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Invariant metrics on convex cones

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Invariant Metrics on Convex Cones (*).

EDOARDO VESENTINI (**)

dedicated to Jean Leray

Let $\mathfrak{R}$ be a locally convex real vector space and let Ω be an open convex cone in $\mathfrak{R}$ containing no non-trivial affine subspace of $\mathfrak{R}$.

In the case in which $\mathfrak{R}$ has finite dimension, an affine-invariant riemannian metric has been introduced in Ω by E. B. Vinberg [16], following closely a similar construction developed earlier by M. Koecher ([11]; cf. also [14]) for the domains of positivity. The invariance of this metric under the action of the group $G(\Omega)$ of affine automorphisms of Ω, coupled with a classical lemma of van Dantzig and van der Waerden, implies that $G(\Omega)$ acts properly on Ω. Furthermore, if Ω is affine-homogeneous, this invariant riemannian metric on Ω is necessarily complete [10, p. 176].

In the general case in which $\mathfrak{R}$ is a locally convex real vector space we will define two metrics on Ω which are invariant with respect to the group $G(\Omega)$ of all continuous affine automorphisms of Ω. After considering a few examples we compute explicitely one of the two metrics in the case in which Ω is the cone of strictly positive hermitian elements of a Banach algebra $\mathcal{A}$ endowed with a locally continuous hermitian involution. Denoting this metric by δ_Ω, we will consider the particular case where $\mathcal{A}$ is a von Neumann algebra and we will suitably extend to this case some of the results of Koecher and Vinberg, proving that δ_Ω is a complete metric (Theorem IV) on Ω and that the action of $G(\Omega)$ is « locally bounded » (Theorem V).

The main idea in the construction of the metric δ_Ω stems from the definition of the Caratheodory invariant metric on a domain of C^n ([1], [2], [3]). A suitable class of real valued functions on the cone Ω takes the place of the bounded holomorphic functions appearing in Caratheodory's definition.

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The crucial role of the Schwarz-Pick lemma in Caratheodory's construction is played here by an elementary property (Lemma 1.1) of the Haar measure of the multiplicative group of real positive numbers.

Let $\mathcal{V}$ be the complexification of $\mathcal{R}$, and let D be the tube domain over Ω: $D = \{z \in \mathcal{V}: \operatorname{Im} z \in \Omega\}$. We will show (Theorem II) that δ_Ω is the restriction to $i\Omega$ of the Caratheodory metric of D.

A new metric has been introduced recently on complex manifolds by S. Kobayashi and has been extensively investigated by him and others ([8], [9]). This metric, besides being of great interest in the theory of value-distribution of holomorphic mappings and in other questions, turns out to be useful in simplifying the construction of Caratheodory's metric. Adapting Kobayashi's ideas to the framework of convex cones and affine mappings, we will define in § 1 the other affine-invariant metric mentioned at the beginning. As in the case of complex manifolds, this metric turns out to be instrumental in the construction of δ_Ω.

1. – A Kobayashi-type invariant metric.

1. – The Haar measure of the multiplicative group $\boldsymbol{R}_*^+$ of positive real number is given, up to a positive constant factor, by $t^{-1}dt$. Assuming as a distance of any two points t_1 and t_2 in $\boldsymbol{R}_*^+$ the measure of the interval determined by t_1 and t_2:

$$\sigma(t_1, t_2) = \left| \int_{t_2}^{t_1} \frac{dt}{t} \right| = \left| \log \frac{t_2}{t_1} \right| , \tag{1.1}$$

we obtain a continuous invariant metric on the group $\boldsymbol{R}_*^+$.

An affine function $f: \boldsymbol{R} \to \boldsymbol{R}$ mapping $\boldsymbol{R}_*^+$ into $\boldsymbol{R}_*^+$ is given by $f(t) = \alpha t + \beta$, with $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta > 0$.

For $t_2 > t_1$ we have

$$\frac{t_2}{t_1} - \frac{\alpha t_2 + \beta}{\alpha t_1 + \beta} = \frac{\beta(t_2 - t_1)}{t_1(\alpha t_1 + \beta)} \geqslant 0 ,$$

equality occurring if, and only if, $\beta = 0$. This proves

LEMMA 1.1. *If f is any real-valued affine function on $\boldsymbol{R}$, mapping $\boldsymbol{R}_*^+$ into $\boldsymbol{R}_*^+$, and if $t_1, t_2 \in \boldsymbol{R}_*^+ (t_1 \neq t_2)$, then*

$$\sigma(f(t_1), f(t_2)) \leqslant \sigma(t_1, t_2) ,$$

equality occurring if, and only if, f is a translation in the multiplicative group $\boldsymbol{R}_*^+$.

2. – Let $\mathfrak{R}$ be a real vector space and let Ω be a convex cone in $\mathfrak{R}$, $\Omega \neq \{0\}$. In the following we shall be mainly concerned with cones satisfying the following condition:

i) If $x \in \Omega$ and if $\mathfrak{L}$ is any affine line in $\mathfrak{R}$ such that $x \in \mathfrak{L}$ and that $\mathfrak{L} \cap \Omega$ contains a half-line, then there is a half-line $\mathfrak{r}_x \subset \mathfrak{L} \cap \Omega$ containing x in its interior.

