since $|v|_\alpha$, $|D_\theta v|_\alpha \leqslant R$ we have that

$$|h(v)|_\alpha \leqslant a_0 R\,, \qquad |D_\theta h_\gamma(v)|_\alpha \leqslant (l+2m)\, a_{|\gamma|+1} R^2\,. \tag{7.1}$$

Assume that $M_0, M_1, \ldots, M_{k-1}$ have been chosen. As in (5.2) we have that $D^k_\theta h_\gamma(v) =$

$$\sum_{j=1}^{l+2m} \frac{\partial}{\partial v_j} h_\gamma(v) D^k_\theta v_j + \sum \frac{\partial^q}{\partial v_{j_1} \ldots \partial v_{j_q}} h_\gamma(v) D^{l_1}_\theta v_{j_1} \ldots D^{l_q}_\theta v_{j_q}\,,$$

where the second summation is taken over $2 \leqslant q \leqslant k$, $1 \leqslant j_1, \ldots, j_q \leqslant l + 2m$, each $l_r \geqslant 1$ and $l_1 + \ldots + l_q = k$. Setting $N = l + 2m$ it follows that

$$\begin{aligned} |D^k_\theta h_\gamma(v)|_\alpha &\leqslant N a_{|\gamma|+1} R |D^k_\theta v|_\alpha + \\ &+ \sum a_{|\gamma|+q} R |D^{l_1}_\theta v_{j_1}|_\alpha \ldots |D^{l_q}_\theta v_{j_q}|_\alpha \leqslant \\ &\leqslant N a_{|\gamma|+1} R^2 M_k + \sum a_{|\gamma|+q} R^{1+q} M_{l_1} \ldots M_{l_q}\,. \end{aligned}$$

For $k \geqslant 2$ the second term on the R.H.S. is bounded by a constant

$$K_{\gamma,k} \equiv K(a_{|\gamma|+2}, \ldots, a_{|\gamma|+k}, M_1, \ldots, M_{k-1}, N, k, R)$$

which depends only on the quantities indicated. We choose $M_k$ such that

$$M_k \geqslant \max\left\{\max_{|\gamma|\leqslant k} K_{\gamma,k}, \sum_{j=2}^{k-1} \binom{k}{j} M_j M_{k-j}\right\}. \tag{7.2}$$

The second of the two conditions in (7.2) assures us that if $f, g \in B^\alpha\{M\}$ then $\|fg\|_M \leqslant 3\|f\|_M \|g\|_M$, so that $B^\alpha\{M\}$ is forced to be closed under multiplication. The first condition, with (7.1), tells us that for $0 < |\gamma| \leqslant k$

$$\frac{|D^k_\theta h_\gamma(v)|_\alpha}{M_k} \leqslant N a_{|\gamma|+1} R^2 + 1\,,$$

which is independent of $k$. Thus for any sequence $M$ constructed in this manner, we have that for each multi-index $\gamma$,

$$\|h_\gamma(v)\|_M \leqslant \max\left\{\max_{k\leqslant|\gamma|} \frac{|D^k_\theta h_\gamma(v)|_\alpha}{M_k},\ N a_{|\gamma|+1} R^2, a_0 R\right\} < \infty \tag{7.3}$$

for $\|v\|_M < R$; hence each $\mathscr{H}_\gamma$ maps $A\{M\}$ boundedly into $B^\alpha_l\{\boldsymbol{M}\}$.

Using the normalized form of the Taylor series with integral remainder, as in the proof of Lemma 5.1, we obtain

$$\|\mathscr{H}_\gamma(v_1) - \mathscr{H}_\gamma(v_2) - D\mathscr{H}_\gamma(v_2)[v_1 - v_2]\|_M \leqslant C_\gamma \|v_1 - v_2\|_M^2 ,$$

where the constant

$$C_\gamma = 9N^2 \sup_{|\beta|=|\gamma|+2} \sup_{v\in A\{M\}} \|\mathscr{H}_\beta(v)\|_M$$

is bounded for $v_1, v_2 \in A\{M\}$ according to (7.3). Therefore each $\mathscr{H}_\gamma$ is differentiable and hence continuous on $A\{M\}$. Since $D\mathscr{H}$ consists of multiplication by a matrix whose entries are the $\mathscr{H}_\gamma$ with $|\gamma| = 1$, and since $B^\alpha\{M\}$ is closed under multiplication, it follows that $D\mathscr{H}: A\{M\} \to L\{B_l^\alpha\{M\} \times \times \mathbf{R}^l \times B_m^\alpha\{M\}, B_l^\alpha\{M\}\}$ and that $\mathscr{H}$ is of class $C^1$. Since the differentials of $\mathscr{H}$ of order $s$ are associated with the $H_\gamma$ for $|\gamma| = s$, it follows that $\mathscr{H}$ is of class $C^\infty$ on $A\{M\}$.

Using the fact that $T$ commutes with $\theta$-derivatives, it is easy to see that $T: B_l^\alpha\{M\} \to B_l^\alpha\{M\}$ is bounded; in fact $\|Tf\|_M \leqslant \|T\|_\alpha \|f\|_M$.

The remainder of the proof of part *a*) now is just like the arguments of section 5 and 6: We have a $C^\infty$ nonlinear mapping $F: A\{M\} \to B_l^\alpha\{M\}$ which satisfies the hypotheses of the implicit function theorem. Hence we obtain a $C^\infty$ solution map $u: U \to B_l^\alpha\{M\}$.

To prove part *b*) it suffices to make the additional requirement in (7.2) that for $k \geqslant 2$ the $M_k$ be chosen to satisfy

$$M_k \geqslant \frac{(k+1)!}{\delta_0^{k+1}} .$$

It then follows from the Cauchy-Schwarz estimates that

$$\|w\|_M \leqslant C(\delta_0)|w|_\alpha$$

and the inclusion map $\mathscr{D}_m^\alpha(D_{\delta_0}) \hookrightarrow B_m^\alpha\{M\}$, being linear, is $C^\infty$. By choosing $|w|_\alpha$ sufficiently small we therefore obtain a neighborhood of the origin $U_0$ in $\mathbf{R}^l \times \mathscr{D}_m^\alpha(D_{\delta_0})$ such that $U_0 \subset U$. This completes the proof of Theorem 7.1.

## 8. – The local family of analytic discs.

Having discussed in sections 5-7 the solvability of the Bishop equation in the $C^{k,\alpha}$, real analytic ($C^\omega$) and $C^\infty$ categories, we now return to the basic problem formulated in section 2: Given a suitably prescribed family of analytic discs in $\mathbf{R}^l \times \mathbf{C}^m$, we want to be able to lift it to a family of analytic discs in $\mathbf{C}^n$ such that each disc in the lifted family has its boundary on $M$. In this section we collect together the previous material and formulate precise theorems in these directions.

According to Theorems 5.1, 6.1 and 7.1 equation (2.1) can be solved by solution maps

$$\begin{array}{lll}
\text{(i) } u\colon U \to C_l^{k,\alpha}, & U \subset \mathbf{R}^l \times \mathscr{D}_m^{k,\alpha} & \\
\text{(ii) } u\colon U \to \mathscr{A}_{l,\delta}^{\alpha} \subset \mathfrak{A}_l, & U \subset \mathbf{R}^l \times \mathscr{D}_m^{\alpha}(D_{\delta_0}) & 0 < \delta < \delta_0 \\
\text{(iii) } u\colon U \to B_l^{\alpha}\{M\} \subset C^{\infty}, & U \subset \mathbf{R}^l \times \mathscr{D}_m^{\alpha}\{M\}, &
\end{array}$$

in each of the categories $C^{k,\alpha}$, $C^{\omega}$ and $C^{\infty}$, respectively. Each solution $u = u(e^{i\theta}, p)$ determines a unique disc

$$g(\zeta, p) = \big(f(\zeta, p), w(\zeta)\big), \qquad p = (c, w) \tag{8.1}$$

in $\mathbf{C}^n$ with boundary on $M$, where $f(\zeta, p)$ has the boundary values $u(e^{i\theta}, p) + iv(e^{i\theta}, p)$ with $v(e^{i\theta}, p) = h\big(u(e^{i\theta}, p), w(e^{i\theta})\big)$, and $\operatorname{Re} f(0, p) = c$.

Let $(c, \mathscr{W})$, $\mathscr{F}$ and $\mathscr{G}$ denote the maps defined on $\bar{D} \times U$ by $(\zeta, p) \mapsto (c, w(\zeta))$, $(\zeta, p) \mapsto f(\zeta, p)$ and $(\zeta, p) \mapsto g(\zeta, p)$, respectively. Their restrictions to $S^1 \times U$ will be denoted by $(c, b\mathscr{W})$, $b\mathscr{F}$ and $b\mathscr{G}$. Thus $\mathscr{G} = (\mathscr{F}, \mathscr{W})$, $\operatorname{Re} \mathscr{F}(0, p) = c$ and there is a commutative diagram

$$\begin{array}{ccccc}
 & & M & \hookrightarrow & \mathbf{C}^n \\
 & \nearrow^{b\mathscr{G}} & & \nearrow^{\mathscr{G}} & \downarrow \\
S^1 \times U & \hookrightarrow & \bar{D} \times U & \xrightarrow{b(c,\mathscr{W})} & \mathbf{R}^l \times \mathbf{C}^m .
\end{array} \tag{8.2}$$

We will speak of $\mathscr{G}$ as being the *lift* of the family $(c, \mathscr{W})$ to $M$, and of $b\mathscr{G}$ as being the lift of its boundary $(c, b\mathscr{W})$.

By a *local family of analytic discs in* $\mathbf{R}^l \times \mathbf{C}^m$, we will mean a family of $(c, \mathscr{W})$ that is sufficiently small (in a sense that depends on which category we are working in) so as to satisfy the hypotheses on $h$ (and hence on $M$) stated in either Theorem 5.1, 6.1 or 7.1. These three cases (the $C^{k,\alpha}$, $C^{\omega}$ and $C^{\infty}$ categories) will in the sequel be referred to by (i), (ii) and (iii), respectively.

In order to state the next theorem, we define the classes $C^{k,s}$ and $C^{k,s;\alpha}$ as follows: A function $f(\zeta, p)$ of two variables is of class $C^{k,s}$ provided

1) $f$ has continuous partial derivatives of all orders which involve at most $k$ derivatives with respect to $\zeta$ and at most $s$ derivatives with respect to $p$, and

2) for any such partial derivative of $f$, the order in which the derivatives are taken doesn't matter.

The function $f$ is of class $C^{k,s;\alpha}$ if, in addition,

3) all the derivatives mentioned above are of class $C^{\alpha}$. Our work up to now can be assembled to obtain:

THEOREM 8.1. *A given local family $(c, \mathscr{W})$ of analytic discs in $\mathbf{R}^l \times \mathbf{C}^m$ has a unique lifting to a corresponding local family $\mathscr{G}$ of analytic discs in $\mathbf{C}^n$ satisfying* (8.2). *Moreover, in each case, we have*:

(i) *If $s \geq 1$ and $h \in C^{k+s+1}$ $(h \in C^{k+s+1,1})$ then $\mathscr{G}$ is of class $C^{k,s}$ (class $C^{k,s;\alpha}$) and $\mathscr{G}(\cdot, p) \in \mathscr{D}_n^{k,\alpha}$ for all $p \in U$.*

(ii) *If $h$ is real analytic then $\mathscr{G}$ is real analytic and $\mathscr{G}(\cdot, p)$ is the restriction to $\bar{D}$ of a real analytic $\mathscr{G}_\delta$ with $\mathscr{G}_\delta(\cdot, p) \in \mathscr{D}_n^{\alpha}(D_\delta)$, for $p \in U$ and $0 < \delta < \delta_0$.*

(iii) *If $h \in C^\infty$ and the parameters $p$ are chosen in $\mathbf{R}^l \times \mathscr{D}_m^{\alpha}\{M\}$ (parameters in $\mathbf{R}^l \times \mathscr{D}_m^{\alpha}(D_{\delta_0})$ then $\mathscr{G}$ is of class $C^\infty$ with respect to $|\zeta| + \|p\|_M$ (with respect to $|\zeta| + \|p\|$) and $\mathscr{G}(\cdot, p) \in \mathscr{D}_n^{\alpha}\{M\}$ for $p \in U$.*

REMARK. (Case $s = 0$) If $h \in C^{k+1,1}$ and $dh(0) = 0$ then the proof of the theorem yields also the following: If $k = 0$ then $D_p \mathscr{G}$ exists strongly at points of the form $\bar{D} \times \{0\}$. If $k \geq 1$ then at such points $\mathscr{G}$ is strongly differentiable and $D_p D_\zeta^j \mathscr{G}$ exists strongly for $0 \leq j \leq k$.

