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Gradient Bounds and Liouville Theorems for Quasilinear Elliptic Equations.

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dedicated to Jean Leray

1. – Introduction.

Let u be a solution of Laplace's equation in the whole of $\boldsymbol{R}^n$. The classical Liouville Theorem states that if u is bounded on one side, say $u \leqslant M$ for all $x \in \boldsymbol{R}^n$, then u must be identically constant. In [8] Serrin has extended this result to entire solutions of the nonlinear Poisson equation

$$\Delta u = f(x, u, Du)\,,$$

where Δ denotes the Laplacian operator in $\boldsymbol{R}^n$, $x=(x_1, \ldots, x_n)$ is a point in $\boldsymbol{R}^n$, and $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ is the gradient of u. His results were based on gradient estimates obtained by means of the maximum principle: to carry the method through, however, it was required that the solution be bounded *both* from above and from below.

In this paper we shall show that for certain classes of quasilinear elliptic equations of the form

$$\mathcal{A}(x, u, Du)D^2 u = \mathcal{B}(x, u, Du)\,, \tag{1.1}$$

where $D^2 u$ denotes the Hessian matrix $(\partial^2 u/\partial x_i\, \partial x_j)$, it is possible to establish gradient bounds and Liouville theorems for solutions which (as in the classical Liouville Theorem) are only bounded on one side. In addition

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we shall prove corresponding results for solutions for which no bounds of any sort are available.

To indicate the scope of our results, we shall formulate a number of theorems for the simpler equation

$$\Delta u = f(u, Du) \tag{1.2}$$

in which we assume that the function $f(u, p)$ is defined on $\boldsymbol{R}\times\boldsymbol{R}^n$ and continuously differentiable on $\boldsymbol{R}\times(\boldsymbol{R}^n\setminus\{0\})$, where $p = (p_1, \dots, p_n)$ denotes a replacement variable for Du.

We first consider solutions of (1.2) for which no a priori bounds are given. To simplify the notation we shall mean by f_u the partial derivative $\partial f/\partial u$ and by f_p the gradient $(\partial f/\partial p_1, \dots, \partial f/\partial p_n)$. When a function $u \in C^3(\boldsymbol{R}^n)$ is a solution of (1.2) in the entire space $\boldsymbol{R}^n$ we shall call it an *entire solution* of (1.2).

THEOREM 1.1. *Let u be an entire solution of* (1.2) *whose gradient Du is bounded. Suppose that for each pair of positive numbers ε and L (with $\varepsilon < L$) there exist positive constants μ and N such that*

$$f_u + \frac{f^2}{n|p|^2} \geqslant \mu \qquad \textit{and} \qquad |f_p| \leqslant N$$

for $u \in \boldsymbol{R}$ and $\varepsilon \leqslant |p| \leqslant L$. Then $u \equiv$ constant.

If f depends only on Du, and if $f(p) \neq 0$ for $p \neq 0$, then the hypotheses of the theorem are automatically satisfied and any entire solution with bounded gradient must be constant. If f depends only on u and if $\partial f/\partial u$ is positive and uniformly bounded from zero then the hypotheses of the theorem are again automatically satisfied so that in this case also any entire solution with bounded gradient is constant. An example in point is the equation

$$\Delta u = cu\,, \qquad c = \text{constant} > 0\,.$$

Finally we observe that Theorem 1.1 implies that if $f(c, 0) \neq 0$ for all $c \in \boldsymbol{R}$ then equation (1.2) can have no entire solutions with bounded gradient.

Theorem 1.1 has already been noted in [9]. It is not yet a Liouville theorem of the ordinary type, however, since one of the required hypotheses is that Du be bounded. In many cases, however, it is possible to obtain a bound for Du by separate means and thus to obtain a standard type Liouville theorem. Such an additional result obviously must take into account the asymptotic behaviour of the function $f(u, p)$ for large p, and accordingly it is natural to consider this aspect of the problem separately. As an example, the following gradient bound can be obtained by our present approach. Here we shall denote the ball with radius R and center at the origin by $B(R)$.

THEOREM 1.2. *Let $u \in C^3(B(R))$ be a solution of (1.2) in $B(R)$. Suppose there exist positive constants ν, $\varkappa$ and L such that either*

$$\left(1 + |p|^2 \left|\frac{f_p}{f}\right|^2\right)|p|^{2\nu} \leqslant \varkappa\left(f_u + \frac{f^2}{2n|p|^2}\right) \tag{1.3}$$

or

$$|p|^{2\nu} + |p|^\nu |f_p| \leqslant \varkappa\left(f_u + \frac{f^2}{n|p|^2}\right) \tag{1.3$'$}$$

for all $u \in \mathbf{R}$ and $|p| > L$. Then there exists a positive constant K depending only on ν, $\varkappa$ and n such that

$$|Du(0)| \leqslant \max\{L, KR^{-1/\nu}\}\,. \tag{1.4}$$

As noted above, Theorems 1.1 and 1.2 together yield a standard Liouville theorem, indeed one with no assumptions whatever concerning the asymptotic behaviour of the solution. As an example, observe that every entire solution of the equation $\Delta u = |Du|^{1+\beta}$ is a constant if $\beta > 0$. It is also worth noting that (1.4) can be used to deduce the behaviour of a solution near an isolated singularity. In particular, if (1.3) holds and if u is a solution of (1.2) with an isolated singularity at the origin, then we have

$$|Du(x)| = O(|x|^{-1/\nu}) \qquad \text{as } |x| \to 0\,.$$

In Section 3 it will be shown by means of the example $\Delta u = |Du|^{1+\beta}$ that this estimate is sharp.

In the above results no bounds for u were required. Next we turn to solutions of equation (1.2) which are bounded on one side. We shall only consider solutions which are bounded above; it is a trivial matter to extend the results to solutions which are bounded below. We define the function

$$M(d) = \sup_{B(d)} u(x)\,.$$

As in [8] we say that the function $f(u, p)$ satisfies *condition H* if for any pair of positive numbers ε and L (with $\varepsilon < L$) there exists a corresponding number C such that

$$|f_p(u, p)| \leqslant C|f(u, p)| \qquad \text{for } u \in \boldsymbol{R},\ \varepsilon \leqslant |p| \leqslant L\,.$$

THEOREM 1.3. *Let u be an entire solution of equation* (1.2), *where*

$$f_u + f^2/2n|p|^2 \geqslant 0$$

and f satisfies condition H. Suppose Du is bounded and that $M(d) = o(d)$ as $d \to \infty$. Then $u \equiv$ constant.

This result strengthens Theorem 3 in [8] by replacing the two-sided condition $u = o(d)$ by the one-sided condition $M(d) = o(d)$. Moreover, just as Theorem 3 of [8] applies unchanged if the Laplace operator in (1.2) is replaced by the quasilinear elliptic operator $\mathcal{A}(Du)D^2u$, so also Theorems 1.1 and 1.3 here remain unchanged if Δu is replaced by $\mathcal{A}(Du)D^2u$ in (1.2).

In particular, it follows that if u is an entire solution of the equation

$$\mathcal{A}(Du)D^2u = 0$$

such that Du is uniformly bounded and $M(d) = o(d)$, then $u \equiv$ constant (1). The growth condition on $M(d)$ is obviously necessary. Conversely, if u is an entire solution of the nonhomogeneous equation

$$\mathcal{A}(Du)D^2u = \mathcal{B}(Du) \tag{1.5}$$

such that Du is uniformly bounded, then $u \equiv$ constant if $\mathcal{B}(p) \neq 0$ for $p \neq 0$. This result obviously does not hold when $\mathcal{B}$ is allowed to vanish for arguments $p \neq 0$, as is clear from Laplace's equation. What is more surprising, however, is that *even a one-sided bound on u does not suffice to ensure that an entire solution of* (1.5) *with bounded gradient is constant, as witness the equation*

$$\Delta u = -\,|1 - |Du|^2|^{\frac{3}{2}}$$

which has the non-constant entire solution $u = -\,(1 + x_1^2)^{\frac{1}{2}}$.

This example shows moreover that Theorem 1.3 cannot be improved by replacing condition H with a straightforward boundedness condition

(1) When $n = 2$ the boundedness condition on the gradient can be dropped; see [1].

on f_p. Should the solution be bounded on *both* sides, however, then finally we do regain the conclusion that an entire solution of (1.5) with gradient must be constant; see Theorem 6 of [8]. These results can be summarized briefly as follows:

Let u be an entire solution of (1.5) *with bounded gradient. If $u = o(\log d)$, then $u \equiv$ constant. If $M(d) = o(d)$ and $\mathfrak{B}$ satisfies condition H (in particular if $\mathfrak{B} \equiv 0$), then again $u \equiv$ constant. Finally if $\mathfrak{B}(p) \neq 0$ for $p \neq 0$ then, without any additional growth hypotheses, we still have $u \equiv$ constant. These results are best possible in the sense that none of the hypotheses can be dropped without altering the conclusion.*

The following two theorems give estimates for the gradient of a solution which is bounded on one side.

THEOREM 1.4. *Let $u \in C^3(B(R))$ be a solution of* (1.2) *in $B(R)$, such that $u \leqslant M$. Suppose that there exist positive constants $\varkappa$ and l such that*

$$f \geqslant -\varkappa|p|\,, \qquad |f_p| \leqslant \varkappa\,, \qquad f_u + f^2/n|p|^2 \geqslant -\varkappa$$

for $u \leqslant M$, $|p| > l$. Then there exists constants K and L, depending only on n, $\varkappa$, l, such that

$$|Du(0)| \leqslant \max\{L, K/R\}\cdot\{M + 1 - u(0)\}\,. \tag{1.6}$$

By strengthening the conditions on f, it is also possible to obtain a gradient estimate similar to (1.6) but in which $L = 0$.

THEOREM 1.5. *Let $u \in C^3(B(R))$ be a solution of* (1.2) *in $B(R)$ such that $u \leqslant M$. Suppose that f satisfies the conditions*

$$f_u + f^2/2n|p|^2 \geqslant 0\,, \qquad p\cdot f_p \leqslant \left(1 + \frac{1}{n}\right) f$$

and that

$$|p||f_p| \leqslant \text{const}\,|f|$$

for $u \leqslant M$ and $p \neq 0$. Then there exists a constant K such that

$$|Du(0)| \leqslant (K/R)[M - u(0)]\,.$$

Clearly Theorem 1.5 immediately yields a Liouville theorem, since if u is an entire solution we have the freedom to choose R arbitrarily large. The new condition in Theorem 1.5, namely that $p\cdot f_p \leqslant (1 + 1/n) f$, is essentially necessary. For consider the example

$$\Delta u = -\,|Du|^{1-1/m}\,,$$

where m is a large odd integer. This satisfies all the conditions of the theorem except the new one (and even satisfies that if the factor $1+1/n$ is replaced by $1-1/m$), yet has the negative entire solution

$$u(x) = -\frac{r^{m+1}}{(m+1)(m+n-1)^m}\,.$$

Theorem 1.5 can also be used to obtain the following Harnack inequality.

THEOREM 1.6. *Let $u \in C^3(\Omega)$ be a solution of* (1.2) *in Ω such that $u \leqslant M$ and let f satisfy the conditions of Theorem* 1.5. *Then for any compact subset Ω' of Ω there exists a constant $\Lambda > 1$ such that for any two points $x, y \in \Omega'$*

$$u(x) \geqslant \Lambda u(y) - (\Lambda - 1)M\,.$$

We have observed that Theorem 1.5 yields a Liouville theorem. In fact for this result it is even possible to replace the one-sided boundedness condition $u \leqslant M$ by a growth condition on the function $M(d)$, as in Theorem 1.3. The precise result is as follows.

THEOREM 1.7. *Let u be an entire solution of equation* (1.2). *Suppose that*

$$f_u + f^2/2n|p|^2 \geqslant 0\,, \qquad p\cdot f_p \leqslant \left(1+\frac{1}{n}\right)f$$

and that for every positive number ε there exists a number $\varkappa$ such that

$$\text{(i)}\ |p||f_p| \leqslant \varkappa|f| \qquad \textit{or} \qquad \text{(ii)}\ |f_p| \leqslant \varkappa$$

for $u \in \mathbf{R}$, $|p| > \varepsilon$. Then $u \equiv$ constant, provided that $M(d) = o(d)$ in case (i) *and $M(d) = o(\sqrt{d})$ in case* (ii).

