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On Nonparametric Surfaces of Constant Mean Curvature. (*)

ROBERT FINN (**) - ENRICO GIUSTI (***)

dedicated to Jean Leray

Let $u(x)$ define a surface of mean curvature $H \equiv 1$ over an n -dimensional domain \mathfrak{D} , that is, let $u(x)$ be a solution in \mathfrak{D} of

$$(1) \quad \operatorname{div} Tu = n, \quad Tu = \frac{1}{W} \nabla u, \quad W = \sqrt{1 + |\nabla u|^2}$$

It was observed by S. Bernstein [1], by E. Heinz [2] and by S. S. Chern [3] that \mathfrak{D} cannot strictly contain a closed ball \bar{B}_R of radius $R \geq 1$; in fact, an integration of (1) over B_R yields

$$(2) \quad n|B_R| = \oint_{\partial B_R} \nu \cdot Tu d\sigma$$

where ν is the exterior directed unit normal. Since $|Tu| < 1$, we find from (2)

$$n\tau R^n < \omega R^{n-1}$$

where τ and ω are volume and surface of the unit n -ball, and thus $R < 1$.

Finn [4] showed that if \mathfrak{D} contains the open ball B_1 then \mathfrak{D} coincides with this ball and $u(x)$ describes a lower hemisphere, $u(x) = u_0 - \sqrt{1 - |x|^2}$; that is, the open ball B_1 admits, essentially, only a single solution of (1).

This result leads naturally to the conjecture that the manifold of such surfaces defined over B_R is progressively more restricted as $R \nearrow 1$. Speci-

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fically, Finn conjectured that if a solution $u(x)$ of (1) is defined over B_R and if $R > 1/n$, then all derivatives of $u(x)$ are bounded at the center of B_R , the bound depending only on R and in no other way on the function $u(x)$. Further, as $R \rightarrow 1$ all solutions in B_R should tend in any fixed $B_{\hat{R}}$, uniformly with all derivatives, to the lower hemisphere.

We note the hypothesis $R > 1/n$ is necessary; the function

$$u = \frac{1}{\cos \alpha} \left\{ x_1 \sin \alpha - \sqrt{\frac{1}{n^2} - \sum_2^n x_i^2} \right\}$$

which defines a cylinder inclined with angle $\alpha \neq \pi/2$ to the plane $u = 0$, covers the ball $B_{1/n}$ for any α , but $|\nabla u(0)| \rightarrow \infty$ as $\alpha \rightarrow \pi/2$.

In the present paper we establish the conjecture in the case $n = 2$ for all R exceeding a critical value $R_0 \sim 0.5654062332 \dots$, and we show the conjecture fails if $R < R_0$.

Our procedure is to compare a given solution $u(x)$ of

$$(3) \quad \operatorname{div} Tu = 2$$

with a particular solution $v(x)$, which will be chosen to majorize the gradient of $u(x)$ at the center $x = 0$. We obtain such a $v(x)$ as solution of a « capillary problem » in a « moon domain » \mathcal{D} as indicated in Figure 1. This solu-

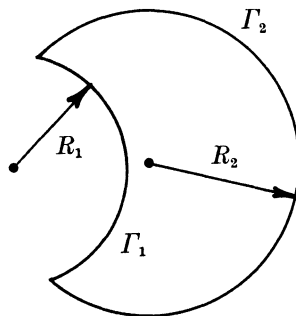


Figure 1

tion has also an independent interest as a solution of (3) in a characteristic domain corresponding to singular data, analogous to the surfaces described by Spruck in [5].

The comparison method was used previously by Finn [6] to obtain a gradient bound for minimal surfaces $z(x, y)$ depending on a bound for $|z|$.

In that case the comparison function was accordingly determined as the solution of a particular Dirichlet problem, depending on the solution z . In the present work we require a « universal » comparison function, suitable for any conceivable solution of (3). The natural defining condition appears in the « capillary problem », in which the solution surface is required to make a prescribed angle with the cylinder projecting onto the boundary of the domain of definition. We prescribe the respective angles 0 and π on the two arcs of \mathfrak{D} , and we show that the corresponding solution $v(x)$ serves as a universal majorant for the gradient.

We prove the existence of the majorizing solution in § III. In § I we apply the solution, in a canonical configuration, to obtain the gradient bound, for any solution of (3) defined in a disk B_R of radius $R > R_0$. Our estimates are, at least in principle, explicit. In § IV we show by a direct construction that the choice R_0 cannot be improved.

The method as described above yields a gradient bound in a disk B_{R-R_0} but provides no information as to what happens outside this disk. We show in § II by an indirect reasoning that as $R \rightarrow 1$ any solution in B_R , together with all its derivatives, must approximate a lower hemisphere in any compact subdomain. Here the convergence estimate from above is obtained simply and explicitly starting with the gradient estimate in B_{R-R_0} . We found the estimate from below, however, to be more difficult.

The results of this paper are evidently related to S. Bernstein's theorem [7] that a minimal surface $z(x, y)$ defined over the (x, y) plane is itself a plane. The result of Finn cited above presents a natural analogue of that theorem for surfaces of prescribed mean curvature; the present paper in turn extends the result to surfaces defined over a domain « close » to the maximal domain of definition.

