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On the Singularities of the Solution of the Cauchy Problem for the Operator with Non Uniform Multiple Characteristics.

YÛSAKU HAMADA (*) - GEN NAKAMURA (**)

dedicated to Jean Leray

We consider the Cauchy problem with singular data for linear partial differential equation in the complex domain.

The Cauchy problem with ramified data has been studied by Y. Hamada, J. Leray and C. Wagschal [5] for the operator with constant multiple characteristics.

In [7], one of the authors has already treated the case for diagonalizable first order system with non uniform multiple characteristics, by applying the method of B. Granoff and D. Ludwig [4] to the operator with variable coefficients. That is, he has studied the propagation of the singularities for the system under the condition that two components of its characteristic variety intersect one another and their intersection is involutive.

In this paper, we shall treat the case for single operator whose principal part satisfies the above conditions, but without Levi's condition. Actually, the geometry of the singular supports of the solution is closely related to the studies of J. Leray [6] and L. Gårding, T. Kotake and J. Leray [3].

Our method is analogous to that of [7]. In the next section, we shall give the precise statement of our results.

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1. – Assumptions and results.

Let X be a neighborhood of the origin in C^{n+1} and (t, x) $(x = (x_1, \ldots, x_n))$ be a point of X.

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We consider a linear partial differential operator of order m with holomorphic coefficients on X:

$$a(t, x, D_t, D_x) = \sum_{|\alpha| \leqslant m} a_\alpha(t, x) D_t^{\alpha_0} D_1^{\alpha_1} \dots D_n^{\alpha_n},$$

where $D_t = \partial/\partial t$, $D_i = \partial/\partial x_i$ $(1 \leqslant i \leqslant n)$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in N^{n+1}$ and $|\alpha| = \sum_{i=0}^{n} \alpha_i$.

We denote by $h(t, x; \lambda, \xi)$ $(\xi = (\xi_1, \dots, \xi_n))$ the characteristic polynomial of $a(t, x, D_t, D_x)$:

$$h(t, x; \lambda, \xi) = \sum_{|\alpha| = m} a_\alpha(t, x) \lambda^{\alpha_0} \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}.$$

We often write $\lambda = \xi_0$.

Let S be the n-dimensional plane $t = 0$ and T be the $(n-1)$-plane $t = x_1 = 0$.

We shall impose on the polynomial $h(t, x; \lambda, \xi)$ the following conditions due to G. Nakamura [7].

ASSUMPTION (A). *Let $\lambda_i(t, x; \xi)$ $(1 \leqslant i \leqslant m-2)$, $\lambda^{\pm}(t, x; \xi)$ be the roots of the equation $h(t, x; \lambda, \xi) = 0$, so*

$$h(t, x; \lambda, \xi) = (\lambda - \lambda^+(t, x; \xi))(\lambda - \lambda^-(t, x; \xi)) \cdot \prod_{i=1}^{m-2} (\lambda - \lambda_i(t, x; \xi)).$$

We assume that these $\lambda_i(t, x; \xi)$ $(1 \leqslant i \leqslant m-2)$, $\lambda^{\pm}(t, x; \xi)$ satisfy the following three conditions:

(i) *$\lambda_i(t, x; \xi)$ $(1 \leqslant i \leqslant m-2)$, $\lambda^{\pm}(t, x; \xi)$ are holomorphic in a neighborhood of $(0, 0; 1, 0, \dots, 0)$.*

(ii) *$\lambda^+(0, 0; 1, 0, \dots, 0) = \lambda^-(0, 0; 1, 0, \dots, 0)$ and $\lambda_i(0, 0; 1, 0, \dots, 0)$ $(1 \leqslant i \leqslant m-2)$, $\lambda^+(0, 0; 1, 0, \dots, 0)$ are distinct.*

(iii) *The Poisson bracket $\{\lambda - \lambda^+(t, x; \xi), \lambda - \lambda^-(t, x; \xi)\}$ vanishes in a neighborhood of $(0, 0; 1, 0, \dots, 0)$. This means that*

$$\lambda_t^+ - \lambda_t^- + \sum_{i=1}^{n} \left(\frac{\partial \lambda^+}{\partial \xi_i} \frac{\partial \lambda^-}{\partial x_i} - \frac{\partial \lambda^+}{\partial x_i} \frac{\partial \lambda^-}{\partial \xi_i} \right) = 0 \tag{1.1}$$

holds in a neighborhood of $(0, 0; 1, 0, \dots, 0)$.

Then there exists m characteristic surfaces K_i $(1 \leqslant i \leqslant m-2)$, $K^{\pm}$ issuing from T.

K_i $(1 \leqslant i \leqslant m-2)$ and $K^{\pm}$ are defined by the equation $\varphi_i(t, x) = 0$ $(1 \leqslant i \leqslant m-2)$ and $\varphi^{\pm}(t, x) = 0$ respectively. Here φ_i $(1 \leqslant i \leqslant m-2)$, $\varphi^{\pm}$ are the solutions of Hamilton-Jacobi's equations

$$D_t \varphi_i = \lambda_i (t, x, \operatorname{grad}_x \varphi_i), \qquad \varphi_i(0, x) = x_1, \ 1 \leqslant i \leqslant m-2$$
$$D_t \varphi^{\pm} = \lambda^{\pm}(t, x, \operatorname{grad}_x \varphi^{\pm}), \qquad \varphi^{\pm}(0, x) = x_1 .$$

Now we consider the solution $\Phi(t, x, \tau)$ of the equation

$$\begin{cases} \Phi_t = \lambda^{-}(t, x, \Phi_x) \\ \Phi(\tau, x, \tau) = \varphi^{+}(\tau, x) . \end{cases}$$

(We shall often write $\Phi_x = \operatorname{grad}_x \Phi$).

Now we assume that

(B) $\Phi(0, 0, \tau) \not\equiv 0$ *holds for* τ *sufficiently small.*

(In § 5, we shall give some remarks on this assumption (B).)

Then the Weierstrass' preparation theorem allows us to write

$$\Phi(t, x, \tau) = p(t, x, \tau) P(t, x, \tau) ,$$

where $p(t, x, \tau)$ is holomorphic in a neighborhood of $(0, 0, 0)$, $p(0, 0, 0) \neq 0$ and $P(t, x, \tau)$ is a distinguished pseudo-polynomial in τ. $P(t, x, \tau)$ is irreducible in τ, because $\Phi(0, x, 0) = x_1$.

Let $\Delta(t, x)$ be the discriminant of $P(t, x, \tau)$. Denote the surface $\Delta(t, x) = 0$ by K_0. This surface is n-dimensional and is characteristic for the operator $a(t, x, D_t, D_x)$ (more precisely, for both $\xi_0 - \lambda^{+}(t, x, \xi)$ and $\xi_0 - \lambda^{-}(t, x, \xi)$), and it is not regular in general and touches $K^{\pm}$.

K_0 is spanned by 2-families of bicharacteristics (also, cf. J. Bony and P. Schapira [1]) which are obtained by integrating successively Hamilton-fields $H_{\xi_0 - \lambda^{+}}$ and $H_{\xi_0 - \lambda^{-}}$.

In Section 5, we shall state precisely the geometrical properties of K_0.

We write $K = \bigcup_{i=0}^{m-2} K_i \cup K^{+} \cup K^{-}$.

Now we consider the Cauchy problem with singular data:

$$(1.2) \qquad \begin{cases} a(t, x, D_t, D_x) u(t, x) = 0 , \\ D_t^h u(0, x) = w_h(x) , \qquad 0 \leqslant h \leqslant m-1 , \end{cases}$$

where $w_h(x)$ $(0 \leqslant h \leqslant m-1)$ have poles along T.

Then our results are stated as follows

THEOREM. *Under the assumptions* (A) *and* (B), *the Cauchy problem* (1.2) *has a unique holomorphic solution on the simply connected covering space over* $V-K$, *where* V *is a neighborhood of the origin.*

More precisely, the solution is expressed by

$$u(t,x)=\sum_{\lambda=1}^{m-2}\left\{\frac{F_\lambda(t,x)}{\varphi_\lambda(t,x)^{p_\lambda}}+G_\lambda(t,x)\log\varphi_\lambda(t\,x)\right\}+\frac{F^+(t,x)}{\varphi^+(t,x)^{p^+}}+G^+(t,x)\log\varphi^+(t,x)+\int_0^t\left\{\sum_{k=1}^{\infty}\frac{F_k(t,x,\tau)}{\Phi(t,x,\tau)^k}+G(t,x,\tau)\log\Phi(t,x,\tau)\right\}d\tau+H(t,x)\,, \tag{1.3}$$

where p_λ, p^+ *are integers* >0 *and* F_λ, F^+, F_k, G_λ, G^+, G, H *are holomorphic in a neighborhood of the origin.*

REMARK 1. The assumption (B) is unnecessary to obtain the expression (1.3). We can easily see that the singularities of the solution lie on K under the assumption (B).

REMARK 2. The solution does not have singularities along K_0 on the sheet of the covering space which contains $S-T$.

The proof of this theorem is performed by a classical method.

In Section 2, we shall prepare some calculations. By using these calculations, we shall construct a formal solution of the problem (1.2) in Section 3. This is based on the method of asymptotic expansion developed by L. Gårding, T. Kotake and J. Leray [3], B. Granoff and D. Ludwig [4], J. Vaillant [9] and G. Nakamura [7]. In Section 4, we shall prove the convergence of this formal solution by the method of majorant function due to C. Wagschal [8], De Paris [2], and Y. Hamada, J. Leray and C. Wagschal [5]. Section 5 is devoted to giving the geometrical properties of the surface K_0 and to giving some remarks on the assumption (B).

2. – Preliminary calculations.

In the preceding section, we have defined Φ as the solution of the equation

$$\begin{cases}\Phi_t=\lambda^-(t,x,\Phi_x)\,,\\ \Phi(\tau,x,\tau)=\varphi^+(\tau,x)\,.\end{cases}\tag{2.1}$$

By the assumption (A), we can see that Φ satisfies

$$(2.2)\qquad \begin{cases} \Phi(t, x, 0) = \varphi^-(t, x)\,, \\ \Phi_t + \Phi_\tau = \lambda^+(t, x, \Phi_x)\,. \end{cases}$$

The first equality is evident and the second is proved in Section 5.