Condition i) is satisfied when every point $x \in \Omega$ is internal (i.e. when Ω is radial at x).

Suppose that $\mathfrak{R} \neq \Omega$, that $\mathfrak{R}$ is generated by Ω and that $\mathfrak{R}$ has finite dimension. Let x be a bounding point of Ω (i.e. both Ω and $\mathfrak{R} \setminus \Omega$ are not radial at x) and let P be a hyperplane of support of Ω at x. Then $\Omega \not\subset P$ and for any $y \in \Omega \setminus P$ the affine line $\{z = x + ty : t \in \boldsymbol{R}\}$ intersects Ω on a half-line with origin x. Hence, if i) holds, Ω is open in $\mathfrak{R}$ for the standard vector topology of $\mathfrak{R}$. Since the converse obviously holds, we have:

LEMMA 2.1. *The convex cone Ω satisfies condition* i) *if, and only if, for every finite-dimensional subspace $\mathfrak{T}$ of $\mathfrak{R}$ such that $\mathfrak{T} \cap \Omega \neq \{0\}$, $\mathfrak{T} \cap \Omega$ is open in the subspace of $\mathfrak{T}$ generated by $\mathfrak{T} \cap \Omega$, for the standard vector topology of $\mathfrak{T}$.*

Throughout § 1 we shall assume that Ω satisfies condition i). If x, y are two points of Ω, $x \neq y$, either x and y are collinear with 0 or x, y and 0 determine a two dimensional space $\mathfrak{S}(x, y)$. In both cases there are two affine half-lines $\mathfrak{r}_x$ and $\mathfrak{r}_y$, starting at x and y, such that

$$\mathfrak{r}_x \subset \mathfrak{S}(x, y) \cap \Omega , \qquad \mathfrak{r}_y \in \mathfrak{S}(x, y) \cap \Omega , \qquad \mathfrak{r}_x \cap \mathfrak{r}_y \neq \emptyset .$$

Let $p^0 = x, p^1, \dots, p^n = y$ be points in Ω, let $a^1, \dots, a^n, b^1, \dots, b^n$ be points in $\boldsymbol{R}_*^+$, and let $f_1, \dots, f_n$ be affine functions of $\boldsymbol{R}$ into $\mathfrak{R}$ mapping $\boldsymbol{R}_*^+$ into Ω, and such that

$$f_j(a^j) = p^{j-1} , \qquad f_j(b^j) = p^j \qquad (j = 1, \dots, n) .$$

Let

$$\gamma_\Omega(x, y) = \inf \{\sigma(a^1, b^1) + \dots + \sigma(a^n, b^n)\} ,$$

where the infimum is taken over all possible choices of $n, p^1, \dots, p^{n-1}$, $a^1, \dots, a^n$, $b^1, \dots, b^n$, $f_1, \dots, f_n$.

Clearly γ_Ω is a pseudo-metric in Ω. The following proposition is a trivial consequence of the definition of γ_Ω.

PROPOSITION 2.2. *Let Ω_1 and Ω_2 be two convex cones in two real vector spaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$, and let $F: \mathfrak{R}_1 \to \mathfrak{R}_2$ be an affine map such that $F(\Omega_1) \subset \Omega_2$.*

If both Ω_1 and Ω_2 satisfy condition i), *then*

$$\gamma_{\Omega_2}(F(x), F(y)) \leqslant \gamma_{\Omega_1}(x, y)$$

for all $x, y \in \Omega_1$.

In particular, if the convex cone Ω satisfies condition i), *the pseudo-metric γ_Ω is invariant under any affine automorphism of Ω.*

By the definition of γ_Ω,

$$\gamma_\Omega(f(a), f(b)) \leqslant \sigma(a, b)$$

for all $a, b \in \boldsymbol{R}_*^+$ and every affine map $f: \boldsymbol{R} \to \mathcal{R}$ such that $f(\boldsymbol{R}_*^+) \subset \Omega$. Actually γ_Ω is the largest pseudo-metric on Ω satisfying the above inequality:

PROPOSITION 2.3. *If Ω satisfies condition* i) *and if γ' is a pseudo-metric on Ω such that*

$$\gamma'(f(a), f(b)) \leqslant \sigma(a, b)$$

for every affine function $f: \boldsymbol{R} \to \mathcal{R}$ such that $f(\boldsymbol{R}_*^+) \subset \Omega$ and all $a, b \in \boldsymbol{R}_*^+$, then

$$\gamma'(x, y) \leqslant \gamma_\Omega(x, y) \quad \text{for all } x, y \in \Omega .$$

PROOF. With the same notations as in the definition of γ_Ω, we have

$$\gamma'(x, y) \leqslant \sum_{j=1}^{n} \gamma'(p^{j-1}, p^j) = \sum_{j=1}^{n} \gamma'(f_j(a^j), f_j(b^j)) \leqslant \sum_{j=1}^{n} \sigma(a^j, b^j),$$

proving our assertion. Q.E.D.