As in the case of Theorem 1.5 the condition $p\cdot f_p \leqslant (1+1/n)f$ is essentially necessary.

Examples of equations which satisfy the conditions of Theorems 1.5 and 1.7 are

$$\Delta u = 0\,, \tag{1.7}$$

$$\Delta u = |Du|\,, \tag{1.8}$$

$$\Delta u = (1+|Du|^2)^{\frac{1}{2}}\,. \tag{1.9}$$

Since for equation (1.9) we have $f(c, 0) \equiv 1$, Theorem 1.7 implies that there exist no entire solutions of (1.9) which are bounded above.

Section 2 is devoted to some preliminary calculations needed for the remaining parts of the paper. Section 3 considers solutions which have no a priori bounds and Section 4 treats solutions which are bounded on one side. The seven theorems noted above are obtained respectively from Theorem 3.4 (with $\theta = 0$), Theorems 3.2 and 3.1 (with $\theta = \frac{1}{2}$ and $\theta = 0$), the corollary of Theorem 4.4 (with $\theta = \frac{1}{2}$), Theorem 4.1 (with $\theta = 0$, $a = \mu = 1$, $s = -1$, $t = 0$), Theorem 4.2 (with $\theta = \frac{1}{2}$, $s = -1$), Theorem 4.3 (with $\theta = \frac{1}{2}$, $s = -1$), and Theorem 4.5 (with $\theta = \frac{1}{2}$, $s = -1$).

Several important variational equations can be treated as special cases of our results, particularly the equation of prescribed mean curvature

$$(1 + |Du|^2)\, \Delta u - \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\, \partial x_j} = n\Lambda(x, u, Du)(1 + |Du|^2)^{\frac{3}{2}}$$

and the Euler-Lagrange equation for the problem

$$\delta \int \{F(|Du|) + G(u)\}\, dx = 0\,.$$

The first of these is considered in Section 4 after Theorem 4.4, and the second in Section 5.

For comparison with the results of this paper, the reader may also note the Liouville-type theorems of Bohn and Jackson [1], Gilbarg and Serrin [3], Ivanov [4], and Redheffer [6], and Tavgelidze [10].

2. – Preliminaries.

We begin by introducing some notation. Let $\mathcal{F}$ be a scalar function of the variables x, u and p. Then we shall write

$$\mathcal{F}_x = (\partial\mathcal{F}/\partial x_1, \ldots, \partial\mathcal{F}/\partial x_n)\,, \qquad \mathcal{F}_u = \partial\mathcal{F}/\partial u\,, \qquad \mathcal{F}_p = (\partial\mathcal{F}/\partial p_1, \ldots, \partial\mathcal{F}/\partial p_n)\,.$$

It will be convenient to define the differential operators δ, δ_1, δ_2 as follows:

$$\begin{aligned} \delta\mathcal{F} &\equiv \mathcal{F}_u + \frac{1}{|p|^2}\, p\cdot\mathcal{F}_x \qquad (p \neq 0)\,,\\ \delta_1\mathcal{F} &\equiv p\cdot\mathcal{F}_p + s\mathcal{F}\,,\\ \delta_2\mathcal{F} &\equiv \mathcal{F}_p + t\mathcal{F}\,. \end{aligned}$$

The quantities s and t are respectively, a scalar and a vector valued function of x, u and p which we are free to choose. Whenever $\mathcal{F}$ is a vector valued or a matrix valued function, we shall continue to employ this notation.

The method we shall use to obtain estimates for the gradient of u is based on one used by Serrin [7] and [8] for solutions which are bounded on both sides. We consider solutions $u(x)$ of the equation

$$\mathcal{A}(x, u, Du)D^2u = \mathcal{B}(x, u, Du)\,, \tag{2.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are respectively a given symmetric positive definite matrix and a given scalar function of the variables indicated, while $\mathcal{A}D^2u$ denotes the natural contraction $\mathcal{A}_{ij}\,\partial^2u/\partial x_i\,\partial x_j$. We shall suppose that $\mathcal{A}(x, u, p)$ is of class C^1 on $\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n$ and that $\mathcal{B}(x, u, p)$ is of class C^1 on $\Omega\times\boldsymbol{R}\times(\boldsymbol{R}^n\setminus\{0\})$. (The reason for using $\boldsymbol{R}^n\setminus\{0\}$ instead of $\boldsymbol{R}^n$ in the differentiability class of $\mathcal{B}$ is so as to include examples of the form $\mathcal{B}(x, u, p) = |p|$, etc.)

Now define the function

$$z(x) = \zeta(x)|Du|^2$$

in which ζ is a cut-off function of the form

$$\zeta(x) = \left(1 - \frac{|x|^2}{d^2}\right)^k, \qquad |x| < d\,,$$

and d and k are positive constants which will be chosen appropriately. It follows from an elementary computation (see [7] or [8]) that in the ball $B(d) = \{x\colon |x| < d\}$ the function z satisfies the equation

$$\begin{aligned}\mathcal{A}D^2z = 2\zeta D^2u\mathcal{A}D^2u - 2(\delta\mathcal{A}D^2u - \delta\mathcal{B})z + (\delta_2\mathcal{A}D^2u - \delta_2\mathcal{B})\frac{D\zeta}{\zeta}z\\ + \left(\frac{\mathcal{A}D^2\zeta}{\zeta} - 2\frac{D\zeta\mathcal{A}D\zeta}{\zeta^2}\right)z + \mathcal{M}\cdot Dz\,,\end{aligned} \tag{2.2}$$

where $\mathcal{M}$ is a continuous vector valued function on $B(d)$ and $D^2u\mathcal{A}D^2u$ denotes the contraction

$$\mathcal{A}_{ij}\frac{\partial^2u}{\partial x_i\,\partial x_k}\,\frac{\partial^2u}{\partial x_j\,\partial x_k}\,.$$

The operators δ and δ_2 are understood to apply only to the quantities immediately following them.

The first term on the right hand side of (2.2) is non-negative and can be used to estimate the terms involving $\delta\mathcal{A}$ and $\delta_2\mathcal{A}$. Indeed we have the inequality

$$D^2u\mathcal{A}D^2u \geqslant \lambda\|D^2u\|^2, \tag{2.3}$$

where λ is the lowest eigenvalue of the positive definite matrix $\mathcal{A}$. We also have

$$D^2u\mathcal{A}D^2u \geqslant (\mathcal{A}D^2u)^2/\mathrm{tr}\,\mathcal{A} = \mathcal{B}^2/\mathrm{tr}\,\mathcal{A}, \tag{2.4}$$

where tr $\mathcal{A}$ denotes the trace of $\mathcal{A}$ (2). Because (2.4) is in some cases stronger than (2.3) we shall only use part of the first term, namely $\theta D^2u\mathcal{A}D^2u$ $(0<\theta\leqslant 1)$, to estimate the terms involving $\delta\mathcal{A}$ and $\delta_2\mathcal{A}$. The remainder $(1-\theta)D^2u\mathcal{A}D^2$ can then be estimated by means of (2.4). It is clear that we may set $\theta=0$ if $\delta\mathcal{A}=\delta_2\mathcal{A}=0$. Thus we obtain eventually (3)

$$\mathcal{A}D^2z \geqslant \mathcal{M}\cdot Dz - 2\mathcal{N}z, \qquad x\in B(d), \tag{2.5}$$

in which

$$\mathcal{N} = -\,\delta\mathcal{B} - (1-\theta)\frac{\mathcal{B}^2}{\mathrm{tr}\,\mathcal{A}|p|^2} + \frac{|p|^2\,\|\delta\mathcal{A}\|^2}{2\theta\lambda} + k|\delta_2\mathcal{B}|\frac{1}{d}\left(\frac{|p|^2}{z}\right)^{1/k} + k(6k+n)\left\{\|\mathcal{A}\| + \frac{|p|^2\,\|\delta_2\mathcal{A}\|^2}{12\theta\lambda}\right\}\frac{1}{d^2}\left(\frac{|p|^2}{z}\right)^{2/k}. \tag{2.6}$$

(2) To see this, note that since $\mathcal{A}$ is positive definite we have

$$\mathcal{A}_{ij}(\xi_{ik}+\lambda\eta_{ik})(\xi_{jk}+\lambda\eta_{jk}) \geqslant 0$$

for any matrices ξ and η and any real λ. Thus, reverting to our vector notation,

$$\xi\mathcal{A}\xi + 2\lambda\xi\mathcal{A}\eta + \lambda^2\eta\mathcal{A}\eta \geqslant 0, \qquad \text{all } \lambda,$$

whence obviously

$$(\xi\mathcal{A}\eta)^2 \leqslant (\xi\mathcal{A}\xi)(\eta\mathcal{A}\eta).$$

Putting $\xi = D^2u$ and $\eta =$ Identity, we get (1.4).

(3) See [8] for the case $k=2$. For the general case it is easiest to suppose $\mathcal{A}=\mathcal{A}'$, $\mathcal{F}=0$ in the calculation of [7], pp. 578-580; note also that in the present paper the operator δ has been defined without the minus sign which appears in [7] and [8]. The introduction of the constant θ has been explained above and causes only minor modifications in these calculations.

Finally note that following misprints in [7]: in formula (13), D^2w should be $\mathcal{A}D^2w$: on page 572, line 7, the inequality should be reversed; on p. 575, middle of page, $\delta\mathcal{B}$ should be $\overline{\delta\mathcal{B}}$; on p. 579, top line, replace second comma by z, and on line 3 from the bottom of the page $4k+n$ should be $4k+2n$ and $1/k$ should be $2/k$. In formula (27), $\delta\mathcal{B}$ should be followed by a $+$ sign and $(|p|^2)^{1/k}$ should be $(|p|^2/z)^{1/k}$; in formula (28), $\delta_2\beta$ should be $\delta_2\mathcal{B}$ and the exponent $2/k$ should be inserted at the end; at the foot of page 580 the following phrase should be added: « (the factor 2 in the denominators of α and γ should moreover be omitted) »; in (29) the exponent 2θ should be -2θ; and, on page 589, one should delete η from (b) and add η to (d).

Now suppose there exist constants l_1 and l_2 such that

$$\mathcal{N} < 0 \quad \text{whenever } |p| > l_1, \quad z > l_2^2. \tag{2.7}$$

Then it is clear that z cannot have an interior maximum in $B(d)$ which exceeds $\max\{l_1^2, l_2^2\}$. Because $z = 0$ on ∂B this implies

$$z \leqslant \max\{l_1^2, l_2^2\} \quad \text{in } B(d).$$

At $x = 0$ we have $\zeta = 1$ and $z = |Du|^2$. Hence

$$|Du(0)| \leqslant \max\{l_1, l_2\}. \tag{2.8}$$

In some cases it is necessary first to transform to a new dependent variable $\bar{u}$ by means of the relation

$$u \geqslant \varphi(\bar{u}) \quad (\text{where } \varphi'(\bar{u}) > 0)$$

before the program outlined above can be successfully carried out. In place of (2.5) we then obtain the principal differential inequality [4]

$$\mathcal{A}D^2\bar{z} \geqslant \overline{\mathcal{M}}\cdot D\bar{z} - 2\overline{\mathcal{N}}\bar{z}, \quad x \in B(d) \tag{2.9}$$

where $\bar{z} = \zeta|D\bar{u}|^2$ and

$$\begin{aligned}\frac{1}{\mathcal{E}}\overline{\mathcal{N}} = \alpha\omega^2 + 2\beta\omega + \gamma - \frac{\omega'}{\varphi'} + k\frac{1}{\mathcal{E}}|\delta_2\mathcal{B} + \omega\delta_2\mathcal{E}|\,\frac{1}{d}\left(\frac{|p|^2}{z}\right)^{1/k} \\ + k(6k+n)\frac{1}{\mathcal{E}}\left\{\|\mathcal{A}\| + \frac{|p|^2\,\|\delta_2\mathcal{A}\|^2}{12\theta\lambda}\right\}\frac{1}{d^2}\left(\frac{|p|^2}{z}\right)^{2/k}.\end{aligned} \tag{2.10}$$

Here

$$\mathcal{E} = p\mathcal{A}p, \quad \omega = -\varphi''/(\varphi')^2$$

(4) See [8], or [7], page 580. The reader should be reminded again that we are taking $\mathcal{A}' = \mathcal{A}$, $\mathcal{F} = 0$ in the calculations of [7], that the constant θ is still to be carried along as noted above, and finally that it is best to include a parameter r in the operator δ (as in [7], page 571), setting $r = 0$ only at the close of the whole calculation.

and

$$
\begin{aligned}
\alpha \; &= \frac{1}{\mathcal{E}}\left\{\delta_1\mathcal{E} - (1-\theta)\frac{\mathcal{E}^2}{\operatorname{tr}\mathcal{A}|p|^2} + \frac{\|\delta_1(|p|\mathcal{A})\|^2}{\theta\lambda}\right\},\\
2\beta &= \frac{1}{\mathcal{E}}\left\{-\delta\mathcal{E} + \delta_1\mathcal{B} - (1-\theta)\frac{2\mathcal{B}\mathcal{E}}{\operatorname{tr}\mathcal{A}|p|^2}\right\},\\
\gamma \; &= \frac{1}{\mathcal{E}}\left\{-\delta\mathcal{B} - (1-\theta)\frac{\mathcal{B}^2}{\operatorname{tr}\mathcal{A}|p|^2} + \frac{|p|^2\|\delta\mathcal{A}\|^2}{\theta\lambda}\right\}.
\end{aligned} \tag{2.11}
$$

As before, $\theta \in (0, 1]$ is a constant; the case $\theta = 0$ is allowed however if $\delta\mathcal{A} = \delta_1(|p|\mathcal{A}) = \delta_2\mathcal{A} = 0$, in which case the terms $\|\delta_1(|p|\mathcal{A})\|^2$ and $\|\delta\mathcal{A}\|^2$ do not appear in the formulas for α, β, γ.

