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Elastic-Plastic Torsion Problem over Multiply Connected Domains (*).

TSUAN WU TING (**)

dedicated to Hans Lewy

0. – Introduction.

Consider a cylindrical pipe with cross-section D. We shall assume that D is bounded internally by distinct Jordan curves $C_1, \ldots, C_n$ and externally by a Jordan curve c_0. The elastic-plastic torsion problem over D is to find a function ψ which is continuous in the closure of D, smooth in D and takes on constant but arbitrary values on each C_j, $j \geqq 0$, such that $|\text{grad}\,\psi|$ is less than or equal to a positive yield constant k in D and that wherever $|\text{grad}\,\psi| < k$, it is twice smooth and satisfies the Poisson equation, $\Delta\psi = -2\mu\theta$, there. Here, μ stands for the (positive) shear modulus and θ the (positive) angle of twice per unit length. Since ψ is determined up to an additive constant we may set $\psi = 0$ on C_0.

For simply connected domains, much information has been obtained through various efforts during the last decade, [1-18]. However, for multiply connected domains, the regularity question remains open when D possesses various types of corners, [10]. It is our objective to answer this question and to derive physically relevant results under minimum assumptions on D. As usual, we formulate the problem as a variational inequality in § 1. The simple idea that leads to the present results is the imbedding technique introduced in § 2. The choice of the upper and lower envelopes for the

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imbedding turns out to be natural, because it leads to the identity of the two minimizers, see § 2. Furthermore, both the upper and the lower envelopes are effective in the sense that the minimizer in question does touch these envelopes for certain values of θ, and certain types of domain D.

By means of imbedding the minimizer, the existence of elastic cores is established in § 4, which is not only physically significant but also an essential step forward in establishing the regularity of the minimizer in the entire domain D. Most interesting of all is the « intersection property » of the plastic zones as given in § 7. It is this intersection property that enables us to locate the unknown elastic plastic boundary.

1. – Formulation of the problem.

Choose a rectangular coordinate system, $x \equiv (x_1, x_2, x_3)$, with the x_3-axis parallel to the generators of the cylindrical surfaces. Our restrictions on D are as follows:

A$_1$) Each of the Jordan curves C_j possesses a parametric representation

$$x_1 = f_j(s)\,, \qquad x_2 = g_j(s)\,, \qquad\qquad j = 0, 1, \ldots, n,\ 0 \leqq s \leqq s_j\,,$$

such that both f_j and g_j are of class C^1 except at a finite of points (corners) where their derivatives suffer finite jump discontinuities.

A$_2$) Between any adjacent corners of each C_j, the functions f_j and g_j are of class C^2 and the curvature $\varkappa_j$ assumes only a finite number of maxima and minima.

Denote by G_j the domain enclosed by C_j, $j = 0, \ldots, n$ and by $\operatorname{dist}(G_j, G_k)$ the distance between G_j and G_k. We shall assume (though unnecessary) that $\operatorname{dis}(G_j, G_k) > 0$ for $j \neq k$. As usual, $C_0^\infty(G_0)$ stands for the class of infinitely smooth functions with compact support in G_0. For u and v in $C_0^\infty(G_0)$, their inner products are defined by

$$(u, v)_0 \equiv \int_{G_0} u(x)\, v(x)\, dx\,, \tag{1.1}$$

$$(u, v)_1 \equiv (u, v) + \int_{G_0} \operatorname{grad} u \cdot \operatorname{grad} v\, dx\,, \tag{1.2}$$

which induce the corresponding norms

$$\|u\|_\alpha \equiv [(u, u)_\alpha]^{\frac{1}{2}}\,, \qquad \alpha = 0, 1\,. \tag{1.3}$$

Here as well as in what follows, we write $dx \equiv dx_1\, dx_2$.

Let $H_0^1(G_0)$ be the completion of $C_0^\infty(G_0)$ under $\|\cdot\|_1$-norm so that it is a separable Hilbert space. Let

$$K = \{u \mid u \in H_0^1(G_0),\ u = \text{const a.e. on } G_j, j > 0\}. \tag{1.4}$$

Then K is a closed subspace of $H_0^1(G_0)$. Let

$$Y = \{u \mid u \in H_0^1(G_0),\ |\text{grad}\, u| \leqq k \text{ a.e. in } G_0\} \tag{1.5}$$

where k is the yield constant. Then Y is a *closed convex* subset of $H_0^1(G_0)$. Hence, $K \cap Y$ is also a closed convex set in $H_0^1(G_0)$, [5].

Finally, the elastic-plastic torsion problem over D is to *find the minimizer of the functional,*

$$I[u] \equiv \int_{G_0} [|\text{grad}\, u|^2 - 4\mu\theta u]\, dx, \tag{1.6}$$

over the closed convex set $K \cap Y$ in $H_0^1(G)$.

By the same reasoning as for the case of simply connected domains, [18, *c*], we have

THEOREM 1.1. *Problem* (1.6) *has a unique solution.*

As we have seen, the weak formulation of the problem provides ready answers to the basic existence and uniqueness questions. Since $|\text{grad}\, \psi| \leqq k$ a.e. in G_0, it is also easy to verify that the minimizer ψ is uniformly Lipschitz continuous in G_0 with the Lipschitz constant $\leqq k$. However, because of the presence of corners in D and the inequality constraint on the gradient of the admissible functions, it is difficult to directly verify that the weak solution actually solves the problem in the sense stated in § 0. To overcome this, we shall imbed the minimizer ψ in a larger admissible class of functions by replacing the yield criterion by a majorant and a minorant function.

2. – Imbedding of the minimizer.

Let ψ be the minimizer of problem (1.6) and let k be the constant value of ψ on G_j, $j \geqq 1$. Consider the set,

$$K^* = \{u;\ u \in H_0^1(G_0),\ u = \psi = k_j \text{ a.e. on } G_j\}. \tag{2.1}$$

Then K^* is a *closed convex* subset of $H_0^1(G_0)$. Hence, there are functions Φ

and φ in $K^* \cap Y$ such that

$$\int_{G_0} \Phi(x)\,dx = \sup_{K^* \cap Y} \int_{G_0} u(x)\,dx\,, \qquad \int_{G_0} \varphi(x)\,dx = \inf_{K^* \cap Y} \int_{G_0} u(x)dx\,. \tag{2.2}$$

In fact, both Φ and φ are unique. Furthermore, if we define

$$\begin{cases} \Phi_0(x) = k\,\mathrm{dist}(x, C_0)\,, \\ \Phi_j(x) = k_j + k\,\mathrm{dist}(x, G_j)\,, \quad j = 1, 2, \ldots, n\,, \end{cases} \tag{2.3}$$

where k is the yield constant. Then

$$\Phi(x) = \min\{\Phi_0(x), \ldots, \Phi_n(x)\}\,. \tag{2.4}$$

Similarly, we have

$$\varphi(x) = \max\{\varphi_0(x), \ldots, \varphi_n(x)\}\,, \tag{2.5}$$

where

$$\begin{cases} \varphi_0(x) = -\,k\,\mathrm{dist}(x, C_0)\,, \\ \varphi_j(x) = k_j - k\,\mathrm{dist}(x, G_j), \quad j = 1, 2, \ldots, n\,. \end{cases} \tag{2.6}$$