Let Ω_1 and Ω_2 be two convex cones in two real vector spaces $\mathcal{R}_1$ and $\mathcal{R}_2$. If both Ω_1 and Ω_2 satisfy condition i), then the convex cone $\Omega_1 \times \Omega_2 \subset \mathcal{R}_1 \times \mathcal{R}_2$ satisfies condition i).

PROPOSITION 2.4. *If both Ω_1* and *Ω_2 satisfy condition* i) *then for $x_1, y_1 \in \Omega_1$, $x_2, y_2 \in \Omega_2$ we have*

$$\max(\gamma_{\Omega_1}(x_1, y_1), \gamma_{\Omega_2}(x_2, y_2)) \leqslant \gamma_{\Omega_1 \times \Omega_2}((x_1, x_2), (y_1, y_2)) \leqslant$$
$$\leqslant \gamma_{\Omega_1}(x_1, y_1) + \gamma_{\Omega_2}(x_2, y_2) .$$

PROOF. The inequality on the right follows from the triangle inequality when we apply Proposition 2.2 to the linear maps $z_1 \mapsto (z_1, x_2)$ and $z_2 \mapsto (y_1, z_2)$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ into $\mathcal{R}_1 \times \mathcal{R}_2$. The inequality on the left follows directly from Proposition 2.2, when we apply it to the canonical projections $\Omega_1 \times \Omega_2 \to \Omega_j$ $(j = 1, 2)$.

3. – We will now construct γ_Ω in a few examples.

Let $\mathfrak{R} = \boldsymbol{R}$ and $\Omega = \boldsymbol{R}_*^+$. With the same notations as in the definition of γ_Ω, we have, by Lemma 1.1 and by the triangle inequality,

$$\sum_{j=1}^{n} \sigma(a^j, b^j) \geqslant \sum_{j=1}^{n} \sigma(f_j(a^j), f_j(b^j)) = \sum_{j=1}^{n} \sigma(p^{j-1}, p^j) \geqslant \sigma(x, y)\,.$$

Thus

$$\gamma_{\boldsymbol{R}_*^+}(x, y) \geqslant \sigma(x, y) \qquad \text{for } x, y \in \boldsymbol{R}_*^+\,.$$

On the other hand, choosing $a^1 = x$, $b^1 = y$, $f_1(t) = t$ in the definition of $\gamma_{\boldsymbol{R}_*^+}$, we have

$$\gamma_{\boldsymbol{R}_*^+}(x, y) \leqslant \sigma(x, y)\,.$$

Hence

$$\gamma_{\boldsymbol{R}_*^+}(x, y) = \sigma(x, y) \qquad \text{for all } x, y \in \boldsymbol{R}_*^+\,. \tag{3.1}$$

We consider next the case where $\mathfrak{R} = \Omega = \boldsymbol{R}$ and we prove that

$$\gamma_{\boldsymbol{R}}(x, y) = 0 \qquad \text{for all } x, y \in \boldsymbol{R}\,. \tag{3.2}$$

Let $x \neq y$. Given any two distinct positive real numbers, a and b, there exists an affine function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ such that $f(a) = x$, $f(b) = y$. Being

$$\gamma_{\boldsymbol{R}}(x, y) \leqslant \sigma(a, b)\,,$$

that proves (3.2).

We say that the cone Ω is *sharp* (or *regular*) if it contains no affine line.

If the cone Ω is not sharp, there is an injective affine map $f\colon \boldsymbol{R} \to \mathfrak{R}$ such that $f(\boldsymbol{R}) \subset \Omega$. Proposition 2.2 and (3.2) imply

LEMMA 3.1. *If the pseudo-metric γ_Ω is a metric, then Ω is sharp.*

We will discuss later on the converse statement.

The following proposition shows that both the upper and lower bounds described by Proposition 2.4 can actually be reached on the same cone.

PROPOSITION 3.2. *Let $\mathfrak{R} = \boldsymbol{R} \times \boldsymbol{R}$, and let $\Omega = \boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$. The distance $\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(x, y)$ of two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $x \neq y$, in $\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$ is given by*

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) = \max(\sigma(x_1, y_1), \sigma(x_2, y_2))\,,$$

or by

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) = \sigma(x_1, y_1) + \sigma(x_2, y_2)\,, \tag{3.3}$$

according as the affine line determined by x and y in $\boldsymbol{R}\times\boldsymbol{R}$ intersects $\boldsymbol{R}_*^+\times\boldsymbol{R}_*^+$ in a half-line or in a finite segment.

We postpone the proof of this Proposition to n. 12.

2. – A Caratheodory-type invariant metric.

4. – Let Ω be a convex cone in a real vector space $\mathfrak{R}$. Let $\mathfrak{F}(\Omega)$ be the collection of all real valued affine functions on $\mathfrak{R}$ mapping Ω into $\boldsymbol{R}_*^+$.

For $x, y \in \Omega$, let δ_Ω be the non-negative (not necessarily finite) number defined by

$$\delta_\Omega(x, y) = \sup\{\sigma(f(x), f(y)) : f \in \mathfrak{F}(\Omega)\}. \tag{4.1}$$

Clearly δ_Ω is a pseudo-metric on Ω whenever $\delta_\Omega(x,y) < \infty$ for all $x, y \in \Omega$.

The following lemma is an obvious consequence of (4.1).