The analogous extension for minimal surfaces could perhaps be regarded as the a priori bound on second derivatives, due originally to Heinz, cf. [8, 9, 10]. This type of bound has been obtained also for surfaces of prescribed mean curvature by Spruck [11, 12]; it does not, however, seem to imply the more precise information contained in the present results.

In the interest of simplicity we present all results for the case of constant mean curvature $H \equiv 1$. The case of general constant $H \neq 0$ is obtained by a similarity transformation. Without essential change a corresponding result can be obtained also for variable $H(x)$ bounded from zero, under reasonable smoothness hypotheses; we shall not go into this matter here in detail. It seems likely that an analogue of the equation class studied in [13, 6] will also be accessible to the method. We have not however investigated this direction.

Most of the results in this paper have been obtained independently by

E. Bombieri. His method, although indirect, extends to any dimension n and yields the same value for R_0 if $n = 2$. His paper will appear elsewhere.

It is a pleasure for us to thank E. Bombieri for a number of helpful comments. We are indebted also to Giovanni Giusti, who generously sacrificed his paper airplane so that his father could read his co-author's proposal for the proof of a theorem.

I. – The gradient estimate.

In this section we assume Theorem 4 and use it to prove the central gradient bound.

LEMMA I.1. *Let $u(x)$ be a solution of (3) in a disk $B_R: |x| < R$. Let \mathcal{D} be a domain with the properties:*

- a) *the boundary $\partial\mathcal{D} = \Gamma_1 + \Gamma_2 + p_0 + q_0$, where Γ_1, Γ_2 are relatively open and p_0, q_0 are points;*
- b) *there is an arc $S \subset \mathcal{D}$ and extending to Γ_1 , such that the maximum distance of any point $p \in S$ to $\partial\mathcal{D}$ is less than R ;*
- c) *there exists a bounded solution $v(x)$ of (3) in \mathcal{D} , such that $v \cdot Tv \rightarrow -1$ for any approach to Γ_1 , $v \cdot Tv \rightarrow +1$ for any approach to Γ_2 .*

Then $|\nabla u(0)| < \min_{p \in S} |\nabla v(p)|$.

We note no condition is imposed on $v(x)$ at p_0, q_0 .

PROOF OF LEMMA I.1. We may assume $\nabla u(0) \neq 0$ as otherwise there is nothing to prove. Let $p \in S$, and place \mathcal{D} over B_R with p at the center. Then by b), $\mathcal{D} \subset B_R$. We may rotate \mathcal{D} about p so that the gradients of u and of v coincide in direction. If $|\nabla u| > |\nabla v|$ at p we move \mathcal{D} continuously so that the origin moves along S in the direction Γ_1 (see Fig. 2). Since

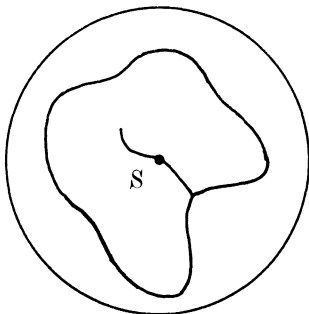


Figure 2

$|\nabla v| = \infty$ on Γ_1 there must be a point $p_1 \in S$ at which (after a possible rotation of \mathcal{D}) $\nabla u = \nabla v$.

Set $w = u - v$. We may assume $w(0) = 0$. There cannot hold $w \equiv 0$ as $|\nabla v| \rightarrow \infty$ at Γ_1 . We assert that in the indicated configuration there is an integer $m \geq 2$ and a neighbourhood of p_1 that is divided by (smooth) level curves $w = 0$ through p_1 into $2m$ distinct regions $\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(2m)}$, and such that $w < 0$ in $\mathcal{D}^{(2i)}$, $w > 0$ in $\mathcal{D}^{(2i+1)}$. The proof of a corresponding assertion for minimal surfaces given in [6] applies without change to the present situation and we shall not repeat it here.

We use the same symbols $\mathcal{D}^{(i)}$ to denote the connected components containing these regions, in which w does not change sign.

Denote by $\Gamma^{(i)}$ the points of $\partial\mathcal{D}^{(i)}$ that lie on either Γ_1 or Γ_2 . Suppose $\Gamma^{(1)} \cap \Gamma_1 \neq \emptyset$, $\Gamma^{(2)} \cap \Gamma_2 \neq \emptyset$, $\Gamma^{(3)} \cap \Gamma_1 \neq \emptyset$. Then clearly $\Gamma^{(4)} \cap \Gamma_2 = \emptyset$ (see Figure 3). We conclude either there is a region $\mathcal{D}^{(2i)}$ with $\Gamma^{(2i)} \cap \Gamma_2 = \emptyset$ or there is a region $\mathcal{D}^{(2i+1)}$ with $\Gamma^{(2i+1)} \cap \Gamma_1 = \emptyset$.