The maximum principle argument given previously can be equally applied to the function $\bar{u}$. This leads to an estimate for $D\bar{u}$ and thus, in turn, to an estimate for Du.

In summary, we have above a program for obtaining gradient bounds for solutions of $\mathcal{A}D^2u = \mathcal{B}$, applying (as we shall see) in each of the following cases: (i) u bounded, (ii) u bounded on one side, and (iii) u unbounded.

Case (i) was discussed in detail in [4] and [8], and will not be considered further here. Our intention is to concentrate on the results obtainable in cases (ii) and (iii). In [9] some preliminary results for case (iii) have already been indicated.

3. – Solutions without a priori bounds.

In this section we shall obtain gradient estimates and a Liouville theorem in the absence of any bounds on the solution, using a method based on inequality (2.8).

If u is a solution in the ball $B(x_0, d) = \{x: |x - x_0| < d\}$, the bound we obtain for $|Du(x_0)|$ will depend on d. We shall exploit this dependence to establish a limit theorem for solutions near an isolated singularity and one for the behavior of solutions as $|x| \to \infty$.

To achieve a certain transparency in the statements of the results, we shall only discuss in detail solutions of the slightly simpler equation

$$
\mathcal{A}(Du)D^2u = \mathcal{B}(x, u, Du)\,. \tag{3.1}
$$

Generalization of these results to solutions of equation (1.1) is entirely straightforward. For convenience we shall also assume that $\operatorname{tr}\mathcal{A} = 1$, as can always be done.

Let us define the quantity

$$X(x, u, p) = \mathcal{B}_u + \frac{1}{|p|^2}\{p \cdot \mathcal{B}_x + \mathcal{B}^2(1-\theta)\}, \qquad p \neq 0,$$

where $\theta \in (0, 1]$. The value of θ will be fixed for any application, though in general one would wish to choose it as small as possible.

THEOREM 3.1. *Let $u \in C^3(\Omega)$ satisfy equation* (3.1). *Assume that* tr $\mathcal{A} = 1$ *and that there exist positive numbers ν, $\varkappa$ and L such that*

$$|p|^{2\nu}\left(1 + \frac{|p|^2}{\lambda}\|\mathcal{A}_p\|^2\right) \leqslant \varkappa X \tag{3.2}$$

and

$$|p|^\nu |\mathcal{B}_p| \leqslant \varkappa X \tag{3.3}$$

for $x \in \Omega$, $u \in \mathbf{R}$, and $|p| > L$. Then for any point $x_0 \in \Omega$ we have

$$|Du(x_0)| \leqslant \max\{L, K d_0^{-1/\nu}\}, \tag{3.4}$$

where d_0 is the distance from x_0 to $\partial\Omega$ and K is a constant which depends only on $\varkappa$, ν, n and *θ.*

PROOF. We consider (3.1) in the ball $B(x_0, d)$, where $d \in (0, d_0)$. Shifting the origin to x_0, we then find that (2.5) holds for the function $z = \zeta|Du|^2$. The coefficient $\mathcal{N}$ in (2.6) can be estimated by

$$\mathcal{N} \leqslant -X + k|\mathcal{B}_p|\frac{1}{d}\left(\frac{|p|^2}{z}\right)^{1/k} + C\left(1 + \frac{|p|^2\|\mathcal{A}_p\|}{\lambda}\right)\frac{1}{d^2}\left(\frac{|p|^2}{z}\right)^{2/k} \tag{3.5}$$

for $|p| > L$. In obtaining (3.5) we have set $t = 0$ in the formula for the differential operator δ_2 and have used the fact that tr $\mathcal{A} = 1$ (hence $\|\mathcal{A}\|^2 < n$) and that $\mathcal{A} = \mathcal{A}(p)$ (hence $\delta\mathcal{A} = 0$ and $\delta_2\mathcal{A} = \mathcal{A}_p$). Clearly the constant C depends only on k, n and θ.

We now choose $k = 2/\nu$ and use conditions (3.2) and (3.3). This yields (with C' a new constant)

$$\mathcal{N} \leqslant X\left\{-1 + C'\left(\frac{1}{dz^{\nu/2}} + \frac{1}{d^2 z^\nu}\right)\right\}, \qquad |p| > L.$$

By condition (3.2), $X > 0$. Hence there exists a positive number K such that if $|p| > L$ and $d^2 z^\nu > K^{2\nu}$, then $\mathcal{N} < 0$. It follows from (2.8), therefore, that

$$|Du(x_0)| \leqslant \max\{L, Kd^{-1/\nu}\}.$$

It is clear that K only depends on C', that is, as one sees, only on ν, $\varkappa$, n and θ. [From another point of view K also depends implicitly on L, this dependence entering via the constant $\varkappa$ in (3.2) and (3.3).]

REMARK. Since $\delta\mathcal{A} = 0$ and $\delta_2\mathcal{A} = \mathcal{A}_p$ in the above calculations, we may take $\theta = 0$ in the formula for X *provided* that $\mathcal{A}$ is constant, and in particular when $\mathcal{A}$ is the Laplace operator (see the remark preceding formula (2.5)).

THEOREM 3.2. *The conclusion of Theorem* 3.1 *still holds if condition* (3.3) *is replaced by*

$$|p|^{2\nu}\frac{1}{\lambda}\left\|\frac{p\mathcal{B}_p}{\mathcal{B}}\right\|^2 \leqslant \varkappa X. \tag{3.6}$$

PROOF. Whereas in the proof of Theorem 3.1 the function t in the differential operator δ_2 was taken to be zero, we now choose $t = -\mathcal{B}_p/\mathcal{B}$ (5). Then $\delta_2\mathcal{B} = 0$ and we obtain instead of (3.5)

$$\mathcal{N} \leqslant -X + C\left\{1 + \frac{|p|^2}{\lambda}\left(\|\mathcal{A}_p\|^2 + \left|\frac{\mathcal{B}_p}{\mathcal{B}}\right|^2\right)\right\}\frac{1}{d^2}\left(\frac{|p|^2}{z}\right)^{2/k}$$

if $|p| > L$. The rest of the proof is the same as before.

The dependence on d in the bound (3.4) for $|Du|$ can be used to obtain results about the limiting behavior of u near an isolated singularity, which for convenience we assume to be at the origin. In this case $d_0 = |x_0|$ as $x_0 \to 0$ and it follows from (3.4) that

$$|Du(x)| = O\big(|x|^{-1/\nu}\big) \quad \text{as } x \to 0. \tag{3.7}$$

For $\nu > 1$, relation (3.7) implies

$$\operatorname*{osc}_{|x|=d} u(x) \to 0 \qquad \text{as} \qquad d \to 0.$$

(5) At any point where $\mathcal{B} = 0$ we define $t = 0$. This convention will also be followed in later proofs.

Moreover since $|Du|$ is radially integrable to the origin when $\nu > 1$ (again by (3.7)) it follows that $\lim_{x\to 0} u(x)$ exists and is finite in this case.

This establishes the following result.

Let $u \in C^3$ be a solution of equation (3.1) *in a deleted neighbourhood of the origin, and let the hypotheses of either Theorem* 3.1 *or* 3.2 *be satisfied. Then* $u = O(|x|^{1-1/\nu})$ *if* $\nu < 1$, *and* u *is continuous at* $x = 0$ *if* $\nu > 1$.

The following example shows that the estimate (3.7) is sharp. Consider the nonlinear Poisson equation

$$\Delta u = a|Du|^{1+\beta}, \tag{3.8}$$

where a and β are positive constants. Here we have (after normalizing to tr $\mathcal{A} = 1$)

$$X = (a/n)^2(1-\theta)|p|^{2\beta}, \qquad \mathcal{A}_p = 0, \ \lambda = 1/n$$

and

$$|p\mathcal{B}_p/\mathcal{B}|^2 = (1+\beta)^2, \qquad |\mathcal{B}_p| = (a/n)(1+\beta)|p|^\beta .$$

Thus conditions (3.2) and (3.3), or (3.2) and (3.6), are satisfied for $\nu = \beta$, any $L > 0$, and an appropriate constant $\varkappa$. It follows from (3.7) that if u is a solution of (3.8) in a deleted neighbourhood of the origin, then

$$|Du(x)| = O(|x|^{-1/\beta}) \qquad \text{as } x \to 0 . \tag{3.9}$$

The function

$$u(x) = \begin{cases} k\displaystyle\int r^{-1/\beta}\, dr , & \beta \neq 1/(n-1) , \\ k\displaystyle\int \left(r \log \frac{1}{r}\right)^{1-n} dr , & \beta = 1/(n-1) ,\ r < 1 , \end{cases}$$

where $r = |x|$, is a solution of (3.8) provided the constant k is chosen appropriately:

$$k = \begin{cases} \operatorname{sign}\big((n-1)\beta - 1\big) \left|\dfrac{(n-1)\beta - 1}{a\beta}\right|^{1/\beta}, & \beta \neq \dfrac{1}{n-1} , \\ \left(\dfrac{n-1}{a}\right)^{n-1} , & \beta = \dfrac{1}{n-1} . \end{cases}$$

Clearly for this solution

$$|Du(x)| = \begin{cases} k|x|^{-1/\beta}, & \beta \neq 1/(n-1), \\ k|x|^{-1/\beta}\left(\log \dfrac{1}{|x|}\right)^{-1/\beta}, & \beta = 1/(n-1), \end{cases}$$

in precise agreement with the conclusion (3.9) when $\beta \neq 1(n-1)$ and satisfying (3.9) when $\beta = 1/(n-1)$. We note also that this solution is continuous at $x = 0$ when $\beta > 1$.

As an example of the application of Theorems 3.1 and 3.2, consider the equation

$$\Delta u = -\,|1 - |Du|^2|^{\frac{3}{2}}. \tag{3.10}$$

Let u be an entire solution of (3.10). We shall show that

$$|Du| \leqslant 1,$$

this result being best possible in view of the fact that $u = -(1 + x_1^2)^{\frac{1}{2}}$ is a solution.

It is clear that for any given $\varepsilon > 0$ the conditions of either Theorem 3.1 or 3.2 are verified when $L = 1 + \varepsilon$, $\nu = 2$, and $\varkappa$ is appropriately chosen. Thus by (3.4)

$$|Du| \leqslant \max\{1 + \varepsilon, Kd^{-\frac{1}{2}}\}$$

at each x in $\boldsymbol{R}^n$, where d is any finite value. Since we may choose d as large as we wish, and K does not depend on d, this gives $|Du| \leqslant 1 + \varepsilon$ in $\boldsymbol{R}^n$. But ε is arbitrary, and the result follows. Another corollary of Theorems 3.1 and 3.2 is the following Liouville theorem.

THEOREM 3.3. *Let u be an entire solution of equation* (3.1), *where we assume that* $\operatorname{tr} \mathcal{A} = 1$.