LEMMA 4.1. *If there is a pseudo-metric δ' on Ω such that*

$$\delta'(x, y) \geqslant \sigma(f(x), f(y)) \tag{4.2}$$

for all $f \in \mathfrak{F}(\Omega)$ and all $x, y \in \Omega$, then δ_Ω is a pseudometric on Ω, and

$$\delta_\Omega(x, y) \leqslant \delta'(x, y) \quad \textit{for all } x, y \in \Omega .$$

According to this lemma, δ_Ω (when it exists) is the smallest pseudo-metric on Ω for which every $f \in \mathfrak{F}(\Omega)$ is distance-decreasing.

Let us assume that Ω satisfies condition i) of n. 2, and let n, a^j, b^j, p^j, f_j $(j = 1, \dots, n)$ be as in the construction of γ_Ω (n. 2). For any $f \in \mathfrak{F}(\Omega)$, $f \circ f_j$ is an affine function $\boldsymbol{R} \to \boldsymbol{R}$ mapping $\boldsymbol{R}_*^+$ into $\boldsymbol{R}_*^+$. Thus by Lemma 1.1

$$\sigma(f \circ f_j(a^j), f \circ f_j(b^j)) \leqslant \sigma(a^j, b^j) .$$

By the triangle inequality

$$\sum_{j=1}^{n} \sigma(a^j, b^j) \geqslant \sum_{j=1}^{n} \sigma(f \circ f_j(a^j), f \circ f_j(b^j)) = \sum_{j=1}^{n} \sigma(f(p^{j-1}), f(p^j)) \geqslant \sigma(f(x), f(y)) ,$$

and therefore

$$\gamma_\Omega(x, y) \geqslant \sigma(f(x), f(y)) .$$

Hence $\delta' = \gamma_\Omega$ satisfies (4.2), and Lemma 4.1 implies

LEMMA 4.2. *If Ω satisfies condition* i) *of* n. 2, *then δ_Ω is a pseudo-metric, and furthermore*

$$\delta_\Omega(x, y) \leqslant \gamma_\Omega(x, y) \quad \textit{for all } x, y \in \Omega .$$

The following propositions are immediate consequences of (4.1) and can be proved imitating similar arguments in n. 2.

PROPOSITION 4.3. *Let Ω_1* and *Ω_2 be convex cones in real vector spaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$ and let $F\colon \mathfrak{R}_1 \to \mathfrak{R}_2$ be an affine map such that $F(\Omega_1) \subset \Omega_2$. If the functions δ_{Ω_1} and δ_{Ω_2} defined by* (4.1) *are pseudo-metrics, then*

$$\delta_{\Omega_2}\big(F(x), F(y)\big) \leqslant \delta_{\Omega_1}(x, y)$$

for all x, y in Ω_1. In particular, if δ_Ω is a pseudo-metric on Ω, then δ_Ω is invariant under any affine automorphism of Ω.

PROPOSITION 4.4. *If δ_{Ω_1} and δ_{Ω_2} are pseudo-metrics on Ω_1 and Ω_2, then the function $\delta_{\Omega_1 \times \Omega_2}$ defined by* (4.1) *on $(\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2)$ is a pseudo-metric on the convex cone $\Omega_1 \times \Omega_2 \subset \mathfrak{R}_1 \times \mathfrak{R}_2$. Furthermore*

$$\max\big(\delta_{\Omega_1}(x_1, y_1), \delta_{\Omega_2}(x_2, y_2)\big) \leqslant \delta_{\Omega_1 \times \Omega_2}\big((x_1, x_2), (y_1, y_2)\big) \leqslant$$
$$\leqslant \delta_{\Omega_1}(x_1, y_1) + \delta_{\Omega_2}(x_2, y_2)$$

for all $x_1, y_1 \in \Omega_1$, $x_2, y_2 \in \Omega_2$.

Suppose that δ_Ω is a pseudo-metric on Ω; let $x \in \Omega$, $r > 0$, and let

$$C(x, r) = \{y \in \Omega\colon \delta_\Omega(x, y) < r\} \tag{4.3}$$

be the ball with center x and radius r for the pseudometric δ_Ω. If $y^1, y^2 \in C(x, r)$, then by (1.1) and (4.1)

$$\exp(-r) < \frac{f(y^j)}{f(x)} < \exp r \qquad (j = 1, 2)$$

for all $f \in \mathfrak{F}(\Omega)$. Since, for $0 \leqslant t \leqslant 1$, $f(ty^1 + (1-t)y^2) = tf(y^1) + (1-t)f(y^2)$, then

$$\exp(-r) < \frac{f\big(ty^1 + (1-t)y^2\big)}{f(x)} < \exp r \qquad (0 \leqslant t \leqslant 1),$$

i.e.

$$\sigma\Big(f\big(ty^1 + (1-t)y^2\big), f(x)\Big) < r \qquad (0 \leqslant t \leqslant 1),$$

showing that $C(x, r)$ is convex.