PROOF. (i) First we show that $\mathscr{W}: \bar{D} \times U \to \mathbf{C}^m$ is of class $C^{k,\infty;\alpha}$. Since $\mathscr{W}$ is a continuous linear operator with respect to $p$, its smoothness is limited only by its smoothness in $\zeta$: For fixed $p$, $\mathscr{W}(\zeta, p)$ is of class $C^{k,\alpha}$ with respect to $\zeta$. If $0 \leq j \leq k$ then

$$
\begin{aligned}
|D_\zeta^j \mathscr{W}(\zeta_1, p_1) - D_\zeta^j \mathscr{W}(\zeta_2, p_2)| &\equiv |w_1^{(j)}(\zeta_1) - w_2^{(j)}(\zeta_2)| \leq \\
&\leq |w_1^{(j)}(\zeta_1) - w_1^{(j)}(\zeta_2)| + |w_1^{(j)}(\zeta_2) - w_2^{(j)}(\zeta_2)| \leq \\
&\leq |w_1|_{k,\alpha} |\zeta_1 - \zeta_2|^\alpha + |w_1 - w_2|_{k,\alpha} \leq \\
&\leq R|\zeta_1 - \zeta_2|^\alpha + |p_1 - p_2|_{k,\alpha} \leq \\
&\leq R 2^{1-\alpha} [|\zeta_1 - \zeta_2| + |p_1 - p_2|_{k,\alpha}]^\alpha,
\end{aligned}
$$

where here and in what follows we assume that $U$ has been chosen so that $p \in U$ implies $|w|_{k,\alpha} \leq R$ with $R \leq 1$. Hence $D_\zeta^j \mathscr{W}$ is jointly continuous, and of class $C^\alpha$. Since $D_p \mathscr{W}(\zeta_0, p_0)$ maps $p$ to $w(\zeta_0)$ an easier version of the above argument shows that, for $0 \leq j \leq k$, $D_\zeta^j D_p \mathscr{W} = D_p D_\zeta^j \mathscr{W}$ is of class $C^\alpha$.

As there are no higher derivatives of $\mathscr{W}$ with respect to $p$ to consider, we have that $\mathscr{W}$ is of class $C^{k,\infty;\alpha}$.

Consider next the map $\mathscr{F}: \bar{D}\times U \to \mathbf{C}^l$. By Proposition 3.1 we have

$$|f(\cdot, p)|^{\bar{D}}_{k,\alpha} \leqslant c|u(\cdot, p)|^{S^1}_{k,\alpha} \tag{8.3}$$

for each fixed $p \in U$. Therefore $\mathscr{G}(\cdot, p) \in \mathscr{D}^{k,\alpha}_n$. Since $|u(\cdot, p)|^{S^1}_{k,\alpha}$ is of class $C^s(C^{s,1})$ with respect to $|p|_{k,\alpha}$, and $f(e^{i\theta}, p) = u(e^{i\theta}, p) + ih(u(e^{i\theta}, p), w(e^{i\theta}))$, it follows by Lemma 5.1 and the equivalence of the norms $|\ |^{S^1}_{k,\alpha}$, $|\ |^{\bar{D}}_{k,\alpha}$ on $\mathscr{D}^{k,\alpha}_l$ that $|f(\cdot, p)|^{\bar{D}}_{k,\alpha}$ is of class $C^s(C^{s,1})$ with respect to $|p|_{k,\alpha}$. Moreover $U$ can be chosen so that $u: U \to C^{k,\alpha}_l$ is Lipschitz continuous with Lipschitz constant $L$, because $s \geqslant 1$. Using (8.3) and the same argument that was applied to $\mathscr{W}$, we obtain

$$|D^j_\zeta \mathscr{F}(\zeta_1, p_1) - D^j_\zeta \mathscr{F}(\zeta_2, p_2)| \leqslant |f(\cdot, p_1)|^{\bar{D}}_{k,\alpha}|\zeta_1 - \zeta_2|^\alpha + |f(\cdot, p_1) - f(\cdot, p_2)|^{\bar{D}}_{k,\alpha} \leqslant$$
$$\leqslant cLR2^{1-\alpha}[|\zeta_1 - \zeta_2| + |p_1 - p_2|_{k,\alpha}]^\alpha$$

for $0 \leqslant j \leqslant k$. Thus each $D^j_\zeta \mathscr{F}$ is jointly continuous, and of class $C^\alpha$.

Let $p_0 \mapsto D_p f(\cdot, p_0) \in L\{P, \mathscr{D}^{k,\alpha}_l\}$ denote the differential of the map $p_0 \mapsto f(\cdot, p_0)$ from $U$ to $\mathscr{D}^{k,\alpha}_l$. For $0 \leqslant_p \leqslant k$ set

$$D_p D^j_\zeta \mathscr{F}(\zeta_0, p_0)[p] \equiv D^j_\zeta(D_p f(\cdot, p_0)[p])(\zeta_0)\,.$$

Then

$$|D^j_\zeta \mathscr{F}(\zeta_0, p) - D^j_\zeta \mathscr{F}(\zeta_0, p_0) - D_p D^j_\zeta \mathscr{F}(\zeta_0, p_0)[p - p_0]| \leqslant$$
$$\leqslant |f(\cdot, p) - f(\cdot, p_0) - D_p f(\cdot, p_0)[p - p_0](\cdot)|_{k,\alpha} = o(|p - p_0|_{k,\alpha})\,,$$

so that $D_p D^j_\zeta \mathscr{F} \in L\{P, \mathbf{C}^l\}$ is the partial differential of the map $D^j_\zeta \mathscr{F}$: $\bar{D}\times U \to \mathbf{C}^l$. To show that it is jointly continuous, we observe that

$$\begin{aligned}
&|D_p D^j_\zeta \mathscr{F}(\zeta_1, p_1)[p] - D_p D^j_\zeta \mathscr{F}(\zeta_2, p_2)[p]| \leqslant \\
&\qquad \leqslant |D_p D^j_\zeta \mathscr{F}(\zeta_1, p_1)[p] - D_p D^j_\zeta \mathscr{F}(\zeta_2, p_1)[p]| + \\
&\qquad\qquad + |D_p D^j_\zeta \mathscr{F}(\zeta_2, p_1)[p] - D_p D^j_\zeta \mathscr{F}(\zeta_2, p_2)[p]| \leqslant \\
&\qquad \leqslant |D^j_\zeta D_p f(\cdot, p_1)[p](\zeta_1) - D^j_\zeta D_p f(\cdot, p_1)[p](\zeta_2)| + \\
&\qquad\qquad + |D^j_\zeta D_p f(\cdot, p_1)[p](\zeta_2) - D^j_\zeta D_p f(\cdot, p_2)[p](\zeta_2)| \leqslant \\
&\qquad \leqslant |D_p f(\cdot, p_1)[p]|_{k,\alpha}\cdot|\zeta_1 - \zeta_2|^\alpha + \\
&\qquad\qquad + |D_p f(\cdot, p_1)[p] - D_p f(\cdot, p_2)[p]|_{k,\alpha} \leqslant \\
&\qquad \leqslant \{\|D_p f(\cdot, p_1)\|_{k,\alpha}\cdot|\zeta_1 - \zeta_2|^\alpha + \\
&\qquad\qquad + \|D_p f(\cdot, p_1) - D_p f(\cdot, p_2)\|_{k,\alpha}\}|p|_{k,\alpha}\,.
\end{aligned}$$

Since $s \geq 1$ there is a $\delta = \delta(\varepsilon)$ for which $|\zeta_1 - \zeta_2|$, $|p_1 - p_2|_{k,\alpha} < \delta$ implies that the last term above is less than $\varepsilon|p|_{k,\alpha}$. (Or, for $h \in C^{k+s+1,1}$, the last term above is bounded by an expression of the form

$$\{C_1|\zeta_1 - \zeta_2|^\alpha + C_2|p_1 - p_2|_{k,\alpha}\}|p|_{k,\alpha} \leq C_3[|\zeta_1 - \zeta_2| + |p_1 - p_2|_{k,\alpha}]^\alpha |p|_{k,\alpha},$$

so in that case $D_p D_\zeta^j \mathscr{F}$ is not only jointly continuous, but of class $C^\alpha$.)

For $0 \leq r \leq s$ let

$$p_0 \mapsto D_p^r f(\cdot, p_0) \in L\{P, \ldots, P; \mathscr{D}_l^{k,\alpha}\} \simeq L\{P, L\{P, \ldots, L\{P, \mathscr{D}_l^{k,\alpha}\}\} \ldots\} \quad (r \text{ times})$$

be the $r$-th differential of the map $p_0 \mapsto f(\cdot, p_0)$, and for $0 \leq j \leq k$ set

$$D_p^r D_\zeta^j \mathscr{F}(\zeta_0, p_0)[p_1, \ldots, p_r] \equiv D_\zeta^j(D_p^r f(\cdot, p_0)[p_1, \ldots, p_r])(\zeta_0).$$

By the same argument as given above for the case $r = 1$, it follows that this $D_p^r D_\zeta^j \mathscr{F} \in L\{P, \ldots, P; \mathbf{C}^l\} \simeq L\{P, L\{P, \ldots, L\{P, \mathbf{C}^l\}\} \ldots\}$ is the $r$-th partial differential of $D_\zeta^j \mathscr{F}$, and that $D_p^r D_\zeta^j \mathscr{F}$ is jointly continuous (or of class $C^\alpha$ if $h \in C^{k+s+1,1}$).

Since $D_\zeta \mathscr{F}$, $D_p \mathscr{F}$ and $D_p D_\zeta \mathscr{F}$ exist and are continuous, a theorem of H. A. Schwarz [2; Theorem 20.15, p. 243] says that $D_\zeta D_p \mathscr{F}$ also exists and $D_\zeta D_p \mathscr{F} = D_p D_\zeta \mathscr{F}$.

To show that $\mathscr{F} \in C^{k,s}$ we proceed by an induction on both $k$ and $s$ as follows: Obviously $\mathscr{F} \in C^{0,0}$. Assume that $\mathscr{F} \in C^{m,n}$ for some $m < k$, $n < s$. To show that $\mathscr{F} \in C^{m,n+1}$ one uses the above theorem of Schwarz to show a separate induction: if $\mathscr{F} \in C^{j,n+1}$ then $\mathscr{F} \in C^{j+1,n+1}$. To show that $\mathscr{F} \in C^{m+1,n}$ one uses a similiar induction: if $\mathscr{F} \in C^{m+1,r}$ then $\mathscr{F} \in C^{m+1,r+1}$. (When $h \in C^{k+s+1,1}$ we get that $\mathscr{F} \in C^{k,s;\alpha}$)

This completes the proof in the $C^{k,\alpha}$ category.