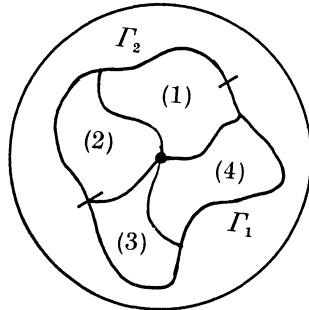


Figure 3

Both cases respond to the same reasoning; it suffices for illustration to consider the situation $\Gamma^{(1)} \cap \Gamma_1 = \emptyset$. Setting

$$\eta(x) = \begin{cases} w(x), & x \in \mathcal{D}^{(1)} \\ 0, & x \notin \mathcal{D}^{(1)} \end{cases}$$

we find immediately

$$(4) \quad \int_{\mathcal{D}} \nabla \eta \cdot \{Tu - Tv\} dx = \int_{\Gamma^{(1)}} \eta \nu \cdot \{Tu - Tv\} d\sigma \leq 0$$

since $\eta \geq 0$ and $\nu \cdot Tv = +1$ on Γ_2 . (We note the singularities of $v(x)$ at p_0, q_0 cause no difficulty in (4), as $|Tf| < 1$ for any function f). The integrand

on the left side of (4) can however be expressed as an integral of positive quadratic forms in the components of $\nabla\eta$, and hence (4) implies $\nabla\eta \equiv 0$ in \mathfrak{D} . This contradiction completes the proof of the lemma.

To apply Lemma I.1 we need to know conditions under which domains \mathfrak{D} and corresponding solutions v can exist. From Theorem 1 of [14] we see that Γ_1, Γ_2 must have curvatures $\kappa_1 > 2, \kappa_2 < 2$, and thus we are led by symmetry considerations to a moon domain \mathfrak{D} as discussed in § III. There must hold the necessary condition, obtained by integrating (3) over \mathfrak{D} ,

$$(5) \quad |I_2| - |I_1| = 2|\mathfrak{D}|.$$

Heuristic considerations indicate that in an extremal situation, Γ_2 will be concentric with B_R and Γ_1 will pass through the center of B_R . In this situation, (5) implies for the radii R_1, R_2 of Γ_1, Γ_2

$$(6) \quad R_2(1 - R_2)(\theta + \pi) - 2R_1\theta = R_2^2 \sin\theta + R_1^2(\sin 2\theta - 2\theta)$$

with

$$(7) \quad \theta = 2 \sin^{-1} \frac{R_2}{2R_1}.$$

One verifies after some calculation that if $R_2(R_1)$ is determined from (6), (7), then $R_2'(R_1) < 0$. Thus, the most favorable geometry occurs if R_1 is as large as possible. If $R_1 = \frac{1}{2}$, then (6), (7) become, setting $R = R_2$,

$$(8) \quad R\sqrt{1-R}(\pi\sqrt{1-R}-\sqrt{1+R}) + [2R(1-R)-1]\sin^{-1}R = 0,$$

which has the solution

$$(9) \quad R_0 = 0.5654062332\dots$$

We refer to the corresponding moon region as \mathfrak{D}_0 .

The configuration \mathfrak{D}_0 is not permitted in Lemma I.1; by Theorem 1 of [14] any comparison solution v would be unbounded. We may, however, choose $R_1 = \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$, in which case $R_2 = R_0 + \eta(\varepsilon)$, $\eta(\varepsilon) \rightarrow 0$ with ε . We show in § III that in the corresponding moon region $\mathfrak{D}_0^\varepsilon$, bounded comparison solutions always exist. Granting this assertion, we choose for p a point on the line L of symmetry of $\mathfrak{D}_0^\varepsilon$ and for S the segment on L joining p to Γ_1 . We obtain

THEOREM 1. *Let $u(x)$ be a solution of (3) in a disk $B_R: |x| < R$. There exists a function $C(R; \varrho) > 0$, non increasing in R , non decreasing in ϱ and*

finite for $R > R_0$, $\varrho < R - R_0$, such that if $R > R_0$ there holds

$$(10) \quad |\nabla u(x)| < C(R; |x|)$$

at all points in the disk B_{R-R_0} .

The considerations of § III permit an explicit calculation of a suitable $C(R; \varrho)$. We show in § IV that the value R_0 cannot be improved.

II. - Behavior as $R \rightarrow 1$.

The proof of Theorem 1 yields information at most in a disk B_{1-R_0} of radius $1 - R_0 \sim 0.434$. We proceed to establish a bound in any fixed $B_{\hat{R}}$, as $R \rightarrow 1$.

II.1. - An upper bound can be found explicitly if $R > (1 + R_0)/2$. We obtain it by comparing the solution with a rotationally symmetric solution of the same equation.

LEMMA II.1. - *There exists a unique (rotationally symmetric) solution $\psi^{(\varepsilon)}(x)$ of (3), defined in the annulus $\mathcal{A}_\varepsilon: \varepsilon < |x| < 1 - \varepsilon$, such that $\nu \cdot T\psi^{(\varepsilon)} = \mp 1$ on the bounding circles $\Gamma_\varepsilon, \Gamma_{1-\varepsilon}$ of \mathcal{A}_ε , and such that $\psi^{(\varepsilon)} = 0$ on Γ_ε . In any fixed $B_{\hat{R}}$, $\psi^{(\varepsilon)}(x)$ tends uniformly to the lower hemisphere $\psi(x) = 1 - \sqrt{1 - r^2}$ as $\varepsilon \rightarrow 0$.*

The proof of the lemma is a formal exercise and we suppress details. A typical $\psi^{(\varepsilon)}(x)$ is indicated in Figure 4. The most general $\psi^{(\varepsilon)}(x)$ can be determined explicitly in terms of elliptic integrals.