Given $f \in \mathfrak{F}(\Omega)$, there is a linear form λ on $\mathfrak{R}$ and a real number $a \geqslant 0$ such that $f(x) = \lambda(x) + a$ for all $x \in \mathfrak{R}$. Clearly $\lambda(x) \geqslant 0$ for all $x \in \Omega$; if $\lambda(x) = 0$ for some $x \in \Omega$, then $a > 0$. Let $x, y \in \Omega$, and let $f(x) > f(y)$. Then $\lambda(x) > \lambda(y)$, and, by (1.1),

$$\sigma(f(x), f(y)) = \log \frac{\lambda(x) + a}{\lambda(y) + a}.$$

Since the function $a \mapsto (\lambda(x) + a)/(\lambda(y) + a)$ is decreasing on $\boldsymbol{R}_*^+$, then (4.1) is equivalent to

$$\delta_\Omega(x, y) = \sup\{\sigma(\lambda(x), \lambda(y)\}, \tag{4.4}$$

where the supremum is taken over all linear forms λ on Ω such that $\lambda(x) > 0$ for every $x \in \Omega$.

5. – Let $\Omega = \boldsymbol{R}_*^+$. Since the identity map of $\boldsymbol{R}$ onto itself belongs to $\mathfrak{F}(\boldsymbol{R}_*^+)$, by (4.1) and (3.1) we have

$$\delta_{\boldsymbol{R}_*^+}(x, y) \geqslant \sigma(x, y) = \gamma_{\boldsymbol{R}_*^+}(x, y),$$

so that, by Lemma 4.2,

$$\delta_{\boldsymbol{R}_*^+}(x, y) = \sigma(x, y) \qquad x, y \in \boldsymbol{R}_*^+. \tag{5.1}$$

If $\Omega = \boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$, then by (4.4)

$$\begin{aligned}\delta_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) &= \sup\left\{\left|\log \frac{a_1 x_1 + a_2 x_2}{a_1 y_1 + a_2 y_2}\right| : a_1 \geqslant 0,\ a_2 \geqslant 0,\ a_1^2 + a_2^2 = 1\right\} \\ &= \sup\left\{\left|\log \frac{x_1 \cos\varphi + x_2 \sin\varphi}{y_1 \cos\varphi + y_2 \sin\varphi}\right| : 0 \leqslant \varphi \leqslant \frac{\pi}{2}\right\}.\end{aligned}$$

Interchanging x and y, if necessary, we may assume $x_2/x_1 \geqslant y_2/y_1$. Since the function $\varphi \mapsto (x_1 \cos\varphi + x_2 \sin\varphi)/(y_1 \cos\varphi + y_2 \sin\varphi)$ is non decreasing for $0 \leqslant \varphi \leqslant \pi/2$, then by (1.1) we have

$$\begin{aligned}\delta_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) &= \max\left(\left|\log \frac{x_1}{y_1}\right|, \left|\log \frac{x_2}{y_2}\right|\right) \\ &= \max\left(\sigma(x_1, y_1), \sigma(x_2, y_2)\right).\end{aligned} \tag{5.2}$$

Comparing the above formula with Proposition 3.2 we see that the two metrics $\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}$ and $\delta_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}$ are different.

If $\Omega = \boldsymbol{R}^+ = \{t \in \boldsymbol{R} : t \geqslant 0\}$, then

$$\delta_{\boldsymbol{R}^+}(x, 0) = \infty \quad \text{for all } x > 0 .$$

This implies that, if the convex cone Ω contains some of its bounding points, then there are points $x, y \in \Omega$ such that $\delta_\Omega(x, y) = \infty$.

Throughout the remainder of this paper we shall consider the case where $\mathfrak{R}$ is a real locally convex (Hausdorff) topological vector space, and Ω is an open convex cone in $\mathfrak{R}$.

For $x, y \in \Omega$, $\mathcal{S}(x, y)$ will denote the vector subspace (of dimension $\leqslant 2$) of $\mathfrak{R}$ determined by x and y. Then $\mathcal{S}(x, y) \cap \Omega$ is an open cone in $\mathcal{S}(x, y)$. By a theorem of M. Krein (cf. e.g. [12, p. 63-64]), every linear form on $\mathcal{S}(x, y)$, which is positive on $\mathcal{S}(x, y) \cap \Omega$ extends to a positive linear form on Ω. By Proposition 4.3 this fact proves

LEMMA 5.1. *If Ω is an open convex cone in a real locally convex vector space $\mathfrak{R}$, then for $x, y \in \Omega$*

$$\delta_\Omega(x, y) = \delta_{\mathcal{S}(x,y) \cap \Omega}(x, y) .$$

COROLLARY 5.2. *Let $\mathfrak{T}$ be a vector subspace of $\mathfrak{R}$. Then for $x, y \in \mathfrak{T} \cap \Omega$*

$$\delta_\Omega(x, y) = \delta_{\mathfrak{T} \cap \Omega}(x, y) .$$

Let $\mathfrak{R}^*$ be the topological dual space of the locally convex space $\mathfrak{R}$, and let $\mathfrak{C}(\Omega) \subset \mathfrak{R}^*$ be the set of all continuous linear forms which are positive on Ω. For $x, y \in \Omega$, let

$$\delta'_\Omega(x, y) = \sup \{\sigma(f(x), f(y)) : f \in \mathfrak{C}(\Omega)\} .$$

Then

$$\delta'_\Omega(x, y) \leqslant \delta_\Omega(x, y) ,$$

while

$$\delta'_{\mathcal{S}(x,y) \cap \Omega}(x, y) = \delta_{\mathcal{S}(x,y) \cap \Omega}(x, y) .$$

Again by Krein's theorem every (continuous) positive linear form on $\Omega \cap \mathcal{S}(x, y)$ extends to a linear form on $\mathfrak{R}$, which is positive on Ω and continuous [6, Lemma 7, p. 417].