We write

$$(2.3)\qquad \begin{cases} h(t, x, \lambda, \xi) = q(t, x, \lambda, \xi)(\lambda - \lambda^+(t, x, \xi))(\lambda - \lambda^-(t, x, \xi))\,, \\ q(t, x, \lambda, \xi) = \prod_{i=1}^{m-2} (\lambda - \lambda_i(t, x, \xi))\,. \end{cases}$$

Let $g(t, x, \lambda, \xi)$ be the homogeneous polynomial of degree $m-1$ in λ, ξ defined by

$$(2.4)\qquad g(t, x, \lambda, \xi) = \tfrac{1}{2} \sum_{i=0}^{n} h_i^{(i)}(t, x, \lambda, \xi)\,,$$

where

$$h^{(0,0)} = \frac{\partial^2}{\partial t\, \partial \lambda} h(t, x, \lambda, \xi)\,, \quad h_i^{(i)} = \frac{\partial^2}{\partial x_i\, \partial \xi_i} h(t, x, \lambda, \xi) \qquad (1 \leqslant i \leqslant n)\,.$$

(We also write

$$h^{(0)}(t, x, \lambda, \xi) = \frac{\partial}{\partial \lambda} h(t, x, \lambda, \xi), \qquad h^{(i)}(t, x, \lambda, \xi) = \frac{\partial}{\partial \xi_i} h(t, x, \lambda, \xi)\,,$$

$$h^{(0,0)} = \frac{\partial^2}{\partial \lambda^2} h, \quad h^{(0,i)} = \frac{\partial^2}{\partial \lambda\, \partial \xi_i} h\,, \quad h^{(i,j)} = \frac{\partial^2}{\partial \xi_i\, \partial \xi_j} h\,, \quad 1 \leqslant i, j \leqslant n\Big)\,.$$

We define the polynomial $H(t, x, \lambda, \xi)$ by

$$(2.5)\qquad H(t, x, \lambda, \xi) = h(t, x, \lambda, \xi) + g(t, x, \lambda, \xi)\,.$$

Furthermore we put

$$(2.6)\qquad c(h, \Phi) = \tfrac{1}{2} \sum_{i,j=0}^{n} h^{(i,j)}(t, x, \Phi_t, \Phi_x) \Phi_{i,j}\,,$$

where

$$\Phi_{0,0} = \Phi_{t,t}\,, \qquad \Phi_{0,i} = \Phi_{t,x_i}\,, \qquad \Phi_{i,j} = \Phi_{x_i,x_j}\,.$$

Then we have the following lemma that plays an important role in constructing a formal solution of (2.1).

LEMMA 1.

$$c(h, \Phi) + g(t, x, \Phi_t, \Phi_x) \equiv 0 \bmod \Phi_\tau \tag{2.7}$$

$$c(h, \varphi^{\pm}) + g(t, x, \varphi_t^{\pm}, \varphi_x^{\pm}) \equiv 0 \bmod (\varphi_t^{\pm} - \lambda^{\mp}(t, x, \varphi_x^{\pm})) . \tag{2.8$^{\pm}$}$$

Here, for holomorphic functions F, G, H, $F \equiv G \bmod H$ means that $F - G = HK$ holds for some holomorphic function K.

PROOF. We can easily see from (2.1) and (2.2) that

$$h^{(0)}(t, x, \Phi_t, \Phi_x) = -q(t, x, \Phi_t, \Phi_x)\Phi_\tau ,$$

$$h^{(i)}(t, x, \Phi_t, \Phi_x) = q(t, x, \Phi_t, \Phi_x)\frac{\partial\lambda^-}{\partial\xi_i}(t, x, \Phi_x)\Phi_\tau .$$

Hence differentiating the above equalities with respect to t and x_i respectively, we obtain for $1 \leqslant i \leqslant n$

$$h_0^{(0)} + h^{(0,0)}\Phi_{0,0} + \sum_{i=1}^n h^{(0,i)}\Phi_{0,i} \equiv -q\,\Phi_{\tau t} \bmod \Phi_\tau ,$$

$$h_i^{(i)} + h^{(i,0)}\Phi_{i,0} + \sum_{j=1}^n h^{(i,j)}\Phi_{i,j} \equiv q\frac{\partial\lambda^-}{\partial\xi_i}(t, x, \Phi_x)\Phi_{\tau,i} \bmod \Phi_\tau .$$

This yields

$$g(t, x, \Phi_t, \Phi_x) + c(h, \Phi) \equiv -\frac{1}{2}q\left(\Phi_{\tau t} - \sum_{i=1}^n \frac{\partial\lambda^-}{\partial\xi_i}\Phi_{\tau i}\right) \bmod \Phi_\tau .$$

On the other hand, differentiating the second equality of (2.1) by τ, we get

$$\Phi_{t\tau} = \sum_{i=1}^n \frac{\partial\lambda^-}{\partial\xi_i}\Phi_{\tau i} .$$

Thus, we have proved (2.7).

By the same procedure, we get

$$g(t, x, \varphi_t^+, \varphi_x^+) + c(h, \varphi^+) \equiv \frac{1}{2}q(t, x, \varphi_t^+, \varphi_x^+)\left(\frac{\partial}{\partial t} - \sum_{i=1}^n \frac{\partial\lambda^+}{\partial\xi_i}\frac{\partial}{\partial x_i}\right)$$

$$\cdot(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+)) \bmod (\varphi_t^+ - \lambda^-(t, x, \varphi_x^+)) .$$

By the assumption (1.1) (or Proposition 5.1), we get

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^n \frac{\partial\lambda^+}{\partial\xi_i}\frac{\partial}{\partial x_i}\right)(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+)) = 0 .$$

This means $(2.8)^+$.

Taking account of $\Phi(t, x, 0) = \varphi^-(t, x)$ and putting $\tau = 0$ in (2.7), we obtain (2.8)$^-$. The proof of the lemma is complete.

We frequently use the following two kinds of Leibniz' formulas

FORMULA 1. *Let $H(x, \xi)$ be a homogeneous polynomial of degree m in ξ with holomorphic coefficients, then we have*

$$H(x, D) f(\varphi(x)) u(x) = f^{(m)}(\varphi) H(x, \varphi_x) u + \sum_{\alpha=1}^{m} f^{(m-\alpha)}(\varphi) L_\alpha u , \tag{2.9}$$

where L_α is differential operator of order α independent of f, u.

FORMULA 2. *Let $P(t, x, D_t, D_x)$ be a differential operator of order m with holomorphic coefficients and $F(t, x, \tau)$ be a holomorphic function. Then we have the following identity*

$$\begin{aligned} P(t, x, D_t, D_x)\int_0^t F(t, x, \tau)\, d\tau \\ = M(t, x, D_t, D_\tau, D_x) F(t, x, \tau)|_{\tau=t} + \int_0^t P(t, x, D_t, D_x) F(t, x, \tau)\, d\tau \end{aligned} \tag{2.10}$$

where $M(t, x, D_t, D_\tau, D_x)$ is the differential operator of order $m-1$ and $M(t, x, \lambda, \mu, \xi)$ is the polynomial of degree $m-1$ in (λ, μ, ξ) defined by

$$M(t, x, \lambda, \mu, \xi) = \mu^{-1}\big(P(t, x, \lambda + \mu, \xi) - P(t, x, \lambda, \xi)\big) .$$

Hereafter we shall use the following notation.

Let $\{U_k\}$ be a series of functions and p, q ($p \leqslant q$) be non-negative integers. Then $\mathcal{O}(p, q;\, U_j)$ is defined by

$$\mathcal{O}(p, q;\, U_j) = \sum_{\alpha=0}^{q-p} L_{p+\alpha} U_{j-\alpha} ,$$

where L_α are differential operators of order α with holomorphic coefficients.

Now let us calculate

$$H(t, x, D_t, D_x) \sum_{j=-\infty}^{\infty} f_j(\varphi^+) V_j .$$

Here $f_j(s)$ ($j \in \boldsymbol{Z}$) are functions of independent variable $s \in \boldsymbol{C}$ which satisfy the relation $df_j/ds = f_{j-1}$ ($j \in \boldsymbol{Z}$).

$\Big($We often omit $\sum_{j=-\infty}^{\infty}$ in the following$\Big)$.

Taking account of $h(t, x, \varphi_t^+, \varphi_x^+) = 0$, we obtain by Leibniz' formula 1

$$H(t, x, D_t, D_x) f_j(\varphi^+) V_j = f_{j-m+1}(\varphi^+)(L_1^+ V_j + \mathcal{O}(2, m; V_{j-1})) , \tag{2.11}$$

where

$$L_1^+ = h^{(0)}(t, x, \varphi_t^+, \varphi_x^+) D_t + \sum_{i=1}^n h^{(i)}(t, x, \varphi_t^+, \varphi_x^+) D_i + c(h, \varphi^+) + g(t, x, \varphi_t^+, \varphi_x^+) .$$

We have by a simple calculation

$$h^{(c)}(t, x, \varphi_t^+, \varphi_x^+) = q(t, x, \varphi_t^+, \varphi_x^+)(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+)) ,$$

$$h^{(i)}(t, x, \varphi_t^+, \varphi_x^+) = q(t, x, \varphi_t^+, \varphi_x^+)(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+)) \left(-\frac{\partial \lambda^+}{\partial \xi_i}(t, x, \varphi_x^+)\right)$$

for $1 \leqslant i \leqslant n$.

Therefore, according to Lemma 1, we can write

$$L_1^+ = q^+(t, x)(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+))\left(D_t - \sum_{i=1}^n \frac{\partial \lambda^+}{\partial \xi_i}(t, x, \varphi_x^+) D_i + c^+(t, x)\right),$$

where $q^+(t, x) = q(t, x, \varphi_t^+, \varphi_x^+) \neq 0$ and $c^+(t, x)$ is holomorphic function in a neighborhood of the origin.

For convenience, we define $P_1^+(t, x, D_t, D_x)$ by

$$P_1^+(t, x, D_t, D_x) = D_t - \sum_{i=1}^n \frac{\partial \lambda^+}{\partial \xi_i}(t, x, \varphi_x^+) D_i + c^+(t, x) . \tag{2.12}$$

Next, let us calculate

$$H(t, x, D_t, D_x)\int_0^t f_j(\Phi) W_j(t, x, \tau) d\tau .$$

By Leibniz' formula 2, we have

$$H(t, x, D_t, D_x)\int_0^t f_j(\Phi) W_j(t, x, \tau) d\tau$$
$$= M(t, x, D_t, D_\tau, D_x) f_j(\Phi) W_j|_{\tau=t} + \int_0^t H(t, x, D_t, D_x) f_j(\Phi) W_j d\tau ,$$

where $M(t, x, D_t, D_\tau, D_\xi)$ is the differential operator of order $m-1$ and

$$M(t, x, \lambda, \mu, \xi) = \mu^{-1}(H(t, x, \lambda + \mu, \xi) - H(t, x, \lambda, \xi)) .$$

Applying Leibniz' formula 1 to the above expression, we have

$$(2.13)\qquad H(t, x, D_t, D_x)\int_0^t f_j(\Phi) W_j(t, x, \tau)\,d\tau$$
$$= f_{j-m+1}(\Phi)\sum_{\alpha=0}^{m-1} M_\alpha W_{j-\alpha}|_{\tau=t} + \int_0^t f_{j-m}(\Phi)\sum_{\alpha=0}^{m} L_\alpha W_{j-\alpha}\,d\tau\,.$$

Here M_α, L_α are differential operators of order α.

We want to compute M_0, M_1, L_0, L_1 more explicitly. First, put

$$\tilde{h}(t, x, \lambda, \mu, \xi) = \mu^{-1}(h(t, x, \lambda+\mu, \xi) - h(t, x, \lambda, \xi))\,.$$

Then, it follows from (2.1), (2.2) that

$$M_0 = \tilde{h}(t, x, \Phi_t, \Phi_\tau, \Phi_x) = \Phi_\tau^{-1}(h(t, x, \Phi_t+\Phi_\tau, \Phi_x) - h(t, x, \Phi_t, \Phi_x)) = 0$$

and

$$(2.14)\qquad M_1 = \frac{\partial\tilde{h}}{\partial\lambda}D_t + \frac{\partial\tilde{h}}{\partial\mu}D_\tau + \sum_{i=1}^{n}\frac{\partial\tilde{h}}{\partial\xi_i}D_i + c(t, x, \tau)\,,$$

where

$$(2.15)\qquad \begin{cases} \dfrac{\partial\tilde{h}}{\partial\lambda}(t, x, \Phi_t, \Phi_\tau, \Phi_x) = \Phi_\tau^{-1}(h^{(0)}(t, x, \Phi_t+\Phi_\tau, \Phi_x) - h^{(0)}(t, x, \Phi_t, \Phi_x))\,, \\ \dfrac{\partial\tilde{h}}{\partial\mu}(t, x, \Phi_t, \Phi_\tau, \Phi_x) = \Phi_\tau^{-1}h^{(0)}(t, x, \Phi_t+\Phi_\tau, \Phi_x)\,, \\ \dfrac{\partial\tilde{h}}{\partial\xi_i}(t, x, \Phi_t, \Phi_\tau, \Phi_x) = \Phi_\tau^{-1}(h^{(i)}(t, x, \Phi_t+\Phi_\tau, \Phi_x) \\ \qquad\qquad - h^{(i)}(t, x, \Phi_t, \Phi_x))\,, \qquad 1\leqslant i\leqslant n\,. \end{cases}$$

We note that these functions are holomorphic since the numerators are evidently equal to 0 modulo Φ_τ.