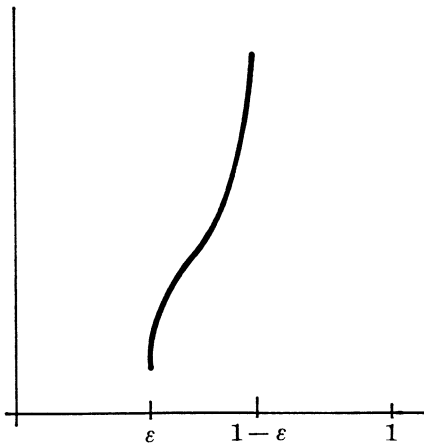


Figure 4

THEOREM 2. *Let $u(x)$ be a solution of (3) in B_R , $R > \frac{1}{2}(1 + R_0)$, and suppose $u(0) = 0$. There exists a continuous decreasing function $\sigma(R)$ defined in $\frac{1}{2}(1 + R_0) < R \leq 1$, with $\sigma(1) = 0$, such that $u(x) \leq \psi(x) + \sigma(R)$ throughout B_R .*

PROOF. By Theorem 1, there holds $|\nabla u| < C(R; \varepsilon)$ in B_ε , for any $\varepsilon < R - R_0$. Hence

$$u(x) < |x|C(R; \varepsilon)$$

in B_ε .

Choose ε in the range $1 - R < \varepsilon < \frac{1}{2}(1 - R_0)$. Then $\mathcal{A}_\varepsilon \subset\subset B_R$, and $u(x) \leq \varepsilon C(R; \varepsilon)$ on Γ_ε .

We show first $u(x) \leq \psi^{(\varepsilon)}(x) + \varepsilon C(R; \varepsilon)$ throughout \mathcal{A}_ε : For if there were an open set $A \subset \mathcal{A}_\varepsilon$ in which $u > \psi^{(\varepsilon)} + \varepsilon C(R; \varepsilon)$ we set

$$\eta = \begin{cases} u - \psi^{(\varepsilon)} - \varepsilon C(R; \varepsilon) & \text{in } A \\ 0 & \text{outside } A \end{cases}$$

and find

$$(11) \quad \int_A \nabla \eta \cdot [Tu - T\psi^{(\varepsilon)}] dx = \int_{\Gamma_{1-\varepsilon}} \eta \nu \cdot [Tu - T\psi^{(\varepsilon)}] d\sigma.$$

As in the proof of Lemma I.1, we are led to a contradiction because of the boundary condition for $\psi^{(\varepsilon)}$. Thus

$$(12) \quad u(x) \leq \psi^{(\varepsilon)}(x) + \varepsilon C(R; \varepsilon)$$

in \mathcal{A}_ε ; defining $\psi^{(\varepsilon)}(x) \equiv 0$ in B_ε , (12) will then hold a fortiori throughout $B_{1-\varepsilon}$.

Now let $\varepsilon \rightarrow 1 - R$; we obtain in the limit

$$(13) \quad u(x) \leq \psi^{(1-R)}(x) + (1 - R)C(R; 1 - R)$$

throughout B_R . Writing

$$(14) \quad \sigma(R) = \max_{\mathcal{A}_{1-R} \cup \Gamma_{1-R}} [\psi^{(1-R)}(x) - \psi(x)] + (1 - R)C(R; 1 - R)$$

we obtain

$$(15) \quad u(x) \leq \psi(x) + \sigma(R),$$

the result to be proved.

II.2. — We obtain a lower bound by an indirect reasoning. Consider a sequence of domains B_{R_j} , $R_j \rightarrow 1$, and a corresponding sequence of solutions $u^j(x)$ in B_{R_j} . We may suppose $u^j(0) = 0$, all j .

Using Theorem 1, we conclude there is a subsequence—which we again denote by $u^j(x)$ —that converges in B_{1-R_0} to a solution of (3).

Denote by Δ the set of all points in B_1 at which some subsequence of the u^j will converge to a finite limit. Let $x \in \Delta$. Then in particular, $u^j(x) > -K > -\infty$ for some subsequence. But by Theorem 2, $u^j(x)$ is bounded above in B_{R_j} . From the Harnack inequality (see Serrin [15]) we conclude there is a disk $D_K(x)$ centered at x and of radius not depending on j , such that $u^j(x) > -K - 1$ in $D_K(x)$. A particular consequence is that Δ is open.

Let $v \subset \subset \Delta$. Then v can be covered by a finite number of disks $D_{K_i}(x^i)$, $i = 1, \dots, m$. Thus, there is a subsequence $u^j(x)$ such that $|u^j(x)| < K_\nu < \infty$ on v .

Let v_k be the set of points $x \in \Delta$ whose distance from $\partial\Delta$ exceeds $1/k$. The general gradient estimates imply that each corresponding sequence u_k^j is equicontinuous in v_{k-1} . Hence a subsequence u^j can be found that converges in all of Δ , uniformly in each v_k , to a solution $U(x)$ of (3) in Δ . Clearly, $U(x) < \psi(x)$ in Δ .