From (2.3) and (2.6), it follows that

$$\varphi_j \leqq \psi \leqq \Phi_j \quad \text{on } G_0 \qquad \text{for } j = 0, 1, \ldots, n\,.$$

Hence, $\varphi \leqq \psi \leqq \Phi$ on G_0 and $\varphi = \psi = \Phi$ on each G_j and C_0. Consequently, the *closed convex* set,

$$H = \{u;\, u \in H_0^1(G_0),\, \varphi \leqq u \leqq \Phi \text{ a.e. in } G_0\}\,, \tag{2.7}$$

is nonempty. Thus, the problem of finding a function ψ^* in H such that it minimizes the functional,

$$I[u] = \int_{G_0} [|\mathrm{grad}\, u|^2 - 4\mu\theta u]\,dx\,, \tag{2.8}$$

over the set H in $H_0^1(G_0)$ is meaningful. In fact, the same reasoning as for Theorem 1.1 ensures the truth of

THEOREM 2.1. *Problem* (2.8) *has a unique solution* ψ^*.

Problem (2.8) was so formulated that the two minimizers ψ and ψ^* are identical in $H_0^1(G_0)$. This fact will be proved as the next theorem. It should be noted that in formulating problem (2.8), we have made use of both the existence and the uniqueness of the minimizer ψ to problem (1.6).

THEOREM 2.2. *The minimizers* ψ *and* ψ^* *of problems* (1.6) *and* (2.8), *respectively, are identical in* $H_0^1(G_0)$.

PROOF. Clearly, $K^* \subset K$. Hence, $I[\psi] = \inf I[u]$ over $K^* \cap Y$. Since $K^* \cap Y \subset H$, we have $I[\psi^*] \leq \inf I[u]$ over $K^* \cap Y$. If we can show that $\psi^* \in Y$, then

$$I[\psi^*] = I[\psi] = \inf I[u] \quad \text{over } K^* \cap Y . \tag{2.9}$$

Now, the non-trivial fact that $\psi^* \in Y$ is assured by the same reasoning as given in [2]. Hence (2.9) holds. It follows from (2.9) and the uniqueness of the minimizer of $I[u]$ over $K^* \cap Y$ that $\psi = \psi^*$ in $H_0^1(G_0)$.

3. – The edges of the enveloping surfaces.

Denote by $\Gamma(\Phi)$ and $\Gamma(\varphi)$, respectively, the set of discontinuities of grad Φ and grad φ in G_0. To establish the regularity of the solution ψ^* of problem (2.8), we first locate the sets $\Gamma(\Phi)$ and $\Gamma(\varphi)$. In addition, it is necessary to assure that these sets are the unions of a finite number of smooth Jordan arcs. In fact, as a consequence of the assumptions A_1) and A_2) in § 1, we have

THEOREM 3.1. *If the bounded domain* D *satisfies assumptions* A_1) *and* A_2) *in* § 1, *then both* $\Gamma(\Phi)$ *and* $\Gamma(\varphi)$ *are the unions of a finite number of smooth Jordan arcs. Hence, the number of branch points and end points in* $\Gamma(\Phi)$ *and* $\Gamma(\varphi)$ *are finite.*

If we denote by $\Gamma(\Phi_j)$ the set of discontinuities of grad Φ_j in G_0, then formula (2.3) shows that $\Gamma(\Phi_j)$ is the « ridge » of the domain G'_j which is the complement of G_j. Accordingly, the statement in the theorem for $\Gamma(\Phi)$ follows from this fact and formula (2.4). The same reasoning applies to $\Gamma(\varphi)$. Although the details of the proof are elementary but rather involved, we omit it here.

REMARK 3.2. Although Theorem 3.1 was stated as a consequence of the restrictions on D, we may simply assume that D is such a domain for which Theorem 3.1 holds. Indeed, the class of such domains is easily seen to be rich. In particular, each C_j can be a polygon.

4. – Existence of elastic cores.

If the domain D is bounded by two concentric circles, then a finite angle of twist per unit length may cause fully plastic torsion, i.e., $|\text{grad}\, \psi| = k$ a.e. in D. However, we are interested in those domains D for which $|\text{grad}\, \psi| < k$

on a set of positive area no matter how large the values of θ may be. In fact, if D possesses a non-reentrant corner on C_0, then the set of Jordan curves

$$\Lambda = D \cap \Gamma(\Phi) \cap \Gamma(\varphi), \tag{4.1}$$

is always non-empty for all values of θ and $|\mathrm{grad}\,\psi| < k$ in a neighbourhood of Λ. In general, the size and the relative position of Λ in D also depends on the values of θ. Our main result is

THEOREM 4.1. *Let D be a bounded domain satisfying Assumptions* A_1 *and* A_2 *in* § 1. *Assume that the set Λ in* (4.1) *is non-empty. Then, for every point x_0 in Λ, there exists a positive number ε_0, depending on x_0, such that if $D(x_0, \varepsilon)$ is an open disk centered at x_0 with radius $\varepsilon \leqq \varepsilon_0$ and if $D(x_0, \varepsilon)$ is contained in D, then the minimizer ψ of problem* (2.8) *satisfies the strict inequalities, $\varphi < \psi < \Phi$, in $D(x_0, \varepsilon)$. Moreover, ψ is analytic and satisfies the Poisson equation $\Delta\psi = -2\mu\theta$ in $D(x_0, \varepsilon)$.*

Instead of directly proving the theorem, we reduce it to Theorem 4.2. Consider a point x_0 on Λ. Choose ε so small that the open disk $D(x_0, \varepsilon)$ is contained in D. There are only three possibilities:

Case a. x_0 is a regular point of Λ. That is, there is uniquely defined tangent to Λ at x_0. Since Λ consists of a finite number of smooth Jordan arcs, we can choose ε so small that

$$\lambda(x_0, \varepsilon) = \Lambda \cap D(x_0, \varepsilon) \tag{4.2}$$

is a single smooth arc passing through x_0.

Case b. x_0 is a branch point on Λ. Then, for all sufficiently small ε, $\lambda(x_0, \varepsilon)$ consists of several smooth arcs with x_0 as a common end point.

Case c. x_0 is an end point on Λ. Then, for all sufficiently small ε, $\lambda(x_0, \varepsilon)$ is a single smooth arc issuing from x_0.