Suppose for $\varepsilon > 0$ that there exist corresponding positive constants $\varkappa$ and ν such that conditions (3.2) *and* (3.3), *or alternately conditions* (3.2) *and* (3.6), *hold for $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}$, and $|p| > \varepsilon$.*

Then $u \equiv$ constant.

PROOF. Let x_0 be any point in $\boldsymbol{R}^n$. We may apply either Theorem 3.1 or Theorem 3.2 with $L = \varepsilon$. By (3.4) we have, for any $d > 0$,

$$|Du(x_0)| \leqslant \max\{\varepsilon, Kd^{-1/\nu}\},$$

where K depends on ε, n and θ. Let $d \to \infty$ and then $\varepsilon \to 0$. This gives $Du(x_0) = 0$. Since x_0 was arbitrary, it follows that $Du \equiv 0$ and $u \equiv$ constant.

Theorem 3.3 applies, for example, to equation (3.8), and more generally to the equation

$$\Delta u = \varphi(u)|Du|^{1+\beta}, \qquad \beta > 0 .$$

In particular if $\varphi' \geqslant 0$ and $\inf(\varphi' + \varphi^2) > 0$ then every entire solution is constant. Note that the last condition on φ is necessary for this conclusion, since otherwise Laplace's equation arises as the special case $\varphi \equiv 0$. Whether the conclusion holds simply when $\varphi' + \varphi^2 > 0$ we do not know.

It is also possible to establish a Liouville theorem under much weaker conditions on $\mathcal{A}$ and $\mathcal{B}$ if we supply a bound for Du, either by fiat or by some alternative procedure.

THEOREM 3.4. *Suppose u is an entire solution of equation* (3.1) *for which Du is uniformly bounded. Assume also that for every pair of positive numbers ε and L (with $\varepsilon < L$) there exist positive constants μ and N such that*

$$\text{(3.12)} \qquad X \geqslant \mu \quad \textit{and} \quad |\mathcal{B}_p| \leqslant N \qquad \textit{for } x \in \Omega,\ u \in \boldsymbol{R},\ \varepsilon \leqslant |p| \leqslant L .$$

Then $u \equiv$ constant.

PROOF. We may suppose $|Du| \leqslant l$ for all $x \in \boldsymbol{R}^n$. By Theorem 3.3 it is enough to show for $\nu = 1$ and for every ε in $(0, l)$ that there exists a positive constant $\varkappa$, possibly depending on ε, such that (3.2) and (3.3) are satisfied for $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}$, $\varepsilon < |p| \leqslant l$. But this follows at once from (3.12) and the fact that $\|\mathcal{A}_p\|^2/\lambda$ is bounded for $|p| \leqslant L$ (recall that $\mathcal{A} = \mathcal{A}(p)$ is positive definite and continuously differentiable).

REMARK. The case $\theta = 0$ is allowed in Theorem 3.4 if $\mathcal{A}$ is constant, and in particular, when $\mathcal{A}$ is the Laplacian.

If $\mathcal{B} = \mathcal{B}(p)$ then the hypothesis (3.12) is satisfied automatically if only $\mathcal{B} \neq 0$ when $p \neq 0$. Thus if u is an entire solution of the equation

$$\mathcal{A}(p)\, D^2 u = \mathcal{B}(p) ,$$

where $\mathcal{B} \neq 0$ when $p \neq 0$, and if the gradient of u is bounded, then u is a constant. A similar conclusion also holds when $\mathcal{B} = \mathcal{B}(u)$ and $\partial\mathcal{B}/\partial u$ is positive and uniformly bounded from zero.

A final corollary of Theorems 3.1 and 3.2 concerns the asymptotic behavior (as $x \to \infty$) of solutions of (3.1) in an exterior domain, i.e. a domain Ω

which contains a full neighborhood of infinity. We suppose that conditions (3.2) and (3.3), or alternately (3.2) and (3.6), hold for $x \in \Omega$, $u \in \boldsymbol{R}$, $|p| > 0$, where ν and $\varkappa$ are fixed positive constants. Then by (3.4) with $L \to 0$ we have

$$|Du(x_0)| \leqslant Kd^{-1/\nu}, \qquad x_0 \in \Omega .$$

We now suppose $x_0 \to \infty$. Then eventually $d_0 > \frac{1}{2}|x_0|$, so that obviously

$$|Du(x)| = O\big(|x|^{-1/\nu}\big) .$$

As in the case of an isolated singularity at a finite point, this estimate leads in turn to the following conclusion.

Let $u \in C^3$ be a solution of equation (3.1) *in an exterior domain Ω. Suppose conditions* (3.2) *and* (3.3), *or alternately* (3.2) *and* (3.6), *hold for $x \in \Omega$, $u \in \boldsymbol{R}$, and $|p| > 0$, where ν and $\varkappa$ are positive constants.*

Then $u = O\big(|x|^{1-1/\nu}\big)$ as $x \to \infty$, when $\nu > 1$. On the other hand, if $\nu < 1$ then $\lim\limits_{x \to \infty} u(x)$ exists and is finite.

4. – Solutions with a one-sided bound.

In Section 3 we made no special assumptions about the solution u beyond requiring it to be of class C^3; on the other hand this generality was bought at the price of assuming $X > 0$, an assumption which rules out the Laplace equation for example. In this section we shall assume that u is bounded on one side, a supposition which will allow us in particular to include equations for which $X = 0$. For definiteness, we formulate our results for solutions which are bounded above, say $u \leqslant M$ for all $x \in \Omega$. It is easy to translate these results into the corresponding ones for solutions which are bounded below.

To exploit the bound on u we first transform to a new dependent variable $\bar{u}$, by means of the relation

$$u = \varphi(\bar{u}) , \qquad \text{where } \varphi'(\bar{u}) > 0 ,$$

and then proceed as before. We thus obtain an elliptic differential inequality for the variable

$$\bar{z} = \zeta |D\bar{u}|^2$$

to which the maximum principle may be applied, as in Section 2.

The transformation φ will be chosen so that $\bar{u}$ is either proportional to a power of $(m-u)$ or to the logarithm of $m-u$, where m is an appropriate parameter which is related to the bound M. More precisely, let m and a be constants, with $a>0$. Then we assume that

$$m-\varphi(t)=\begin{cases} a^{-1}\{(1-a)\,t\}^{a/(a-1)} & \text{if } a\neq 1 \\ e^{-t} & \text{if } a=1\,, \end{cases}$$

the domain of definition being $(-\infty, 0)$ if $a>1$, $(0,\infty)$ if $0<a<1$, and $(-\infty,\infty)$ if $a=1$. Obviously this implies at once that

$$\varphi<m\,.$$

Moreover by direct calculation the following properties hold:

(a) $\varphi'=\{a(m-\varphi)\}^{1/a}$,

(b) $\omega\equiv-\varphi''/\varphi'^2=\{a(m-\varphi)\}^{-1}$,

(c) $\omega=(\varphi')^{-a}$,

(d) $d\omega/d\varphi=a\omega^2$.

Note in particular that φ' and ω are positive.

We begin by establishing a number of results for the full equation (1.1). Theorems 1.3 to 1.7 in the introduction are obtained by specializing these to the case $\mathcal{A}=I$, $\mathcal{B}=f(u,p)$. For this special case we have from (2.11)

$$\mathcal{E}=p\mathcal{A}p=|p|^2$$

$$\lambda=1\,,\qquad \delta\mathcal{A}=\delta_1(|p|\mathcal{A})=0\,,\qquad \delta_2\mathcal{A}=tI$$

$$\alpha=1-(1-\theta)/n$$

$$2|p|^2\beta=p\cdot f_p-\big(1+2(1-\theta)/n\big)f$$

$$|p|^2\gamma=-\{f_u+(1-\theta)f^2/n|p|^2\}\,,$$

where we have taken $s=-1$, and where $\theta\in(0,1]$ [6]. The reader may use these explicit formulae to give more immediate meaning to the following calculations, if so desired.

[6] If we wish to take $\theta=0$ it is additionally necessary to have $\delta_2\mathcal{A}=0$ and thus $t=0$.

We define the quantities

$$\alpha^* = \sup\{\alpha(x, u, p) : x \in \Omega,\ u \leqslant M,\ |p| > l\}$$

$$\beta^* = \sup\{\beta(x, u, p)|p|^a : x \in \Omega,\ u \leqslant M,\ |p| > l\}$$

$$\gamma^* = \sup\{\gamma(x, u, p)|p|^{2a} : x \in \Omega,\ u \leqslant M,\ |p| > l\},$$

where a and l are positive constants.

THEOREM 4.1. *Let $u \in C^3$ be a solution of equation* (1.1) *in Ω such that $u \leqslant M$. Suppose that for some positive constants a and l we have*

$$\alpha^* < a, \qquad \beta^* < \infty, \qquad \gamma^* < \infty.$$

Moreover, assume that there exist constants $\varkappa$ and $\varkappa'$ such that

$$|p|^{2a}\|\mathcal{A}\| \leqslant \varkappa \mathcal{E}, \qquad |p|^a|\delta_2\mathcal{E}| \leqslant \varkappa\mathcal{E},$$

$$|p|^{1+a}\|\delta_2\mathcal{A}\| \leqslant \varkappa\sqrt{\lambda\mathcal{E}}, \qquad |p|^{2a}|\delta_2\mathcal{B}| \leqslant \varkappa'\mathcal{E},$$

for $x \in \Omega$, $u \leqslant M$, and $|p| \geqslant l$.

Let $x_0 \in \Omega$, and let d_0 be the distance from x_0 to $\partial\Omega$. Then

$$|Du(x_0)| \leqslant \max\{L, Kd_0^{-1/a}\}[M + \mu - u(x_0)]^{1/a} \tag{4.1}$$

for any $\mu > 0$. The constants K and L depend only on n, a, μ, θ, $\varkappa$, $\varkappa'$, l, α^*, β^* and γ^*.

PROOF. We consider (1.1) in the ball $B(x_0, d)$, where $d \in (0, d_0)$. Shifting the origin to x_0 as in the proof of Theorem 3.1, we then obtain inequality (2.9) for $\bar{z}$. The function $\overline{\mathcal{N}}$ in (2.10) can be estimated by

$$\begin{aligned}\frac{1}{\mathcal{E}}\overline{\mathcal{N}} \leqslant \left(a\omega^2 - \frac{d\omega}{d\varphi}\right) - (a - \alpha^*)\omega^2 + 2\beta^*\omega|p|^{-a} + \gamma^*|p|^{-2a} + \\ + k\frac{1}{\mathcal{E}}|\delta_2\mathcal{B} + \omega\delta_2\mathcal{E}|\frac{1}{d}\left(\frac{|p|^2}{z}\right)^{1/k} + k(6k+n)\frac{1}{\mathcal{E}}\left\{\|\mathcal{A}\| + \frac{|p|^2\|\delta_2\mathcal{A}\|^2}{12\lambda\theta}\right\}\frac{1}{d^2}\left(\frac{|p|^2}{z}\right)^{2/k}\end{aligned} \tag{4.2}$$

for $|p| > l$, $u \leqslant M$, $x \in B(d)$. We now choose φ from the family of functions defined above, with $m = M + \mu$. Using property (d) of φ and the given asymptotic behavior of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ we obtain (with $k = 2/a$)

$$(4.3) \qquad \frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant \omega^2\Big[-(a-\alpha^*) + 2\frac{\beta^*}{\omega|p|^a} + \frac{\gamma^*}{\omega^2|p|^{2a}} + \\ + C\Big\{\Big(\frac{\varkappa'}{\omega|p|^a}+1\Big)\frac{1}{d\omega z^{a/2}} + \frac{1}{d^2\omega^2 z^a}\Big\}\Big],$$

for $|p| > l$ and $u \leqslant M$, where the generic constant C depends only on n, a, θ, and $\varkappa$.