Next, let us observe the second term of the right hand side of (2.13).

Now we can easily see that

$$L_0 = h(t, x, \Phi_t, \Phi_x) = 0\,,$$
$$L_1 = h^{(0)}(t, x, \Phi_t, \Phi_x)D_t + \sum_{i=1}^{n} h^{(i)}(t, x, \Phi_t, \Phi_x)D_i + c(h, \Phi) + g(t, x, \Phi_t, \Phi_x)\,.$$

An application of Lemma 1 allows us to rewrite

$$L_1 = \Phi_\tau\{\Phi_\tau^{-1}h^{(0)}(t, x, \Phi_t, \Phi_x)D_t + \sum_{i=1}^{n} \Phi_\tau^{-1}h^{(i)}(t, x, \Phi_t, \Phi_x)D_i + c'(t, x, \tau)\},$$

where $c'(t, x, \tau)$ is a holomorphic function in a neighborhood of the origin.

In connection with the operator L_1, we define the differential operator

$$(2.16) \qquad J_1 = \Phi_\tau^{-1}h^{(0)}(t, x, \Phi_t, \Phi_x)D_t + \sum_{i=1}^{n} \Phi_\tau^{-1}h^{(i)}(t, x, \Phi_t, \Phi_x)D_i + c'(t, x, \tau),$$

then J_1 is a holomorphic differential operator, because we obviously know $h^{(i)}(t, x, \Phi_t, \Phi_x) \equiv 0 \mod \Phi_\tau$ for $0 \leqslant i \leqslant n$.

Now, an integration by parts yields

$$(2.17) \qquad \int_0^t f_{j-m+1}(\Phi) L_1 W_j = \int_0^t f_{j-m+1}(\Phi) \Phi_\tau J_1 W_j \, d\tau$$

$$= f_{j-m+2}(\Phi) J_1 W_j \Big|_{\tau=0}^{\tau=t} + \int_0^t f_{j-m+2}{}'(\Phi)' - D_\tau J_1 W_j) \, d\tau .$$

Hence, it follows from (2.13) and (2.17) that

$$H(t, x, D_t, D_x) \int_0^t f_j(\Phi) W_j(t, x, \tau) \, d\tau$$

$$= f_{j-m+2}(\Phi) \sum_{\alpha=1}^{m-1} N_\alpha^+ W_{j-\alpha+1}|_{\tau=t} + f_{j-m+2}(\Phi) N_1^- W_j|_{\tau=0}$$

$$+ \int_0^t f_{j-m+2}(\Phi) \sum_{\alpha=2}^{m} K_\alpha W_{j-\alpha+2} \, d\tau ,$$

where $N_1^+ = M_1 + J_1$, $N_1^- = -J_1$ and $K_2 = L_2 - D_\tau J_1$.

By making use of (2.14), (2.15) and (2.16), we can express N_1^+, N_1^- in the following forms

$$N_1^+ = q(t, x, \Phi_t + \Phi_\tau, \Phi_x)Q_1^+ , \qquad N_1^- = q(t, x, \Phi_t, \Phi_x)Q_1^- ,$$

where the differential operators Q_1^+ and Q_1^- are defined by

$$Q_1^+ = D_t + D_\tau - \sum_{i=1}^{n} \frac{\partial \lambda^+}{\partial \xi_j} (t, x, \Phi_x) D_i + b^+(t, x, \tau)$$

$$Q_1^- = D_t - \sum_{i=1}^{n} \frac{\partial \lambda^-}{\partial \xi_i} (t, x, \Phi_x) D_i + b^-(t, x, \tau) .$$

Next, in order to see the form of differential operator K_2 more explicitly, we compute the principal part of K_2. This is really of the form

$$\begin{aligned}\frac{1}{2}\Big\{h^{(0,0)}(t, x, \Phi_t, \Phi_x)D_t^2 &+ 2\sum_{i=1}^{n} h^{(0,i)}(t, x, \Phi_t, \Phi_x)D_tD_i \\ &+ \sum_{i,j=1}^{n} h^{(i,j)}(t, x, \Phi_t, \Phi_x)D_iD_j\Big\} - \Phi_\tau^{-1}h^{(0)}(t, x, \Phi_t, \Phi_x)D_tD_\tau \\ &- \sum_{i=1}^{n}\Phi_\tau^{-1}h^{(i)}(t, x, \Phi_t, \Phi_x)D_iD_\tau + \textit{lower order terms} \\ &\equiv q(t, x, \Phi_t, \Phi_x)Q_1^+Q_1^- + \textit{lower order terms} \bmod \Phi_\tau .\end{aligned}$$

By integrating by parts, we may write K_2 as

$$K_2 = R_0(D_t + D_\tau)D_t + R_2 ,$$

where R_0 and R_2 are the differential operators of order 0 and 2 respectively, and R_2 does not involve the terms of D_t^2, D_tD_τ, D_τ^2.

As we proceed to construct a formal solution of the Cauchy problem (1.2), it is preferable to replace $W_j(t, x, \tau)$ by $\tilde{W}_j(t-\tau, x, \tau)$, and for simplicity we write again $\tilde{W}_j(t, x, \tau)$ by $W_j(t, x, \tau)$.

Consequently, summarizing the calculations made above, we obtain

$$\begin{aligned}&H(t, x, D_t, D_x)\Big\{f_j(\varphi^+)V_j + \int_0^t f_j(\Phi)W_j(t-\tau, x, \tau)\,d\tau\Big\} \\ &= f_{j-m+1}(\varphi^+)L_1^+V_j + f_{j-m+2}(\varphi^+)\mathcal{O}(2, m; V_j) \\ &+ f_{j-m+2}(\varphi^+)\{(\tilde{N}_1^+W_j + \mathcal{O}(2, m-1; W_{j-1}))(t-\tau, x, \tau)\}_{\tau=t} \\ &+ f_{j-m+2}(\varphi^-)\{(\tilde{N}_1^-W_j + \mathcal{O}(2, m-1; W_{j-1}))(t-\tau, x, \tau)\}_{\tau=0} \\ &+ \int_0^t f_{j-m+2}(\Phi)\{\tilde{K}_2W_j + \mathcal{O}(3, m; W_{j-1})\}\,(t-\tau, x, \tau)\,d\tau .\end{aligned}$$

Here we have set

$$\begin{aligned}L_1^+ &= q^+(t, x)(\varphi_t^+ - \lambda^-(t, x, \varphi_x^+))P_1^+ , \\ P_1^+ &= D_t - \sum_{i=1}^{n}\frac{\partial\lambda^+}{\partial\xi_i}(t, x, \varphi_x^+)D_i + c^+(t, x) , \\ \tilde{N}_1^+ &= q(t, x, \Phi_t + \Phi_\tau, \Phi_x)\tilde{Q}_1^+ , \qquad \tilde{N}_1^- = q(t, x, \Phi_1, \Phi_x)\tilde{Q}_1^- , \\ \tilde{Q}_1^+ &= D_t - \sum_{i=1}^{n}\frac{\partial\lambda^+}{\partial\xi_i}(t, x, \Phi_x)\,D_i + c^+(t, x, \tau) , \\ \tilde{Q}_1^- &= D_t - \sum_{i=1}^{n}\frac{\partial\lambda^-}{\partial\xi_i}(t, x, \Phi_x)D_i + c^-(t, x, \tau) ,\end{aligned}$$

$\tilde{K}_2 = \tilde{R}_0 D_t D_\tau + \tilde{R}_2$, $\tilde{R}_0$ and $\tilde{R}_2$ are the differential operators of order 0 and 2 respectively, and $\tilde{R}_2$ does not contain the second order derivatives D_t^2, D_τ^2, $D_t D_\tau$.

3. – Construction of formal solution of the Cauchy problem.

Making use of the results obtained in the preceding Section, we shall construct a formal solution of the Cauchy problem:

$$\left\{\begin{array}{l} a(t, x, D_t, D_x)u(t, x) = 0 \,, \\ D_t^l u(0, x) = w_l(x) \qquad 0 \leqslant l \leqslant m-1 \,, \end{array}\right. \tag{3.1}$$

where $w_l(x)$ $(0 \leqslant l \leqslant m-1)$ have poles along T.

By the principle of superposition, it is enough to consider the simple case $w_l(x) = f_{-l}(x_1) w_l(x)$ $(0 \leqslant l \leqslant m-1)$, where $f_0(s) = 1/s^p$ (p: integer > 0).

We split $a(t, x, D_t, D_x)$ into two parts:

$$a(t, x, D_t, D_x) = H(t, x, D_t, D_x) - b(t, x, D_t, D_x) \,,$$

where $b(t, x, D_t, D_x)$ is a differential operator of order $m-1$.

Let $u^{(k)}$ $(k \geqslant 0)$ be the successive solutions of the Cauchy problems:

$$\left\{\begin{array}{l} H(t, x, D_t, D_x)u^{(0)} = 0 \,, \\ D_t^l u^{(0)}(0, x) = f_{-l}(x_1) w_l(x) \qquad 0 \leqslant l \leqslant m-1 \,, \end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{l} H(t, x, D_t, D_x)u^{(k)} = b(t, x, D_t, D_x)u^{(k-1)} \\ u^{(k)}(t, x) = O(t^m) \,. \end{array}\right. \tag{3.3}$$

Then $u = \sum_{k=0}^{\infty} u^{(k)}$ is obviously a formal solution of (3.1).

We shall seek each $u^{(k)}$ in the form

$$u^{(k)} = \sum_{j=-\infty}^{\infty} \left\{ \sum_{\lambda=1}^{m-2} f_j(\varphi_\lambda)\, U_{j,\lambda}^{(k)} + f_j(\varphi^+) V_j^{(k)} + \int_0^t f_j(\Phi)\, W_j^{(k)}(t-\tau, x, \tau)\, d\tau \right\} .$$

First, we want to find the conditions that $u^{(k)}$ satisfy the differential equations of (3.2) and (3.3).