(ii) Now $U \subset \mathbf{R}^l \times \mathscr{D}_m^\alpha(D_{\delta_0})$ and for any $0 < \delta < \delta_0$, $\mathscr{W}: \bar{D} \times U \to \mathbf{C}^m$ is the restriction to $\bar{D} \times U$ of a map $\mathscr{W}_\delta: \bar{D}_\delta \times U \to \mathbf{C}^m$. The argument we have given above shows that $\mathscr{W}_\delta$ is differentiable, and that its differential is complex linear. Hence $\mathscr{W}_\delta$, and in particular $\mathscr{W}$, is holomorphic.

Next consider the holomorphic map $\tilde{u}: \tilde{U} \to \mathscr{A}_{l,\delta}^\alpha$, where $\tilde{U} \subset \mathbf{C}^l \times \mathscr{A}_{2m,\delta}^\alpha$, that was constructed in the proof of Theorem 6.1. The value of $\tilde{u}(c, w, w^*)$ at $\zeta \in \bar{A}_\delta$ will be denoted by $\tilde{u}(\zeta, c, w, w^*)$. For fixed $(c, w, w^*) \in \tilde{U}$ the function

$$\tilde{u}(\zeta, c, w, w^*) + i\tilde{h}(\tilde{u}(\zeta, c, w, w^*), w(\zeta), w^*(\zeta)) \tag{8.4}$$

is holomorphic in $\zeta$ for $\zeta \in A_\delta$. Let $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$, $c = c_1 + ic_2$, $\tilde{h} = \tilde{h}_1 + i\tilde{h}_2$

be the real and imaginary parts. Since $\tilde{u}$ is a solution of (6.5) we have

$$(8.5) \qquad \tilde{u}_1 = c_1 - \tilde{T}[\tilde{h}_1(\tilde{u}_1 + i\tilde{u}_2, w, w^*)], \qquad \tilde{u}_2 = c_2 - \tilde{T}[\tilde{h}_2(\tilde{u}_1 + i\tilde{u}_2, w, w^*)].$$

Recalling that $\tilde{T} = T$ on $S^1$, we see from (8.5) that there exist holomorphic functions $\tilde{f}_1$, $\tilde{f}_2$ in $D$ which have the real analytic boundary values $\tilde{u}_1 + i\tilde{h}_1(\tilde{u}_1 + i\tilde{u}_2, w, w^*)$, $\tilde{u}_2 + i\tilde{h}_2(\tilde{u}_1 + i\tilde{u}_2, w, w^*)$ on $S^1$, respectively. But the holomorphic function $\tilde{f} = \tilde{f}_1 + i\tilde{f}_2$ agrees with (8.4) on $S^1$; hence it has an extension to a holomorphic function $\tilde{f} = \tilde{f}(\zeta, c, w, w^*)$ on $D_\delta$ which belongs to the space $\mathscr{D}_l^\alpha(D_\delta)$. Define $\tilde{\mathscr{F}}: \bar{D}_\delta \times \tilde{U} \to \mathbf{C}^l$ by $(\zeta, c, w, w^*) \mapsto \tilde{f}(\zeta, c, w, w^*)$. Once again, we apply to $\tilde{\mathscr{F}}$ the same type of arguments that was applied to $\mathscr{F}$ in (i): We obtain that $\tilde{\mathscr{F}}$ is jointly continuous, and that the differentials $D_\zeta \tilde{\mathscr{F}}$, $D_c \tilde{\mathscr{F}}$, $D_w \tilde{\mathscr{F}}$, $D_{w^*} \tilde{\mathscr{F}}$ exist and are complex linear. Hence $\tilde{\mathscr{F}}$ is a holomorphic map.

Going back to the proof of Theorem 6.1, we defined there a continuous linear map $w \mapsto w^*$ taking $\mathscr{D}_m^\alpha(D_{\delta_0})$ to $\mathscr{A}_{m,\delta}^\alpha$. Thus there is a real analytic map $r: U \to \tilde{U}$ defined by $(c, w) \mapsto (c, w, w^*)$. Let $\mathscr{F}: \bar{D} \times U \to \mathbf{C}^l$ be the restriction of $\mathscr{F}_\delta$, where $\mathscr{F}_\delta$ is defined by the commutative diagram

$$\begin{array}{ccc} & & \mathbf{C}^l \\ & \overset{\mathscr{F}_\delta}{\nearrow} & \uparrow \tilde{\mathscr{F}} \\ \bar{D}_\delta \times U & \xrightarrow[(id, r)]{} & \bar{D}_\delta \times \tilde{U}. \end{array}$$

It follows that $\mathscr{G} = (\mathscr{F}, \mathscr{W})$ is real analytic, and that $\mathscr{G}_\delta(\cdot, p) \in \mathscr{D}_n^\alpha(D_\delta)$, where $\mathscr{G}_\delta = (\mathscr{F}_\delta, \mathscr{W}_\delta)$. This completes the proof in the real analytic category.

(iii) When $h \in C^\infty$ we take the parameters $p = (c, w) \in \mathbf{R}^l \times \mathscr{D}_m^\alpha\{M\}$. Clearly $\mathscr{W}(\cdot, p) \in \mathscr{D}_m^\alpha\{M\}$ for each $p \in U$ and $\mathscr{W}: \bar{D} \times U \to \mathbf{C}^m$ is a $C^\infty$ map with respect to the norm $|\zeta| + \|p\|_M$. Since for $u: U \to B_l^\alpha\{M\}$, we know that $\|u\|_M$ is $C^\infty$ with respect to $\|p\|_M$, we have also that $\|f\|_M$ is $C^\infty$ with respect to $\|p\|_M$. But there are constants $c(k)$ such that $|f|_{k,\alpha}^{\bar{D}} \leq c(k)\|f\|_M$. Hence the same argument that was used in (i) shows that $\mathscr{F}: \bar{D} \times U \to \mathbf{C}^l$ is a $C^\infty$ map. Also $\mathscr{F}(\cdot, p) \in \mathscr{D}_l^\alpha\{M\}$ for each $p \in U$. If instead, we take parameters $p = (c, w) \in \mathbf{R}^l \times \mathscr{D}_m^\alpha(D_{\delta_0})$, then we obtain that $\mathscr{F}$ is $C^\infty$ with respect to the norm $|\zeta| + \|p\|$ defined in $b$) of Theorem 7.1. This completes the proof of Theorem 8.1.

For any set $S \subset \mathbf{C}^n$ we denote the algebra of germs of holomorphic functions on $S$ by $\mathcal{O}(S)$ and equip it with the inductive limit topology

$$\mathcal{O}(S) = \varinjlim_{S \subset \Omega} \mathcal{O}(\Omega), \qquad \Omega^{\text{open}} \subset \mathbf{C}^n,$$

where each $\mathcal{O}(\Omega)$ has its usual Fréchet-Schwartz topology of uniform convergence on compact subsets, and the $\Omega$ are partially ordered by inclusion.

Consider a local family $\mathcal{G}$ of analytic discs in $\mathbf{C}^n$ which is the lift to $M$ of a local family of analytic discs $(c, \mathcal{W})$ in $\mathbf{R}^l \times \mathbf{C}^m$. For any subset $\Sigma \subset U$ we will denote by

$$\mathcal{G}(\Sigma) = \mathcal{G}(\bar{D} \times \Sigma)$$

$$\boldsymbol{b}\mathcal{G}(\Sigma) = b\mathcal{G}(S^1 \times \Sigma)$$

the images in $\mathbf{C}^n$ of those analytic discs and their boundaries in the family $\mathcal{G}$ which correspond to the subspace of parameters $p \in \Sigma$. Note that the image of a disc in $\mathcal{G}$ consists of a single point iff it corresponds to a parameter disc which is also a point; i.e. one of the form $(c_0, 0)$, $c_0 \in \mathbf{R}^l$. Such a disc will be called a *degenerate disc.*

The following is a refinement of a theorem of Wells [22].

THEOREM 8.2. Let $\Sigma \subset U$ be any subset such that

i) *$\Sigma$ contains a degenerate disc,*

ii) $\pi_0(\Sigma) = \pi_1(\Sigma) = 0$.

*Then the map*

$$r\colon \mathcal{O}(\mathcal{G}(\Sigma)) \to \mathcal{O}(\boldsymbol{b}\mathcal{G}(\Sigma)),$$

*induced by restriction, is a topological isomorphism.*

COROLLARY 8.1. *If $S \subset M$ is any connected subset such that $\boldsymbol{b}\mathcal{G}(\Sigma) \subset S$ for some such choice of $\Sigma$, then every holomorphic function on $S$ has a unique holomorphic extension to $\mathcal{G}(\Sigma)$.*

PROOF OF THE THEOREM. Let $\omega$ be a connected open neighborhood of $\boldsymbol{b}\mathcal{G}(\Sigma)$ in $\mathbf{C}^n$. We will show that there exists an open connected neighborhood $\Omega = \Omega(\omega)$ of $\mathcal{G}(\Sigma)$, with $\Omega \supset \omega$, such that the restriction map

$$r^{\Omega}_{\omega}\colon \mathcal{O}(\Omega) \to \mathcal{O}(\omega)$$

is an isomorphism.

Consider an arbitrary point $z \in \mathcal{G}(\Sigma)$. First we will show that there is a connected open neighborhood $\Omega(z)$ of $z$, with $\Omega(z) \supset \omega$, such that every $f \in \mathcal{O}(\omega)$ has a unique extension to an $F \in \mathcal{O}(\Omega(z))$. Without loss of generality we can assume that the image in $\mathbf{R}^l \times \mathbf{C}^m$ of the degenerate disc in $\Sigma$, as well as its lift to $M$, is the origin in $\mathbf{C}^n$. Choose some $p_0 \in \Sigma$ with $z \in \mathcal{G}(\bar{D} \times \{p_0\})$. Since $\pi_0(\Sigma) = 0$ there is a continuous path $\gamma\colon [0,1] \to \Sigma$ such that $\gamma(0) = 0$ and $\gamma(1) = p_0$. For $0 \leqslant \tau \leqslant 1$ set $\boldsymbol{\Gamma}(\tau) = \mathcal{G}\big(\bar{D} \times \gamma([0, \tau])\big)$.

We will analytically continue $f$ along $\mathbf{\Gamma}(\tau)$ from $\mathbf{\Gamma}(0) \subset \omega$ to $\mathbf{\Gamma}(1) \supset \{z\}$.

Let $\sigma$ denote the subset of $[0, 1]$ consisting of those $\tau$ for which there is an open connected neighborhood $\Omega(\tau)$ of $\mathbf{\Gamma}(\tau)$, with $\Omega(\tau) \supset \omega$, such that every holomorphic function in $\omega$ has a unique holomorphic extension to $\Omega(\tau)$. It is clear from the definitions that $\sigma \neq \emptyset$ and $\sigma$ is open. To show that $\sigma$ is closed, let $\{\tau_j\}$ be a sequence in $\sigma$ converging to a point $\tau_0$. Without loss of generality we may assume that $\tau_j \nearrow \tau_0$, and that $\Omega(\tau_j) \subset \Omega(\tau_{j+1})$. Setting $\Omega_0 = \bigcup_{j=1}^{\infty} \Omega(\tau_j)$, we have that $\Omega_0$ is an open connected set with $\Omega_0 \supset \omega$, such that $\Omega_0$ is a neighborhood of $\mathbf{\Gamma}(t)$ for every $0 \leqslant t < \tau_0$, and such that every holomorphic function in $\omega$ has a unique holomorphic extension to $\Omega_0$. Thus in order to show that $\tau_0 \in \sigma$, we need to extend $\Omega_0$ to an open connected set $\Omega_1 \supset \Omega_0$ such that $\Omega_1 \supset \mathbf{\Gamma}(\tau_0)$ and such that every holomorphic function in $\Omega_0$ extends holomorphically to $\Omega_1$.