Having fixed a point x_0 on Λ and the corresponding disk $D(x_0, \varepsilon)$, we now consider three auxiliary problems. Specifically, let Φ_ε, ψ_ε, φ_ε be, respectively, the solutions of the Dirichlet problems:

$$\Delta\Phi_\varepsilon = -2\mu\theta \quad \text{in } D(x_0, \varepsilon), \qquad \Phi_\varepsilon = \Phi \quad \text{on } \partial D(x_0, \varepsilon), \tag{4.3}$$

$$\Delta\psi_\varepsilon = -2\mu\theta \quad \text{in } D(x_0, \varepsilon), \qquad \psi_\varepsilon = \psi \quad \text{on } \partial D(x_0, \varepsilon), \tag{4.4}$$

$$\Delta\varphi_\varepsilon = -2\mu\theta \quad \text{in } D(x_0, \varepsilon), \qquad \varphi_\varepsilon = \varphi \quad \text{on } \partial D(x_0, \varepsilon). \tag{4.5}$$

We assert that to prove Theorem 4.1 it suffices to show that *there exists*

$\varepsilon_0 > 0$ *such that*

$$\varphi \leqq \psi_\varepsilon \leqq \Phi \quad \text{in } D(x_0, \varepsilon) \qquad \text{for all } \varepsilon \leqq \varepsilon_0 . \tag{4.6}$$

Indeed, if this has been proved, then it follows from Dirichlet's principle that both ψ and ψ_ε minimizes the same functional,

$$I[u] \equiv \int_{D(x_0, \varepsilon)} [|\text{grad } u|^2 - 4\mu\theta u]\, dx$$

under the same constraints, namely

$$u \in H^1(D(x_0, \varepsilon)) , \qquad \varphi \leqq u \leqq \Phi \quad \text{in } D(x_0, \varepsilon) , \qquad u = \psi \quad \text{on } \partial D(x_0, \varepsilon) .$$

Accordingly, the uniqueness of the minimizer demands that $\psi_\varepsilon = \psi$ in $D(x_0, \varepsilon)$ for all $\varepsilon \leqq \varepsilon_0$. But then Theorem 4.1 follows.

Since both $\Phi_\varepsilon - \psi_\varepsilon$ and $\psi_\varepsilon - \varphi_\varepsilon$ are harmonic in $D(x_0, \varepsilon)$ and are non-negative on $\partial D(x_0, \varepsilon)$, it follows from the strong maximum principle that

$$\varphi_\varepsilon \leqq \psi_\varepsilon \leqq \Phi_\varepsilon \quad \text{in } D(x_0, \varepsilon) . \tag{4.7}$$

From (4.6) and (4.7) we see that Theorem 4.1 will be proved if we can establish

THEOREM 4.2. *For every point x_0 on Λ, there is a positive number ε_0 depending on x_0 such that if Φ_ε and φ_ε are, respectively, the solutions of problems* (4.3) *and* (4.5), *then for all $\varepsilon \leqq \varepsilon_0$,*

$$\Phi_\varepsilon \leqq \Phi , \qquad \varphi_\varepsilon \geqq \varphi \quad \text{in } D(x_0, \varepsilon) . \tag{4.8}$$

The remainder of this section is devoted to the proof of Theorem 4.2. First, we observe that $\Lambda \cap \Gamma(\Phi) = \Lambda \cap \Gamma(\varphi)$ and that along Λ the surfaces $x_3 = \Phi(x_1, x_2)$ and $x_3 = \varphi(x_1, x_2)$ must be of Λ-shape and of V-shape, respectively. In fact, in a small neighbourhood of Λ, one surface is just mirror image of the other. Consequently, we need only to prove the Theorem for Φ and Φ_ε. To do this, we first calculate the Laplacian of Φ.

For any regular point x_0 on $\lambda(x_0, \varepsilon)$ defined in (4.2), let grad $\Phi(x^+)$ and grad $\Phi(x^-)$ be, respectively, the limiting values of grad $\Phi(y)$ as $y \to x$ from the left and from the right of $\lambda(x_0, \varepsilon)$. It is not difficult to show that the unique tangent to $\lambda(x_0, \varepsilon)$ at x bisects the angle between the directions grad $\Phi(x^+)$ and grad $\Phi(x^-)$. Let β be the non-obtuse angle between grad $\Phi(x^+)$ and the tangent to $\lambda(x_0, \varepsilon)$ at x. Then β varies continuously along each branch of $\lambda(x_0, \varepsilon)$, issuing from x_0. Moreover, $0 < \beta \leqq \pi/2$. However, all we need in what follows is that β be continuous and bounded.

LEMMA 4.3. *If x_0 is a regular or a branch point of Λ and if $D(x_0, \varepsilon) \subset D$, then, as a distribution on $C_0^\infty(D(x_0, \varepsilon))$,*

$$\Delta\Phi(x) = -k[\sin\beta(x)]\delta(\lambda(x_0, \varepsilon)) + f(x) \quad \text{in } D(x_0, \varepsilon)\,, \tag{4.9}$$

where f is a bounded piecewise continuous function in $D(x_0, \varepsilon)\setminus\Lambda$ and where $\delta(\lambda(x_0, \varepsilon))$ is the Dirac measure concentrated along $\lambda(x_0, \varepsilon)$. Specifically,

$$\langle[\sin\beta(x)]\delta(\lambda(x_0, \varepsilon)), \eta(x)\rangle = \int_{\lambda(x_0, \varepsilon)} \sin\beta(x)\eta(x)\, ds_x \tag{4.9'}$$

for all η in $C_0^\infty(D(x_0, \varepsilon))$, where ds_x stands for the element of arc length along $\lambda(x_0, \varepsilon)$.

PROOF. For every point x in the multiply connected domain D, $\Phi(x) = k\,\text{dis}(x, \partial D) +$ some constant. This constant may take the values $0, k_1, \ldots, k_n$. Let s be a point on ∂D such that $\text{dist}(x, s) = \text{dist}(x, \partial D)$. If s is a regular point on ∂D and if $x \notin \Lambda$, then simple computation gives the formula,

$$f(x) = \Delta\Phi(x) = -k\varkappa(s)/[1 - t\varkappa(s)]\,, \tag{4.10}$$

where k is the yield constant, $t \equiv \text{dist}(x, s)$ and $\varkappa(s)$ the curvature of ∂D at s. On the other hand, if s is an reentrant corner of D, then

$$f(x) = \Delta\Phi(x) = k/t \quad \text{with } t \equiv \text{dist}(x, s)\,. \tag{4.10'}$$

Formulas (4.10) and (4.10′) assure us that $\Delta\Phi$ is continuous in $(D\setminus\Gamma(\Phi))$ except along the *extreme* inward normals at the reentrant corners of D, where it suffers jump discontinuities. For our purpose, we may assign $f(x)$ any finite values on $D \cap \Gamma(\Phi)$ as well as on the extreme inward normals.

To derive the distributional part of $\Delta\Phi$, we note that as a distribution on $C_0^\infty(D(x_0, \varepsilon))$, $\Delta\Phi$ is defined by the rule,

$$\langle\Delta\Phi, \eta\rangle = -\int_{D(x_0, \varepsilon)} \text{grad}\,\Phi\cdot\text{grad}\,\eta\, dx\,, \tag{4.11}$$

for all η in $C_0^\infty(D(x_0, \varepsilon))$. Since grad Φ is piecewise smooth in $D(x_0, \varepsilon)$, an application of the divergence theorem gives

$$\int_{D(x_0, \varepsilon)} \text{grad}\,\Phi\cdot\text{grad}\,\eta\, dx = \int_{D(x_0, \varepsilon)\setminus\lambda(x_0, \varepsilon)} \text{grad}\,\Phi\cdot\text{grad}\,\eta\, dx \tag{4.12}$$

$$= \int_{\lambda(x_0, \varepsilon)} \left(\frac{\partial}{\partial n}\Phi\right)\eta\, ds - \int_{D(x_0, \varepsilon)\setminus\lambda(x_0, \varepsilon)} (\Delta\Phi)\eta\, dx = k\int_{\lambda(x_0, \varepsilon)} [\sin\beta(x)]\eta(x)\, ds_x - \int_{D(x_0, \varepsilon)\setminus\lambda(x_0, \varepsilon)} f(x)\eta(x)\, dx\,,$$

where the notations are the usual ones. Now, formulas (4.9) and (4.9′) follow directly from (4.11) and (4.12). The proof is now completed.