Next we use property (c) of φ. Since $p = \varphi'\bar{p}$, where $\bar{p} = D\bar{u}$, we find that

$$\omega|p|^a = |\bar{p}|^a$$

and

$$\omega z^{a/2} = \omega\zeta^{a/2}|p|^a = \zeta^{a/2}|\bar{p}|^a = \bar{z}^{a/2}.$$

Now by (a)

$$\varphi'(\bar{u}) = \{a(M+\mu-u)\}^{1/a} \geqslant (a\mu)^{1/a}.$$

Therefore

$$|p| = \varphi'|\bar{p}| \geqslant (a\mu)^{1/a}|\bar{p}|.$$

It follows that if

$$|\bar{p}|^a \geqslant L_1^a = \max\left\{\frac{8\beta^*}{a-\alpha^*},\, 2\left(\frac{\gamma^*}{a-\alpha^*}\right)^{\frac{1}{2}},\, \varkappa'\right\} \text{ [7]}$$

and also

$$|\bar{p}| \geqslant l/(a\mu)^{1/a},$$

then $|p| \geqslant l$ and

$$\frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant -\omega^2\left\{\frac{1}{2}(a-\alpha^*) - C\left(\frac{2}{d\bar{z}^{a/2}} + \frac{1}{d^2\bar{z}^a}\right)\right\}.$$

From this estimate, it is apparent that $\overline{\mathcal{N}} < 0$ provided we have, in addition,

$$d\bar{z}^{a/2} \geqslant K_1 = 5\max\left(\left(\frac{C}{a-\alpha^*}\right)^{\frac{1}{2}},\, \frac{C}{a-\alpha^*}\right).$$

[7] If either β^* or γ^* are negative, they should be omitted.

Then by (2.8) applied in the barred variables

$$|D\bar{u}(x_0)| \leqslant \max\left(L_1, \frac{l}{(a\mu)^{1/a}}, \left(\frac{K_1}{d}\right)^{1/a}\right).$$

Finally, using the relation $p = \varphi'\bar{p}$ once again, we get

$$|Du(x_0)| \leqslant \max(L, Kd^{-1/a})[M + \mu - u(x_0)]^{1/a},$$

where

$$L = \max(L_1 a^{1/a}, l\mu^{-1/a}), \qquad K = (aK_1)^{1/a}.$$

As a simple application of Theorem 4.1, consider the linear elliptic equation

$$a_{ij}(x)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + b_i(x)\frac{\partial x_i}{\partial u} + c(x)u = f(x),$$

where $\lambda\xi^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda\xi^2$ and, of course, the standard summation convention is used. Here one finds easily that

$$\alpha \leqslant 1 - \frac{\lambda}{2n\Lambda},$$

$$2|p|\beta \leqslant \frac{1}{\lambda}\left\{\left\|\frac{\partial a_{ij}}{\partial x}\right\| + \frac{1}{n}|b_i| + \frac{1}{n|p|}|f - cu|\right\},$$

$$|p|^2\gamma \leqslant \frac{1}{\lambda}\left\{\frac{2}{\lambda}\left\|\frac{\partial a_{ij}}{\partial x}\right\|^2 + \left\|\frac{\partial b_i}{\partial x}\right\| + |c| + \frac{1}{|p|}\left|\frac{\partial f}{\partial x} - u\frac{\partial c}{\partial x}\right|\right\},$$

where we have put $s = -1$, $\theta = \frac{1}{2}$. Consequently, if u is *bounded*, say $|u| \leqslant M$, then the first set of displayed conditions in Theorem 4.1 holds with

$$a = 1, \qquad l = M + \sup|f| + \sup|\partial f/\partial x|$$

provided that $\lambda \geqslant \text{constant} > 0$ and the quantities

$$\|a_{ij}\|, \ |b_i|, \ |c|, \left\|\frac{\partial a_{ij}}{\partial x}\right\|, \ \left\|\frac{\partial b_i}{\partial x}\right\|, \ \left|\frac{\partial c}{\partial x}\right|$$

are bounded. The second set of displayed conditions likewise holds for some constants $\varkappa$, $\varkappa'$ depending only on λ and $\|a_{ij}\|$ and $|b_i|$. If we now choose $\mu = l$, then the condition of Theorem 4.1 yields the gradient estimate

$$|Du(x)| \leqslant \text{const}\left(1 + \frac{1}{d}\right)\left(M + \sup|f| + \sup\left|\frac{\partial f}{\partial x}\right|\right),$$

where the constant depends only on λ and on bounds for the coefficients a_{ij}, b_i, c and their first derivatives. Of course, this is no new result but it does indicate clearly the strength of Theorem 4.1. The result moreover shows that when $c \equiv 0$ there is no need to assume the solution bounded on two sides; indeed if we suppose in this case that $u \geqslant 0$ then we obtain the estimate

$$|Du(x)| \leqslant \text{const}\left(1 + \frac{1}{d}\right)\left(\sup |f| + \sup \left|\frac{\partial f}{\partial x}\right| + u(x)\right),$$

the constant now depending only on λ and on a_{ij}, b_i and their derivatives.

It follows from Theorem 4.1 that

$$[M + \mu - u(x_0)]^{-1/a}|Du(x_0)| = O(d^{-1/a}) \qquad \text{as } d \to 0 .$$

Hence, if Ω is a deleted neighbourhood of an isolated singularity x_0 and $u \in C^3(\Omega)$ is a solution of (1.1) which is uniformly bounded above in Ω, then

$$[M + \mu - u(x)]^{-1/a}|Du(x)| = O\left(|x - x_0|^{-1/a}\right) \qquad \text{as } x \to x_0 .$$

If $a > 1$ integration shows that u must be bounded, while if $a = 1$ the singularity can be no worse than algebraic. A more precise result is stated in Theorem 4.9 below.

THEOREM 4.2. *Let $u \in C^3$ be a solution of equation* (1.1) *in Ω, such that $u \leqslant M$. Suppose that* (1.1) *is uniformly elliptic and that there exist positive constants $\hat{\alpha} < 1$ and $\varkappa$ such that*

$$\begin{gathered} \alpha \leqslant \hat{\alpha}, \qquad \beta \leqslant 0, \qquad \gamma \leqslant 0 \\ |p|\|\mathcal{A}_p\| \leqslant \varkappa\|\mathcal{A}\|, \qquad |p||\mathcal{B}_p| \leqslant \varkappa|\mathcal{B}| \end{gathered} \tag{4.4}$$

for $x \in \Omega$, $u \leqslant M$, $p \neq 0$. Then there exists a constant K, depending only on n, θ, $\hat{\alpha}$, $\varkappa$, and the ellipticity modulus, such that

$$|Du(x_0)| \leqslant K d_0^{-1}(M - u(x_0)) . \tag{4.5}$$

If (4.4) *is replaced by*

$$|p|\|\mathcal{A}_p\| + |\mathcal{B}_p| \leqslant \|\mathcal{A}\|$$

then the conclusion holds in the weaker form

$$|Du(x_0)| \leqslant K \max(d_0^{-\frac{1}{2}}, d_0^{-1})(M - u(x_0)) . \tag{4.6}$$

REMARK. If $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where

$$|p||\mathcal{B}_{1p}| \leqslant \varkappa|\mathcal{B}_1| \qquad \text{and} \qquad |\mathcal{B}_{2p}| \leqslant \varkappa\|\mathcal{A}\|$$

(roughly speaking, combining the two parts of the theorem) then the conclusion (4.6) still holds. We omit the proof, which is quite similar to the following.

PROOF OF THEOREM 4.2. Consider (1.1) in the set $B(x_0, d)$, where $d \in (0, d_0)$, and shift the origin to x_0. Then as in (4.2) and (4.3) we get (with $a = 1$, $k = 2$, and $t = -\mathcal{B}_p/\mathcal{B}$ so that $\delta_2\mathcal{B} = 0$)

$$\frac{1}{\mathcal{E}}\overline{\mathcal{N}} \leqslant -(1-\hat{\alpha})\omega^2 + \frac{2\omega}{\mathcal{E}}|\delta_2\mathcal{E}|\frac{1}{d}\frac{|p|}{z^{\frac12}} + \\ + \frac{2(12+n)}{\mathcal{E}}\left\{\|\mathcal{A}\| + \frac{|p|^2\|\delta_2\mathcal{A}\|^2}{12\theta\lambda}\right\}\frac{1}{d^2}\frac{|p|^2}{z}, \tag{4.7}$$

valid for $p \neq 0$ and $u \leqslant M$. Since $\mathcal{A}$ is uniformly elliptic we have also

$$|p|^2\|\mathcal{A}\| \leqslant \text{const.}\ |p|^2\lambda \leqslant \text{const}\ \mathcal{E}$$

$$|p|^4\|\mathcal{A}_p\|^2 \leqslant \varkappa^2|p|^2\|\mathcal{A}\|^2 \leqslant \text{const}\ \lambda\mathcal{E}$$

$$|p||\mathcal{E}_p| \leqslant 2|p|^2\|\mathcal{A}\| + |p|^3\|\mathcal{A}_p\| \leqslant (2+\varkappa)|p|^2\|\mathcal{A}\| \leqslant \text{const}\ \mathcal{E}\,.$$

Hence, since $|t| \leqslant \varkappa|p|^{-1}$, we find from (4.7) that

$$\frac{1}{\mathcal{E}}\overline{\mathcal{N}} \leqslant -\omega^2\left\{(1-\hat{\alpha}) - C\left(\frac{1}{d\omega z^{\frac12}} + \frac{1}{d^2\omega^2 z}\right)\right\}, \tag{4.8}$$

where C depends only on n, θ, $\varkappa$ and the ellipticity modulus. The rest of the proof is essentially the same as that of Theorem 4.1, except that we can take $L = L_1 = 0$. This gives the estimate

$$|Du(x_0)| \leqslant Kd^{-1}\{M + \mu - u(x_0)\}\,.$$

Since K is independent of μ in the present case, as one easily sees, we can finally let μ tend to zero, completing the proof of the first part of the theorem.

To prove the final part we note that if $t = 0$ (rather than $t = -\mathcal{B}_p/\mathcal{B}$)

the inequality (4.7) becomes

$$\frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant -(1-\hat{\alpha})\,\omega^2 + \frac{2}{\mathcal{E}}\,|\mathcal{B}_p + \omega\mathcal{E}_p|\,\frac{1}{d}\,\frac{|p|}{z^{\frac{1}{2}}} + \\ + \frac{2(12+n)}{\mathcal{E}}\left\{\|\mathcal{A}\| + \frac{|p|^2\,\|\mathcal{A}_p\|^2}{12\theta\lambda}\right\}\frac{1}{d^2}\,\frac{|p|^2}{z},$$

valid for $p \neq 0$, $u \leqslant M$. Using the given conditions on $\mathcal{A}$ and $\mathcal{B}$, we get in place of (4.8)

$$\frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant -\omega^2\left\{(1-\hat{\alpha}) - C\left[\left(\frac{1}{\omega|p|}+1\right)\frac{1}{d\omega z^{\frac{1}{2}}} + \frac{1}{d^2\omega^2 z}\right]\right\}.$$

Now $|p| \geqslant z^{\frac{1}{2}}$ so that (with $\tilde{d} = \min(d, d^2)$)

$$\frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant -\omega^2\left[(1-\hat{\alpha}) - C\left(\frac{1}{\tilde{d}^{\frac{1}{2}}\omega z^{\frac{1}{2}}} + \frac{1}{\tilde{d}\omega^2 z}\right)\right].$$

The rest of the proof is the same as before.

REMARK. A result similar to Theorem 4.2 can also be obtained for non-uniformly elliptic equations, but is omitted here because the statement of the hypotheses would be more involved. This comment applies equally to the following Theorems 4.4, 4.5 and 4.6.

Theorem 4.2 can be used to obtain a Harnacki nequality for a class of uniformly elliptic equations. Let $u \in C^3$ be a solution of (1.1) in Ω such that $u \leqslant M$. Let us write

$$v(x) = M - u(x).$$

The following lemma provides a bound for $v(x)$ in a sufficiently small ball around a point $x_0 \in \Omega$ in terms of $v(x_0)$.

LEMMA 4.1. *Let $B(x_0, d) \subset \Omega$, and let the conditions of the first part of Theorem 4.2 be satisfied. Then*

$$v(x) \leqslant v(x_0)\{1-(r/d)\}^{-K}, \qquad 0 < r < \tfrac{1}{2}d,$$

where $r = |x - x_0|$ and K is the constant in (4.5).

PROOF. Let $x - x_0 = r\xi$, where $r = |x - x_0|$. Keeping ξ fixed we write

$$v(r) = v(x) \qquad \text{and} \qquad v'(r) = \xi\cdot Dv(x).$$

Because $B(x, d-r) \subset B(x_0, d)$ for $0 \leqslant r \leqslant \frac{1}{2}d$, it follows from Theorem 4.2 that

$$|v'(r)| \leqslant K \frac{v}{d-r}, \qquad 0 \leqslant r \leqslant \tfrac{1}{2} d .$$

In particular

$$v'(r) \leqslant K \frac{v}{d-r}$$

and by integration

$$v(r) \leqslant v(0)\{1-(r/d)\}^{-K}$$

for $0 \leqslant r \leqslant \frac{1}{2}d$.