The parameter point $\gamma(\tau_0) \in \Sigma$ corresponds to a specific analytic disc $g_0 \colon \overline{D} \to \mathbf{C}^n$ with compact image $g_0(\overline{D})$. Since $\mathscr{G}(\zeta, \gamma(\tau_j)) \to \mathscr{G}(\zeta, \gamma(\tau_0)) = g_0(\zeta)$ uniformly as $\tau_j \nearrow \tau_0$ it will suffice, by compactness, to show that each point $z_1 \in g_0(\overline{D})$ has a neighborhood $N(z_1)$ such that all holomorphic functions in $\Omega_0$ have analytic continuations to $\Omega_0 \cup N(z_1)$. Let $K$ be a compact neighborhood of $g_0(S^1)$ with $K \subset \omega$, and let $\Delta$ be the distance from $K$ to the complement of $\omega$, measured in the maximum norm. For $j_0$ chosen sufficiently large we have $b\mathscr{G}(S^1 \times \gamma(\tau_{j_0})) \subset K$, and there is a point $z_0 \in \mathscr{G}(\overline{D} \times \gamma(\tau_{j_0}))$ such that $z_1$ belongs to the open polydisc of radius $\Delta$ centered about $z_0$. If $f \in \mathcal{O}(\Omega_0)$ then $|f(z_0)| \leqslant \max_K |f|$; hence $z_0$ belongs to the holomorphic hull

$$\hat{K} = \left\{ z \in \Omega_0 \,\big|\, |f(z)| \leqslant \max_K |f| \text{ for all } f \in \mathcal{O}(\Omega_0) \right\}$$

taken with respect to $\Omega_0$. Now we use the classical argument of the Cartan-Thullen theorem: Consider any $f \in \mathcal{O}(\Omega_0)$, any $0 < R < \Delta$, and any $z' \in K$. For any multi-index $\beta$ we have

$$|D_z^\beta f(z')| \leqslant M(f) \frac{\beta!}{R^\beta}, \tag{8.6}$$

where $M(f)$ denotes the maximum of $|f|$ taken over the closure of an $R$-neighborhood of $K$. But since $D_z^\beta f \in \mathcal{O}(\Omega_0)$ it follows from the definition of $\hat{K}$ that (8.6) also holds at the point $z_0 \in \hat{K}$. Hence the power series expansion of $f$ about $z_0$ converges in the polydisc about $z_0$ of radius $R'$, for any $0 < R' < R$. Thus each $f$ has an analytic extension $F$ to a fixed neighborhood $N(z_1)$ of $z_1$.

We have shown that $\sigma = [0, 1]$ and that the desired neighborhood $\Omega(z)$

of the original point $z$ exists. Set

$$\Omega = \Omega(\omega) = \bigcup_{z \in \mathcal{G}(\Sigma)} \Omega(z) .$$

Since $\pi_1(\Sigma) = 0$, the argument from the monodromy theorem shows that the procedure described above gives a well-defined analytic continuation of an $f \in \mathcal{O}(\omega)$ to an $F \in \mathcal{O}(\Omega)$.

The restriction map $r$ is injective and is obviously continuous, since each $r_\omega^\Omega$ is continuous. Let $r^{-1}$ be defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(\Omega(\omega)) & \hookrightarrow & \mathcal{O}(\mathcal{G}(\Sigma)) \\ \Big\uparrow {\scriptstyle (r_\omega^\Omega)^{-1}} & & \Big\uparrow {\scriptstyle r^{-1}} \\ \mathcal{O}(\omega) & \hookrightarrow & \mathcal{O}(b\mathcal{G}(\Sigma)) . \end{array}$$

Since $r_\omega^\Omega$ is surjective, it follows by the open mapping theorem that $(r_\omega^\Omega)^{-1}$ is continuous. Therefore $r^{-1}$ is continuous, since it is continuous on each $\mathcal{O}(\omega)$. This completes the proof of Theorem 8.2.

## 9. – On going up one dimension.

Now that we have at our disposal a suitably ample family of local analytic discs with boundaries on $M$, we turn to our first main application: The problem is the one discussed in the Introduction; namely, if the Levi form of $M$ at the origin does not vanish identically, we will show how to construct a local manifold $\tilde{M}$ near p, of one real dimension greater than $M$, which is nicely attached along $M$ in such a way that $M$ is the (partial) boundary of $\tilde{M}$ in the good sense of a « differential manifold with boundary ». For a more precise statement of this result, see Theorem 9.1 below.

Recall that $M = M^{n+m} \subset \mathbf{C}^n$ is a generic real $n + m$ dimensional manifold embedded in $\mathbf{C}^n$. Let

$$B_p \colon T_p(M) \times T_p(M) \to N_p(M)$$

be the *second fundamental form* of $M$ at $p \in M$, where $N_p(M)$ denotes the normal space to $M$ at $p$. It assigns to each pair of tangent vectors $u$, $v$ the normal vector

$$B_p(X, Y) = (\nabla_X Y)^N ,$$

where $\nabla$ denotes the standard connection in $\mathbf{R}^{2n} \cong \mathbf{C}^n$, and where $X$, $Y$ are local tangent vector fields near $p$ with $X(p) = u$, $Y(p) = v$. By the *Levi* *form* of $M$ at $p$ we will mean the vector valued Hermitian form

$$L_p\colon HT_p(M) \to N_p(M)$$

defined by

$$L_p(Z) = B_p(X, X) + B_p(JX, JX)\,,$$
$$Z = X - iJX\,,$$

where $J$ denotes the almost complex tensor of multiplication by $\sqrt{-1}$ in $\mathbf{C}^n$. Thus $L_p$ assigns to each complex holomorphic tangent vector $Z(p)$ the normal vector $L_p(Z)$. We will call $L_p(Z)$ the *Levi vector associated to the holomorphic tangent vector* $Z(p)$ at $p$.

Suppose $p = 0$ and let $M = \{z\colon \varrho(z) = 0\}$ be local defining equations for $M$, where each component of $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_l)$ is a real valued function, and $d\varrho_1 \wedge d\varrho_2 \wedge \ldots \wedge d\varrho_l|_0 \neq 0$. Frequently we will identify $N_0(M)$ with the space of covectors spanned by $d\varrho_1(0)$, $d\varrho_2(0), \ldots, d\varrho_l(0)$. Furthermore we shall assume that $d\varrho_1(0)$, $d\varrho_2(0), \ldots, d\varrho_l(0)$ are *orthonormal.* Let

$$Z(0) = \sum_{j=1}^{n} \zeta_j\, 2 \frac{\partial}{\partial z_j}\,, \quad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbf{C}^n$$

where $2(\partial/\partial z_j) = (\partial/\partial x_j) - i(\partial/\partial y_j)$. Then a calculation (see [5]) leads to the following expression for the Levi form of $M$ at the origin:

$$(9.1) \qquad \begin{cases} L_0(Z) = \displaystyle\sum_{i=1}^{l} \left[ -\sum_{j,k=1}^{n} 4\, \frac{\partial^2 \varrho_i}{\partial z_j\, \partial \bar{z}_k}(0) \zeta_j \bar{\zeta}_k \right] d\varrho_i(0)\,, \\ \displaystyle\sum_{j=1}^{n} \frac{\partial \varrho_i}{\partial z_j}(0) \zeta_j = 0\,, \quad 1 \leqslant i \leqslant l\,. \end{cases}$$

Here the second equation in (9.1) is just the requirement that $Z(0) \in HT_0(M)$.

We can now state the principle result to be proved in this section:

THEOREM 9.1. *Let $M$ be a generic real $n + m$ dimensional manifold embedded in $\mathbf{C}^n$, and at some $p \in M$, let $\xi \neq 0$ be a normal vector in the range of the Levi form $L_p$. Then (with the precise differentiability assumptions stated below) there exists a local embedded generic real manifold-with-boundary, $\tilde{M}$, of real dimension $n + m + 1$, with the boundary of $\tilde{M}$ equal to an open neighborhood of $p$ in $M$, and with $T_p(\tilde{M}) =$* span $\{T_p(M), \xi\}$. *Moreover $\tilde{M}$ is foliated by a real $n + m - 1$ parameter family of complex one dimensional analytic discs with boundaries on $M$:*

(i) *If $M$ is of class $C^k$ and $k \geqslant 5$ then $\tilde{M}$ is of class* $C^{[(k-2)/3],\frac{1}{2}}$.

(ii) *If $M$ is real analytic then $\tilde{M}$ is real analytic; moreover $\tilde{M}$ has a « border » $\tilde{M}_\delta - \tilde{M}$ in the sense that $\tilde{M}$ extends real analytically to a slightly larger $\tilde{M}_\delta$ such that $M \cap \tilde{M}_\delta$ forms an embedded real analytic hypersurface in $\tilde{M}_\delta$:*

(iii) *If $M$ is of class $C^\infty$ then $\tilde{M}$ is of class $C^\infty$.*

REMARK. Our proof actually shows more than is stated; e.g. in the differentiability-up-to-the-boundary of $\tilde{M}$, in (i), we lose only $1 + \varepsilon$ derivatives except along an exceptional set which is an $n + m - 2$ dimensional submanifold of $M$.

PROOF OF THEOREM 9.1. As usual we take $p = 0$. Let $\xi \neq 0$ be a covector in the range of the Levi form. Without any loss of generality we can assume that a system of holomorphic coordinates

$$(z_1, \ldots, z_l, z_{l+1}, \ldots, z_n) = (z_1, \ldots, z_l, w_1, \ldots, m_m) = (z, w) \in C^l \times \mathbf{C}^m = \mathbf{C}^n$$

has been introduced so that $\xi = dy_1$ and $\xi = L_0(\partial/\partial w_1)$. It follows from (9.1), upon setting $\varrho_i(z, w) = h_i(x, w) - y_i$, that

$$\text{(9.2)} \qquad \frac{\partial^2 h_1}{\partial w_1\, \partial \overline{w}_1}(0) = 1 \quad \text{and} \quad \frac{\partial^2 h_i}{\partial w_1\, \partial \overline{w}_1}(0) = 0 \quad \text{for } i \neq 1 .$$

Moreover we can assume that the Taylor expansion of $h(x_1, \ldots, x_l, z_{l+1}, 0, \ldots, 0)$ about the origin has the form

$$\text{(9.3)} \qquad \sum_{j,k=1}^{l} a_{jk} x_j x_k + \sum_{j=1}^{l} b_j x_j z_{l+1} + \sum_{j=1}^{l} \overline{b}_j x_j \overline{z}_{l+1} + d|z_{l+1}|^2 + O(3) .$$

Here $d$ is a column vector whose transpose ${}^t d$ is the $l$-tuple $(1, 0, \ldots, 0)$, the $a_{jk} = a_{kj}$ are real column vectors, and $O(n)$ denotes terms bounded by a constant times $\|(x_1, \ldots, x_l, z_{l+1}, 0, \ldots, 0)\|^n$ in a suitably small neighborhood of the origin.

Indeed, since $dh(0) = 0$, the Taylor expansion of $h(x_1, \ldots, x_l, z_{l+1}, 0, \ldots, 0)$ out to terms of the second order would look like (9.3) except, possibly, for additional terms of the form

$$\text{(9.4)} \qquad c z_{l+1}^2 + \overline{c} \overline{z}_2^{l+1} ,$$

with ${}^t c = (c_1, c_2, \ldots, c_l)$. But then, in a neighborhood of the origin, the biholomorphic transformation

$$\begin{aligned} z_j' &= z_j - i2c_j z_l^{2+1} \qquad (1 \leqslant j \leqslant l) \\ w_k' &= w_k \qquad\qquad\qquad (1 \leqslant k \leqslant m) \end{aligned}$$

would carry the local defining functions $\varrho(z, w) = h(x, w) - y$ into new $\varrho'(z', w') = h'(x', w') - y'$ in which the terms (9.4) disappear, and the Taylor expansion of $h'$ would have the desired form (9.3). It should be noted also that this change of variables would not destroy the relations described in the paragraph above.