Of course, x_0 being a regular or a branch point of Λ is the general case. For the general case, it follows from Lemma 4.3 that

$$\Delta(\Phi - \Phi_\varepsilon) = -k[\sin\beta(x)]\delta(\lambda(x_0, \varepsilon)) + f(x) + 2\mu\theta \tag{4.9''}$$

in the sense of distribution. Applying the principle of super position, we write

$$\Phi - \Phi_\varepsilon = \chi_1 + \chi_2 \tag{4.13}$$

such that

$$\Delta\chi_1 = f(x) + 2\mu\theta \quad \text{in } D(x_0, \varepsilon), \qquad \chi_1 = 0 \quad \text{on } \partial D(x_0, \varepsilon), \tag{4.13'}$$

$$\Delta\chi_2 = -k[\sin\beta(x)]\delta(\lambda(x_0, \varepsilon)) \tag{4.13''}$$

as a distribution on $C_0^\infty(D(x_0, \varepsilon))$ and $\chi_2 = 0$ on $\partial D(x_0, \varepsilon)$.

As we shall see, both (4.13′) and (4.13″) are well-posed problems. Our immediate objective is to derive a point-wise upper bound for $|\chi_1|$. To this end, we shall make use of

LEMMA 4.4. *Let D be a bounded domain in R^2. Let f_α, $\alpha = 1, 2$ be given functions in L^2 such that $f_2 \geqq f_1$ a.e. in D. Suppose that u_α, $\alpha = 1, 2$,* mini*mizes the functional*

$$J[w] = \int_D [|\operatorname{grad} w|^2 - 2f_\alpha w]\, dx,$$

over $H_0^1(D)$. Then, $u_2 \geqq u_1$ a.e. on D.

LEMMA 4.5. *Let χ_1 be the solution of* (4.13′). *Then there is a constant C independent of ε such that*

$$|\chi_1(x)| \leqq C[\varepsilon^2 - |x - x_0|^2] \quad \textit{in } D(x_0, \varepsilon), \tag{4.14}$$

provided $D(x_0, \varepsilon)$ is contained in D.

PROOF. Formulas (4.10) and (4.10′) assure us that $f(x) + 2\mu\theta$ is less than some positive constant C in absolute value over all disks $D(x_0, \varepsilon)$ which are contained in D. Hence the estimate (4.14) follows directly from Dirichlet's principle and Lemma 4.4.

Next, we derive a pointwise lower bound for the solution χ_2 in (4.13″). To this end, we first establish

LEMMA 4.6. *Problem (4.13″) has a unique solution and it is given by the formula,*

$$\chi_2(x) = k \int_{\lambda(x_0, \varepsilon)} G(x, \xi) \sin \beta(\xi)\, ds_\xi , \tag{4.15}$$

where $G(x, \xi)$ is the Green function of the Laplacian operator in $D(x_0, \varepsilon)$ and ds_ξ is the element of arc length along $\lambda(x_0, \varepsilon)$.

For the proof, we simply verify that χ_2 so defined satisfies equation (4.13″). Since the details are similar to that in Lemma 4.3, we omit them here.

We wish to derive certain estimate for χ_2 from the integral representation in (4.15). To do this, we recall [8, p. 248] that

$$G(x, \xi) = \frac{1}{2\pi} \log \left\{ \frac{|x_0 - \xi|}{\varepsilon} \frac{|x - \xi'|}{|x - \xi|} \right\} \tag{4.16}$$

where $|x - x'| \equiv \mathrm{dist}(x, x')$, $\xi' - x_0 \equiv \varepsilon^2(\xi - x_0)/|\xi - x_0|^2$.

Now, choose a plane polar coordinate system (r, ω) with the origin at x_0 and with the initial line containing segment $x_0 x$. Then, $G(x, \xi)$ can be written as

$$\begin{aligned} g(x, \xi) &= \\ &= \frac{1}{2\pi} \log \left\{ \frac{|\xi - x|}{\varepsilon} \left[\frac{|x - x_0|^2 + |\xi' - x_0|^2 - 2|x - x_0||\xi' - x_0| \cos \omega}{|x - x_0|^2 + |\xi - x_0|^2 - 2|x - x_0||\xi - x_0| \cos \omega} \right] \right\} . \end{aligned} \tag{4.17}$$

For fixed x in $D(x_0, \varepsilon)$, $G(x, \xi)$ is a function of the variable ξ alone. Hence, if x is fixed, then G is a function of the variable $|\xi - x_0|$ and ω. For our purpose, we like to know for fixed x where $G(x, \xi)$ achieves it maxima and minima on each of the circles $|\xi - x_0| = \text{constant}$.

LEMMA 4.7. *For arbitrarily fixed x in $D(x_0, \varepsilon)$, let r_0 be any number such that $|x - x_0| \neq r_0 < \varepsilon$. Then, the restriction of $G(x, \xi)$ to the circle $|\xi - x_0| = r_0$ achieves its maximum at the point $\omega = 0$ and its minimum at $\omega = \pi$.*

PROOF. Based on the explicit expression in (4.17), direct differentiation with respect to the variable ω give

$$\begin{aligned} \frac{\partial}{\partial \omega} G(x; |\xi - x_0|, \omega) = \frac{1}{2\pi} \frac{|x - x_0|}{|x - \xi|^2 |x - \xi'|} [|\xi - x_0| - |\xi' - x_0|] \cdot \\ \cdot [|\xi - x_0||\xi' - x_0| - |x - x_0|^2] \sin \omega , \end{aligned}$$

which vanishes only at $\omega = 0$ and $\omega = \pi$, because

$$|x - \xi| \neq 0 , \quad |\xi - x_0| < \varepsilon , \quad |\xi - x_0| - |\xi' - x_0| < 0$$

$$|\xi - x_0||\xi' - x_0| = \varepsilon^2 > |x - x_0| .$$

Consequently, on the circle $|\xi - x_0| = r_0$, $G(x; |\xi - x_0|, \omega)$ achieves the extreme values at $\omega = 0$ and $\omega = \pi$. It is a matter of computation to check that

$$G(x; |\xi - x_0|, 0) > G(x; |\xi - x_0|, \pi) .$$

The lemma is justified.