Thus

$$\delta'_{\mathcal{S}(x,y) \cap \Omega}(x, y) = \delta'_\Omega(x, y) ,$$

so that, by Lemma 5.1, $\delta_\Omega(x, y) = \delta'_\Omega(x, y)$, i.e.

$$\delta_\Omega(x, y) = \sup\{\sigma(f(x), f(y)) : f \in \mathfrak{C}(\Omega)\} \quad \text{for all } x, y \in \Omega . \tag{5.3}$$

Suppose now that the open convex cone Ω in $\mathfrak{R}$ is sharp. For $x, y \in \Omega$, $x \neq y$, $\mathcal{S}(x, y) \cap \Omega$ is either an open half-line or a sharp open cone in a plane. In both cases

$$\delta_{\mathcal{S}(x,y)\cap\Omega}(x, y) > 0$$

and therefore, by Lemma 5.1, $\delta_\Omega(x, y) > 0$. Hence, if Ω is sharp, δ_Ω is a metric, and thus, by Lemma 4.2, also γ_Ω is a metric.

In view of Lemma 3.1, we conclude with

THEOREM I. *Let Ω be an open convex cone in a locally convex (Hausdorff) real vector space $\mathfrak{R}$. The cone Ω is sharp if, and only if, at least one of the two pseudometrics γ_Ω and δ_Ω is a metric. If one of them is a metric, the other too is a metric.*

By (5.3) δ_Ω is the upper envelope of a family of continuous functions on $\Omega \times \Omega$. Hence, *the relative topology of Ω in $\mathfrak{R}$ is finer than the topology defined by δ_Ω.*

6. – Let $\mathfrak{V}$ be a locally convex (Hausdorff) vector space over the complex field, and let $\mathfrak{R}$ be a real subspace of $\mathfrak{V}$ such that $\mathfrak{V}$ is the complexification of $\mathfrak{R}$.

Let Ω be an open convex sharp cone in $\mathfrak{R}$, and let D be the tube domain in $\mathfrak{V}$ defined by

$$D = \{x + iy : x \in \mathfrak{R},\ y \in \Omega\} .$$

Let Π^+ be the upper half-plane in $\boldsymbol{C}$:

$$\Pi^+ = \{\zeta \in \boldsymbol{C} : \operatorname{Im} z > 0\} .$$

The distance between two points $\zeta_1, \zeta_2 \in \Pi^+$ with respect to the Poincaré metric in Π^+ is

$$\omega(\zeta_1, \zeta_2) = \log \frac{1 + \left|\dfrac{\zeta_2 - \zeta_1}{\zeta_2 - \bar{\zeta}_1}\right|}{1 - \left|\dfrac{\zeta_2 - \zeta_1}{\zeta_2 - \bar{\zeta}_1}\right|} .$$

The Caratheodory distance $c_D(z_1, z_2)$ of two points $z_1, z_2 \in D$ is given by the formula

$$c_D(z_1, z_2) = \sup \omega(f(z_1), f(z_2)) ,$$

where the supremum is taken over all holomorphic maps $f: D \to \Pi^+$.

If λ is any continuous linear form on $\mathfrak{R}$, positive on Ω, the function $x + iy \mapsto \lambda(x) + i\lambda(y)$ is a holomorphic map: $D \to \Pi^+$.

Since, for $t_1, t_2 \in \mathbf{R}_*^+$,

$$\omega(it_1, it_2) = \sigma(t_1, t_2) ,$$

then

$$\delta_\Omega(y_1, y_2) \leqslant c_D(iy_1, iy_2) \qquad (y_1, y_2 \in \Omega) . \tag{6.1}$$

If $D = \Pi^+$, then ([9, Proposition 2.4, p. 51])

$$c_{\Pi^+}(\zeta_1, \zeta_2) = \omega(\zeta_1, \zeta_2) \qquad (\zeta_1, \zeta_2 \in \Pi^+) .$$

Thus, by (5.1),

$$\delta_{\mathbf{R}_*^+}(y_1, y_2) = c_{\Pi^+}(iy_1, iy_2) \qquad (y_1, y_2 \in \mathbf{R}_*^+) . \tag{6.2}$$

Similarly, if $D = \Pi^+ \times \Pi^+$, then [9, example 1, p. 51]

$$c_{\Pi^+ \times \Pi^+}((\zeta_1', \zeta_1''), (\zeta_2', \zeta_2'')) = \max(\omega(\zeta_1', \zeta_2'), \omega(\zeta_1'', \zeta_2'')) .$$