Now that the coordinate system $(z, w)$ has been fixed, let us describe some notation to be used in the proof: $t = (t_1, t_2, \ldots, t_l) \in R^l$; $s = (s_2, s_3, \ldots, s_m) \in \in \mathbf{C}^{m-1}$; $\sigma \in \mathbf{C}$; $\tau, r \in \mathbf{R}^+ = \{x \in \mathbf{R} | x \geqslant 0\}$; $\zeta \in \bar{D}$ where $D$ is the open unit disc in $\mathbf{C}$; $Q$ is the infinite *salad bowl*

$$Q = \{(\tau, \sigma) \in \mathbf{R}^+ \times \mathbf{C} | \tau = |\sigma|^2\},$$

and

$$\tilde{Q} = \{(\tau, \sigma) \in \mathbf{R}^+ \times \mathbf{C} | \tau \geqslant |\sigma|^2\}$$

is the *full* salad bowl. Since $l + 2m = n + m$ we have that $\mathbf{R}^l \times Q \times \mathbf{C}^{m-1}$ and $\mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$ are real analytically diffeomorphic to copies of $\mathbf{R}^{n+m}$ and $\mathbf{R}^{n+m} \times \mathbf{R}^+$, respectively.

The manifold $\tilde{M}$ of real dimension $n + m + 1$ which we are going to construct will be exhibited as the image in $\mathbf{C}^n$ of a map $\tilde{G}$ of the form

$$\tilde{G}(t, \tau, \sigma, s) = \big(\tilde{F}(t, \tau, \sigma, s), \sigma, s\big), \tag{9.5}$$

defined for $(t, \tau, \sigma, s)$ in a suitable neighborhood of the origin in $R^l \times \tilde{Q} \times \mathbf{C}^{m-1}$, with $\tilde{F}$ taking values in $\mathbf{C}^l$. The restriction

$$G(t, \tau, \sigma, s) = \big(F(t, \tau, \sigma, s), \sigma, s\big) \tag{9.6}$$

of $\tilde{G}$ to $\mathbf{R}^l \times Q \times \mathbf{C}^{m-1}$ will be such that it provides a nonsingular parametrization of a corresponding neighborhood of the origin in $M$.

The map $\tilde{F}(t, \tau, \sigma, s)$ is constructed as follows: To each point $(t, r, s) \in \in \mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1}$ we associate the parameter disc $p(t, r, s) = \big(t, w(r, s)\big)$, where $w(r, s): \bar{D} \to \mathbf{C}^m$ is defined by $\zeta \mapsto (r\zeta, s_2, s_3, \ldots, s_m)$. Note that $p(t, r, s)$ defines an analytic disc in $\mathbf{R}^l \times \mathbf{C}^m$. Since

$$|p(t, r, s)|_{k,\alpha} = |t| + |w(r, s)|_{k,\alpha} \leqslant |t| + 4r + |s|,$$

we will have that $p(t, r, s) \in U(k, \alpha) \subset P = \mathbf{R}^l \times \mathscr{D}_m^{k,\alpha}$, provided that $|t|$, $|s|$ and $r$ are kept sufficiently small. Then by Theorem 5.1 we can solve the functional equation

$$u(e^{i\theta}) = t - T\big[h\big(u(e^{i\theta}), w(r, s)(e^{i\theta})\big)\big] \tag{9.7}$$

and thereby lift each parameter disc $p(t, r, s)$ to an analytic disc

$$g(t, r, s)(\zeta) = (f(t, r, s)(\zeta), w(r, s)(\zeta)) = (f(t, r, s)(\zeta), r\zeta, s)$$

in $\mathbf{C}^n$ with boundary on $M$. The center of each lifted disc is given by

$$g(t, r, s)(0) = (t + i\,\mathrm{Im}\, f(t, r, s)(0), 0, s)\,, \tag{9.8}$$

and when $r = 0$ we have the degenerate disc

$$g(t, 0, s)(\zeta) = (t + ih(t, 0, s), 0, s)\,. \tag{9.9}$$

Finally let

$$\tilde{G}(t, \tau, \sigma, s) = g(t, \tau^{\frac{1}{2}}, s)(\tau^{-\frac{1}{2}}\sigma) \qquad \tilde{F}(t, \tau, \sigma, s) = f(t, \tau^{\frac{1}{2}}, s)(\tau^{-\frac{1}{2}}\sigma) \tag{9.10}$$

denote the pullbacks via the map $\mu\colon \mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1} \times \mathbf{C} \to \mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$ defined by $(t, r, s, \zeta) \mapsto (t, \tau, \sigma, s)$ with $\tau = r^2, \sigma = r\zeta$. Note that the restriction (9.6) does map a neighborhood of the origin in $\mathbf{R}^l \times Q \times \mathbf{C}^{m-1}$ into a neighborhood of the origin on $M$.

We claim that $\tilde{F}$, and hence $\tilde{G}$, is a sufficiently smooth map defined on a neighborhood of the origin in $\mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$. Actually this investigation of the amount of smoothness of $\tilde{F}$ is the most tedious part of the proof. Therefore we postpone it until the end, and give the remainder of the proof under the assumption that $\tilde{F}$ is as smooth as is needed.

To start with we show that

$$\frac{\partial \tilde{G}}{\partial \tau}(0) \equiv \lim_{\tau \downarrow 0} \tau^{-1}[\tilde{G}(0, \tau, 0, 0) - \tilde{G}(0, 0, 0, 0)] = {}^t(i, 0, \ldots, 0) \in \mathbf{C}^n\,. \tag{9.11}$$

In view of (9.8)-(9.10), it will suffice to show that

$$\tilde{F}(0, \tau, 0, 0) = id\tau + O(\tau^{\frac{3}{2}})$$

as $\tau \downarrow 0$; or equivalently, that

$$f(0, r, 0)(0) = idr^2 + O(r^3) \tag{9.12}$$

as $r \downarrow 0$, where ${}^t d = (1, 0, \ldots, 0)$. Now $f(0, r, 0)(\zeta) - idr^2$ has on $S^1$ the boundary values $u(e^{i\theta}) + iv(e^{i\theta})$, where $u(e^{i\theta})$ denotes the solution of (9.7) corresponding to $(t, s) = (0, 0)$, and where

$$v(e^{i\theta}) \equiv h(u(e^{i\theta}), re^{i\theta}, 0, \ldots, 0) - dr^2 .$$

Therefore

$$f(0, r, 0)(0) - idr^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\, d\theta + \frac{i}{2\pi}\int_{-\pi}^{\pi} v(e^{i\theta})\, d\theta \equiv I + II .$$

We obtain from Theorem 5.1, by comparing $u(e^{i\theta})$ with the trivial solution corresponding to $r = 0$, an estimate of the form $|u|_\alpha \leqslant C|w(r, 0)|_\alpha \leqslant Cr$. (Here, and in what follows, $C$ will denote a generic constant that does not depend on any variables relevant to the argument at hand). In particular $|u|_{L^2}$, $|w(r, 0)|_{L^2} \leqslant Cr$. Now in a ball in $\mathbb{R}^l \times \mathbb{C}$ of radius $R \leqslant Cr$, we have an estimate on the Lipschitz constant for $h(x_1, \ldots, x_l, z_{l+1}, 0, \ldots, 0)$ of the form $\mathrm{Lip}(h) \leqslant Cr$. Thus

$$|u|_{L^2} \leqslant |T[h(u, w(r, 0))]|_{L^2} \leqslant \|T\|_{L^2} \mathrm{Lip}(h)[|u|_{L^2} + |w(r, 0)|_{L^2}] ,$$

and by keeping $r$ so small that

$$\|T\|_{L^2} \mathrm{Lip}(h) = \mathrm{Lip}(h) < \tfrac{1}{2} ,$$

we can arrange that $|u|_{L^2} \leqslant Cr^2$. Now using (9.3) we obtain that

$$\begin{aligned} |h(u, w(r, 0)) - dr^2|_{L^2} &\leqslant |\Sigma a_{jk} u_j u_k|_{L^2} + |\Sigma b_j u_j w(r, 0)|_{L^2} + \\ &+ |\Sigma \bar{b}_j u_j \overline{w}(r, 0)|_{L^2} + O(3) \leqslant \\ &\leqslant Cr^3 . \end{aligned}$$

Since $u = T[h(u, w(r, 0)) - dr^2]$ we obtain finally that

$$|u|_{L^2} \leqslant Cr^3 . \tag{9.13}$$

Then by the Schwarz inequality we have

$$|I| , \qquad |II| \leqslant Cr^3$$

and the proof of (9.12) is complete.

Next we show that the Jacobian matrix of the map $\tilde{G}$, evaluated at the origin in $\mathbb{R}^l \times \tilde{Q} \times \mathbb{C}^{m-1}$, has maximal (column) rank equal to $n + m + 1$. Because of the special form (9.5) of $\tilde{G}$, it will suffice to prove that the (partial)

Jacobian matrix $J(0)$ of the map

$$\tilde{F}(t, \tau, 0, 0): \mathbf{R}^l \times \mathbf{R}^+ \to \mathbf{C}^l = \mathbf{R}^{2l}$$

evaluated at the origin, has maximal column rank equal to $l + 1$. In fact we will show that

$$J(0) = \begin{bmatrix} I & id \\ I & -id \end{bmatrix}, \tag{9.14}$$

where $I$ is the $l \times l$ identity matrix, and where ${}^t d = (1, 0, \ldots, 0)$. Now the last column in (9.14) is just

$$\begin{bmatrix} id \\ \\ -id \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \tilde{F}}{\partial \tau}(0) \\ \overline{\dfrac{\partial \tilde{F}}{\partial \tau}(0)} \end{bmatrix},$$

as was shown in (9.11). Thus all that remains to prove is that

$$\frac{\partial \tilde{F}_j}{\partial t_k}(0) = \delta_{jk} \qquad (1 \leqslant j, k \leqslant l). \tag{9.15}$$

But

$$\tilde{F}(t, 0, 0, 0) = f(t, 0, 0)(0) = t + ih(t, 0),$$

so (9.15) follows since $h$ vanishes to second order at the origin.

Our map $\tilde{G}$ is defined and will be at least of class $C^1$ in some neighborhood of the origin in $\mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$. Therefore there is a neighborhood of the origin, which we can take to be of the form

$$\tilde{N} = \{t, \tau, \sigma, s) \in \mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1} |\, |t| < \delta_1, |s| < \delta_2, \ 0 \leqslant \tau < \tau_0\}$$

for $\delta_1$, $\delta_2$, $\tau_0 > 0$ sufficiently small, such that $\tilde{M} = \tilde{G}(\tilde{N})$ is an embedded manifold in $\mathbf{C}^n$, with boundary, of real dimension $n + m + 1$. The boundary of $\tilde{M}$ is given by $M = G(N)$ where

$$N = \{(t, \tau, \sigma, s) \in \mathbf{R}^l \times Q \times \mathbf{C}^{m-1} |\, |t| < \delta_1, |s| < \delta_2, 0 \leqslant \tau < \tau_0\}.$$

The tangent space to $\tilde{M}$ at the origin is the space spanned by $T_0(M) = \mathbf{R}^l \times \mathbf{C}^m$ and the Levi vector $\xi$. Since $\tilde{M}$ has real dimension $n + m + 1$ the holomorphic tangent space at each point of $\tilde{M}$ must have complex dimension $m + 1$, which is minimal; hence $\tilde{M}$ is automatically generic. Moreover,

$\tilde{M} - M$ is foliated by (locally closed) complex analytic submanifolds of $\mathbf{C}^n$ of complex dimension one: a typical leaf in the foliation is determined by setting $(t, \tau, s) = \text{const.}$ in (9.5).