PROOF OF THEOREM 4.2 FOR THE GENERAL CASE. We are now ready to prove the theorem when x_0 is a regular or a branch point of Λ. From formula (4.15), we have

$$\chi_2(x) = k \sin\bar{\beta} \cdot \int_{\lambda(x_0, \varepsilon)} G(x, \xi)\, ds_\xi , \tag{4.18}$$

where $\bar{\beta}$, $0 < \bar{\beta} \leq \pi/2$, is determined by the mean-value theorem for the integrals. Note that we have made use of the continuity and the positivity of $\beta(x)$ along $\lambda(x_0, \varepsilon)$. In the plane polar coordinate system (ϱ, ω) with the origin at x_0, we consider the « *circular projection* » of the point $(|\xi - x_0|, \omega)$ on $\lambda(x_0, \varepsilon)$ onto the point $(|\xi - x_0|, \pi)$ along the circular arc $\varrho = |\xi - x_0|$. By means of such a circular projection, the elements of the arcs ds_ξ in (4.18) becomes $d\varrho$ with $\varrho = |\xi - x_0|$ and $G(x; |\xi - x_0|, \omega)$ becomes $G(x; \varrho, \pi)$. Of course, all the intersection points of $\lambda(x_0, \varepsilon)$ with $\varrho = |\xi - x_0|$, which are only finite many, are carried into a single point (ϱ, π) by means of the circular projection. It is this fact together with Lemma 4.7 that leads to the useful estimate

$$\chi_2(x) \geq k(\sin\bar{\beta}) \int_0^\varepsilon G(x; \varrho, \pi)\, d\varrho = \frac{1}{2\pi} k(\sin\bar{\beta}) \int_0^\varepsilon \log \frac{\varrho(r + \varrho')}{\varepsilon(r + \varrho)}\, d\varrho ,$$

where $r \equiv |x - x_0|$, $\varrho \equiv |\xi - x_0|$ and $\varrho' \equiv |\xi' - x_0|$. Performing an integration by parts on the last integral, we find

$$\begin{aligned} \chi_2(x) &\geq \frac{1}{2\pi} k(\sin\bar{\beta}) \int_0^\varepsilon \frac{\varrho')\varepsilon^2 - r^2}{(r + \varrho)(r + \varrho')}\, d\varrho \\ &= \frac{1}{2\pi} k(\sin\bar{\beta}) \int_0^\varepsilon \frac{\varepsilon^2 - r^2}{(r + \varrho)(r\varrho + \varepsilon^2)}\, d\varrho \\ &\geq \frac{1}{2\pi} k(\sin\bar{\beta}) \frac{\varepsilon^2 - r^2}{4\varepsilon^3} \int_0^\varepsilon \varrho\, d\varrho = \frac{k}{16\pi} (\sin\bar{\beta}) \left(\frac{\varepsilon^2 - r^2}{\varepsilon}\right). \end{aligned} \tag{4.19}$$

Finally, it follows from (4.13), (4.14) and (4.19) that in the disk $D(x_0, \varepsilon)$

$$\Phi - \Phi_\varepsilon = \chi_1 + \chi_2 \geq (\varepsilon^2 - |x - x_0|^2) \left[\frac{1}{\varepsilon} \frac{k}{16\pi} \sin\bar{\beta} - C\right]. \tag{4.20}$$

Since x_0 is not an end point of Λ, β is greater than some positive constant β_0 on $\lambda(x_0, \varepsilon)$ and the lower bound β_0 can be chosen to be independent of all small ε. Hence, $\sin\bar{\beta} > \sin\beta_0 > 0$. On the other hand, Lemma 3.5 ensures that the constant C in (4.20) is independent of ε too. Thus, the value of the last expression in (4.20) will be positive provided

$$\varepsilon \leqq k \sin\beta_0/16C\pi \equiv \varepsilon_0 .$$

Theorem 4.2 is now proved for the general case.

PROOF OF THEOREM 4.2 FOR THE PARTICULAR CASE. We consider the particular case for which x_0 is an end point of Λ. Let s_0 be a point on ∂D such that $\mathrm{dist}(x_0, s_0) = \mathrm{dist}(x_0, \partial D)$. As was mentioned in the general case, for all points along the segment s_0x_0,

$$\Delta\Phi(x) = -k\varkappa(s)/[1 - t\varkappa(s)], \tag{4.21}$$

where $t \equiv \mathrm{dist}(x, s_0)$ and $\varkappa(s_0) > 0$ is the curvature of ∂D at s_0. In view of assumption A_2) in § 1, there are only two possibilities:

Subcase (i). $\varkappa(s_0)$ is a proper local maximum. That is, $\varkappa(s_0) > \varkappa(s)$ for all points s sufficiently close to s_0. Under this circumstances, $\Delta\Phi \to \infty$ as $x \to x_0$ along the segment s_0x_0, because $\mathrm{dist}(x_0, s_0) = 1/\varkappa(s_0)$. On the other hand, the angle $\beta(x)$ goes to zero along the arc $\lambda(x_0, \varepsilon)$. However, if ε is less than some fixed number δ_0 which depends on x_0, then $D(x_0, \varepsilon)$ is completely covered by the inwards to ∂D near the point s_0. In fact, the radii of curvatures at all points s near s_0 are greater than $1/\varkappa(s_0)$ and these inward normals meet along the single smooth arc $\lambda(x_0, \varepsilon)$ issuing from x_0. Consequently, if $\varepsilon < \delta_0$ is small enough, we conclude from the continuity of $\varkappa(s)$ and from the fact that $\mathrm{dist}(x_0, s_0) = 1/\varkappa(s_0)$ that

$$\Delta\Phi < -k/2\varepsilon \quad \text{in } D(x_0, \varepsilon). \tag{4.22}$$

Since $\Phi - \Phi_\varepsilon$ satisfies equation (4.9″) in the sense of distribution and since $\Phi - \Phi_\varepsilon$ vanishes on $\partial D(x_0, \varepsilon)$, we have

$$\Phi(x) - \Phi_\varepsilon(x) = k\int_{\lambda(x_0, \varepsilon)} G(x, \xi)\sin\beta(\xi)\, ds_\xi \tag{4.23}$$

$$+\int_{D(x_0, \varepsilon)} G(x, \xi)\Delta\Phi(\xi)\, d\xi - \mu\theta(\varepsilon^2 - |x - x_0|^2) \geqq \frac{k}{2\varepsilon}\int_{D(x_0, \varepsilon)} G(x, \xi)\, d\xi - \mu\theta(\varepsilon^2 - |x - x_0|^2),$$

because $\beta(\xi) > 0$, $G(x, \xi) > 0$ and $\Delta\Phi(\xi) < -k/2\varepsilon$ in $D(x_0, \varepsilon)$. Applying

Lemma 4.7 and the similar estimate for deriving (4.19), we find

$$\int_{D(x_0,\varepsilon)} G(x,\xi)\,d\xi \geqq \frac{1}{16\pi\varepsilon}\left(\varepsilon^2-|x-x_0|^2\right). \tag{4.24}$$

It follows from (4.23) and (4.24) that

$$\Phi(x)-\Phi_\varepsilon(x) \geqq \left(\varepsilon^2-|x-x_0|^2\right)\left[\frac{k}{32\pi\varepsilon}-2\mu\theta\right] \tag{4.25}$$

which is clearly positive if $\varepsilon < k/32\pi\mu\theta$. Thus, Theorem 4.2 is justified for subcase (i).