To this end, taking account of the results of the preceding section, we see by a simple calculation that it is sufficient that $U_{j,\lambda}^{(k)}(t, x)$, $V_j^{(k)}(t, x)$ and

$W_j^{(k)}(t, x, \tau)$ satisfy the relations:

$$
(3.4)\quad \begin{cases}
L_1 U_{j,\lambda}^{(k)} = \mathcal{O}(2, m;\, U_{j-1,\lambda}^{(k)}) + \mathcal{O}(0, m-1;\, U_{j,\lambda}^{(k-1)}), \\
P_1^+ V_j^{(k)} = 0, \\
(\tilde{Q}_1^+ W_j^{(k)})(0, x, \tau)|_{\tau=t} = \big(\mathcal{O}(2, m-1;\, W_{j-1}^{(k)}) \\
\quad + \mathcal{O}(0, m-2;\, W_j^{(k-1)})\big)(0, x, \tau)|_{\tau=t} + \mathcal{O}(2\ m;\, V_j^{(k)}) \\
\quad + \mathcal{O}(0, m-1;\, V_{j+1}^{(k-1)}), \\
(\tilde{Q}_1^- W_j^{(k)})(t, x, 0) = \mathcal{O}(2, m-1;\, W_{j-1}^{(k)})(t, x, 0), \\
D_t D_\tau W_j^{(k)} + R_2 W_j^{(k)} = \mathcal{O}(3, m;\, W_{j-1}^{(k)}) + \mathcal{O}(0, m-1;\, W_{j+1}^{(k-1)}).
\end{cases}
$$

Additionally, we put here these equations (3.4) in the following forms which are convenient to discuss the convergence of the formal solution:

$$
(3.5)\quad \begin{cases}
D_t U_{j,\lambda}^{(k)} = L_1' U_{j,\lambda}^{(k)} + \mathcal{O}(2, m;\, U_{j-1,\lambda}^{(k)}) + \mathcal{O}(0, m;\, U_{j,\lambda}^{(k-1)}), \\
D_t V_j^{(k)} = L_1' V_j^{(k)}, \\
D_\tau W_j^{(k)}(0, x, \tau)|_{\tau=t} = \{(N_1' W_j^{(k)} + \mathcal{O}(2, m-1;\, W_{j-1}^{(k)}) \\
\quad + \mathcal{O}(0, m-2;\, W_j^{(k-1)})((0, x, \tau)\}_{\tau=t} + \mathcal{O}(2, m;\, V_j^{(k)}) \\
\quad + \mathcal{O}(0, m-1;\, V_{j+1}^{(k-1)}), \\
D_t W_j^{(k)}(t, x, 0) = (N_1'' W_j^{(k)} + \mathcal{O}(2, m-1;\, W_{j-1}^{(k)})((t, x, 0), \\
D_t D_\tau W_j^{(k)} = R_2' W_j^{(k)} + \mathcal{O}(3, m;\, W_{j-1}^{(k)}) + \mathcal{O}(0, m-1;\, W_{j+1}^{(k-1)}).
\end{cases}
$$

Here L_1', N_1' and N_1'' are differential operators of order 1 and do not contain the term of D_t, D_τ and D_t respectively.

Furthermore R_2' is differential operator of order 2 which does not contain D_t^2, D_τ^2, $D_t D_\tau$.

Finally, we shall observe the initial conditions of (2.3) and (2.4).

We have by Leibniz' formula 1

$$
D_t^l u^{(k)}|_{t=0} = f_{j-l}(x_1)\Big[\sum_{\lambda=1}^{m-2} (D_t\varphi_\lambda)^l U_{j,\lambda}^{(k)} + (D_t\varphi^+)^l V_j^{(k)}
$$

$$
+ \sum_{\lambda=1}^{m-2} \mathcal{O}(1, l;\, U_{j-1,\lambda}^{(k)}) + \mathcal{O}(1, l;\, V_{j-1}^{(k)}) + \{\alpha_l^+ W_{j-1}^{(k)} + \mathcal{O}(1, l-1;\, W_{j-2}^{(k)})\}_{\tau=t}\Big]_{t=0},
$$

where

$$\alpha_l^+ = \Phi_\tau^{-1}((\Phi_t + \Phi_\tau)^l - \Phi_t^l), \quad \text{so } \alpha_l^+\Big|_{\substack{t=\tau=0\\x=0}} = l(D_t\varphi^+)^{l-1}\Big|_{\substack{t=\tau=0\\x=0}}.$$

Observe that the determinant

$$\begin{vmatrix} 1 & \cdots & 1 & 1 & 0 \\ D_t\varphi_1 & \cdots & D_t\varphi_{m-2} & D_t\varphi^+ & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ (D_t\varphi_1)^{m-1} & \cdots & (D_t\varphi_{m-2})^{m-1} & (D_t\varphi^+)^{m-1} & (m-1)(D_t\varphi^+)^{m-2} \end{vmatrix}$$

does not vanish at $t = x = 0$.

Then u satisfies the initial condition if the following relations hold.

$$(3.6) \qquad U_{0,\lambda}^{(0)}(0, x), \quad V_0^{(0)}(0, x), \quad W_{-1}^{(0)}(0, x, 0) = \sum_{l=0}^{m-1} \Delta_l(x) w_l(x)$$

$$(3.7) \qquad U_{j,\lambda}^{(k)}(0, x), \quad V_j^{(k)}(0, x), \quad W_{j-1}^{(k)}(0, x, 0)$$

$$= \sum_{\lambda=1}^{m-2} \mathcal{O}(1, m-1; U_{j-1,\lambda}^{(k)}) + \mathcal{O}(1, m-1; V_{j-1}^{(k)}) + \mathcal{O}(1, m-2; W_{j-2}^{(k)})|_{t=\tau=0},$$

where Δ_l are holomorphic in a neighborhood of the origin.

Thus we can determine successively $U_{j,\lambda}^{(k)}(t, x)$, $V_j^{(k)}(t, x)$ and $W_j^{(k)}(t, x, \tau)$ from (3.4), (3.6) and (3.7).

Indeed, we first determine $U_{j,\lambda}^{(k)}$, $V_j^{(k)}$ from (3.4), (3.6) and (3.7). Secondly, we calculate $W_j^{(k)}(0, x, t)$ and $W_j^{(k)}(t, x, 0)$ by (3.4), (3.6) and (3.7). Then $W_j^{(k)}(t, x, \tau)$ can be determined by solving Goursat problem for the last equation of (3.4).

We can see that

$$U_{j,\lambda}^{(k)}, \quad V_j^{(k)} = 0 \quad \text{for } j < 0,$$

$$W_j^{(k)} = 0 \quad \text{for } j < -k-1,$$

$$W_j^{(k)} = O((t\tau)^{-j-1}) \quad \text{for } -k-1 \leqslant j \leqslant -1.$$

Thus we have obtained a formal solution of the Cauchy problem (3.1):

$$(3.8) \qquad u(t, x) = \sum_{\lambda=1}^{m-2}\sum_{j=0}^{\infty} f_j(\varphi_\lambda) F_{j,\lambda}(t, x) + \sum_{j=0}^{\infty} f_j(\varphi^+) G_j(t, x)$$
$$+ \sum_{j=-\infty}^{\infty} \int_0^t f_j(\Phi) S_j(t-\tau, x, \tau)\, d\tau,$$

where

$$F_{j,\lambda}(t, x) = \sum_{k=0}^{\infty} U_{j,\lambda}^{(k)}(t, x),$$

$$G_j(t, x) = \sum_{k=0}^{\infty} V_j^{(k)}(t, x),$$

$$S_j(t-\tau, x, \tau) = \sum_{k=l}^{\infty} W_j^{(k)}(t-\tau, x, \tau)$$

with $l = \mathrm{Max}(-j-1, 0)$.

4. – Convergence of formal solution.

In this section, we shall prove the convergence of the formal solution obtained in the above section. In order to do so, we shall employ the method of majorant ameliorated in a convenient form for this type of problem by the introduction of the majorant functions: C. Wagschal [8], De Paris [2], Y. Hamada, J. Leray and C. Wagschal [5].

Let f, g be holomorphic functions in a neighborhood of the origin. We say that g is a majorant of f, symbolized by $f \ll g$, if $|D^\alpha f(0)| \leqslant D^\alpha g(0)$, $\forall \alpha \in \boldsymbol{N}^{n+1}$.

We recall the functions $\theta^{(k)}(r, z)$ (k: integer) which play a fundamental role to prove the convergence of the formal solution. That is:

$$\begin{cases} \theta^{(k)}(r, z) = \sum_{s=0}^{\infty} \dfrac{s!}{(k+s)!} \dfrac{z^{k+s}}{r^{k+s}} \\ \theta^{(-k)}(r, z) = \dfrac{k!}{(r-z)^{k+1}} \end{cases} \qquad \text{for } k \geqslant 0,$$

where $r > 0$ is a constant.

We know that $\theta^{(k)}$ satisfies the following properties:

1) *For every* k, $d\theta^{(k)}/dz = \theta^{(k-1)}$.

2) *For every* k, $\theta^{(k)}(r, z) \ll r\theta^{(k-1)}(r, z)$.

3) *Let* $u(t, x) \ll \theta^{(k)}(r, z)$ *hold for some* k *and* $\varrho > 0$ $\left(z = \varrho t + \sum_{i=1}^{n} x_i\right)$, *then we have for* $\forall \beta \in \boldsymbol{N}^{n+1}$ $(\beta = (\beta_0, \beta'), \beta' \in \boldsymbol{N}^n)$

$$D_t^{\beta_0} D_x^{\beta'} u(t, x) \ll \varrho^{\beta_0} \theta^{(k-|\beta|)}(r, z).$$

4) *For r, R constants* $(0 < r < R)$ *there exists a constant* $A > 0$ *such that*

$$\frac{R}{R-z}\theta^{(k)}(r, z) \ll A^k\theta^{(k)}(r, z) \qquad k > 0,$$

$$\frac{1}{R-z}\theta^{(-k)}(r, z) \ll \frac{1}{R-r}\theta^{(-k)}(r, z) \qquad k \geqslant 0.$$

J. Leray gave an elegant proof of the first part of 4) with $A = \exp[R/(R-r)]$.

In the terms of these functions $\theta^{(k)}$, we shall put for $0 < r < R$ and $k, j \geqslant 0$

$$\Theta_j^k(r, R, z) = \left(\frac{d}{dz}\right)^j \left[\frac{R}{R-z}\theta^{(k)}(r, z)\right].$$

Then the functions Θ_j^k have the following properties:

1) $(d/dz)\Theta_j^k = \Theta_{j+1}^k$.

2) $\Theta_j^k \ll r\Theta_{j+1}^k$.

3) $\Theta_j^k \ll \Theta_{j+1}^{k+1}$.

4) *Let* $u(t, x) \ll \Theta_j^k(r, R, z)$ *with* $z = \varrho t + \sum_{i=1}^n x_i$ $(\varrho > 0)$, *then we have*

$$D_t^{\beta_0} D_x^{\beta'} u \ll \varrho^{\beta_0}\Theta_{j+|\beta|}^k \quad \text{for } \forall\beta = (\beta_0, \beta') \in N^{n+1}.$$

5) *For* $0 < r < R < R'$, *we have*

$$\frac{1}{R'-z}\Theta_j^k(r, R, z) \ll \frac{1}{R'-R}\Theta_j^k(r, R, z).$$

By taking account of these properties, we have two lemmas (c.f. [8], [2], [5]).

LEMMA 4.1. *Let* $Q(t, x, \tau, D_t, D_\tau, D_x)$ *be a differential operator of order m with holomorphic coefficients on* $|t|, |\tau|, |x_i| \leqslant R'$ $(0 < r < R < R')$, *and the order of Q with respect to* D_t, D_τ *is* $\leqslant m_0$.

Then there exists a constant $B > 0$ *depending only on* Q, r, R, R' *such that, if*

$$u(t, x, \tau) \ll \Theta_j^k(r, R, z) \quad \textit{for } z = \varrho(t+\tau) + \sum_{i=1}^n x_i, \ \varrho \geqslant 1,$$

then we have

$$Qu \ll B\varrho^{m_0}\Theta^k_{j+m}\,.$$

LEMMA 4.2. *Let $H(t, x, D_t, D_x)$ be a differential operator with holomorphic coefficients in a neighborhood of 0. Its order with respect to D_t is $< p$.*

We consider the Cauchy problem

$$\begin{cases} D_t^p u = Hu + f\,, \\ D_t^h u(0, x) = w_h(x) \quad 0 \leqslant h < p\,, \end{cases}$$

where f, $w_h(x)$ $(0 \leqslant h < p)$ are holomorphic in 0.