Now we turn to the task of justifying our claim that $\tilde{F}$ is a sufficiently smooth map.

First of all Theorem 8.1 gives us a precise amount of smoothness for the map $\mathscr{G}: \bar{D} \times U \to \mathbf{C}^n$. Since the map from $\bar{D} \times \mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1}$ to $\bar{D} \times U \subset \subset \bar{D} \times \mathbf{R}^l \times \mathscr{D}_m^{k,\alpha}$ defined by $(\zeta, t, r, s) \mapsto (\zeta, p(t, r, s))$ is real analytic, the composition $\mathscr{G}(\zeta, p(t, r, s))$, which is defined on $\bar{D} \times \{$a neighborhood of the origin in $\mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1}\}$, has as much smoothness as $\mathscr{G}(\zeta, p)$ does. The connection between the notation of this section and that of section 8 is

$$f(t, r, s)(\zeta) \equiv \mathscr{F}(\zeta, p(t, r, s)) , \qquad \tilde{F}(t, \tau, \sigma, s) \equiv \mathscr{F}(\tau^{-\frac{1}{2}}\sigma, p(t, \tau^{\frac{1}{2}}, s)) .$$

Now the map $\mu: \bar{D} \times \mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1} \to \mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$, defined by $(\zeta, t, r, s) \mapsto \mapsto (t, \tau, \sigma, s)$ with $\tau = r^2$, $\sigma = r\zeta$, is a local real analytic diffeomorphism at points where $r > 0$ (or $\tau > 0$). Hence at such points the pullback $\tilde{F}$ has as much smoothness as $\mathscr{G}$ does. Thus the whole issue here is that of investigating how smooth $\tilde{F}$ is along the locus $\tau = 0$.

Let $|\ |_{k,\alpha}^r$ denote the $C^{k,\alpha}$-norm taken over the closed disc of radius $r$, with $r \leqslant 1$. Consider any function $f(\zeta)$ defined for $\zeta \in \bar{D}$ and define $\tilde{f}(\sigma)$ by

$$\tilde{f}(\sigma) = f(\zeta) , \qquad \sigma = r\zeta . \tag{9.16}$$

Then for $f \in \mathscr{D}^{k,\alpha}$ one has

$$|\tilde{f}|_{k,\alpha}^r \leqslant r^{-(k+\alpha)} |f|_{k,\alpha}^{\bar{D}} \leqslant c(k, \alpha) r^{-(k+\alpha)} |Re f|_{k,\alpha}^{S^1} ,$$

where the last inequality comes from Proposition 3.1.

For $k, j \geqslant 0$ let $D_\sigma^k D_{t,s}^j \tilde{F}$ denote any partial derivative of $\tilde{F}$, of order $k$ with respect to $\sigma$, and of total order $j$ with respect to the variables $(t, s)$, where the mixed partial $D_{t,s}^j \tilde{F}$ is taken in any order with respect to those variables. Consider the continuity of $D_\sigma^k D_{t,s}^j \tilde{F}$ at an arbitrary point $(t_0, 0, 0, s_0)$:

$$\begin{cases} |D_\sigma^k D_{t,s}^j \tilde{F}(t, \tau, \sigma, s) - D_\sigma^k D_{t,s}^j \tilde{F}(t_0, 0, 0, s_0)| \leqslant \\ \leqslant |D_\sigma^k D_{t,s}^j \tilde{F}(t, \tau, \sigma, s) - D_\sigma^k D_{t,s}^j \tilde{F}(t, \tau, 0, s)| + \\ + |D_\sigma^k D_{t,s}^j \tilde{F}(t, \tau, 0, s) - D_\sigma^k D_{t,s}^j \tilde{F}(t, 0, 0, s)| + \\ + |D_\sigma^k D_{t,s}^j \tilde{F}(t, 0, 0, s) - D_\sigma^k D_{t,s}^j F(t_0, 0, 0, s_0)| \equiv \\ \equiv I + II + III . \end{cases} \tag{9.18}$$

Now

$$\tilde{F}(t, 0, 0, s) = t + ih(t, 0, s)\,, \tag{9.19}$$

so $III = 0$ unless $k = 0$, in which case

$$|III| \leq |t - t_0| + C\{|t - t_0| + |s - s_0|\}\,, \tag{9.20}$$

provided $h \in C^{j,1}$. Let $\tilde{f}_{k,j}(\sigma)$ be a temporary notation for the terms $D^k_\sigma D^j_{t,s}\tilde{F}(t, \tau, \sigma, s) - D^k_\sigma D^j_{t,s}\tilde{F}(t, 0, 0, s)$, let $f_{k,j}$ be related to $\tilde{f}_{k,j}$ as in (9.16), and $f_j = f_{0,j}$. In what follows we set $r = \tau^{\frac{1}{2}} > 0$. On $S^1$ we have

$$|Ref_j|_{k,\alpha} = |D^j_{t,s}u - D^j_{t,s}t|_{k,\alpha} \leq C|r\zeta|_{k,\alpha} \leq Cr\,, \tag{9.21}$$

provided $h \in C^{k+j+1,1}$, according to Theorem 5.1. Therefore

$$|\tilde{f}_{k,j}|^r_\alpha \leq \tau^{-\alpha/2}|f_{k,j}|^{\overline{D}}_\alpha \leq \tau^{-\alpha/2}|f_j|^{\overline{D}}_{k,\alpha} \leq \tau^{-\alpha/2}c(k, \alpha)|Ref_j|^{S^1}_{k,\alpha} \leq C\tau^{(1-\alpha)/2}\,. \tag{9.22}$$

But then

$$|I| = |\tilde{f}_{k,j}(\sigma) - \tilde{f}_{k,j}(0)| \leq |\tilde{f}_{k,j}|^r_\alpha|\sigma|^\alpha \leq C\tau^{\frac{1}{2}}\,, \qquad (|\sigma| < r) \tag{9.23}$$

and

$$|II| = |\tilde{f}_{k,j}(0)| \leq |f_{k,j}|^{\overline{D}}_\alpha \leq |f_j|^{\overline{D}}_{k,j} \leq c(k, \alpha)|Ref_j|^{S^1}_{k,\alpha} \leq C\tau^{\frac{1}{2}}\,.$$

Thus $D^k_\sigma D^j_{t,s}\tilde{F}$ is not only continuous at $(t_0, 0, 0, s_0)$, but is of class $C^{\frac{1}{2}}$ in a neighborhood of the origin in $\mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$. Finally, using the theorem of H. A. Schwarz and arguing by induction (as in Section 8), we obtain the existence of a partial derivative (taken in any order) of order $k$ in $\sigma$ and total order $j$ in $(t, s)$, as well as its equality with $D^\sigma_k D^j_{t,s}\tilde{F}$.

Unfortunately the simple argument given above does not apply to a partial derivative of $\tilde{F}$ which involves some differentiations with respect to $\tau$. Therefore we are forced to develop a different approach.

First suppose that $h = h(w)$ is a (vector-valued) polynomial in the variables $w = (w_1, w_2, \ldots, w_m) = (w_1, s)$ and $\overline{w} = (\overline{w}_1, \overline{s})$ alone. Then $h$ can be written as

$$h(w) = \sum_{m=0}^{N} \sum_{\substack{j+k=m \\ j \geq k}} Re\{a^m_{jk}(s)w^j_1\overline{w}^k_1\}\,, \tag{9.25}$$

with coefficients $a^m_{jk}(s)$ that are complex (vector-valued) polynomials in $s$ and $\overline{s}$, and where $a^{2k}_{k,k}(s)$ is real. In this case (9.7) becomes an explicit expression

$$u = t - \sum_{m=0}^{N} r^m \sum_{\substack{j+k=m \\ j>k}} \operatorname{Im}\{a_{jk}^m(s)\exp[i(j-k)\theta]\} \tag{9.26}$$

for the solution, and it is easy to verify that

$$f(t, r, s)(\zeta) = t + i \sum_{m=0}^{N} r^m \sum_{\substack{j+k=m \\ j\geqslant k}} a_{jk}^m(s)\zeta^{j-k} . \tag{9.27}$$

Since $m-(j-k) = 2k$ if $j+k=m$ we obtain

$$\tilde{F}(t, \tau, \sigma, s) = t + i \sum_{m=0}^{N} \sum_{k=0}^{[m/2]} \tau^k a_{m-k,k}^m(s)\sigma^{m-2k} , \tag{9.28}$$

which is a polynomial in $t, \tau, \sigma, s$ and $\bar{s}$.

Second suppose that $h = h(u, w)$ is a polynomial in the full set of variables $u$, $w$ and $\bar{w}$. Then in multi-index notation we have

$$h(u, w) = \sum_{|\lambda|\leqslant N} h^\lambda(w)u^\lambda , \qquad (u^\lambda = u_1^{\lambda_1}u_2^{\lambda_2}\dots u_l^{\lambda_l}) \tag{9.29}$$

where each $h^\lambda(w)$ is a polynomial of the form (9.25) with coefficients $a_{jk}^{\lambda m}(s)$ and sum on $m$ up to $N(\lambda)$. In this case we will show that $\tilde{F}$ is real analytic in the variables $t$, $\tau$, $\sigma$, $s$ and $\bar{s}$ at the origin.

Let the sequence of functions $u_n$ be defined recursively by $u_0(e^{i\theta}) = 0$ and

$$u_{n+1}(e^{i\theta}) = t - T\big[h\big(u_n(e^{i\theta}), w(r, s)(e^{i\theta})\big)\big] . \tag{9.30}$$

Obviously $u_1$ has the form (9.26); i.e. it is a real polynomial in $re^{i\theta}$ and $re^{-i\theta}$. Because of (9.29) it follows by induction that each succeeding $u_n$ is a polynomial of the form (9.26). It follows that a corresponding $f_n(t, r, s)(\zeta)$ can be defined by (9.27); hence each $\tilde{F}_n(t, \tau, \sigma, s)$ is a polynomial of the form (9.28).

We now pass to the complexifications of $\mathbb{R}^l \times \mathbb{R}^+ \times \mathbb{C}^{m-1}$ and $S^1$, and adopt the notation from Section 6. Thus $\tilde{h}(u, w, w^*)$ is the holomorphic extension of the real analytic function $h(u, w, \bar{w})$, etc. Set $w(r, s)(\zeta) = (r\zeta, s)$, $w^*(r, s^*)(\zeta) = (r\zeta^{-1}, s^*)$, and let $u_n(\zeta) = u_n(t, r, s, s^*)(\zeta)$ denote the holomorphic extension of $u_n(e^{i\theta})$, which is defined for *complex* $t$, $r$, $s$, $s^*$ and $\zeta \in \bar{A}_\delta$. The holomorphic extension of (9.30) is

$$u_{n+1}(\zeta) = t - \tilde{T}\big[\tilde{h}\big(u_n(\zeta), w(r, s)(\zeta), w^*(r, s)(\zeta)\big)\big] . \tag{$\widetilde{9.30}$}$$

Let $u(\zeta) = u(t, r, s, s^*)(\zeta) \equiv$ the value of $\tilde{u}(t, w(r, s), w^*(r, s^*))$ at $\zeta \in \bar{A}_\delta$, denote the holomorphic solution of

$$u = t - \tilde{T}[\tilde{h}(u, w(r, s), w^*(r, s^*))] \tag{9.31}$$

that was constructed at the end of Section 6. We will show that, as $n \to \infty$, $u_n \to u$ uniformly for $\zeta \in \bar{A}_\delta$ and for $(t, r, s, s^*)$ kept in a suitably small complex neighborhood $\tilde{C}$ of the origin in $\mathbf{C}^l \times \mathbf{C} \times \mathbf{C}^{m-1} \times \mathbf{C}^{m-1}$.