Subcase (ii). ∂D contain a circular arc with the point x_0 as its center. Denote the arc length of ∂D also by s. Then, there are two numbers s_1 and s_2 such that $s_1 < s_0 < s_2$ and that

$$\varkappa(s) = \varkappa(s_0) \quad \text{for all } s \text{ in } [s_1, s_2].$$

This implies that the limit $\beta(x_0)$ of $\beta(x)$ as $x \to x_0$ along $\lambda(x_0, \varepsilon)$ is strictly positive and hence $\beta(x) \geqq \beta_0 > 0$ on $\lambda(x_0, \varepsilon)$. Moreover, β_0 can be chosen to be independent of ε. Furthermore, there is a sector Σ_0 in $D(x_0, \varepsilon)$ which contains the sector bounded by the segments $x_0 s_1$ and $x_0 s_2$ such that

$$0 > \Delta\Phi = -\frac{k\varkappa(s)}{1-t\varkappa(s)} \geqq \frac{k\varkappa(s_0)}{1-t\varkappa(s_0)} \quad \text{in } \Sigma_0 \tag{4.26}$$

and that for some constant independent of ε,

$$|\Delta\Phi| \leqq \text{constant in } D(x_0,\varepsilon)\setminus\Sigma_0. \tag{4.27}$$

It follows from (4.9″), (4.26) and (4.27) that

$$\begin{aligned}\Phi(x)-\Phi_\varepsilon(x) &= k\int_{\lambda(x_0,\varepsilon)} g(x,\xi)\sin\beta(\xi)\,ds_\xi \\ &\quad -\int_{D(x_0,\varepsilon)} G(x,\xi)\,\Delta\Phi(\xi)\,d\xi - \mu\theta\left(\varepsilon^2-|x-x_0|^2\right) \\ &\geqq k\int_{\lambda(x_0,\varepsilon)} G(x,\xi)\sin\beta(\xi)\,ds_\xi - O\left(\varepsilon^2-|x-x_0|^2\right),\end{aligned} \tag{4.28}$$

where C is a constant independent of ε. By similar estimate for deriving (4.19), we have from (4.28) that in $D(x_0, \varepsilon)$,

$$(4.29)\qquad \Phi(x)-\Phi_\varepsilon(x)\geqq \frac{k}{16\pi}(\sin\beta_0)\left[\frac{1}{\varepsilon}\left(\varepsilon^2-|x-x_0|^2\right)\right]-C\left(\varepsilon^2-|x-x_0|^2\right)$$
$$=\left(\varepsilon^2-|x-x_0|^2\right)\left[\frac{k}{16\pi\varepsilon}\sin\beta_0-C\right].$$

The estimates in (4.20), (4.25) and (4.29) now assure us that Theorem 4.2 holds in all possible cases and that the proof is complete.

Since Theorem 4.2 is applicable to every point on the set Λ defined in (4.1), there exists a complete neighbourhood N of Λ in which ψ is analytic and satisfies the Poisson equation $\Delta\psi = -2\mu\theta$ in N. This means that the *material in the elastic core N does obey Hook's law.* In the next section, we shall prove that ψ is smooth in the whole domain D. It will then follow that $|\text{grad}\,\psi|$ is strictly less than k in the elastic core N. Thus, *the material in N also stays below the yield point*, which means that *N is the elastic core in the usual sense.*

Note that for given domain D, the location of $\Gamma(\Phi)$ and $\Gamma(\varphi)$ can be determined without knowing the precise values of ψ on G_j. Accordingly, the relative position of the elastic core in D can be, roughly, determined without precise information about the solution ψ.

5. – Continuity of stress components.

For this problem the only non-vanishing components of the stress are the gradient of the minimizer ψ. Consequently, to assure the continuity of the stress in the entire domain D, it suffices to establish the smoothness of ψ in D. Since we have proved the existence of the elastic core and the analyticity of ψ there, it is enough to show that ψ is smooth in the complement of the elastic core, in D. Since the smoothness of a function is a local problem, we need only to show that for every point x_0 in $D\setminus N$, there exists $\varepsilon_0 > 0$ such that ψ is smooth in the disk $D(x_0, \varepsilon)$ centered at x_0 with radius $\varepsilon \leqq \varepsilon_0$. Of course, $D(x_0, \varepsilon_0)$ should be contained in D.

5.1. *Smoothness of ψ along the edges of the enveloping surfaces.*

Since ψ is known to be smooth along Λ, it suffices to prove that ψ is smooth along

$$\Lambda'(\Phi)\equiv\left(D\cap\Gamma(\Phi)\right)\setminus\Lambda\,,\qquad \Lambda'(\varphi)\equiv\left(D\cap\Gamma(\varphi)\right)\setminus\Lambda\,.$$

Since the proof is the same for both of them, we only carry it out for $\Lambda'(\Phi)$.

THEOREM 5.1. *For every point $x_0 \in \Lambda'(\Phi)$, there exists a positive number ε_0 depending on x_0 such that if $D(x_0, \varepsilon)$ is a disk contained in D centered at x_0 with radius $\varepsilon \leqq \varepsilon_0$, then $\psi < \Phi$ in $D(x_0, \varepsilon)$. Moreover,*

(i) *ψ is analytic and satisfies the equation, $\Delta\psi = -2\mu\theta$, in $D(x_0, \varepsilon)$ provided $\psi(x_0) > \varphi(x_0)$,*

(ii) *ψ is smooth in $D(x_0, \varepsilon)$ if $\psi(x_0) = \varphi(x_0)$.*

PROOF OF (i). To be specific, let $\psi(x_0) - \varphi(x_0) = h$. Since both $\varphi(x)$ and $\psi(x)$ are uniformly Lipschitz continuous in G_0, there exists a positive number δ such that

$$\psi(x) - \varphi(x) > h/2 \quad \text{and} \quad |\varphi(x) - \varphi(x_0)| < h/4 \tag{5.1}$$

provided $|x - x_0| < \delta$. Now, let $D(x_0, \varepsilon)$ be a disk contained in D and centered at x_0 with radius ε. Let ψ_ε be the solution of the problem,

$$\Delta\psi_\varepsilon = -2\mu\theta \quad \text{in } D(x_0, \varepsilon), \qquad \psi_\varepsilon = \psi \quad \text{on } \partial D(x_0, \varepsilon). \tag{5.2}$$

By Theorem 4.1, there is a positive number $\varepsilon_0(x_0)$ such that $\psi_\varepsilon < \Phi$ in $D(x_0, \varepsilon)$ provided $\varepsilon \leqq \varepsilon_0$. On the other hand, as the solution of (5.2),

$$\psi_\varepsilon(x) = \mu\theta\big(\varepsilon^2 - |x_0 - x|^2\big) + \zeta_\varepsilon(x) \quad \text{in } D(x_0, \varepsilon), \tag{5.3}$$

where $\zeta_\varepsilon(x)$ is harmonic in $D(x_0, \varepsilon)$ and equals to ψ on $\partial D(x_0, \varepsilon)$. According to the minimum principle for harmonic functions, there is a point x_m on $\partial D(x_0, \varepsilon)$ such that

$$\zeta_\varepsilon(x_m) = \min_{|x - x_0| = \varepsilon} \zeta_\varepsilon(x) = \min_{|x - x_0| \leqq \varepsilon} \zeta_\varepsilon(x). \tag{5.4}$$