Let $\mathfrak{H}(t, x, D_t, D_x)$ be the differential operator obtained by replacing the coefficients of H by their majorant functions — we call it a majorant differential operator of H —. Let $f \ll F$ and $w_h \ll W_h$ $(0 \leqslant h \leqslant p-1)$.

If a majorant function U verifies

$$\begin{cases} D_t^p U \gg \mathfrak{H} U + F \\ D_t^h U(0, x) \gg W_h(x) \qquad \text{for } 0 \leqslant h < p\,, \end{cases}$$

then it follows that

$$u \ll U\,.$$

We can obtain an analogous lemma for Goursat problem. But we do not formulate it in particular.

By using Lemma 4.1, we have the following

LEMMA 4.3. *Let $\{U_j\}$ be a series of holomorphic functions and let C, ϱ, K be constants independent of j, l, s $(C \geqslant \varrho \geqslant 1,\ K > 0)$.*

If $\{U_j\}$ satisfies

$$U_j(t, x) \ll KC^j\Theta^s_{j+l}(r, R, z)\,, \qquad z = \varrho t + \sum_{i=1}^n x_i\,,$$

then we can find a positive constant B independent of j, l, s K, C such that

$$\mathcal{O}(p, q;\, U_j) \ll BK\varrho^p C^j \Theta^s_{j+l+p} \qquad \textit{for } p,\ q \leqslant m\,,$$

where m is a positive integer.

By making use of these lemmas, we can obtain an estimate of $U^{(k)}_{j,\lambda}$, $V^{(k)}_j$ and $W^{(k)}_j$.

PROPOSITION 4.4. *There exist constants* C, ϱ $(C \geqslant \varrho \geqslant 1)$ *independent of* k, j *such that*

$$U^{(k)}_{j,\lambda}(t, x), \quad V^{(k)}_j(t, x), \quad W^{(k)}_{j-1}(t, x, \tau) \ll \|w\| C^{j+k+1} \Theta^{2k}_{j+k}(r, R, z), \qquad z = \varrho(t+\tau) + \sum_{i=1}^{n} x_i,$$

where $\|w\| = \max_{\substack{|x| \leqslant r \\ 0 \leqslant h \leqslant m-1}} |w_h(x)|$.

PROOF. We set $\Gamma^k_j = \|w\| C^{j+k+1} \Theta^{2k}_{j+k}(r, R, z)$ and denote by $\mathfrak{L}'_1$, $\mathfrak{N}'_1$, $\mathfrak{N}''_1$, $\mathfrak{R}'_2$, $\tilde{\mathfrak{O}}(p, q;\ )$ the majorant differential operators of L'_1, N'_1, N''_1, R_2, $\mathcal{O}(p, q;\ )$ in (3.5) and (3.7).

By virtue of the successive application of Lemma 4.2, we see that it is sufficient to prove that the following majorant equations of (3.5), (3.6) and (3.7) hold;

$$(\widetilde{3.5}) \quad \begin{cases} D_t \Gamma^k_j \quad \gg \mathfrak{L}'_1 \Gamma^k_j + \tilde{\mathfrak{O}}(2, m; \Gamma^k_{j-1}) + \tilde{\mathfrak{O}}(0, m; \Gamma^{k-1}_j), \\ D_t \Gamma^k_j \quad \gg \mathfrak{L}'_1 \Gamma^k_j, \\ D_\tau \Gamma^k_{j+1} \quad \gg \mathfrak{N}'_1 \Gamma^k_{j+1} + \tilde{\mathfrak{O}}(2, m-1; \Gamma^k_j) + \tilde{\mathfrak{O}}(0, m-2; \Gamma^{k-1}_{j+1}) \\ \quad + \hat{\mathfrak{O}}(2, m; \Gamma^k_j) + \tilde{\mathfrak{O}}(0, m-1; \Gamma^{k-1}_{j+1}), \\ D_t \Gamma^k_{j+1} \quad \gg \mathfrak{N}''_1 \Gamma^k_{j+1} + \tilde{\mathfrak{O}}(2, m-1; \Gamma^k_j), \\ D_t D_\tau \Gamma^k_{j+1} \gg \mathfrak{R}'_2 \Gamma^k_{j+1} + \tilde{\mathfrak{O}}(3, m; \Gamma^k_j) + \tilde{\mathfrak{O}}(0, m-1; \Gamma^{k-1}_{j+2}), \end{cases}$$

$$(\widetilde{3.6}) \qquad \Gamma^0_0 \gg \sum_{l=0}^{m-1} \Delta_l(x) w_l(x),$$

$$(\widetilde{3.7}) \qquad \Gamma^k_j \gg \tilde{\mathfrak{O}}(1, m-1; \Gamma^{(k)}_{j-1}).$$

We shall show, for example, that the 3-rd majorant equation of $(\widetilde{3.5})$ holds, because the validity of the other equations is shown by the same procedure.

In view of Lemma 4.1, 4.3 and the properties of Θ^k_j, the right hand side of this equation is majorized by

$$\|w\| B C^{j+k+2} \left(1 + \frac{\varrho^2}{C} + \frac{1}{C}\right) \Theta^{2k}_{j+k+2}$$

with a suitable constant $B > 0$.

Therefore, if we choose C, ϱ $(C \geqslant \varrho \geqslant 1)$ so that

$$\varrho > \left(1 + \frac{\varrho^2}{C} + \frac{1}{C}\right) B,$$

the 3-rd majorant equation of $(\widetilde{3.5})$ is verified. This completes the proof of Proposition 4.4.

In view of this proposition and the properties of $\theta^{(k)}$, we have

$$U_{j,\lambda}^{(k)}, \quad V_j^{(k)}, \quad W_{j-1}^{(k)} \ll \|w\| C^j B^{k+1} \theta^{(k-j)} \tag{4.1}$$

with suitable constant $B > 0$.

Now let us prove the convergence of the formal solution (3.8).

We shall estimate $S_j(t, x, \tau) = \sum_{k=l}^{\infty} W_j^{(k)}(t, x, \tau)$, $l = \mathrm{Max}(-j-1, 0)$.

In order to do so, we consider two cases.

The first case (I): $j \geqslant 0$.

We see by (4.1)

$$S_j(t, x, \tau) = \sum_{k=0}^{\infty} W_j^{(k)} \ll \|w\| B C^j \sum_{k=0}^{\infty} B^k \theta^{(k-j-1)}$$

with a constant $B > 0$.

Now we have

$$\theta^{(k-j)} \ll r^k \frac{(j-k)!}{j!} \theta^{(-j)} \qquad \text{for } 1 \leqslant k \leqslant j. \tag{4.2}$$

Then, taking account of this fact,

$$\sum_{k=0}^{j} B^k \theta^{(k-j-1)} \ll \sum_{k=0}^{j} \frac{(j+1-k)!}{(j+1)!} (Br)^k \theta^{(-j-1)} \ll \sum_{k=0}^{j} \frac{(Br)^k}{k!} \theta^{(-j-1)} \ll \exp[Br] \theta^{(-j-1)}.$$

We also have

$$\theta^{(k)} \ll \frac{r^k}{k!} \theta^{(0)} \text{ for } k \geqslant 0. \tag{4.3}$$

By using this, we obtain

$$\begin{aligned} \sum_{k=j+1}^{\infty} B^k \theta^{(k-j-1)} &= B^{j+1} \sum_{k=0}^{\infty} B^k \theta^{(k)} \ll B^{j+1} \sum_{k=0}^{\infty} \frac{(Br)^k}{k!} \theta^{(0)} \ll B^{j+1} \exp[Br] \theta^{(0)} \\ &\ll \frac{(Br)^{j+1}}{(j+1)!} \exp[Br] \theta^{(-j-1)} \ll \exp[2Br] \theta^{(-j-1)}. \end{aligned}$$

Thus we have obtained an estimate of $S_j(t, x, \tau)$ for $j \geqslant 0$:

$$S_j(t, x, \tau) \ll \|w\| B^{j+1} (j+1)! \frac{1}{(r-z)^{j+2}} \qquad \text{for } j \geqslant 0$$

with a suitable constant $B \not> 0$.

The second case (II): $j \leqslant -1$.

Write $j = -s$ $(s \geqslant 1)$, then we have

$$S_j(t, x, \tau) = \sum_{k=s-1}^{\infty} W_j^{(k)} \ll \|w\| C^{-s} \sum_{k=s-1}^{\infty} B^{k+1} \theta^{(k+s-1)}$$

with a constant $B > 0$.

By using (4.3) once more, we have

$$\sum_{k=s-1}^{\infty} B^{k+1} \theta^{(k+s-1)} = B^s \sum_{k=0}^{\infty} B^k \theta^{(2s-2+k)} \ll B^s \sum_{k=0}^{\infty} B^k \frac{r^{2s-2+k}}{(2s-2+k)!} \theta^{(0)}$$

$$\ll \frac{B^s r^{2s-2}}{(2s-2)!} \sum_{k=0}^{\infty} \frac{(Br)^k}{k!} \theta^{(0)} \ll \frac{B^s r^{2s-2}}{(2s-2)!} \exp[Br] \theta^{(0)} .$$

Thus, with a suitable constant B independent of s, we have obtained an estimate of $S_j(t, x, \tau)$ for $j = -s \leqslant -1$:

$$S_{-s}(t, x, \tau) \ll \|w\| \frac{B^{s+1} C^{-s}}{(2s-2)!} \frac{1}{r-z} \qquad \text{for } s \geqslant 1 .$$

Therefore, taking account of the results of the two cases (I) and (II), we see that $\sum_{j=-\infty}^{\infty} f_j(\Phi) S_j(t-\tau, x, \tau)$ converges uniformly in a neighborhood of 0 except on $\Phi(t, x, \tau) = 0$.

The proof of the convergence of $\sum_{j=0}^{\infty} f_j(\varphi_\lambda) F_{j,\lambda}(t, x)$ and $\sum_{j=0}^{\infty} f_j(\varphi^+) G_j(t, x)$ can be performed by the same procedure.

Thus, we have proved the exactness of the formal solution (3.8).

This completes the proof of our theorem.

5. – Geometrical properties of characteristic surface K_0.

In this section, we shall state the geometrical properties of the characteristic surface K_0 and give some remarks on the assumption (B). This geometry is closely related to the studies of J. Leray [6] and L. Gårding, T. Kotake and J. Leray [3].

Let $\mathfrak{V}$ be a neighborhood of the point $(0, 0; 1, 0, \ldots, 0)$ in (t, x, ξ)-space: $C^{n+1} \times (C^n - 0)$.