Let p_l be a sequence of points such that $p_l \in \Delta$ and $p_l \rightarrow p \in B_1$, and suppose $U(p_l) > -K > -\infty$ for all l . For each p_l there is an index $j_0(l)$ such that $u^j(p_l) > -K$ for all $j > j_0(l)$. Applying again the Harnack inequality, we find p_l is the center of a disk $D_K(p_l)$, of radius ρ_K depending only on K and on p (and not on l), in which $u^j(x) > -K - 1$ for $j > j_0(l)$. Choosing l sufficiently large that $|p_l - p| < \rho_K$, and then $j > j_0(l)$, we find $u^j(p) > -K - 1$, and hence $p \in \Delta$. We conclude that if $p \in B_1 \cap \partial\Delta$ then $\lim U(x) = -\infty$ for any approach to p from within Δ .

We have $U(0) = \psi(0) = 0$. Thus the set $\Delta_1 \subset \Delta$ defined by $U(x) - \psi(x) > -1$ is non empty; by the above remarks, the function

$$\eta(x) = \begin{cases} U(x) - \psi(x) + 1, & x \in \Delta_1 \\ 0, & x \notin \Delta_1 \end{cases}$$

is a continuous Lipschitz function in \mathcal{A}_ε , and

$$(16) \quad \int_{\mathcal{A}_\varepsilon} \nabla \eta \cdot [TU - T\psi^{(\varepsilon)}] dx = \int_{I_\varepsilon} \eta \nu \cdot [TU - T\psi^{(\varepsilon)}] d\sigma + \int_{I_{1-\varepsilon}} \eta \nu \cdot [TU - T\psi^{(\varepsilon)}] d\sigma.$$

The second integral on the right is non positive, as $\eta \geq 0$ and $\nu \cdot T\psi^{(\varepsilon)} \equiv 1$,

$v \cdot TU < 1$ on $\Gamma_{1-\varepsilon}$. We have also, by Theorem 2 and by the properties of $\psi^{(\varepsilon)}$,

$$\left| \int_{\Gamma_\varepsilon} \eta v \cdot [TU - T\psi^{(\varepsilon)}] d\sigma \right| < 4\pi\varepsilon C$$

since $|TU| < 1$, $|T\psi^{(\varepsilon)}| < 1$; here C is independent of ε .

Letting $\varepsilon \rightarrow 0$ we find $\mathcal{A}_\varepsilon \rightarrow B_1$, and

$$\int_{B_1} \nabla \eta \cdot [TU - T\psi] dx \leq 0.$$

Once again, the integrand is non negative and vanishes only if $\nabla \eta = 0$. We conclude $U(x) \equiv \psi(x)$ in the component $\Delta^{(0)}$ of Δ containing the origin.

Since $U(x) \rightarrow -\infty$ at any boundary point of $\Delta^{(0)}$ interior to B_1 , we conclude also $\Delta \equiv B_1$. With a standard reasoning we now obtain

THEOREM 3. *There exists a function $\varphi(\hat{R})$ defined in $R_0 < \hat{R} < 1$ and satisfying $\hat{R} < \varphi(\hat{R}) < 1$, and a function $\sigma^*(R; \hat{R})$ defined in $\varphi(\hat{R}) < R < 1$ and satisfying $\lim_{R \rightarrow 1} \sigma^*(R; \hat{R}) = 0$, such that if $u(x)$ is a solution of (3) in B_R and $u(0) = 0$, then $|u(x) - \psi(x)| < \sigma^*(R; \hat{R})$ in $B_{\hat{R}}$.*

That is, if R is close to 1, a solution of (3) in B_R cannot differ significantly from a lower hemisphere in compact subdomains.

We note Theorem 3 does not include Theorem 2, as the bound given in Theorem 2 is uniform throughout B_R . It seems dubious that a bound of that type would hold from below.

The methods of this section extend without essential change to any number n of dimensions. The results, however, depend on Theorem 1, the proof of which does not seem to extend as given.

III. - Existence of the comparison surfaces.

We establish here the existence of the surfaces $v(x)$ in domains $\mathfrak{D}_0^\varepsilon$, on which the discussion in § I is based.

THEOREM 4. *Let \mathfrak{D} be a moon domain (see Figure 1), satisfying the necessary condition*

$$(17) \quad |\Gamma_2| - |\Gamma_1| = 2|\mathfrak{D}|$$

and for which the respective radii R_1, R_2 satisfy $R_1 < \frac{1}{2}, R_2 > \frac{1}{2}$. Then the

problem

$$(18) \quad \begin{cases} \operatorname{div} Tu = 2 & \text{in } \mathfrak{D} \\ v \cdot Tu = \begin{cases} -1 & \text{on } \Gamma_1 \\ +1 & \text{on } \Gamma_2 \end{cases} \end{cases}$$

has a (bounded) solution $v(x)$, unique up to an additive constant.

We shall reduce the proof of Theorem 4 to the general existence results of [20]. We introduce first some notation.

For any function $u(x) \in L_1$ in an open set A we define

$$(18) \quad \int_A |Du| = \sup \left\{ \int u \operatorname{div} g \, dx ; g \in C_0^1(A), |g| \leq 1 \right\}.$$

If $\int_A |Du| < \infty$ we say that $u \in BV(A)$, the space of functions of bounded variation over A . $BV(A)$ is a Banach space with the norm

$$\|u\| = \int_A |u| + \int_A |Du|.$$

If φ_E is the characteristic function of a Borel set E , we call $\int_A |D\varphi_E|$ the *perimeter of E in A* ; if $A = \mathbf{R}^n$ we refer to the *perimeter $P(E)$ of E* . Sets with finite perimeter will be called *Caccioppoli sets*.