Hence, by (5.2),

$$\delta_{\mathbf{R}_*^+ \times \mathbf{R}_*^+}((y_1', y_1''), (y_2', y_2'')) = c_{\Pi^+ \times \Pi^+}((iy_1', iy_1''), (iy_2', iy_2'')) \qquad (y_1', y_2', y_1'', y_2'' \in \mathbf{R}_*^+)) . \tag{6.3}$$

Going back to the general case, let $y_1, y_2 \in \Omega$, and let $\mathcal{S}(y_1, y_2)$ denote, as before, the subspace of $\mathfrak{R}$ spanned by y_1, y_2. Let $\mathcal{S}(y_1, y_2)^{\mathbf{C}} = \mathcal{S}(y_1, y_2) + i\mathcal{S}(y_1, y_2)$ be its complexification. By (6.2) and (6.3) we have

$$\delta_{\Omega \cap \mathcal{S}(y_1, y_2)}(y_1, y_2) = c_{D \cap \mathcal{S}(y_1, y_2)^{\mathbf{C}}}(iy_1, iy_2) \qquad (y_1, y_2 \in \Omega) . \tag{6.4}$$

Since [9, Proposition 2.2, p. 50]

$$c_D(iy_1, iy_2) \leqslant c_{D \cap \mathcal{S}(y_1, y_2)^{\mathbf{C}}}(iy_1, iy_2) ,$$

Lemma 5.1 and (6.4) yield

$$\delta_\Omega(y_1, y_2) \geqslant c_D(iy_1, iy_2) \qquad (y_1, y_2 \in \Omega),$$

and in conclusion, by (6.1),

$$\delta_\Omega(y_1, y_2) = c_D(iy_1, iy_2) \qquad (y_1, y_2 \in \Omega).$$

This proves

THEOREM II. *Let $\mathcal{V}$ be a complex locally convex (Hausdorff) vector space, and let $\mathcal{R}$ be a real subspace of $\mathcal{V}$ such that $\mathcal{V}$ is the complexified of $\mathcal{R}$. If Ω is an open, convex sharp cone in $\mathcal{R}$ and if D is the tube domain over Ω,*

$$D = \{z \in \mathcal{V} : \operatorname{Im} z \in \Omega\},$$

then the metric δ_Ω is the restriction to $i\Omega$ of the Caratheodory metric of D.

3. – The cone of positive hermitian elements.

7. – Let $\mathcal{A}$ be a complex Banach algebra with an identity element e. In the following $\mathcal{A}$ will be assumed to be endowed with an involution $*$ which is locally continuous (i.e. continuous on every maximal commutative $*$ subalgebra of $\mathcal{A}$) and with respect to which $\mathcal{A}$ is symmetric. The latter hypothesis implies [13, p. 233] that the involution $*$ is hermitian, i.e. every hermitian element of $\mathcal{A}$ has a real spectrum.

Let $\mathcal{R} = \mathcal{H}_\mathcal{A}$ be the real linear subvariety of $\mathcal{A}$ consisting of all the hermitian elements of $\mathcal{A}$; $\mathcal{H}_\mathcal{A}$ is a (real) closed subspace of $\mathcal{A}$ if, and only if, the involution is continuous. For any $x \in \mathcal{A}$, let $\operatorname{Sp} x$ denote the spectrum of x in $\mathcal{A}$. An element $h \in \mathcal{H}_\mathcal{A}$ is *positive*, $h \geqslant 0$, if $\operatorname{Sp} h \subset \boldsymbol{R}^+$. Let Ω_0 be the cone in $\mathcal{H}_\mathcal{A}$ consisting of all the positive hermitian elements in $\mathcal{A}$.

Since $\mathcal{A}$ is symmetric, if $h \geqslant 0$, $k \geqslant 0$, then $h + k \geqslant 0$. Thus the cone Ω_0 is convex. Let Ω be the interior part of Ω_0 in $\mathcal{H}_\mathcal{A}$. If $h \in \Omega_0$ and if $0 \in \operatorname{Sp} h$, for every positive integer ν, $h - (1/\nu)e \notin \Omega_0$: Since $h - (1/\nu)e$ tends to h as $\nu \to +\infty$, then $h \notin \Omega$. On the other hand, if $\operatorname{Sp} h \subset \boldsymbol{R}^+_*$, by the upper semicontinuity of the map $x \mapsto \operatorname{Sp} x$ $(x \in \mathcal{A})$ [13, pp. 35-36], there is a neighborhood of h in $\mathcal{H}_\mathcal{A}$ all of whose points have their spectra in $\boldsymbol{R}^+_*$.