Consequently, for all $\varepsilon \leqq \min(\varepsilon_0, \delta)$, we have from (5.1)-(5.4) that for all x in $D(x_0, \varepsilon)$

$$\begin{aligned}
\psi_\varepsilon(x) - \varphi(x) &= \mu\theta(\varepsilon^2 - x - x_0^2) + \zeta_\varepsilon(x) - \varphi(x) \\
&\geqq \zeta_\varepsilon(x_m) - \varphi(x) \\
&= [\zeta_\varepsilon(x_m) - \varphi(x_m)] + [\varphi(x_m) - \varphi(x)] \\
&\geqq [\psi(x_m) - \varphi(x_m)] - \tfrac{1}{4}h \\
&\geqq \tfrac{1}{2}h - \tfrac{1}{4}h = \tfrac{1}{4}h > 0.
\end{aligned}$$

Thus, if $\varepsilon \leqq \min(\varepsilon_0, \delta)$, then the solution ψ_ε of (5.2) satisfies the inequality $\varphi < \psi_\varepsilon < \Phi$ in $D(x_0, \varepsilon)$. Hence, as was mentioned in the proof of Theorem 4.2, both ψ and ψ_ε minimize the same functional,

$$I[u] = \int_{D(x_0,\varepsilon)} [|\text{grad}\, u|^2 - 4\mu\theta u]\, dx\,,$$

over the same closed convex set,

$$\{u | u \in H_0^1(D(x_0, \varepsilon)),\ u = \psi \text{ on } \partial D(x_0, \varepsilon),\ u \geqq \varphi \text{ in } D(x_0, \varepsilon)\}\,,$$

in $H_0^1(D(x_0, \varepsilon))$. It follows from the uniqueness of the minimizer that $\psi = \psi_\varepsilon$ in $D(x_0, \varepsilon)$. Thus, assertion (i) is proved.

PROOF OF (ii). First, we note that Φ is strictly greater than φ in the multiply connected domain D. In particular,

$$\Phi(x_0) - \varphi(x_0) = \Phi(x_0) - \psi(x_0) = h_0 > 0\,.$$

By continuity of $\varphi(x)$, $\Phi(x)$ and $\psi(x)$ in D, there exists a number $\delta > 0$ such that

$$\Phi(x) - \psi(x) > \tfrac{1}{2} h_0\,, \qquad \psi(x) - \varphi(x) \leqq \tfrac{1}{4} h_0$$

provided $|x - x_0| \leqq \delta$. Let $D(x_0, \varepsilon)$ be a disk contained in D centered at x_0 with radius $\varepsilon \leqq \delta$. With ε so chosen, we keep it fixed and for simplicity in notation, we write $D(x_0) \equiv D(x_0, \varepsilon)$. Clearly,

$$\Phi(x) > \varphi(x) + \tfrac{1}{2} h_0 \geqq \psi(x) \geqq \varphi(x) \quad \text{ in } D(x_0)\,.$$

Accordingly, as a minimizer of problem (2.8), $\psi(x)$ must also minimize the functional.

$$I[u] = \int_{D(x_0)} [|\text{grad}\, u|^2 - 4\mu\theta u]\, dx\,, \tag{5.5}$$

over the closed convex set,

$$\{u | u \in H^1(D(x_0)),\ u = \psi \text{ on } \partial D(x_0),\ \varphi \leqq u \leqq \varphi + \tfrac{1}{2} h_0 \text{ in } D(x_0)\}\,, \tag{6.5'}$$

in $H^1(D(x_0))$.

Now, we restrict our attention to the restriction of ψ to $D(x_0)$. Then, part (ii) of Theorem 6.1 is nothing but

LEMMA 5.2. *As the solution of problem* (6.5), *the first derivatives of* ψ *are Hölder-continuous in* $D(x_0)$.

This regularity result is a special case of the general regularity theorem for the solutions of variational inequalities which has attracted a great deal of attention [1, 3, 7, 13, 16, 18]. Nevertheless, for the simple case here, the proof can be made more elementary as was carried out in [18, *c*].

5.2. *Smoothness of ψ off the edges of the enveloping surfaces.*

Theorems 4.1 and 5.1 assure us that ψ is smooth in a complete neighbourhood N of $D \cap \Gamma(\Phi)$ and $D \cap \Gamma(\phi)$. We may assume that ψ is smooth in $\bar{N}$, the closure of N.

THEOREM 5.2. *For every point x_0 in $D \setminus \bar{N}$, there exists a positive number ε_0 such that if $D(x_0, \varepsilon)$ is a disk contained in D centered at x_0 with radius $\varepsilon \leqq \varepsilon_0$, then*

(i) *ψ is analytic and satisfies the equation $\psi = -2\mu\theta$ in $D(x_0, \varepsilon)$ if $\varphi(x_0) < \psi(x_0) < \Phi(x_0)$,*

(ii) *ψ is smooth in $D(x_0, \varepsilon)$ if either $\psi(x_0) = \varphi(x_0)$ or $\psi(x_0) = \Phi(x_0)$.*

The proof for both part (i) and part (ii) is essentially the same as that for Theorem 5.1 with slight modifications. Hence, it is omitted.

6. – Existence of a strict solution.

We are now in a position to prove

THEOREM 6.1. *The minimizer ψ of problem* (1.6) *is a solution of the problem stated in the introduction.*

PROOF. Theorem 2.2 ensures that problems (1.6) and (2.8) have identical solution and Theorems 4.1, 5.1 and 5.2 guarantee that the minimizer ψ is smooth in the entire domain D. Hence, as was required in problem (1.6),

$$|\mathrm{grad}\,\psi| \leqq k \quad \text{everywhere in } D. \tag{6.1}$$

Consider the open set,

$$E = \{x | x \text{ in } D,\ \varphi(x) < \psi(x) < \Phi(x)\}, \tag{6.2}$$

in D. According to Theorems 4.1, 5.1 and 5.2, ψ is analytic and satisfies the equation,

$$\Delta\psi = -2\mu\theta \quad \text{in } E. \tag{6.3}$$

On the other hand, if we define

$$P = D \setminus E, \tag{6.4}$$

then either $\psi(x) = \varphi(x)$ or $\psi(x) = \Phi(x)$ in P. Hence, if the interior of P is non-empty, then either

$$\begin{aligned} |\text{grad}\,\psi| &= |\text{grad}\,\varphi| = k \quad \text{in } P \text{ or} \\ |\text{grad}\,\psi| &= |\text{grad}\,\Phi| = k \quad \text{in } P. \end{aligned} \tag{6.5}$$

From the reasoning given in §§ 4 and 5, it is not hard to see that the inward normal derivatives of ψ at all regular points of ∂D are uniquely defined and their absolute values are bounded from above by k. At the reentrant corners of D, the derivatives of ψ along each inward normal there are also uniquely defined and bounded from above by k in absolute value. At the non-reentrant corners of D, the Dini derivatives of ψ there are also bounded from above by k in absolute value.