For detailed background information we refer the reader to [22, 21]. We remark that if the boundary ∂E of E is a piecewise smooth curve, then the perimeter of E is the length of the part of E that lies in A .

We shall obtain the solution to (18) as a minimum for the functional

$$(19) \quad \mathcal{F}(u) \equiv \int_{\mathfrak{D}} \sqrt{1 + |Du|^2} + 2 \int_{\mathfrak{D}} u \, dx + \int_{\Gamma_1} u \, d\sigma - \int_{\Gamma_2} u \, d\sigma$$

We wish to apply the results of [20], Chapter 4. To do so, we must show there is a positive constant α_0 such that for every Caccioppoli set $E \subset \mathfrak{D}$ we have

$$(20) \quad \left| 2|E| + \int_{\Gamma_1} \varphi_E \, d\sigma - \int_{\Gamma_2} \varphi_E \, d\sigma \right| \leq \int_{\mathfrak{D}} |D\varphi_E| - \alpha_0 \min(|E|, |\mathfrak{D} - E|).$$

It suffices to prove

$$(21) \quad 2|E| + \int_{\Gamma_1} \varphi_E \, d\sigma - \int_{\Gamma_2} \varphi_E \, d\sigma \leq \int_{\mathfrak{D}} |D\varphi_E| - \alpha_0 \min(|E|, |\mathfrak{D} - E|),$$

we obtain the remaining inequality by replacing E by $\mathfrak{D} - E$ and using (17).

Since every Caccioppoli set can be approximated by piecewise regular sets (see [22]), it suffices to prove (21) for sets E with piecewise smooth boundary. For such sets (21) reads

$$(22) \quad |\partial_1 E| \leq |\partial_2 E| - 2|E| - \alpha_0 \min(|E|, |\mathcal{D} - E|)$$

where

$$\begin{aligned} \partial_1 E &= \partial E \cap \Gamma_1 \\ \partial_2 E &= \partial E - \partial_1 E. \end{aligned}$$

We observe also that we can always suppose $\partial_1 E = \Gamma_1$; otherwise we could add to E an ε -neighbourhood of Γ_1 :

$$E_\varepsilon = E \cup \{x \in \mathcal{D} \mid \text{dist}(x, \Gamma_1) < \varepsilon\}$$

and then pass to the limit as $\varepsilon \rightarrow 0^+$.

We may now note that for fixed $|E|$, the quantity $|\partial_2 E|$ attains its minimum when $\partial_2 E$ is an arc of a circle passing through the endpoints of Γ_1 , and it will suffice to prove (22) for that configuration.

Consider the situation in Figure 5. Here d is a fixed number, $0 < d < \frac{1}{2}$,

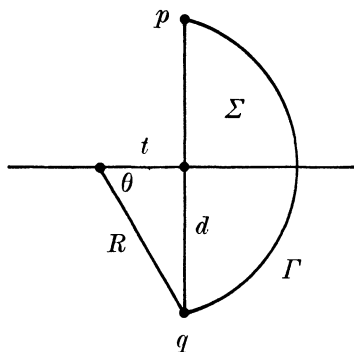


Figure 5

and a circle passing through the points p and q is determined by the position of its center on the t axis (note that in Figure 5 we have $t < 0$). We denote by $l(t)$ the length of the arc Γ and by $A(t)$ the area of the region Σ .

We have

$$\begin{aligned} l(t) &= 2R(t)\theta(t) \\ A(t) &= R^2(t)\theta(t) + td \end{aligned}$$

and

$$R(t) = \sqrt{t^2 + d^2}$$

$$\theta(t) = \operatorname{arctg}(-d/t).$$

If we set

$$(23) \quad f(t) = l(t) - 2A(t)$$

we find

$$f'(t) = 2R^{-1}(1 - 2R)(d + \theta t);$$

Thus $f' = 0$ only at $t = t_0 = \sqrt{\frac{1}{4} - d^2}$. The function $f(t)$ is illustrated in Figure 6.

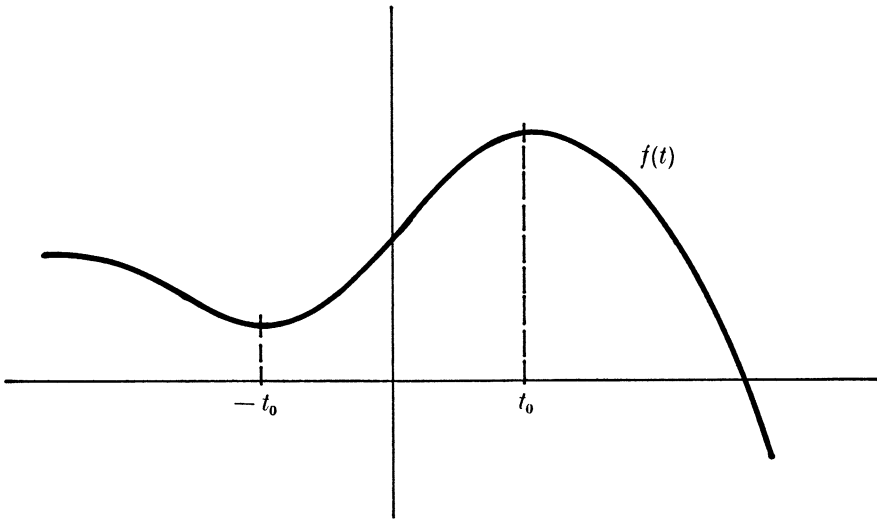


Figure 6

Let t_1 and t_2 be the values corresponding to the radii R_1 and R_2 of \mathfrak{D} . The necessary condition (17) becomes