We set

$$\Pi^{\pm} = \left\{ \begin{array}{l} (t, x, \xi_0, \xi) \in C^{n+1} \times (C^{n+1} - 0); \quad (t, x, \xi) \in \mathfrak{V} \\ \xi_0 - \lambda^{\pm}(t, x, \xi) = 0 \end{array} \right\}$$

and set

$$\Pi^{\pm}_{(t,x)} = \{(\xi_0, \xi) \in C^{n+1} - 0,\ (t, x, \xi_0, \xi) \in \Pi^{\pm}\}.$$

The condition (1.1) of (A) (that is, the Poisson bracket $\{\xi_0 - \lambda^+(t, x, \xi), \xi_0 - \lambda^-(t, x, \xi)\}$ vanishes in a neighborhood of $(0, 0; 1, 0, \dots, 0)$) implies that $\Pi^+ \cap \Pi^-$ is involutive in (t, x, ξ_0, ξ)-space. However we do not assume that $d(\xi_0 - \lambda^+(t, x, \xi))$ and $d(\xi_0 - \lambda^-(t, x, \xi))$ are linearly independent, where d is the differential with respect to t, x, ξ_0, ξ and also ξ_0, ξ. Hence $\Pi^+ \cap \Pi^-$ is not always regular in (t, x, ξ_0, ξ)-space and also $\Pi^+_{(t,x)} \cap \Pi^-_{(t,x)}$ is not necessarily of codimension 2 in (ξ_0, ξ)-space.

The bicharacteristic strip corresponding to $(\xi_0 - \lambda^{\pm})$ (we call it the $(\xi_0 - \lambda^{\pm})$-bicharacteristic strip) is defined by the solution of Hamilton system

$$(5.1)^{\pm} \qquad \begin{cases} \dfrac{dt}{d\sigma} = 1 & \dfrac{d\xi_0}{d\sigma} = \lambda^{\pm}_t \\ \dfrac{dx_i}{d\sigma} = -\dfrac{\partial \lambda^{\pm}}{\partial \xi_i} & \dfrac{d\xi_i}{d\sigma} = \dfrac{\partial \lambda^{\pm}}{\partial x_i} \end{cases} \qquad 1 \leqslant i \leqslant n.$$

Then the curve $(t(\sigma), x(\sigma))$ (the projection of the bicharacteristic strip on (t, x)-space) is the $(\xi_0 - \lambda^{\pm})$-bicharacteristic curve.

The next proposition can be easily seen from the condition (1.1).

PROPOSITION 5.1. *$\xi_0 - \lambda^+(t, x, \xi)$ (resp. $\xi_0 - \lambda^-(t, x, \xi)$) is constant along the $(\xi_0 - \lambda^-)$ (resp. $(\xi_0 - \lambda^+)$)-bicharacteristic strip.*

Then as a direct consequence, we have the following Proposition.

PROPOSITION 5.2. *The $(\xi_0 - \lambda^{\pm})$-bicharacteristic strip issuing from a point of $\Pi^+ \cap \Pi^-$ remains in $\Pi^+ \cap \Pi^-$.*

Now we set

$$\Omega = \left\{ \begin{array}{l} (t, y) \in T;\ t = 0,\ y = (0, y'),\ y' = (y_2, \dots, y_n) \\ \lambda^+(0, 0, y';\ 1, 0, \dots, 0) = \lambda^-(0, 0, y'; 1, 0, \dots, 0) \end{array} \right\},$$

and let $\tilde{\Omega}$ be the set of the characteristic elements for $\xi_0 - \lambda^{\pm}$ in Ω, that is;

$$\tilde{\Omega} = \left\{ \begin{array}{l} (t, y;\ \xi_0, \xi);\ (t, y) \in \Omega,\ \xi_0 = \lambda^+(0, 0, y';\ 1, 0, \dots, 0) \\ \xi = (1, 0, \dots, 0) \end{array} \right\}.$$

Obviously

$$\tilde{\Omega} \subset \Pi^+ \cap \Pi^-.$$

Next, we set

$$\Lambda^{\pm} = \{(t, x) \in K^{\pm};\ \lambda^{+}(t, x, \varphi_x^{\pm}(t, x)) = \lambda^{-}(t, x, \varphi_x^{\pm}(t, x))\} .$$

$\Lambda^{\pm}$ is the subvariety of $K^{\pm}$ and $K^{\pm}$ is characteristic for both $(\xi_0 - \lambda^{+})$ and $(\xi_0 - \lambda^{-})$ on $\Lambda^{\pm}$.

We also set

$$\tilde{\Lambda}^{\pm} = \{(t, x, \xi_0, \xi);\ (t, x) \in \Lambda^{\pm},\ \xi_0 = \varphi_t^{\pm}(t, x),\ \xi = \varphi_x^{\pm}(t, x)\} .$$

It is evident that $\tilde{\Lambda}^{\pm} \subset \Pi^{+} \cap \Pi^{-}$ and if $(t, x, \xi_0, \xi) \in \tilde{\Lambda}^{\pm}$, (t, x, ξ_0, ξ) is the contact element of $K^{\pm}$.

The following Proposition follows from Proposition 5.1 and the definition of $\Lambda^{\pm}$, $\tilde{\Lambda}^{\pm}$.

PROPOSITION 5.3. *$\tilde{\Lambda}^{\pm}$ is generated by the $(\xi_0 - \lambda^{\pm})$-bicharacteristic strips issuing from $\tilde{\Omega}$. Hence $\Lambda^{\pm}$ is generated by the $(\xi_0 - \lambda^{\pm})$-bicharacteristic curves issuing from $\tilde{\Omega}$. Moreover we have*

$$\Lambda^{\pm} \cap S = \Omega \subset T .$$

In general, $\Omega \subset T$. When $T = \Omega$, we have

PROPOSITION 5.4. *The following three assertions are equivalent.*

1) $\Omega = T$

2) $K^{+} = \Lambda^{+}$ *(or* $K^{-} = \Lambda^{-}$*)*

3) $K^{+} = K^{-}$.

PROOF. In fact, it follows from Proposition 5.3 that 1) is equivalent to 2). Next, 2) means that K^{+} is characteristic for $(\xi_0 - \lambda^{-})$. Since the characteristic surface for $(\xi_0 - \lambda^{-})$ passing T is K^{-}, we have $K^{+} = K^{-}$. Conversely 3) means evidently 2).

PROPOSITION 5.5. *K^{+} and K^{-} intersect not only on T, but also outside T. Thus, $K^{+} \cap K^{-} \supsetneq T$.*

PROOF. We can write $\varphi^{+}(t, x) = x_1 + ta(t, x)$ and $\varphi^{-}(t, x) = x_1 + tb(t, x)$, where a, b are holomorphic in a neighborhood of the origin. Obviously,

$$a(0, x) = \varphi_t^{+}(0, x) = \lambda^{+}(0, x, 1, 0, \dots, 0) ,$$

$$b(0, x) = \varphi_t^{-}(0, x) = \lambda^{-}(0, x, 1, 0, \dots, 0) .$$

Hence, if we set

$$a(t, x) - b(t, x) = C_1(x') + x_1 C_2(x) + tC_3(t, x) \qquad (x' = (x_2, \dots, x_n)) ,$$

then we have

$$C_1(x') + x_1 C_2(x) = \lambda^+(0, x, 1, 0, \dots, 0) - \lambda^-(0, x, 1, 0, \dots, 0)$$

and $C_1(0) = 0$.

In the case of $C_1(x') \equiv 0$, we have $K^+ = K^-$ by Proposition 5.4, which means that our Proposition is valid.

In the case of $C_1(x') \not\equiv 0$, we have evidently

$$\{(t, x);\ C_1(x') + x_1 C_2(x) + tC_3(t, x) = 0,\ x_1 + ta(t, x) = 0\} \neq \{(t, x);\ t = x_1 = 0\} .$$

This completes our Proposition.

Now let us observe $\Phi(t, x, \tau)$.

PROPOSITION 5.6. *Let $\Phi(t, x, \tau)$ be the solution of* (2.1), *then it satisfies* (2.2).

REMARK. Φ_τ is constant along the $(\xi_0 - \lambda^-)$-bicharacteristic curves associated with Φ, as will be seen in the proof of this Proposition.

PROOF. Differentiate $\Phi(\tau, x, \tau) = \varphi^+(\tau, x)$ by τ, then we have

$$\Phi_t(\tau, x, \tau) + \Phi_\tau(\tau, x, \tau) = \varphi_t^+(\tau, x) = \lambda^+(\tau, x, \Phi_x(\tau, x, \tau)) .$$

Now let $t = s + \tau$, $x = x(s, y, \tau)$, $\xi_i = \xi_i(s, y, \tau)$ $(0 \leqslant i \leqslant n)$ be the $(\xi_0 - \lambda^-)$-bicharacteristic strip with the initial data $t(0) = \tau$, $x_i(0) = y_i$ $(1 \leqslant i \leqslant n)$, $\xi_0(0) = \lambda^-(\tau, y, \varphi_x^+(\tau, y))$, $\xi_i(0) = \varphi_{x_i}^+(\tau, y)$ $(1 \leqslant i \leqslant n)$.

Then we have by the definition of Φ,

$$\Phi_t(s + \tau, x(s, y, \tau), \tau) = \xi_0(s, y, \tau)$$
$$\Phi_{x_i}(s + \tau, x(s, y, \tau), \tau) = \xi_i(s, y, \tau) \qquad (1 \leqslant i \leqslant n) .$$

Now by Proposition 5.1, $\lambda^+(t, x, \Phi_x) - \lambda^-(t, x, \Phi_x)$ is constant along this $(\xi_0 - \lambda^-)$-bicharacteristic curve.

On the other hand, $\Phi_\tau(t, x, \tau)$ is also constant along this curve.

Indeed, we have

$$\frac{d}{ds} \Phi_\tau(t, x, \tau) = \Phi_{\tau t} + \sum_{i=1}^n \Phi_{\tau x_i} \frac{dx_i}{ds} = \Phi_{\tau t} - \sum_{i=1}^n \Phi_{\tau x_i} \frac{\partial \lambda^-}{\partial \xi_i}(t, x, \Phi_x) .$$

By differentiating (2.1) by τ, we have

$$\Phi_{\tau t}=\sum_{i=1}^{n}\frac{\partial\lambda^{-}}{\partial\xi_{i}}(t, x, \Phi_{x})\Phi_{\tau x_{i}},$$

so this shows that Φ_{τ} is constant along this curve.

Since (2.2) is valid for $t=\tau$, the above fact means that (2.2) holds for any t. Thus Proposition 5.4 has been proved.

PROPOSITION 5.7. *Let a point (t, x) satisfy $\Phi(t, x, \tau)=\Phi_{\tau}(t, x, \tau)=0$ with some τ, then (t, x) lies on the $(\xi_{0}-\lambda^{-})$-(resp. $(\xi_{0}-\lambda^{+})$)-bicharacteristic curve issuing from the point of $\tilde{\Lambda}^{+}$ (resp. $\tilde{\Lambda}^{-}$). The converse is also valid.*

PROOF. This results from the remark of Proposition 5.6 and the fact that $\Phi(t, x, \tau)$ is constant along the $(\xi_{0}-\lambda^{-})$-bicharacteristic curves associated with Φ.