This is so, because $\psi = \varphi = \Phi$ on ∂D and $\varphi < \psi < \Phi$ near these corners. From these observations and the smoothness of ψ in D, we conclude that

$$|\text{grad}\,\psi| \leqq k \quad \text{everywhere along } \partial E.$$

Since ψ satisfies the Poisson equation in E, direct computation gives

$$\Delta |\text{grad}\,\psi|^2 > 2(\psi_{11}^2 + 2\psi_{11}\psi_{22} + \psi_{22}^2) \quad \text{in } E, \tag{6.6}$$

where $\psi_{ii} = \partial^2\psi/\partial x_i \partial x_i$. Consequently, the maximum principle applied to $|\text{grad}\,\psi|$ demands that

$$|\text{grad}\,\psi| < k \quad \text{everywhere in } E. \tag{6.7}$$

The result in (6.1)-(6.7) now completes our justification.

7. – Elastic-plastic partition of the cross section.

According to the theory of elasticity and plasticity, a set E in D is said to be elastic if the material in E obeys Hook's law and the modulus of the stress stays below the yield point; while a set P in D is plastic if the modulus of the stress is identically equal to k. Accordingly,

$$E = \{x | x \in D,\ |\mathrm{grad}\,\psi| < k \text{ and } \Delta\psi = -2\mu\theta\}\,,$$
$$P = \{x | x \in D,\ |\mathrm{grad}\,\psi| = k\}\,.$$

Looking back at the proof of Theorem 6.1, we find

7.1. *An equivalent characterization for E and P.*

THEOREM 7.1.

$$E = \{x | x \in D,\ \varphi(x) < \psi(x) < \Phi(x)\}\,,$$
$$P = \{x | x \in P,\ p(x) = \varphi(x) \text{ or } \psi(x) = \Phi(x)\}\,. \tag{7.1}$$

It is this characterization that leads to interesting informations about the sets E and P. First, Theorems 4.1, 5.1 and 5.2 imply that the elastic cores are all contained in the elastic zone E and hence *they are elastic in the sense as was just stated.* Secondly, for a given multiply connected domain the common lines of discontinuities $\Gamma(\Phi) \cap \Gamma(\varphi)$, can be located without knowing the precise values of the minimizer ψ on the subdomains G_i's. Consequently, the existence of the elastic cores which contain $D \cap \Gamma(\Phi) \cap \Gamma(\varphi)$ tells us the *relative position of E in D as well as its extent in D.*

7.2. *Adherence of P to ∂D.*

Since $\varphi = \psi = \Phi$ on ∂D, if we add ∂D to the plastic zone P, then P is a closed set. Consider an inward normal at a regular point of ∂D or any one of the inward normals at a reentrant corner of ∂D.

THEOREM 7.2. *The intersection of P with any inward normal to ∂D is either a single point on ∂D or a single segment with one end point on ∂D.*

PROOF. For if this were not the case, then, as a consequence of the mean-value theorem in differential calculus, we would have a contradiction with the fact that $|\mathrm{grad}\,\psi| \leqq k$ in D and $|\mathrm{grad}\,\psi| < k$ in E by considering the variations of ψ along the inward normals to ∂D.

We proceed to derive the consequences of this intersection property of P with the inward normals of D. Clearly, it demands that *P always adheres to ∂D.* In particular, it implies that *yielding starts from the boundary during the «loading» process, which is what we would expect.* As another consequence of Theorem 7.2 we have

7.3. *A parametric representation of the elastic-plastic boundary.*

Denote the arc length of ∂D and a point on ∂D by the same letter s. If the inward normal to ∂D at s interesects P in a segment, then we denote the length of this segment by $R(s)$. By elastic-plastic boundary, we mean the set of points,

$$\Sigma \equiv \partial P \cap D \equiv \partial E \cap D . \tag{7.2}$$

This set is always non-empty, if θ is sufficiently large.

We proceed to decompose Σ *into a finite number of disjoint Jordan arcs.* From the characterization of P in (7.1), we have the decomposition,

$$\Sigma = \Sigma^+ + \Sigma^- \tag{7.3}$$

such that $\psi = \Phi$ on Σ^+ and $\psi = \varphi$ on Σ^-. Since $\varphi < \Phi$ in D, $\Sigma^+ \cap \Sigma^-$ is, indeed, empty.

Consider the edges $\Gamma(\Phi)$ of the upper enveloping surface defined by Φ. Since the inward normals to ∂D that meet along $\Gamma(\Phi)$ completely cover D without overlap, it follows that $D \setminus \Gamma(\Phi)$ consists of a finite number of components, say

$$D_1^+, D_2^+, \ldots, D_m^+ .$$

By Theorems 4.1 and 5.1, ψ is strictly less than Φ along $D \cap \Gamma(\Phi)$. This means that $\Sigma^+ \cap (D \cap \Gamma(\Phi))$ is empty. Consequently, the sets

$$\Sigma^+ \cap D_1^+ , \quad \Sigma^+ \cap D_2^+ , \ldots, \Sigma^+ \cap D_m^+ , \tag{7.4}$$

are mutually disjoint, i.e., any two of them are disjoint. With obvious notations and by the same reasoning, the sets

$$\Sigma^- \cap D_1^- , \quad \Sigma^- \cap D_2^- , \ldots, \Sigma^- \cap D_p^- , \tag{7.4$'$}$$

are also mutually disjoint. Of course, any two sets, one from (7.4) and the other from (7.4′) are also disjoint.

Having established the decompositions in (7.3), (7.4) and (7.4′), we restrict our attention to any one of these sets, say $\Sigma^+ \cap D_1^+$. For any point $x \equiv (x_1, x_2)$ on $\Sigma^+ \cap D_1^+$, there is a point s on ∂D such that

$$R(s) \equiv \text{dist}\,(x, s) = \text{dist}\,(x, \partial D) .$$

Let $\partial D_1^+ \cap \partial D$ be defined by the equations,

$$x_1 = f(s) , \quad x_2 = g(s) . \tag{7.5}$$

Then, in view of Theorem 7.2, $\Sigma^+ \cap D_1^+$ has the parametric representation,

$$x_1 = f(x) + R(s)n_1(s), \quad x_2 = g(x) + R(s)n_2(s), \tag{7.6}$$

where $n_1(s)$ and $n_2(s)$ are the components of the unit inward normal to ∂D at s.

By assumptions A_1) and A_2) in § 1, the functions f and g in (7.6) are, of course, piece-wise twice differentiable. Moreover, the same proof given in [8, (*b*), pp. 546-550] for the continuity of the function $R(s)$ in (7.6), can be applied here without any changes. In fact, in that proof, no convexity of the domain has been used. Thus, we have

THEOREM 7.3. *The elastic-plactic boundary Σ consists of a finite number of Jordan arcs.*

In fact, the continuity of each component of the sets in (7.4) and (7.4′) follows from the formulas in (7.6) and the continuity of $R(s)$ as a function of s. The fact that each set in (7.4) and (7.4′) consists of finite number of components can be proved in the same way as for the lower semi continuity of $R(s)$ given in [18, (*b*)].

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