$$(24) \quad f(t_1) = f(t_2).$$

To satisfy the conditions of Theorem 4 we must have $t_2 > t_0$, $-t_0 < t_1 < t_0$. For $t_1 < t < t_2$ we have

$$(25) \quad f(t) > f(t_1) = f(t_2).$$

Further

$$(26) \quad f'(t_1) > 0, \quad f'(t_2) < 0.$$

The proof of (22) now reduces to showing

$$(27) \quad f(t_1) < f(t) - \alpha_0 \min \{A(t) - A(t_1), A(t_2) - A(t)\}$$

for every t , $t_1 < t < t_2$.

We obtain this inequality from

LEMMA III.1. *Let $f(t)$ be a $C^{(1)}$ function in the interval $[t_1, t_2]$, satisfying (25) and (26), and let $g(t)$ be a Lipschitz-continuous function, with $g(t_1) = g(t_2) = 0$. Then there exists a positive α_0 such that*

$$(28) \quad f(t_1) < f(t) - \alpha_0 g(t)$$

for every t , $t_1 < t < t_2$.

PROOF. Since $f'(t_1) > 0$, there exist $\varepsilon_1 > 0$ and $\bar{t}_1 > t_1$ such that

$$f'(t) \geq \varepsilon_1, \quad t_1 < t < \bar{t}_1$$

and hence

$$f(t) \geq f(t_1) + \varepsilon_1(t - t_1), \quad t_1 < t < \bar{t}_1.$$

Similarly, we have

$$f(t) \geq f(t_2) + \varepsilon_2(t - t_2), \quad \bar{t}_2 < t < t_2.$$

Let $m > f(t_1)$ be the minimum of $f(t)$ in the interval $[\bar{t}_1, \bar{t}_2]$, let M and L be the maximum of $|g|$ and the Lipschitz constant of g , respectively. Then (28) will hold if α_0 is chosen so that

$$\alpha_0 L < \min(\varepsilon_1, \varepsilon_2)$$

$$\alpha_0 M < m - f(t_1)$$

which proves the lemma, and hence also (20).

We can now apply the results of [20], Chapter 4, which yield the existence of a minimum for the functional (19). The singularities of $\partial\mathfrak{D}$ cause no essential difficulty, as one sees by a simple modification of the reasoning in § 4E. The minimizing function $u(x)$ is unique up to an additive constant and is bounded and regular in \mathfrak{D} . It remains only to show that the boundary data are achieved strictly.

Let us start with Γ_1 . Since

$$(29) \quad \mathcal{F}(u + c) = \mathcal{F}(u), \quad c \in \mathbf{R},$$

we can suppose $u \geq 1$ in \mathfrak{D} . On the other hand, for positive functions $v(x)$ we have

$$(30) \quad \mathcal{F}(v) = \int_{\mathfrak{D}} \sqrt{1 + |Dv|^2} dx + 2 \int_{\mathfrak{D}} v dx + \int_{\Gamma_1} |v| d\sigma - \int_{\Gamma_2} v d\sigma,$$

and thus the function $u(x)$ minimizes the functional

$$(31) \quad \mathfrak{G}(v) = \int_{B_2} \sqrt{1 + |Dv|^2} dx + 2 \int_{B_2} v dx - \int_{\Gamma_1} v d\sigma$$

in the class

$$(32) \quad Q = \{v \in BV(B_2); v \geq 0, v = 0 \text{ in } B_2 - \mathfrak{D}\}$$

where B_2 denotes the ball of radius R_2 .

It follows as in [23] that the set

$$U = \{(x, y) \in B_2 \times \mathbf{R}; y > u(x)\}$$

minimizes the functional

$$(33) \quad \int |D\varphi_U| dx + 2 \int \varphi_U dx$$

in the sense that for every set V which coincides with U outside some compact set $K \subset B_2 \times \mathbf{R}^+$, and which contains $(B_2 - \mathfrak{D}) \times \mathbf{R}^+$ we have

$$\int_K |D\varphi_U| + 2 \int_K \varphi_U dx \leq \int_K |D\varphi_V| + 2 \int_K \varphi_V dx.$$

This means that U minimizes the functional (33) in $B_2 \times \mathbf{R}^+$, with obstacle $L = (B_2 - \mathfrak{D}) \times \mathbf{R}^+$. From a result of Miranda [24] there follows that ∂U is a $C^{(1)}$ surface in a neighbourhood of L . On the other hand, since $u \geq 1$ in \mathfrak{D} , ∂U contains at least the vertical surface $\Gamma_1 \times (0, 1)$. This implies

$$\frac{x}{|x|} \cdot Tu(x) \rightarrow -1$$

as $x \rightarrow \Gamma_1$ and establishes the first of the boundary conditions in (18). To obtain the other one, we note that since $u(x)$ is bounded (say $u < M$) the function u minimizes also the functional

$$\mathcal{H}(u) \equiv \int_{\mathfrak{D}} \sqrt{1 + |Du|^2} + 2 \int_{\mathfrak{D}} u dx + \int_{\Gamma_1} u d\sigma + \int_{\Gamma_2} |M - u| d\sigma$$

among all functions $v \in BV(\mathfrak{D})$, $v \leq M$. We can then repeat the previous reasoning, obtaining the regularity at Γ_2 .