Since $\tilde{h}$ vanishes to second order at the origin, there is a neighborhood $\tilde{B}$ of the origin in $\mathbf{C}^l \times \mathbf{C}^m \times \mathbf{C}^m$ such that

$$\theta \equiv \|\tilde{T}\|_{\alpha,\delta} \cdot \sup_{\tilde{B}} |d\tilde{h}| < \tfrac{1}{2} ;$$

then $\tilde{C}$ can be chosen such that

$$M \equiv 2 \sup_{\tilde{C}} \left\{ |t| + \theta \|w(r, s)\|_{\alpha,\delta} + \theta \|w^*(r, s^*)\|_{\alpha,\delta} \right\} < \text{the radius of } \tilde{B} .$$

For $(t, r, s, s^*) \in \tilde{C}$ we have

$$\|u_{n+1}\|_{\alpha,\delta} \leqslant M/2 + \theta \|u_n\|_{\alpha,\delta} ,$$

from which it follows that for all $n$

$$\|u_n\|_{\alpha,\delta} \leqslant M/2(1 + \theta + \theta^2 + \ldots + \theta^{n-1}) \leqslant \frac{M}{2(1-\theta)} \leqslant M . \tag{9.32}$$

Now bounded sets in $C^\alpha$ are precompact in $C^{\alpha'}$ for any $0 < \alpha' < \alpha$. Thus any subsequence of the $u_n$ has a subsequence which converges in $\underset{\sim}{A}{}^{\alpha'}_{\delta,l}$ to a limit $u'$. But by passing to the limit in (9.30) and using the argument of the uniqueness theorem, we can identify $u'$ with $u(t, r, s, s^*)(\zeta)$. Consequently the entire sequence $u_n \to u$ uniformly for $\zeta \in \bar{A}_\delta$ and $(t, r, s, s^*) \in \tilde{C}$.

Corresponding to $u_n$ and $u$ we have, as in Section 8, the functions $\tilde{f}_n(t, r, s, s^*)(\zeta)$ and $\tilde{f}(t, r, s, s^*)(\zeta)$, respectively; they are holomorphic on $\tilde{C} \times D_\delta$ and $\tilde{f}_n \to \tilde{f}$ uniformly there. Since, in a compact neighborhood of the origin, each partial derivative of $\tilde{f}$ is the limit of the same partial derivative of the $\tilde{f}_n$, it follows that the Taylor expansion about the origin of the real analytic function $f(t, r, s)(\zeta)$ is an infinite series having the general form of (9.27). Hence $\tilde{F} = \tilde{F}(t, \tau, \sigma, s)$ is real analytic in a neighborhood of the origin.

Third we suppose that $h = h(u, w)$ is a function of class $C^{l+2}$ with $l \geqslant 0$. Let $h^{(n)} = h^{(n)}(u, w)$ be a sequence of polynomials, like (9.29), such that in some compact neighborhood of the origin in $\mathbf{R}^l \times \mathbf{C}^m$, $h^{(n)} \to h$ in the $C^{l+2}$-norm, and $h^{(n)}(0) = dh^{(n)}(0) = 0$ for all $n$. In what follows the superscript $n$ will refer to solutions $u^{(n)}$ etc., of (9.7) corresponding to $h^{(n)}$.

Let $D^j$ denote temporarily any mixed partial derivative of total order $j$ taken with respect to the variables $t$, $r$, $s$, $\theta$, which involves at most $j-1$ differentiations with respect to $\theta$. We will show that $D^j u^{(n)} \to D^j u$ in $C_l^\alpha(S^1)$ for $0 \leqslant j \leqslant l+1$. When $j=0$ this is a consequence of our stability theorem 5.2. For the induction step, it is convenient to first observe that

$$D^j h(u, w) = h_u(u, w)\cdot D^j u + h_w(u, w)\cdot D^j w + R(h, u, w)$$

where $R(h, u, w)$ is a universal finite linear combination of terms of the form

$$D^{p,q}_{u,w} h(u, w) D^{\lambda_1} u_{\beta_1} \dots D^{\lambda_p} u_{\beta_p} D^{\mu_1} w_{\gamma_1} \dots D^{\mu_q} w_{\gamma_q}$$

where $D^{p,q}_{u,w}$ denotes any partial derivative of total order $p$ with respect to $u$ and of total order $q$ with respect to $w$, and in which $\lambda_1 + \dots + \lambda_p + \mu_1 + \dots + \mu_q = j$; $p+q = 2, \dots, j$; $1 \leqslant \beta_i \leqslant l$, $1 \leqslant \gamma_j \leqslant m$ and $\lambda_i$, $\mu_j \geqslant 1$. Therefore with $w = w(r, s)(e^{i\theta})$ we have

$$\begin{aligned}(9.34)\qquad |D^j u^{(n)} - D^j u|_\alpha &\leqslant \|T\|_\alpha |D^j h^{(n)}(u^{(n)}, w) - D^j h(u, w)|_\alpha \leqslant \\ &\leqslant \|T\|_\alpha \{|h_u^{(n)}(u^{(n)}, w)\cdot D^j u^{(n)} - h_u(u, w)\cdot D^j u|_\alpha + \\ &+ |h_w^{(n)}(u^{(n)}, w) - h_w(u, w)|_\alpha |D^j w|_\alpha + \\ &+ |R(h^{(n)}, u^{(n)}, w) - R(h, u, w)|_\alpha\} \equiv \\ &\equiv \|T\|_\alpha \{I + II + III\}.\end{aligned}$$

Now

$$\begin{aligned}(9.35)\qquad I \leqslant |h_u^{(n)}(u^{(n)}, w)|_\alpha |D^j u^{(n)} - D^j u|_\alpha + |h_u^{(n)}(u^{(n)}, w) - h_u(u^{(n)}, w)|_\alpha |D^j u|_\alpha + \\ + |h_u^{(n)}(u^{(n)}, w) - h_u(u, w)|_\alpha |D^j u|_\alpha .\end{aligned}$$

Since $h^{(n)} \to h$ in at least the $C^2$-norm, there is a compact neighborhood $B$ of the origin in $\mathbf{R}^l \times \mathbf{C}^m$ and there is a constant $C$ such that in $B$

$$|\mathrm{Lip}\, h_u^{(n)}|, \qquad |\mathrm{Lip}\, h_w^{(n)}| \leqslant C$$

for all $n$. By using our induction hypothesis (for $j=0$) we can find a bound of the form

$$(9.36)\qquad |h_u^{(n)}(u^{(n)}, w)|_\alpha \leqslant C\{|t| + r + |s|\} \leqslant \tfrac{1}{2}\|T\|_\alpha^{-1},$$

provided $(t, r, s)$ is kept in a suitably small compact neighborhood $K$ of the origin in $\mathbf{R}^l \times \mathbf{R}^+ \times \mathbf{C}^{m-1}$. Likewise the last two terms on the right in (9.35), as well as $II$, can be bounded by an expression of the form

$$(9.37)\qquad C\{|h_u^{(n)} - h_u|_{0,1} + |h_w^{(n)} - h_w|_{0,1} + (|h_u|_{1,1} + |h_w|_{1,1})|u^{(n)} - u|_\alpha\}.$$

As for $III$, each term in it can by repeated use of the triangle inequality, and the induction hypothesis, be bounded by expressions of the form

$$(9.38)\qquad C\{|D_{u,w}^{p,q} h^{(n)} - D_{u,w}^{p,q} h|_{0,1} + |D_{u,w}^{p,q} h|_{1,1} \cdot |D^\lambda u^{(n)} - D^\lambda u|_\alpha\}$$

where $0 \leqslant \lambda \leqslant j - 1$. Combining (9.34)-(9.38) we obtain an estimate for $|D^j u^{(n)} - D^j u|_\alpha$ which tends to zero as $n \to \infty$, uniformly for $(t, r, s) \in K$, provided $j \leqslant l + 1$.

Consider the functions $f^{(n)}(t, r, s)(\zeta)$ and $f(t, r, s)(\zeta)$ which correspond to $u^{(n)}$ and $u$. By applying Proposition 3.1 in the standard way we obtain that

$$(9.39)\qquad |D^j f^{(n)} - D^j f|_\alpha^{\bar{D}} \to 0$$

as $n \to \infty$, for $0 \leqslant j \leqslant l + 1$, where here $D^j$ now denotes any partial derivative of total order $j$ taken with respect to the variables $t$, $r$, $s$, $\zeta$, which involves at most $j - 1$ differentiations with respect to $\zeta$; the convergence is uniform for $(t, r, s) \in K$.

Finally we consider a partial derivative of the form $D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F}$ of total order $p + i + q$, where $2p + i + 3q \leqslant l$.

Since

$$D_\tau \tilde{F} = (\tfrac{1}{2}\tau^{-\frac{1}{2}}) D_r f + (-\tfrac{1}{2}\tau^{-1} \cdot \tau^{-\frac{1}{2}}\sigma) D_\zeta f,$$

it follows by repeated use of the chain rule that

$$D_\tau^q \tilde{F} = \sum_{|\gamma| \leqslant q} a_\gamma (\tau^{-\frac{1}{2}})^{\alpha_\gamma} (\tau^{-\frac{1}{2}}\sigma)^{\beta_\gamma} D^\gamma f,$$

where the $a_\gamma$ are numerical coefficients, $q \leqslant \alpha_\gamma \leqslant 2q$, $0 \leqslant \beta_\gamma \leqslant q$, and where the $D^\gamma f$ denote certain partial derivatives with respect to $r$ and $\zeta$ of order $|\gamma|$. Therefore

$$(9.40)\qquad D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F} = \Sigma a_\nu (\tau^{-\frac{1}{2}})^{\lambda_\nu} (\tau^{-\frac{1}{2}}\sigma)^{\mu_\nu} D^\nu f,$$

with $p + i \leqslant |\nu| \leqslant p + i + q$, $p + q \leqslant \lambda_\nu \leqslant p + 2q$, $0 \leqslant \mu_\nu \leqslant q$, and where now $D^\nu f = D_\zeta^p D_{t,s}^i D^\gamma f$ represents a derivative of total order $|\nu| = p + i + |\gamma| \leqslant$

$\leqslant p+i+q$. Each $D^\nu f$ has a Taylor expansion of the form

$$D^\nu f(t, r, s)(\zeta) = \sum_{k=0}^{p+2q} \frac{r^k}{k!} D_r^k D^\nu f(t, 0, s)(\zeta) + r^{p+2q+1} \cdot R(f, \nu, q) \tag{9.41}$$

in which $R(f, \nu, q)$ can be expressed, via the Lagrange form of the remainder applied to each of the $2l$ components of $D^\nu f$, in terms of $D_r^{p+2q+1} D^\nu f$ evaluated at intermediate points in $(0, r)$. Hence we can write

$$D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F} = \sum_{l=1}^{p+2q} \tau^{-l/2} A_l(f) + \sum_{l=0}^{q+1} \tau^{l/2} B_l(f)\,, \tag{9.42}$$

where $A_l(f)$ is a linear combination of terms of the form $(\tau^{-\frac{1}{2}}\sigma)^{\mu_\nu} D_r^k D^\nu f(t, 0, s) \cdot \cdot (\tau^{-\frac{1}{2}}\sigma)$ with $k \leqslant p+2q-1$, and where $B_l(f)$ is a linear combination of certain terms like those in $A_l(f)$, with $k \leqslant p+2q$, plus the $R(f, \nu, q)$'s.