It is now easy to prove the following Harnack inequality.

THEOREM 4.3. *Let* $u \in C^3(\Omega)$ *be a solution of equation* (1.1), *and let the conditions of Theorem* 4.2 *be satisfied. Let* Ω' *be a compact subdomain of* Ω. *Then there exists a constant* $\Lambda > 1$ *such that for any points* $x, y \in \Omega'$

$$u(x) \geqslant \Lambda u(y) - (\Lambda - 1) M . \tag{4.9}$$

PROOF. From Lemma 4.1 and a standard compactness argument (e.g. see [5], p. 109) there exists a constant $\Lambda > 1$ such that for any two points $x, y \in \Omega'$,

$$v(x) \leqslant \Lambda v(y) .$$

Recalling that $v(x) = M - u(x)$, we obtain (4.9).

Another corollary of Theorem 4.2 is a Liouville theorem. If $u \in C^3$ is a solution of (1.1) in $\boldsymbol{R}^n$, we may let $d_0 \to \infty$ and it follows then that $Du(0) = 0$. Since any point $x \in \boldsymbol{R}^n$ may be chosen as origin we get $Du(x) \equiv 0$ and hence $u \equiv$ constant.

Next we shall prove a Liouville theorem of a slightly different type, in that we assume that Du is uniformly bounded in $\boldsymbol{R}^n$. Such a bound may be obtained by means of Theorem 4.1 or by entirely different methods.

THEOREM 4.4. *Suppose equation* (1.1) *is uniformly elliptic for any bounded range of the argument* Du. *Let* u *be an entire solution such that* Du *is bounded. Assume also that for each pair of positive numbers* ε *and* L (*with* $\varepsilon < L$) *there exist constants* $\hat{\alpha}$ *and* $\varkappa$ *such that*

$$\alpha \leqslant \hat{\alpha}, \quad \beta \leqslant 0, \quad \gamma \leqslant 0,$$

and

$$\|\mathcal{A}_p\| \leqslant \varkappa \|\mathcal{A}\|, \qquad |\mathcal{B}_p| \leqslant \varkappa |\mathcal{B}|, \tag{4.10}$$

for all $x \in \boldsymbol{R}^n$ $u \in \boldsymbol{R}$, $\varepsilon \leqslant |p| \leqslant L$. *Then if* $M(d) = o(d)$ *as* $d \to \infty$, *we have* $u \equiv$ *constant.*

If (4.10) *is replaced by*

$$\|\mathcal{A}_p\| + |\mathcal{B}_p| \leqslant \varkappa \|\mathcal{A}\| \tag{4.11}$$

then the conclusion again holds, provided $M(d) = o(\sqrt{d})$ *as* $d \to \infty$.

Proof. Without loss of generality we may suppose that $M(d) > 0$ for all $d > 0$. Choose

$$a = \max(0, \hat{\alpha}) + 1$$

and

$$m = 2M(d)$$

in the selection of the function φ. Then inequality (2.9) holds for $\bar{z}$ in $B(d)$. The coefficient $\overline{\mathcal{N}}$ given by (2.10) can be estimated by setting $t = -\mathcal{B}_p/\mathcal{B}$ (hence $\delta_2 \mathcal{B} = 0$) and using property (d) of φ. This yields

$$\frac{1}{\mathcal{E}} \overline{\mathcal{N}} \leqslant -\omega^2 + k\omega \frac{|\delta_2 \mathcal{E}|}{\mathcal{E}} \frac{1}{d} \left(\frac{|p|^2}{z}\right)^{1/k} + k(6k+n) \frac{1}{\mathcal{E}} \left\{\|\mathcal{A}\| + \frac{|p|^2 \|\delta_2 \mathcal{A}\|^2}{12\theta\lambda}\right\} \frac{1}{d^2} \left(\frac{|p|^2}{z}\right)^{2/k}$$

for $x \in B(d)$. Now set $k = 2/a$. Then by (4.10) and the same calculations which led to (4.8) we get

$$\frac{1}{\mathcal{E}} \overline{\mathcal{N}} \leqslant -\omega^2 \left\{1 - C\left(\frac{1}{d\omega z^{a/2}} + \frac{1}{d^2 \omega^2 z^a}\right)\right\} \tag{4.12}$$

for $|p| > \varepsilon$ and $x \in B(d)$. Here C depends only on l (the given bound for Du), on ε (through $\varkappa$ and $a = \max(0, \hat{\alpha}) + 1$), on n and θ, and on the ellipticity modulus of $\mathcal{A}$ over the range $|p| \leqslant l$, $u \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$.

Now as in the proof of Theorem 4.1 we find

$$\varphi'(\bar{u}) = [a(m-u)]^{1/a} \geqslant [aM(d)]^{1/a}$$

since $m = 2M(d)$. Thus

$$|p| \geqslant [aM(d)]^{1/a} |\bar{p}|.$$

Consequently, if $|\bar{p}| \geqslant \varepsilon [aM(d)]^{-1/a}$ we have $|p| \geqslant \varepsilon$ and (4.8) holds.

If also $d^2\bar{z}^a \geq K^2$ for some appropriate constant K, then $\overline{\mathcal{N}} < 0$ as in the proof of Theorem 4.1. Hence by (2.8) applied in the barred variables

$$|D\bar{u}(0)| \leq \max\{\varepsilon[aM(d)]^{-1/a},\ (K/d)^{1/a}\}\,.$$

Consequently

$$|Du(0)| = \varphi'(\bar{u}(0))|D\bar{u}(0)| \leq \max\left\{\varepsilon\left(\frac{2M(d)-u(0)}{M(d)}\right)^{1/a},\ \left(aK\,\frac{2M(d)-u(0)}{d}\right)^{1/a}\right\}.$$

Here we may let $d \to \infty$. Since $M(d) = o(d)$ it follows that

$$|Du(0)| \leq \varepsilon\left(2-\frac{u(0)}{M^*}\right)^{1/a}$$

where $M^* = \lim M(d)$, possibly infinite. Now we may let $\varepsilon \to 0$. The exponent a can only increase because $\hat{\alpha}$ serves as an upper bound for α when $|p|$ ranges over the increasingly large interval $[\varepsilon, l]$. Hence the factor

$$\left(2-\frac{u(0)}{M^*}\right)^{1/a}$$

remains bounded, and we conclude that $|Du(0)| = 0$. The origin, however, may be arbitrarily shifted to any point; hence $Du \equiv 0$ and $u \equiv$ constant. This proves the first half of the theorem.

The second part follows the same pattern, using however the calculations of the last part of the proof of Theorem 4.2.

COROLLARY. *Suppose $\mathcal{A} = \mathcal{A}(p)$ in* (1.1). *Let u be an entire solution such that Du is bounded. Assume also that for each pair of positive numbers ε and L (with $\varepsilon < L$) there exists a constant $\varkappa$ such that* (8)

$$X \geq 0\,, \qquad |\mathcal{B}_p| \leq \varkappa|\mathcal{B}|$$

for $x \in \mathbf{R}^n$, $u \in \mathbf{R}$, $\varepsilon \leq |p| \leq L$. Then if $M(d) = o(d)$ we have $u \equiv$ constant.

PROOF. This is essentially the same as before. We choose

$$s = -\frac{p\cdot\mathcal{B}_p}{\mathcal{B}} + 2(1-\theta)\frac{\mathcal{E}}{\operatorname{tr}\mathcal{A}|p|^2}$$

so that $\beta = 0$ (recall here that $\mathcal{A} = \mathcal{A}(p)$, $\mathcal{E} = \mathcal{E}(p)$). It is also easy to see that the ellipticity modulus of $\mathcal{A}$ is bounded, and $\|\mathcal{A}_p\| \leq \varkappa\|\mathcal{A}\|$ for $|p| \leq L$.

(8) Here X is defined at the beginning of Section 3, except that now since tr $\mathcal{A}$ is no longer assumed equal to 1 we must replace the term $\mathcal{B}^2$ by $\mathcal{B}^2/\operatorname{tr}\mathcal{A}$.

Finally, since s is bounded for $\varepsilon \leqslant |p| \leqslant L$ it is clear that so also is α. Because $\gamma = -X/\mathcal{E}$ when $\mathcal{A} = \mathcal{A}(p)$ we have $\gamma \leqslant 0$, and accordingly Theorem 4.4 applies.

As an application of this result, consider the equation representing surfaces whose mean curvature is a given function $\Lambda = \Lambda(x, u, Du)$, namely

$$(1 + |Du|^2)\Delta u - \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\,\partial x_j} = n\Lambda(x, u, Du)(1 + |Du|^2)^{\frac{3}{2}} .$$

Suppose the following conditions hold:

$$\Lambda_u \geqslant 0 , \qquad |\Lambda_x| \leqslant \begin{cases} \sqrt{3}\Lambda^2 & n = 2 \\ \dfrac{n}{n-1}\Lambda^2 & n < 2 , \end{cases} \qquad |\Lambda_p| \leqslant \varkappa |\Lambda|$$

when p lies in any bounded set, say $|p| \leqslant l$. With the identification $\mathcal{B}(x, u, p) = n\Lambda(x, u, p)(1 + |p|^2)^{\frac{3}{2}}$ it is clear that

$$|\mathcal{B}_p| \leqslant (\varkappa + 3/2)|\mathcal{B}|$$

and

$$X = n\frac{(1 + |p|^2)^{\frac{3}{2}}}{|p|^2}\left\{|p|^2\frac{\partial \Lambda}{\partial u} + p\cdot\frac{\partial \Lambda}{\partial x} + \frac{n}{n-1}\Lambda^2\left(\frac{(n-1)(1 + |p|^2)^{\frac{3}{2}}}{n + (n-1)|p|^2}(1-\theta)\right)\right\}.$$

Choosing

$$\begin{aligned} &1 - \frac{2}{3}\sqrt{2} && n = 2 , \\ &1 - \frac{l(n + (n-1)l^2)}{(n-1)(1 + l^2)^{\frac{3}{2}}} && n > 2 , \end{aligned}$$

it is easily checked that $X \geqslant 0$ for $|p| \leqslant l$. Thus by the corollary of Theorem 4.4, any solution with bounded gradient which also satisfies $M(d) = o(d)$ must be constant. (We believe the constant $n/(n-1)$ in the principal condition is best possible, though algebraic complications make it difficult to create an explicit counterexample.)

Additionally, if we suppose that

$$|\Lambda| \geqslant \mu > 0$$

for some constant μ and for $\varepsilon \leqslant |p| \leqslant l$, then we can apply Theorem 3.4 to show that any entire solution with a bounded gradient must be constant. An even stronger result holds if Λ = constant $\neq 0$, for then there can be no entire solution swhatever, as shown by Finn [2]. Finally, if $\Lambda_u \geqslant 0$, $\Lambda_x = 0$, then any entire solution with bounded gradient which is $o(d)$ must be constant, according to [8], Theorem 3.

The next result extends the conclusion of Theorem 4.4 to solutions without a priori bounds on the gradient. Unfortunately, considerably stronger conditions are required of the coefficient matrix $\mathcal{A}$.

THEOREM 4.5. *Assume that* (1.1) *is uniformly elliptic, and let u be an entire solution. Suppose for each positive ε there exist positive constants $\hat{\alpha} < 1$ and $\varkappa$ such that*

$$\alpha \leqslant \hat{\alpha}\,, \qquad \beta \leqslant 0\,, \qquad \gamma \leqslant 0\,,$$

and

$$|p|\,\|\mathcal{A}_p\| \leqslant \varkappa \|\mathcal{A}\|\,, \qquad |p|\,|\mathcal{B}_p| \leqslant \varkappa |\mathcal{B}| \tag{4.13}$$

for all $x \in \mathbf{R}^n$, $u \in \mathbf{R}$, and $|p| > \varepsilon$. Then if $M(d) = o(d)$ as $d \to \infty$ we have $u \equiv$ constant.

If (4.13) *is replaced by*

$$|p|\,\|\mathcal{A}_p\| + |\mathcal{B}_p| \leqslant \varkappa \|\mathcal{A}\| \tag{4.14}$$

then the conclusion remains true provided $M(d) = o(\sqrt{d})$.