IV. - R_0 is best possible.

LEMMA IV.1. *Let B_R be a ball of radius R centered at 0, and let B be a ball of radius $\frac{1}{2}$ passing through 0. Let $E = B_R - B$. If $R < R_0$, the set E is the unique minimum for the functional*

$$(34) \quad \mathcal{H}(E) = \int_{B_R} |D\varphi_E| + 2 \int_{B_R} \varphi_E dx + \int_{\partial B_R} |\varphi_E - \varphi| d\sigma$$

where

$$\varphi = \begin{cases} 1 & \text{in } \Gamma_2 \\ 0 & \text{in } \Gamma_1 \end{cases} \quad (\text{see Figure 7}).$$

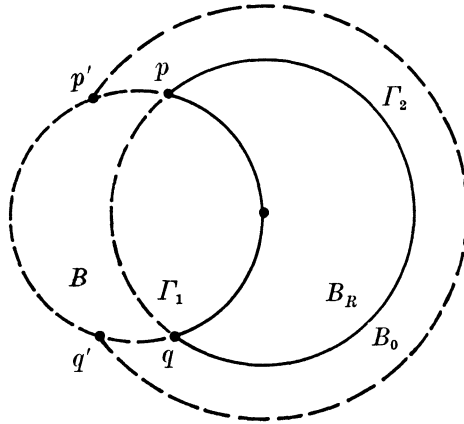


Figure 7

PROOF. It is equivalent to show that B is the unique minimum of

$$(35) \quad \int |D\varphi_B| - 2 \int \varphi_B dx$$

in the class

$$Q = \{E; E \subset B \cup B_R; E - B_R = B - B_R\}.$$

As in the proof of Theorem 4, it suffices to compare B with domains E that coincide with B outside B_R and are bounded by a circular arc passing through the endpoints of Γ_1 .

If Γ is the circular arc bounding E one has to show that

$$(36) \quad |\Gamma_1| - 2|B \cap B_R| < |\Gamma| - 2|E \cap B_R|$$

unless $E = B$.

We consider separately two cases.

(I) $|E| < |B|$. In this case the arc Γ lies in B and one verifies readily (using the function (23) $f(t)$) that (36) holds.

(II) $|E| > |B|$. Let B_0 be a ball concentric with B_R and with radius R_0 , and let p', q' be the intersections of ∂B_0 with ∂B . Let $\Gamma'_1 = \partial B \cap B_0$.

Let E' be the domain coinciding with B outside B_R , bounded by a circular arc Γ' passing through p' and q' , and such that $|E'| = |E|$. We then have

$$(37) \quad |\Gamma'| \leq |\Gamma| + |\Gamma'_1 - \Gamma|.$$

On the other hand, since $|E'| = |E| \leq |B_R \cup B| < |B_0 \cup B|$, we have (cf. Figure 7)

$$(38) \quad |\Gamma'_1| - 2|B_0 \cap B| < |\Gamma'| - 2|B_0 \cap E'|.$$

On the other hand

$$\begin{aligned} |B_0 \cap E'| &= |B_0 \cap E| = |B_R \cap E| + |(B_0 - B_R) \cap B| \\ |B_0 \cap B| &= |B_R \cap B| + |(B_0 - B_R) \cap B| \end{aligned}$$

and hence (37), (38) imply (36). This proves the lemma.

To show R_0 is best possible, we set

$$F = B_0 - (B_R \cup B)$$

and, for $k = 1, 2, \dots$, let

$$\Phi_k = \begin{cases} 0 & \text{in } F \\ k & \text{in } B_0 - F. \end{cases}$$

Let u_k minimize the functional

$$\int \sqrt{1 + |Du|^2} + 2 \int u \, dx$$

in the class

$$\{u \in BV(B_0); u = \Phi_k \text{ in } B_0 - B_R\}.$$

The sequence u_k is obviously nondecreasing; proceeding as in [25], we conclude that the set

$$P = \{x \in B_0; \lim_{k \rightarrow \infty} u_k(x) = +\infty\}$$

minimizes $\mathcal{H}(E)$. Hence from Lemma IV.1 we have $P = E$, and therefore $0 \in \partial P$.

Suppose there exist $\varepsilon > 0$ and $M < \infty$ such that

$$|Du_k(x)| \leq M \quad \text{for } |x| < \varepsilon.$$

Then if $0 \in P$, the ball $B_\varepsilon = \{|x| < \varepsilon\}$ would lie in P ; if $0 \notin P$ we would have $P \cap B_\varepsilon = \emptyset$. In either case $0 \notin \partial P$, a contradiction. Thus, the sequence u_k cannot have bounded gradient in any neighborhood of 0.

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