PROPOSITION 5.8. *If $T=\Omega$, then $\Phi(t, x, \tau)=a(t, x, \tau)\ \varphi^{+}(t, x)$ with a holomorphic function $a(t, x, \tau)$ $(a(0,0,0)=1)$. Therefore, the assumption* (B) *does not hold.*

PROOF. In this case, by Proposition 5.4 we have $K^{+}=K^{-}$. Let (t, x) be an arbitrary point of K^{+}. Then the $(\xi_{0}-\lambda^{-})$-bicharacteristic curve issuing from $(t, x, \varphi_{t}^{-}(t, x), \varphi_{x}^{-}(t, x))$ is contained in K^{+}. Denote by $(\tau, y(t, x, \tau))$ for any τ the point on this curve, so $(\tau, y(t, x, \tau))\in K^{+}$. At this point, we have $\left(\varphi_{t}^{+}(\tau, y(t, x, \tau)), \varphi_{x}^{+}(\tau, y(t, x, \tau))\right)=\alpha(t, x, \tau)\left(\varphi_{t}^{-}(\tau, y(t, x, \tau)), \varphi_{x}^{-}(\tau, y(t, x, \tau))\right)$ with a holomorphic function $\alpha(t, x, \tau)$ $(\alpha(0,0,0)=1)$. Therefore the $(\xi_{0}-\lambda^{-})$-bicharacteristic curve issuing from $(\tau, y(t, x, \tau), \varphi_{t}^{+}(\tau, y(t, x, \tau)), \varphi_{x}^{+}(\tau, y(t, x, \tau))$ coincides with the above $(\xi_{0}-\lambda^{-})$-bicharacteristic curve. Φ is constant on this curve, so $\Phi(t, x, \tau)=\varphi^{+}(\tau, y(t, x, \tau))=0$ for any τ. Namely, $\Phi(t, x, \tau)=0$ on K^{+}, which means that $\Phi(t, x, \tau)=a(t, x, \tau)\ \varphi^{+}(t, x)$. Obviously, $a(0, x, 0)=1$. Thus Proposition 5.8 has been proved.

Before we observe the properties of the surface K_{0}, let us recall the definition of K_{0}. According to the assumption (B), we have written

$$\Phi(t, x, \tau)=p(t, x, \tau)P(t, x, \tau),$$

where $p(t, x, \tau)$ is a non zero holomorphic function defined in a neighborhood of the origin and $P(t, x, \tau)$ is a distinguished pseudo-polynomial. From the fact that $\Phi(0, x, 0)=x_{1}$, we see that $P(t, x, \tau)$ is irreducible in τ.

$\Delta(t, x)$ is the discriminant of $P(t, x, \tau)$ and K_{0} is defined by $\{(t, x);\ \Delta(t, x)=0\}$. Hence, K_{0} is n-dimensional, and $K_{0}\neq S$.

EXAMPLE 5.1. $h = D_t^2 + 2x_2 D_t D_1 + D_t D_2$.

$\lambda^+ = -2x_2\xi_1 - \xi_2$, $\lambda^- = 0$, $\Phi = x_1 - 2x_2\tau + \tau^2$, $K^+: x_1 - 2x_2 t + t^2 = 0$,

$K^-: x_1 = 0$, $K_0: x_1 - x_2^2 = 0$.

PROPOSITION 5.9. *Assume that* (B) *holds. Then we have*

1) $K_0 = \{(t, x); \exists\tau, \Phi(t, x, \tau) = \Phi_\tau(t, x, \tau) = 0\}$.

2) $K^\pm \underset{\neq}{\supset} \Lambda^\pm$ *(resp.)*, $T \underset{\neq}{\supset} \Omega$ $(\dim \Lambda^\pm = n-1,\ \dim \Omega = n-2)$.

3) $K_0 \neq K^\pm$.

4) K_0 *is generated by the* $(\xi_0 - \lambda^-)$ *(resp.* $(\xi_0 - \lambda^+)$*)-bicharacteristic curves issuing from* $\tilde{\Lambda}^+$ *(resp.* $\tilde{\Lambda}^-$*).*

5) K_0 *is tangent to* $K^\pm$ *on* $\Lambda^\pm$. K^+, K^-, K_0 *touch one another on* Ω.

6) K_0 *is characteristic for* $(\xi_0 - \lambda^\pm)$.

In general, the surface K_0 *is not regular.*

REMARK. It is not always valid that $K_0 \cap K^\pm = \Lambda^\pm$, as will be seen in Example 5.3 below.

PROOF. 1) results immediately from the definition of K_0. 2) results from Proposition 5.8 and 5.4. 4) follows from 1) and Proposition 5.7. Next, $(\xi_0 - \lambda^-)$-bicharacteristic curve issuing from $\tilde{\Lambda}^+$ is obviously tangent to K^+, which implies 5).

Now we pass to the proof of 6). Let $\tau = \tau(t, x)$ be the solution of $\Phi(t, x, \tau) = 0$, then $\tau = \tau(t, x)$ is algebroid function. Let $f(t, x)$ be an irreducible factor of $\Delta(t, x)$, then $\{f(t, x) = 0\}$ is an irreducible branch of K_0. We consider a regular point (t, x) of $\{f = 0\}$, thus $(f_t, f_x) \neq 0$. Suppose for example $f_t \neq 0$, then we can find holomorphic function $t(x)$ such that $t(0) = 0$ and $f(t(x), x) = 0$. Since $\tau(x) = \tau(t(x), x)$ is also algebroid, it is holomorphic outside ω (ω is $(n-1)$ dimensional variety) and satisfies

$$\Phi(t(x), x, \tau(x)) = \Phi_\tau(t(x), x, \tau(x)) = 0 .$$

By differentiating these equations by x_i, we have $\Phi_t f_{x_i} = \Phi_{x_i} f_t$. This means that (f_t, f_x) is proportional to (Φ_t, Φ_x). On the other hand, we have $\Phi_t - \lambda^\pm(t, x, \Phi_x) = 0$ on $\{f = 0\}$. Hence $\{f = 0\}$ is characteristic for $(\xi_0 - \lambda^\pm)$, which proves 6).

Finally, let $K_0 = K^{\pm}$, then it follows from 6) that K^+ is characteristic for $(\xi_0 - \lambda^{\pm})$. Hence we have $K^+ = \Lambda^+$ by the definition of Λ^+. This is contrary to 2), which implies 3). Thus we have achieved the proof of our Proposition.

EXAMPLE 5.2. $h = D_t^2 - 3(x_2^2 - x_3)D_tD_1 + D_tD_2$.

$$\lambda^+ = 3(x_2^2 - x_3)\xi_1 - \xi_2\,, \qquad \lambda^- = 0\,, \qquad \Phi = x_1 - 3x_3\tau + x_2^3 + (\tau - x_2)^3\,,$$

K_0: $(x_1 - 3x_2x_3 + x_2^3)^2 = 4x_3^3$. K_0 is not regular on $\{(t, x); x_3 = 0, x_1 + x_2^3 = 0\}$. We shall again observe the singularities of K_0 for this example with relation to Proposition 5.15.

By combining 4) of Proposition 5.9 with Proposition 5.3, we also have

PROPOSITION 5.10. *We suppose that* (B) *is fulfilled.*

We consider the integral manifold $\tilde{K}_0$ *passing* $\tilde{\Omega}$ *for Hamilton fields* $H_{\xi_0-\lambda^+}$, $H_{\xi_0-\lambda^-}$ *(that is, we first consider the* $(\xi_0 - \lambda^+)$*-bicharacteristic strips issuing from* $\tilde{\Omega}$ *and next from the points of these strips we draw* $(\xi_0 - \lambda^-)$*-bicharacteristic strips.* $\tilde{K}_0$ *is spanned by these strips: 2-families of bicharacteristics which are obtained by integrating successively Hamilton fields* $H_{\xi_0-\lambda^+}$, $H_{\xi_0-\lambda^-}$ *with initial conditions* $(0, y, \xi_0, \xi) \in \tilde{\Omega}$).

Obviously, $\tilde{K}_0 \subset \Pi^+ \cap \Pi^-$, *and* K_0 *is the projection of* $\tilde{K}_0$ *on* (t, x)*-space.*

Now we give an example which shows the dependence of K_0 on Ω along 2-families of bicharacteristics.

EXAMPLE 5.3. $h = D_t^2 - 3(x_2 + 2x_3)x_2D_tD_1 + D_tD_2$,

$$\Phi = x_1 + 3(x_2 + 2x_3)x_2\tau - 3(x_2 + x_3)\tau^2 + \tau^3\,,$$

$$\Phi_\tau = 3(\tau - x_2 - 2x_3)(\tau - x_2)\,.$$

K_0 consists of two regular surfaces. This follows from the fact that Ω decomposes into a reunion of two regular varieties:

$$t = x_1 = x_2 + 2x_3 = 0 \quad \text{and} \quad t = x_1 = x_2 = 0\,.$$

Now we seek a sufficient condition that K_0 is regular at the origin. We easily see that

$$\Phi_{\tau\tau}(0, 0, 0) = \sum_{i=1}^{n}\left(\frac{\partial\lambda^+}{\partial\xi_i} - \frac{\partial\lambda^-}{\partial\xi_i}\right)\left(\frac{\partial\lambda^+}{\partial x_i} - \frac{\partial\lambda^-}{\partial x_i}\right)\Bigg|_{(0,0;1,0,\dots,0)}$$

According to this, we have

PROPOSITION 5.11. *Suppose*

$$\sum_{i=1}^{n}\left(\frac{\partial\lambda^+}{\partial\xi_i}-\frac{\partial\lambda^-}{\partial\xi_i}\right)\left(\frac{\partial\lambda^+}{\partial x_i}-\frac{\partial\lambda^-}{\partial x_i}\right)\neq 0 \quad \text{for } (0, 0; 1, 0, \dots, 0)\,. \tag{5.2}$$

Then, the assumption (B) *is verified and the surface* K_0 *is regular at the origin. Incidentally,* Ω *is of course regular at the origin.* K_0 *touches* $K^\pm$ *with the order* 2 *of contact.*

REMARK. The geometrical meaning of (5.2) will be also explained in Remark of Proposition 5.14.

PROOF. By the hypothesis, we have $\Phi_{\tau\tau}(0, 0, 0)\neq 0$. Hence, it is evident that (B) is verified. Furthermore, we can find holomorphic function $\tau=\tau(t, x)$ such that $\Phi_\tau(t, x, \tau(t, x))=0$ and $\tau(0, 0)=0$. We set $\psi(t, x)=$ $=\Phi(t, x, \tau(t, x))$, then $\psi(t, x)$ is holomorphic in a neighborhood of the origin and $(\psi_t, \psi_x)\neq(0, 0)$, because $\psi_{x_1}(0, 0)=\Phi_{x_1}(0, 0, 0)=1$. Since $K_0=\{\psi(t, x)=0\}$, K_0 is regular at the origin.

Finally, let $(\tau(\sigma), x(\sigma))$ be the $(\xi_0-\lambda^-)$-bicharacteristic curves issuing from $\tilde{\Lambda}^+$, then we have by a simple calculation $d^2\varphi^+/d\sigma^2))(t(\sigma), x(\sigma))|_{\sigma=0}\neq 0$ at the origin. This implies the last assertion of our Proposition.

Now we want to construct directly the surface $K_0\cap S$.