PROOF. This is exactly the same as the proof of Theorem 4.4 with the condition $|Du| \leqslant L$ omitted. It is this omission which accounts for the appearance of the factor $|p|$ in the displayed hypotheses and the fact that we must have $\hat{\alpha} < 1$ in order to take $a = 1$.

In the results we have established so far the dependence on x of the functions $\mathcal{A}(x, u, p)$ and $\mathcal{B}(x, u, p)$ has not played a significant rôle. Nevertheless it is known from linear theory that this dependence can be important. For instance suppose $u \in C^3$ is a solution of the equation

$$\Delta u = b(x) \cdot Du$$

in the whole of $\mathbf{R}^n$, and $b(x)$ is uniformly bounded on $\mathbf{R}^n$. Then if u is bounded on one side and $b = O(|x|^{-1})$ as $|x| \to \infty$, u must be a constant [3, p. 324].

In the following two results we shall similarly exploit corresponding conditions on the asymptotic behaviour of $\mathcal{A}$ and $\mathcal{B}$ as $|x| \to \infty$.

We define the following functions

$$\overline{M}(d) = \sup\{u(x) \colon \tfrac{1}{2} d \leqslant |x| \leqslant \tfrac{3}{2} d\}$$

and

$$\bar{\alpha}(d) = \sup\{\alpha(x, u, p): \quad \tfrac{1}{2}d < |x| < \tfrac{3}{2}d,\ u \leqslant \bar{M}(d),\ p \neq 0\}$$

$$\bar{\beta}(d) = \sup\{\beta(x, u, p)|p|: \tfrac{1}{2}d < |x| < \tfrac{3}{2}d,\ u \leqslant \bar{M}(d),\ p \neq 0\}$$

$$\bar{\gamma}(d) = \sup\{\gamma(x, u, p)|p|^2: \tfrac{1}{2}d < |x| < \tfrac{3}{2}d,\ u \leqslant \bar{M}(d),\ p \neq 0\}.$$

In what follows we shall tacitly assume that these quantities are finite.

Our first result concerns solutions of (1.1) in an exterior domain Ω and is closely related to Theorem 4.2, though the conditions $\beta \leqslant 0$, $\gamma \leqslant 0$ in that theorem are here replaced by growth conditions on $\bar{\beta}$, $\bar{\gamma}$ as $d \to \infty$. As we are mainly interested in the convergence of Du to zero, we assume a uniform bound on Du. For convenience we shall also suppose that the complement of Ω contains a ball $B(R)$, $R > 0$.

THEOREM 4.6. *Let $u \in C^3$ be a solution of equation* (1.1) *in an exterior domain Ω. Suppose* (1.1) *is uniformly elliptic, that*

$$\sup \bar{\alpha}(d) < 1,$$

and that

$$\bar{\beta}(d) = O(d^{-1}), \qquad \bar{\gamma}(d) = O(d^{-2})$$

as $d \to \infty$. Assume also that there exists a constant $\varkappa$ such that

$$|p|\,\|\mathcal{A}_p\| \leqslant \varkappa \|\mathcal{A}\|, \qquad |p|\,|\mathcal{B}_p| \leqslant \varkappa |\mathcal{B}| \tag{4.15}$$

for $x \in \Omega$, $u \in \mathbf{R}$, $p \neq 0$. Then there exists a constant K such that

$$|Du(x_0)| \leqslant K|x_0|^{-1}[\bar{M}(|x_0|) - u(x_0)]$$

for any point x_0 such that $|x_0| > 2R$.

If (4.15) *is replaced by*

$$|p|\,\|\mathcal{A}_p\| + |x|\,|\mathcal{B}_p| \leqslant \varkappa \|\mathcal{A}\| \tag{4.16}$$

the conclusion continues to hold.

PROOF. For $d > 2R$ the ball $B(x_0, \tfrac{1}{2}d)$ is in Ω and also in the set $\{\tfrac{1}{2}d < |x| < \tfrac{3}{2}d\}$. Proceeding as in the proof of Theorem 4.2, we obtain

$$\frac{1}{\mathcal{E}}\,\bar{\mathcal{N}} \leqslant \omega^2 \left\{-(1 - \sup \bar{\alpha}) + \frac{2\beta}{\omega} + \frac{\gamma}{\omega^2} + C\left(\frac{1}{d\omega z^{\frac{1}{2}}} + \frac{1}{d^2\omega^2 z}\right)\right\}$$

for some constant C. In view of the asymptotic behaviour of β as $d \to \infty$, we have, using property (c) of φ' as in the proof of Theorem 4.1,

$$\frac{2\beta}{\omega} \leqslant \frac{C}{d\omega|p|} \leqslant \frac{C}{d\bar{z}^{\frac{1}{2}}}$$

and similarly for γ. Hence, for $p \neq 0$

$$\frac{1}{\mathcal{E}}\,\overline{\mathcal{N}} \leqslant -\,\omega^2 \left\{(1-\sup\bar{\alpha}) - C\left(\frac{1}{d\bar{z}^{\frac{1}{2}}} + \frac{1}{d^2\bar{z}}\right)\right\},$$

which leads to the desired result.

Of special interest is the case when solutions cannot have an interior maximum. Then we can use a method from [3] to obtain a Liouville theorem and limit theorems. We shall need the following standard maximum principle.

LEMMA 4.2. *Let u be a non-constant solution of equation* (1.1), *and let*

$$\mathcal{B}(x, c, 0) \geqslant 0 \qquad (\leqslant 0)$$

for all c in the range of u. Then u can not have an interior maximum (*minimum*).

THEOREM 4.7. *Let u be an entire solution of equation* (1.1) *such that $u \leqslant M$. Suppose the conditions of Theorem* 4.6 *are satisfied and assume that*

$$\mathcal{B}(x, c, 0) = 0$$

for all $c \leqslant M$. Then $u \equiv$ constant.

PROOF. Let $\{x_k\}$ be a sequence such that

$$M - u(x_k) < k^{-1}$$

(we may suppose that $M = \sup u$ without loss of generality). In view of Lemma 4.2, $|x_k| \to \infty$.

For a given $k \geqslant 1$, set $|x_k| = d$. Denote by $S(d)$ the sphere with center at the origin and radius d. Let $y \in S(d)$, and let Γ be the shortest arc on $S(d)$ connecting x_k and y. If s measures distance along Γ such that $s = 0$ at $x = x_k$, and $s = s_0$ at $x = y$, then, by Theorem 4.6,

$$|v'(s)| \leqslant (K/d)\,v \tag{4.17}$$

where $v(s) = M - u(x)$ in an obvious notation, and $v' = dv/ds$.

Integration of (4.17) yields

$$v(s) \leqslant v(0) \exp (Ks/d)$$

or

$$M - u(y) \leqslant \frac{1}{k} \exp (Ks_0/d) \leqslant \frac{1}{k} \exp (\pi K) , \qquad y \in S(d) .$$

Applying Lemma 4.2 again we obtain

$$M - u(x) \leqslant \frac{1}{k} \exp (\pi K) , \qquad x \in B(d) .$$

Now, given $x_0 \in \boldsymbol{R}^n$ and $\varepsilon > 0$, we choose k so that (i) $|x_k| > |x_0|$ and (ii) $k^{-1} \exp (\pi K) < \varepsilon$. Then $M - u(x_0) < \varepsilon$. Since for any $x_0 \in \boldsymbol{R}^n$ and any $\varepsilon > 0$ such a k can be found, it follows that $u(x) \equiv M$ in $\boldsymbol{R}^n$.

If Ω is an exterior domain, Theorem 4.6 can be used to obtain a limit theorem for $u(x)$ as $x \to \infty$. Let

$$M = \sup_{\Omega} u(x) < \infty$$

and let the conditions of Theorem 4.6 be satisfied. Moreover, let $\mathcal{B}(x, c, 0) \leqslant 0$ for all $c \leqslant M$. Suppose

$$\limsup_{x \to \infty} u(x) = u_0 . \tag{4.18}$$

If $u_0 = -\infty$, then $\lim_{x \to \infty} u(x) = -\infty$, and the limit exists. Suppose, therefore, that $u_0 > -\infty$.

It follows from (4.18) that, given any $\varepsilon > 0$, there exists a radius $\tilde{R} = \tilde{R}(\varepsilon)$ such that

$$u(x) \leqslant u_0 + \varepsilon \qquad \text{for } |x| \geqslant \tilde{R} .$$

If $u_0 = M$, we have, of course, $u(x) \leqslant u_0$. Thus

$$u(x) \leqslant M_\varepsilon \qquad \text{for } |x| \geqslant \tilde{R} ,$$

where $M_\varepsilon = M$ if $u_0 = M$ and $M_\varepsilon = u_0 + \varepsilon$ if $u_0 < M$. Throughout we shall choose ε so small that $M_\varepsilon \leqslant M$.

It also follows from (4.18) that there exists a sequence $\{x_k\}$, $x_k \to \infty$, such that $\lim_{k \to \infty} u(x_k) = u_0$. Hence there exists a number N such that

$$u(x_k) \geqslant u_0 - \varepsilon \qquad \text{if } k \geqslant N .$$

We now apply Theorem 4.6, evaluating the bounds on α, β, γ and the constant $\varkappa$ for $u \in (-\infty, M]$. Choosing k such that $k \geqslant N$ and $d_k = |x_k| > 2\tilde{R}$, we obtain

$$|Du(x)| \leqslant K d_k^{-1}\big(M_\varepsilon - u(x)\big)$$

for all $x \in S(d_k)$, where K does not depend on ε.

We now follow the proof of Theorem 4.7. Then for $x \in S(d_k)$

$$\begin{aligned} u(x) &\geqslant M_\varepsilon - \exp(\pi K)\big(M_\varepsilon - u(x_k)\big) \\ &\geqslant M_\varepsilon - \exp(\pi K)(M_\varepsilon - u_0 + \varepsilon)\,. \end{aligned} \tag{4.19}$$

By Lemma 4.2, (4.19) holds on any spherical shell $B(d_l)\diagdown B(d_k)$, where $d_l > d_k > 2\tilde{R}$ and k, $l > N$. Hence

$$\liminf_{x\to\infty} u(x) \geqslant M_\varepsilon - \exp(\pi K)(M_\varepsilon - u_0 + \varepsilon)\,.$$

If $u_0 = M$, we have $M_\varepsilon = M = u_0$ and so

$$\liminf_{x\to\infty} u(x) \geqslant u_0 - \varepsilon \exp(\pi K)$$

and if $u_0 < M$ then $M_\varepsilon = u_0 + \varepsilon$ and

$$\liminf_{x\to\infty} u(x) \geqslant u_0 - \varepsilon\big\{2\exp(\pi K) - 1\big\}\,.$$

Since ε may be chosen arbitrarily small, it follows in both cases that

$$\liminf_{x\to\infty} u(x) \geqslant u_0\,.$$

Thus $\lim\limits_{x\to\infty} u(x)$ exists, and we have proved

THEOREM 4.8. *Let $u \in C^3$ be a solution of* (1.1) *in an exterior domain Ω such that $u \leqslant M$. Suppose the conditions of Theorem* 4.6 *are satisfied and that*

$$\mathcal{B}(x, c, 0) \leqslant 0$$

for all $c \leqslant M$. Then $\lim\limits_{x\to\infty} u(x)$ exists.

A similar result can be proved for the behavior of $u(x)$ near an isolated singularity. Let Ω be a deleted neighbourhood of an isolated singularity, which we choose as the origin.

THEOREM 4.9. *Let $u \in C^3$ be a solution of* (1.1) *in* Ω, *such that* $u \leqslant M$. *Suppose the conditions of Theorem* 4.1 *are satisfied and that* $\mathcal{B}(x, c, 0) \leqslant 0$ *for all* $c \leqslant M$. *Then* $\lim_{x \to 0} u(x)$ *exists.*

As an example consider the linear equation

$$\Delta u + b(x) \cdot Du = 0 .$$

Then with $s = -1$ we have $\alpha = 1 - (1-\theta)/n$ and

$$\beta = \frac{1-\theta}{n|p|^2} b \cdot p , \qquad \gamma = \frac{1}{|p|^4} \left\{ p \frac{\partial b}{\partial x} p - \frac{1-\theta}{n} (b \cdot p)^2 \right\} .$$

If $b = O(|x|^{-1})$ and $\partial b / \partial x = O(|x|^{-2})$ as $x \to \infty$ then

$$\bar{\beta}(d) = O(1/d) , \quad \bar{\gamma}(d) = O(1/d^2)$$

and

$$|x||\mathcal{B}_p| = O(1) .$$

Consequently the hypotheses of Theorem 4.6, and in particular (4.16), hold.