The universal formulas (9.40)-(9.42) are also valid with $\tilde{F}$ and $f$ replaced by $\tilde{F}^{(n)}$ and $f^{(n)}$, respectively. Since each $\tilde{F}^{(n)}$ is known to be real analytic at the origin, and $|\tau^{-\frac{1}{2}}\sigma| \leqslant 1$, it follows that $A_l(f^{(n)}) = 0$ for all $l$ and all $n$. As $n \to \infty$ we know that $A_l(f^{(n)}) \to A_l(f)$, by (9.39); hence $A_l(f) = 0$ for all $l$. But our assumptions imply that each of the terms $B_l(f)$ remains bounded as $\tau \downarrow 0$, so that

$$\lim_{\tau \downarrow 0} D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F} = B_0(f) \tag{9.43}$$

exists in a neighborhood of the origin in $\mathbf{R}^l \times \tilde{Q} \times \mathbf{C}^{m-1}$.

This argument shows that $D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F}$ is continuous, and in fact of class $C^{\frac{1}{2}}$, at $\tau = 0$. Using the theorem of H. A. Schwarz as before, we obtain the continuity of the mixed partials taken in a different order, and their equality with $D_\sigma^p D_{t,s}^i D_\tau^q \tilde{F}$. Thus $\tilde{F}$ is of class $C^{[l/3],\frac{1}{2}}$ in a neighborhood of the origin. This completes the proof of Theorem 9.1 in the $C^{k,\alpha}$ category.

Now for the proof in the real analytic category: Assume that $h$ is real analytic in a neighborhood of the origin in $\mathbf{R}^l \times \mathbf{C}^m$. Then we can use Theorem 6.1 in place of Theorem 5.1, and the proof we have given above goes through without change down to the point of investigating the smoothness of $\tilde{F}(t, \tau, \sigma, s)$ at $\tau = 0$. We will show that, under our current assumption, $\tilde{F}$ is a real analytic function of all of its variables at the origin in $\mathbf{R}^l \times \mathbf{R} \times \mathbf{C} \times \mathbf{C}^{m-1}$.

The function $h$, although no longer a polynomial, still has a convergent multiple power series expansion about the origin that can be written in the form of (9.29) and (9.25). Let the functions $u_n$ be defined recursively as in (9.30) and $(\widetilde{9.30})$. Since $u_0 \equiv 0$ it follows that $u_1$, and hence each succeeding $u_n$, has a multiple power series expansion that has the form of (9.26). Now the rest of the argument following $(\widetilde{9.30})$ applies, since it did not rely on $h$ being a polynomial. Hence we can conclude that the

$$u_n(t, r, s, s^*)(\zeta) \to u(t, r, s, s^*)(\zeta)$$

and the corresponding

$$\tilde{f}_n(t, r, s, s^*)(\zeta) \to \tilde{f}(t, r, s, s^*)(\zeta),$$

uniformly for $\zeta \in \bar{A}_\delta$ and $(t, r, s, s^*) \in \tilde{C}$, where $\tilde{C}$ is a neighborhood of the origin in $\mathbf{C}^l \times \mathbf{C} \times \mathbf{C}^{m-1} \times \mathbf{C}^{m-1}$. Therefore the multiple power series expansion of the real analytic function $f(t, r, s)(\zeta)$ has the form of (9.27), and it follows that $\tilde{F}(t, \tau, \sigma, s)$ is real analytic at the origin with a power series expansion having the form (9.28).

In particular $\tilde{F}$ is well-defined for sufficiently small *negative* values of $\tau$. Therefore for some $\delta > 0$ we can take $\tilde{M}_\delta = \tilde{G}(\tilde{N}_\delta)$ with

$$\tilde{N}_\delta = \{(t, \tau, \sigma, s) \in \mathbf{R}^l \times \tilde{Q}_\delta \times \mathbf{C}^{m-1} \,|\, |t| < \delta_1, |s| < \delta_2, -\delta < \tau < \tau_0\},$$

where

$$\tilde{Q}_\delta = \{(\tau, \sigma) \in \mathbf{R} \times \mathbf{C} \,|\, \tau > |\sigma|^2 - \delta\}.$$

This completes the proof in the real analytic category.

Finally for the proof in the $C^\infty$ category: Let $h \in C^\infty$ and let $B$ be a compact neighborhood of the origin in $\mathbf{R}^l \times \mathbf{C}^m$. Then by the diagonal trick we can find a sequence of polynomials $h^{(n)}(u, w)$, with $h^{(n)}(0) = 0$ and $dh^{(n)}(0) = 0$, such that $h^{(n)} \to h$ in $C^k(B)$ for all $k \geq 0$. Let the sequence of functions $u_n$ be defined recursively by $u_0 \equiv 0$ and

$$u_{n+1} = t - T[h^{(n)}(u_n, w(r, s))]. \tag{9.44}$$

As before each $u_n$ is a polynomial of the form (9.26), and corresponding $f_n$'s can be defined by (9.27), so that each $\tilde{F}_n$ has the form (9.28). Since $h^{(n)} \to h$ in at least the $C^2$-norm we can, without loss of generality, assume that $B$ has been chosen so small that

$$\theta \equiv \|T\|_\alpha \cdot \sup_B |dh^{(n)}| < \tfrac{1}{2}$$

for all $n$. Then the argument leading to (9.32) applies.

Hence there is a neighborhood $V$ of the origin in $\mathbb{R}^l \times \mathbb{R}^+ \times \mathbb{C}^{m-1}$ such that

$$|u_n|_\alpha \leqslant M \tag{9.45}$$

for all $n$ and for $(t, r, s) \in V$. Thus any subsequence of the $u_n$ has a subsubsequence which converges to a limit $u'$ in $C_l^{\alpha'}$ for any $0 < \alpha' < \alpha$. For simplicity we drop the prime on $\alpha$. Once we verify that $u'$ is a solution of (9.7) on some neighborhood of the origin in $\mathbb{R}^l \times \mathbb{R}^+ \times \mathbb{C}^{m-1}$, we can without loss of generality assume that $V$ has been chosen such that the original sequence $u_n \to u$ in $C_l^\alpha$ uniformly for $(t, r, s) \in V$, where $u$ is the $C^\infty$ solution of (9.7) constructed in Theorem 7.1.

To verify that $u'$ solves (9.7) we write

$$\begin{aligned}|u' - t + T[h(u', w(r, s))]|_\alpha \leqslant |u' - u_{n+1}|_\alpha + |T[(h - h^{(n)})(u_n, w(r, s))]|_\alpha + \\ + |T[h(u_n, w(r, s)) - h(u', w(r, s))]|_\alpha ,\end{aligned}$$

and let $n$ run through the subsubsequence. The second term on the R.H.S. is bounded by

$$C \operatorname{Lip}(h - h^{(n)})$$

which tends to zero as $n \to \infty$. The third term on the R.H.S. tends to zero because $\mathscr{H}$ is continuous on $C^\alpha$.

Next we show that $D^j u_n \to D^j u$ in $C_l^\alpha$ for any derivative $D^j$ of total order $j$, taken with respect to the variables $t$, $r$, $s$ and $\theta$, uniformly on $V$. The proof, by induction on $j$, is exactly as in (9.34)-(9.38) except for the fact that two additional terms arise in (9.34):

$$|D^j u_n - D^j u|_\alpha \leqslant \|T\|_\alpha \{I + II + III + IV + V\} . \tag{9.46}$$

The terms $I$, $II$, $III$ are exactly as before, and

$$\begin{aligned} IV &= |D^j h^{(n)}(u_n, w) - D^j h^{(n)}(u_{n-1}, w)|_\alpha , \\ V &= |D^j h^{(n)}(u_{n-1}, w) - D^j h^{(n-1)}(u_{n-1}, w)|_\alpha . \end{aligned}$$

Since $h^{(n)} \to h$ in $C^{j+2}$, we have that

$$IV \leqslant |D^j h^{(n)}|_{1,1} \cdot |u_n - u_{n-1}|_\alpha \leqslant C |u_n - u_{n-1}|_\alpha \to 0$$

as $n \to \infty$. Likewise

$$V \leqslant C |D^j h^{(n)} - D^j h^{(n-1)}|_{0,1} \to 0 .$$

Therefore $D^j u_n \to D^j u$.

The rest of the proof following (9.39) goes exactly as before. Therefore $\tilde{F} \in C^\infty$ in a neighborhood of the origin in $\mathbb{R}^l \times \tilde{Q} \times \mathbb{C}^{m-1}$. This completes the proof of Theorem 9.1.

## 10. – On the Hans Lewy extension phenomenon.

Now that we have the results of Section 9, it would be appropriate to consider in some detail the problem analogous to the Hans Lewy extension phenomenon. Due to limitations of time and space we confine ourselves here to a simple illustration of the type of theorem that can be obtained. We plan to discuss these matters more fully in a future publication.

Let $CR(M)(CR(\tilde{M}))$ denote the algebra of $C^\infty$ functions on $M(\tilde{M})$ which are annihilated by the tangential Cauchy Riemann equations to $M(\tilde{M})$. In what follows $M$, $\tilde{M}$, $\tilde{M}_\delta$ will be as in part (ii) of Theorem 9.1, and $M$ will be chosen so that $M = \partial\tilde{M}$.

THEOREM 10.1. *Let $M$ be real analytic and satisfy the hypothesis of Theorem* 9.1. *Then the restriction map*

$$CR(\tilde{M}) \simeq CR(M)$$

*is a topological isomorphism.*

PROOF. We employ some of the notation of [7] and [1]: $\bar{\partial}_{\tilde{M}}$ will denote the tangential Cauchy Riemann operator to $\tilde{M}(\tilde{M}_\delta)$, $H^*_{\tilde{M}}(\tilde{M}_\delta)$ is the cohomology of the $\bar{\partial}_{\tilde{M}}$-complex on $\tilde{M}_\delta$ with supports in $\tilde{M}$, and $B_\delta = \tilde{M}_\delta -$ {the interior of $\tilde{M}$ in $\tilde{M}_\delta$} will denote the border. There is by [1] an exact sequence

$$0 \to H^0_{\tilde{M}}(\tilde{M}_\delta) \to H^0_{\tilde{M}}(B_\delta) \oplus H^0_{\tilde{M}}(\tilde{M}) \to H^0_{\tilde{M}}(M) \to H^1_{\tilde{M}}(\tilde{M}_\delta) \to \dots,$$

because $M$ is a noncharacteristic hypersurface in $\tilde{M}_\delta$ for $\bar{\partial}_{\tilde{M}}$. Since $\tilde{M}_\delta$ is real analytic, the operator $\bar{\partial}_{\tilde{M}}$ has real analytic coefficients. By the argument of the Holmgren uniqueness theorem it follows that $\bar{\partial}_{\tilde{M}}$ has the unique continuation property on $\tilde{M}_\delta$. Hence $H^0_{\tilde{M}}(\tilde{M}_\delta) = H^0_{\tilde{M}}(B_\delta) = 0$. By definition $H^0_{\tilde{M}}(\tilde{M}) = CR(\tilde{M})$ and $H^0_{\tilde{M}}(M) = CR(M)$. Therefore it will suffice to show that $H^1_{\tilde{M}}(\tilde{M}_\delta) = 0$. But that statement is the content of Lemma 1 of [7], because $(\tilde{M}, \tilde{M}_\delta)$ has a top hat foliation, as defined in [7]. This completes the proof of Theorem 10.1.

One can work in other categories beside $C^\infty$. For example, by employing the results of [8] it is possible to treat these questions in the hyperfunction category, see [17], [18], [20], [19].

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