Consider the equation

$$\Phi_\tau(0, x, \tau)=\lambda^+(0, x, \Phi_x(0, x, \tau))-\lambda^-(0, x, \Phi_x(0, x, \tau))$$

with the initial condition $\Phi(0, x, 0)=x_1$, and the associated equation of the $(\lambda^+-\lambda^-)$-bicharacteristic strip

$$\begin{cases} \dfrac{d\tau}{d\sigma}=1\,, & \dfrac{d\eta}{d\sigma}=0\,, \\[2ex] \dfrac{dx_i}{d\sigma}=-\dfrac{\partial(\lambda^+-\lambda^-)}{\partial\xi_i}(0, x, \xi)\,, & \dfrac{d\xi_i}{\partial\sigma}=\dfrac{\partial(\lambda^+-\lambda^-)}{\partial x_i}(0, x, \xi) \\ & \qquad 1\leqslant i\leqslant n\,. \end{cases} \tag{5.3}$$

Let

$$\begin{cases} \tau=\sigma\,, & \eta=\eta(\sigma, y)=\lambda^+(0, y; 1, 0, \dots, 0) \\ & \qquad -\lambda^-(0, y; 1, 0, \dots, 0)\,, \\ x=x(\sigma, y)\,, & \xi=\xi(\sigma, y)\,. \end{cases} \tag{5.4}$$

be the solution of (5.3) with the initial condition

$$\begin{cases} \tau(0)=0\,, & \eta(0)=\lambda^+(0,y;1,0,\dots,0)-\lambda^-(0,y;1,0,\dots,0)\,,\\ x(0)=y\,, & \xi(0)=(1,0,\dots,0)\,. \end{cases}$$

Then $\Phi(0,x,\tau)$ and $\Phi_\tau(0,x,\tau)$ are constant along this curve.
Therefore we have

PROPOSITION 5.12. *Let* $U=\{(0,x);\ \exists\tau;\ \Phi(0,x,\tau)=\Phi_\tau(0,x,\tau)=0\}$. *Then* U *is generated by* $(\lambda^+-\lambda^-)$*-bicharacteristic curves* (5.4) *issuing from* Ω.

The following Proposition is obtained by Proposition 5.8 and 5.12.

PROPOSITION 5.13. *Suppose that* (B) *holds. Then* $K_0\cap S=U\underset{\neq}{\supset}\Omega$ *and* $\dim U=n-1$.

By virtue of the existence theorem of the initial value problem for non linear first order partial differential equation, we have

PROPOSITION 5.14. *Assume that* (5.2) *is fulfilled. Then the equation*

$$\lambda^+(0,x,\alpha_x(x))-\lambda^-(0,x,\alpha_x(x))=0 \tag{5.5}$$

with the initial condition

$$\alpha(x)=x_1 \qquad \text{on } \Gamma=\left\{\begin{matrix}(0,x); & \lambda^+(0,x;1,0,\dots,0)\\ & -\lambda^-(0,x;1,0,\dots,0)=0\end{matrix}\right\}$$

have a unique holomorphic solution $\alpha(x)$, *and* U *is given by* $U=\{(0,x);\ \alpha(x)=0\}$. *Moreover* U *is regular at the origin.*

REMARK. The condition (5.2) implies that the initial condition $\alpha(x)=x_1$ on Γ $\big($thus $(\alpha_{x_1}(x),\dots,\alpha_{x_n}(x))=(1,0,\dots,0)$ on $\Gamma\big)$ is non characteristic for (5.5) in a neighborhood of the origin. More precisely,

$$\left\{(0,x)\in\Gamma;\quad \left\{\sum_{i=1}^n\frac{\partial(\lambda^+-\lambda^-)}{\partial\xi_i}\frac{\partial(\lambda^+-\lambda^-)}{\partial x_i}\right\}(0,x;1,0,\dots,0)=0\right\}$$

is the set of the characteristic points of the initial condition for (5.5).

PROOF. In fact, we see from the assumption (5.2) that Γ is regular and the initial condition is non characteristic for (5.5). Hence this Cauchy problem has a unique holomorphic solution. The second part of Proposition follows from Proposition 5.12.

According to Propositions 5.10, 5.13 and 5.14 we have the following

PROPOSITION 5.15. *We suppose that* (5.2) *is fulfilled.*
Let $\beta(t, x)$ *be the solution of the Cauchy problem*:

$$\beta_t(t, x) = \lambda^+(t, x, \beta_x(t, x)) \qquad (or\ \lambda^-(t, x, \beta_x(t, x))$$

with the initial condition $\beta(0, x) = \alpha(x)$.
Then K_0 *is given by* $\{(t, x);\ \beta(t, x) = 0\}$.

REMARK OF EXAMPLE 5.2. In this example, $\Gamma = \{(0, x); x_2^2 - x_3 = 0\}$ is characteristic for the operator $3(x_2^2 - x_3)D_1 - D_2$ on the $\{(0, x); x_2 = x_3 = 0\}$. The solution of (5.5): $\alpha(x) = x_1 - 3x_2x_3 + x_2^3 + 2x_3^{\frac{3}{2}}$ is ramified around the vairety $\{(0, x); x_3 = 0\}$ generated by the $(\lambda^+ - \lambda^-)$-bicharacteristic curves (5.3) issuing from $\{(0, x); x_2 = x_3 = 0\}$. Therefore, $\beta(t, x) = \alpha(x)$ is ramified around the variety $\{(t, x); x_3 = 0\}$. Incidentally, K_0 is not regular along the subvariety $\{(t, x); x_3 = x_1 + x_2^3 = 0\}$ of K_0.

Finally we give a simple criterion that the assumption (B) does not hold.

PROPOSITION 5.16.

1) *If* $\Pi^+_{(0,0)} = \Pi^-_{(0,0)}$, *then* (B) *does not hold.*

2) *If* $T = \Omega$, *then* (B) *is not verified. In this case, we have* $\Phi(t, x, \tau) = a(t, x, \tau)\varphi^+(t, x)$ *with a holomorphic function* $a(t, x, \tau)$ $(a(0, 0, 0) = 1)$. *Therefore, the singular support of the solution of the Cauchy problem* (1.2) *is contained in* $\bigcup_{i=1}^{m-2} K_i \cup K^+$. *In particular,* $\Phi(t, x, \tau) = \varphi^+(t, x)$ *for operator with constant coefficients.*

PROOF. 1) is evident, because $\Phi_\tau(0, 0, \tau) = \lambda^+(0, 0, \Phi_x(0, 0, \tau)) - \lambda^-(0, 0, \Phi_x(0, 0, \tau)) \equiv 0$.

2) has been already shown in Proposition 5.8.

EXAMPLE 5.4. $h = D_t^2 - (x_1\ a(x))^2(D_1^2 + \dots + D_n^2)$, $(a(x)$ is holomorphic at $x = 0)$.

$$\lambda^+ = -\lambda^- = x_1 a(x)\sqrt{\xi_1^2 + \dots + \xi_n^2}\,.$$

For this operator we have $T = \Omega$. Hence by 2) of Proposition 5.16 the Cauchy problem (1.2) has singularities only on K^+: $x_1 = 0$.

Exceptional point ... The origin is said to be exceptional for operator h *when the assumption* (B) *does not hold and* $T \neq \Omega$.

In this case, the various phenomena happen. Here, we shall only give some examples of these phenomena.

EXAMPLE 5.5. $h = D_t^2 + x_2 D_t D_1$, $\lambda^+ = -x_2\xi_1$, $\lambda^- = 0$, $\Phi = x_1 - x_2\tau$. $K^+: x_1 - x_2 t = 0$, $K^-: x_1 = 0$. The origin is exceptional for h.

We consider the Cauchy problem:

$$\begin{cases} hu = 0\,, \\ u(0, x) = 0\,, \qquad D_t u(0, x) = \dfrac{1}{x_1}\,. \end{cases}$$

The solution is given by

$$u(t, x) = \frac{1}{x_2} \log \frac{x_1}{x_1 - x_2 t}\,,$$

which has singularities on K^+, K^- and the surface $x_2 = 0$.

As for the $(\xi_0 - \lambda^\pm)$-characteristic surface $x_2 = 0$, we interpret that it is generated by $(\xi_0 - \lambda^-)$-bicharacteristic curves issuing from $(\xi_0 - \lambda^-)$-characteristic points of K^+ at infinity $t = \infty$. Thus we may have to discuss the singularities of the solution from a viewpoint of its global existence domain (as A. Takeuchi has pointed out it).

EXAMPLE 5.6. $h = D_t^2 - x_2^2(D_1^2 + \dots + D_n^2)$

$$\lambda^+ = x_2\sqrt{\xi_1^2 + \dots + \xi_n^2}\,, \qquad \lambda^- = -x_2\sqrt{\xi_1^2 + \dots + \xi_n^2}\,,$$

$$K^+ : x_1 + \frac{x_2}{2}(\exp[t] - \exp[-t]) = 0\,, K^- : x_1 - \frac{x_2}{2}(\exp[t] - \exp[-t]) = 0\,,$$

$$\Phi = x_1 + \frac{x_2}{2}(\exp[2\tau - t] - \exp[t - 2\tau])\,.$$

The solution of the Cauchy problem (1.2) for this operator h may have singularities on K^+, K^- and the surface $x_2 = 0$. The surface $K^\pm$ has the essential singularities at infinity $t = \infty$ and accumulates to the $(\xi_0 - \lambda^\pm)$-characteristic surface $x_2 = 0$ as t tends to ∞.

EXAMPLE 5.7. $h = D_t^2 + 2x_2 D_t D_1 + x_3 D_t D_2$.

$$\lambda^+ = -2x_2\xi_1 - \xi_2\,, \quad \lambda^- = 0\,, \quad K^+: x_1 - 2x_2 t + x_3 t^2 = 0\,, \quad K^-: x_1 = 0\,,$$

$$\Lambda^+: x_1 = x_3 t^2\,, \quad x_2 = x_3 t\,, \quad \Phi = x_1 - 2x_2\tau + x_3\tau^2\,.$$

The solution of the Cauchy problem (1.2) for h has singularities on K^+, K^- and the surfaces $\{(t, x); x_3 = 0\}$, $\{(t, x); x_1x_3 - x_2^2 = 0\}$. The $(\xi_0 - \lambda^\pm)$-characteristic surface $x_3 = 0$ is generated by the $(\xi_0 - \lambda^-)$-bicharacteristic curves issuing from the $(\xi_0 - \lambda^-)$-characteristic points of K^+ at infinity $t = \infty$, while the $(\xi_0 - \lambda^\pm)$-characteristic surface $x_1x_3 - x_2^2 = 0$ is spanned by $(\xi_0 - \lambda^-)$-bicharacteristic curves issuing from Λ^+.

REFERENCES

[1] J. M. BONY - P. SCHAPIRA, *Propagation des singularités analytiques pour les solutions des équations aux dérivées partielles*, Ann. Inst. Fourier. Grenoble, **26** (1976), pp. 81-140.

[2] J.-CL. DE PARIS, *Problème de Cauchy analytique à données singulières pour un opérateur differentiel bien décomposable*, J. Math. pures et appl., **51** (1972), pp. 465-488.

[3] L. GÅRDING - T. KOTAKE - J. LERAY, *Uniformisation et développement asymptotiques de la solution du problème de Cauchy linéaire à données holomorphes*, Bull. Soc. math. France, **92** (1964), pp. 263-361.

[4] B. GRANOFF - D. LUDWIG, *Propagation of singularities along characteristics with nonuniform multiplicities*, J. Math. Anal. Appl., **21** (1968), pp. 556-574.

[5] Y. HAMADA - J. LERAY - C. WAGSCHAL, *Systèmes d'équations aux dérivées partielles à caractéristiques multiples: problème de Cauchy ramifié, hyperbolicité partielle*, J. Math. pures et appl., t. **55** (1976), p. 297-352.

[6] J. LERAY, *Problème de Cauchy I*, Bull. Soc. Math. France, **85** (1957), pp. 389-429.

[7] G. NAKAMURA, *The singularities of solutions of the Cauchy problems for systems whose characteristic roots are non uniform multiple*, to appear.

[8] C. WAGSCHAL, *Problème de Cauchy analytique à données méromorphes*, J. Math. pure et appl., **51** (1972), pp. 465-488.

[9] J. VAILLANT, *Solutions asymptotiques d'un système à caractéristiques de multiplicité variable*, J. Math. pures et appl., **53** (1974), pp. 71-98.