It then follows from Theorem 4.7 that any entire solution which is bounded on one side must be constant, and from Theorem 4.8 that a solution defined in an exterior domain and bounded on one side must tend to a limit at infinity. This gives an alternate proof of a result of Gilbarg and Serrin [3], pages 323-324, when the principal part of the equation is the Laplace operator. On the other hand the present result is in a different way stronger than that of [3], since (as is readily apparent from the proof technique) it continues to hold for nonlinear operators $\mathcal{A}(p)$ which are suitably close to the Laplacian.

Rather than making asymptotic assumptions on the behavior of the coefficient $b(x)$, let us instead suppose that the matrix $\partial b / \partial x$ is non-positive definite. Then if $\theta = 1$ we find that $\beta = 0$ and $\gamma \leqslant 0$. Now assume also that b is bounded. Then the hypothesis (4.14) of Theorem 4.5 applies, and we see that every entire solution such that $M(d) = o(\sqrt{d})$ must be a constant.

5. – A variational example.

Consider the Euler-Lagrange equation associated with the variational problem

$$\delta \int F(|Du|)\, dx = 0 .$$

This has the form

$$\Delta u + \frac{g(|Du|) - 1}{|Du|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \, \partial x_j} = 0 , \tag{5.1}$$

where

$$g(t) = \frac{tF''(t)}{F'(t)} ,$$

the primes denoting differentiation with respect to the (real) argument of F. The corresponding matrix $\mathcal{A}$ is then

$$\mathcal{A} = I + (g(|p|) - 1)\sigma\sigma ,$$

where I is the identity, $\sigma = p/|p|$, and $\sigma\sigma$ is the associated dyadic.

In order that the integrand be a smooth function of Du near $Du = 0$ we suppose that $F'(0) = 0$; ellipticity is then guaranteed by the assumption $F'' > 0$ (thus $g > 0$).

THEOREM 5.1. *Suppose that equation* (5.1) *is uniformly elliptic and that*

$$\lim_{t \to \infty} tg'(t) \text{ exists} . \tag{5.2}$$

Then every entire solution, such that $M(d) = o(d)$ is constant.

PROOF. It is enough to show that the gradient of u is bounded when $M(d) = o(d)$, for then $u \equiv$ constant by the corollary to Theorem 4.4.

The idea of the proof is basically to apply Theorem 4.1 with $a = 1$; indeed it is not hard to verify that the hypotheses of that result are all satisfied *with the exception of the condition* $\alpha^* < 1$. This difficulty, however, must be overcome in a circuitous manner. We use the following lemma.

Suppose the matrix $\mathcal{A}(x, u, p)$ *in Theorem* 4.1 *has the form*

$$\mathcal{A} = I + \mathcal{F}p + p\mathcal{F} ,$$

where $\mathcal{F}$ is a vector function and $\mathcal{F}p$ and $p\mathcal{F}$ are dyadics. Then Theorem 4.1 *remains valid if $\mathcal{A}$ is replaced by I in the computation of α, β, γ and λ and in the displayed hypotheses* (*but $\mathcal{E}$ still equal to $p\mathcal{A}p$*), *and if also the condition*

$$|p|^a\{|p|^{a+1}|\mathcal{F}| + |\delta_1(|p|^2\mathcal{F})| + |p|^{a+2}\|\delta_2 \mathcal{F}\|\} \leqslant \varkappa\mathcal{E} \tag{5.3}$$

is added to the list of hypotheses.

This result may be proved in exactly the same way as Theorem 4.1, except that, at the start, instead of (2.10) we use the more sophisticated formula noted on p. 580 of [7], setting $\mathcal{A}' = I$. Since the term $\delta_1(|p|^2\mathcal{F})$ appears there in conjunction with $\delta_2 \mathcal{E}$ and the term $|p|^2\delta_2\mathcal{F}$ in conjunction with $\|\mathcal{A}\|$, and since $\|\mathcal{A}\| \leqslant \|I\| + 2|p||\mathcal{F}|$, it is apparent that the additional condition (5.3) is precisely what is required to carry through the proof.

The lemma being shown, we next verify that its hypotheses are satisfied in the case at hand, where

$$\mathcal{F}(p) = \frac{1}{2}\frac{g(|p|) - 1}{|p|^2}p\,, \qquad \mathcal{B} = 0\,.$$

By direct calculation, with $s = -1$, $t = 0$, $\theta = 0$ we find

$$\begin{aligned}
&\mathcal{E} \quad = p\mathcal{A}p = |p|^2 g(|p|)\\
&\delta_1\mathcal{E} = \big(h(|p|) + 1\big)\,\mathcal{E}\\
&\delta_2\mathcal{E} = \big(h(|p|) + 2\big)\,\mathcal{E}p/|p|^2\\
&2\delta_1(|p|^2\mathcal{F}) \;= h(|p|)\,\mathcal{E}p/|p|^2\\
&2|p|^4\delta_2\mathcal{F} = (2pp - |p|^2 I) + \{|p|^2 I + (h(|p|) - 2)pp\}\,\mathcal{E}/|p|^2
\end{aligned}$$

and

$$\alpha = 1 + h - g/n\,, \qquad \beta = \gamma = 0\,,$$

where

$$h(t) = tg'(t)/g(t)\,.$$

We shall show that, when $a = 1$, the conditions of Theorem 4.1 and the lemma hold. Using the above formulae, it is easy to check that the inequality $\alpha^* < a = 1$ reduces to

$$h - g/n \leqslant \text{const} < 0$$

and that the remaining conditions are satisfied if

$$g \geqslant \text{const} > 0\,, \qquad h \text{ bounded}\,,$$

where of course we need only consider values $|p| > l$, where l is as large as we please.

Now by hypothesis equation (5.1) is uniformly elliptic. Thus g is bounded, both from zero and infinity. Moreover

$$g(t) = \int^t \frac{\tilde{h}(\tau)}{\tau}\, d\tau\,, \qquad \tilde{h}(t) = tg'(t)\,.$$

Since $\tilde{h}(t)$ tends to a limit as $t \to \infty$ and since $g(t)$ is bounded it follows that $\lim_{t\to\infty} \tilde{h}(t) = 0$. But $\tilde{h} = gh$ so that

$$\lim_{t\to\infty} h(t) = 0\,.$$

Thus the required conditions for the lemma are demonstrated.

It follows from (4.1) therefore that at any (fixed) point x we have

$$|Du(x)| \leqslant \max\{L, Kd^{-1}\}[\bar{M}(d) + \mu - u(x)]\,, \tag{5.4}$$

where d is arbitrary and $\bar{M}(d) = \sup\{u(y)\colon |y - x| < d\}$. Here the constant L has the specific form (see the proof of Theorem 4.1, noting that $\varkappa' = \beta^* = \gamma^* = 0$ in the present case):

$$L = l/\mu\,,$$

while the constant K depends only on $\varkappa$, n and the difference $1 - \alpha^*$. If we choose $\mu = \bar{M}(d)$, then (5.4) becomes

$$|Du(x)| \leqslant \max\left\{\left(\frac{2\mu - u(x)}{\mu}\right) l,\ \text{const.}\,\frac{2\mu - u(x)}{d}\right\}.$$

Here we may let $d \to \infty$. Since $\mu = \bar{M}(d) = o(d)$ this yields

$$|Du(x)| \leqslant 2l$$

as required (if $\bar{M}(d)$ is bounded, choose $\mu = \sqrt{d}$ and the result follows equally). This completes the proof.

We note that the hypothesis of uniform ellipticity as well as condition (5.2) can be weakened to the form

$$\begin{cases} g \geqslant \text{constant} > 0 \\ h \text{ bounded, } \limsup h(t) \leqslant 0 \end{cases} \tag{5.5}$$

(when $t \to \infty$) without affecting the proof; recall here that

$$g = \frac{tF''(t)}{F'(t)}, \qquad h = \frac{tg'(t)}{g(t)}.$$

Moreover, if the growth condition on u is strengthened to $u = o(d)$ it is enough in place of (5.5) merely to assume

$$(5.6) \qquad g \geqslant \text{constant} > 0, \quad h \text{ bounded}$$

(see Theorem 5.2(c), below). Unfortunately, the minimal surface equation, corresponding to the area integrand $F(t) = (1 + t^2)^{\frac{1}{2}}$, is not covered by either set of conditions, since $g = 1/(1 + t^2)$ and $\lim g = 0$. The integrand

$$F(t) = (1 + t^2)^{\alpha/2}, \quad \alpha > 1,$$

on the other hand, satisfies the conditions of Theorem 5.1 since $g = 1 + (\alpha - 2)t^2/(1 + t^2)$ is bounded from both zero and infinity, and $tg' = 2(\alpha - 2)t^2/(1 + t^2)^2$ tends to zero as $t \to \infty$. The same holds true for any integrand of the form

$$F(t) = t^\alpha \tilde{F}(1/t) \quad \alpha > 1$$

where $\tilde{F}(\tau)$ is a three times continuously differentiable function of τ with $\tilde{F}(0) = 0$, provided of course that $F'' > 0$ and $F'(0) = 0$.

The equation

$$(5.7) \qquad (I + Du\, Du)D^2 u = 0$$

is another example of interest, being non-uniformly elliptic with the same ellipticity ratio (namely $\lambda_2/\lambda_1 = 1 + |p|^2$) as the minimal surface equation. Here $g = 1 + t^2 \geqslant 1$, $|h| = 2t^2/(1 + t^2) \leqslant 2$ and $\alpha = 1 + h - g/n \to -\infty$ as $t \to \infty$, so that the result of Theorem 5.1 clearly holds: any entire solution of (5.7) satisfying the condition $M(d) = o(d)$ is constant.

The situation of the related variational problem

$$(5.8) \qquad \delta \int \{F(|Du|) + G(u)\}\, dx = 0$$

is somewhat more complicated. For example if $F(t) = t^2$ and $G(u) = u^2$ the Euler-Lagrange equation is $\Delta u = u$, which has the negative entire solution $u = -\exp(x_1)$. Consequently we cannot expect a one-sided Liouville theorem to hold merely when $G''(u) \geqslant 0$, though of course that condition must play a major role in any considerations. The following result gives an idea of the conclusions which can be obtained.

THEOREM 5.2. *Consider the Euler-Lagrange equation*

$$\Delta u + \frac{g(|Du|) - 1}{|Du|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \, \partial x_j} = \frac{|Du|}{F'(|Du|)} G'(u)$$

associated with the variational problem (5.8), *and suppose that* $F'(0) = 0$, $F''(t) > 0$ *and* $G''(u) \geqslant 0$.

(*a*) *If* $G' > 0$ *there are no entire solutions satisfying* $M(d) = o(d)$;

(*b*) *If* $G' \geqslant 0$ *and if conditions* (5.5) *hold, then any entire solution satisfying* $M(d) = o(d)$ *is constant.*

(*c*) *If conditions* (5.6) *hold, then any entire solution satisfying* $u = o(d)$ *is constant.*

REMARK. Case (*b*) includes Theorem 5.1 as the special case $G' \equiv 0$. We note also that the condition $G''(u) \geqslant 0$ together with the ellipticity guarantees that solutions of the Dirichlet problem in a bounded region are unique.

PROOF OF THEOREM 5.2. By [9], Theorem 3, or more specifically by the proof of that result together with the remark following the proof, it is clear that $G'(c) \leqslant 0$ for each value c in the range of an entire solution u satisfying $M(d) = o(d)$. This proves (*a*), and shows moreover in case (*b*) that u is an entire solution of (5.1). But then by Theorem 5.1, or more precisely by the remark at the end of the proof, we get $u \equiv$ constant.

In case (*c*) the result of [9], Theorem 3, again shows that $G'(c) = 0$ for each c in the range of u, so that once more u is a solution of (5.1). The proof is then essentially the same as that of Theorems 3 and 4 in [7], the only difference being that the Laplace operator is replaced by $I + (g-1)\sigma\sigma$; the changes which result from this are exactly the same as those already treated in the proof of Theorem 5.1. Finally, note that the condition $\alpha^* < 1$ is not required in [8] so that we no longer need to assume lim sup $h \leqslant 0$. This completes the proof.

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