Thus

$$\Omega = \{h \in \mathcal{H}_\mathcal{A} : \operatorname{Sp} h \subset \boldsymbol{R}^+_*\}. \tag{7.1}$$

Given any $x \in \Omega$, let $x^{\frac{1}{2}}$ be its positive square root. By the spectral mapping theorem $\operatorname{Sp} x^{\frac{1}{2}}$ is the image of $\operatorname{Sp} x$ by the map $t \mapsto \sqrt{t}$. Thus $x^{\frac{1}{2}} \in \Omega$. Let $x^{-\frac{1}{2}} = (x^{\frac{1}{2}})^{-1}$. The map

$$T_x : z \mapsto x^{-\frac{1}{2}} z x^{-\frac{1}{2}} \qquad (x \in \Omega) \tag{7.2}$$

is a bounded linear automorphism of the Banach space $\mathcal{A}$ mapping $\mathcal{H}_{\mathcal{A}}$ onto itself. If $y \in \Omega$, then $x^{-\frac{1}{2}} y x^{-\frac{1}{2}}$ is invertible; since

$$x^{-\frac{1}{2}} y x^{-\frac{1}{2}} = x^{-\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}} x^{-\frac{1}{2}} = (y^{\frac{1}{2}} x^{-\frac{1}{2}})^* (y^{\frac{1}{2}} x^{-\frac{1}{2}}) \in \Omega_0 ,$$

then $x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \in \Omega$. Thus T_x maps Ω onto itself. Being $T_x(x) = e$, then the group $\{T_x : x \in \Omega\}$ acts transitively on Ω. Hence the cone Ω is affine-homogeneous.

NOTE. We will now discuss briefly the case where the $*$-algebra $\mathcal{A}$ does not have an identity. For the sake of simplicity we will consider only the case where $\mathcal{A}$ is a C^* algebra. As it is well known, the involution $*$ extends naturally to the Banach algebra $\mathcal{A}_1 = \mathcal{A} \times \boldsymbol{C}$ obtained from $\mathcal{A}$ by adjoining the identity e, in such a way that $\mathcal{A}_1$ is a C^* algebra. For any $x \in \mathcal{A}$, let $\operatorname{Sp} x$ be the spectrum of $(x, 0)$ in $\mathcal{A}_1$. Let Ω_0 be the convex cone in $\mathcal{H}_{\mathcal{A}}$ consisting of all positive hermitian elements of $\mathcal{A}$. Then $h \in \Omega_0$ if, and only if, $\operatorname{Sp} h \subset \boldsymbol{R}^+$. However, since $0 \in \operatorname{Sp} x$ for every $x \in \mathcal{A}$, the interior part Ω of Ω_0 in $\mathcal{H}_{\mathcal{A}}$ does not satisfy (7.1).

The following lemma holds

LEMMA 7.1. *If $h \in \Omega$, then 0 is an isolated point of* $\operatorname{Sp} h$.

PROOF. The closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by h is a C^* algebra which is $*$-isomorphic and isometric to the uniform algebra $C_0(\operatorname{Sp} h)$ of all continuous complex valued functions on $\operatorname{Sp} h$ vanishing at 0, the element h corresponding to the restriction to $\operatorname{Sp} h$ of the function $\zeta \mapsto \zeta$ ($\zeta \in \boldsymbol{C}$).

If 0 is not an isolated point of $\operatorname{Sp} h$, for every $\varepsilon > 0$ there is a real-valued continuous function k_ε on $\operatorname{Sp} h$, vanishing at 0, and a neighborhood V of 0 in $\operatorname{Sp} h$, such that

$$\max_{t \in \operatorname{Sp} h} |k_\varepsilon(t) - t| < \varepsilon \quad \text{and} \quad k_\varepsilon(t) < 0 \qquad \text{for any } t \in V \setminus \{0\} .$$

Denoting by the same symbol k_ε the element of $\mathcal{B}$ whose Gelfand transform is k_ε, we have

$$(-\operatorname{Sp} k_\varepsilon) \cap \boldsymbol{R}_*^+ \neq \emptyset \quad \text{and} \quad k_\varepsilon \to h \text{ as } \varepsilon \to 0 .$$

Hence h is not an interior point of Ω. Q.E.D.

According to a theorem of Hille [7, p. 684], 0 is an isolated point of $\operatorname{Sp} h$ if, and only if, there is a subalgebra of $\mathcal{A}$ which has an identity and contains h as an invertible element.

As a consequence of Lemma 7.1, if $\mathcal{A}$ is the C^* algebra of all compact linear operators on an infinite dimensional Hilbert space, then $\Omega = \emptyset$.

8. – Let $\mathfrak{C}$ be the collection of all positive linear forms $\not\equiv 0$ on $\mathcal{A}$. All elements of $\mathfrak{C}$ are continuous linear forms. Since Ω is open, every form in $\mathfrak{C}$ assumes positive values at all points of Ω. If $f \in \mathfrak{C}$, then $f(e) > 0$, and $(1/f(e))f$ is a state of $\mathcal{A}$. Let $\mathfrak{P}$ be the set of all states of $\mathcal{A}$, and let $\mathfrak{Q}$ be the set of all pure states.

For any $h \in \mathcal{K}_{\mathcal{A}}$, let

$$a(h) = \inf\{t : t \in \operatorname{Sp} h\}, \quad b(h) = \sup\{t : t \in \operatorname{Sp} h\}. \tag{8.1}$$

By definition, $h \in \Omega_0$ or $h \in \Omega$ if, and only if, $a(h) \geqslant 0$ or $a(h) > 0$, respectively. Furthermore

$$b(h) = \varrho(h) \quad \text{if } h \in \Omega_0, \tag{8.2}$$

where $\varrho(h)$ denotes the spectral radius of h, and

$$a(h) = \frac{1}{\varrho(h^{-1})} \quad \text{if } h \in \Omega. \tag{8.3}$$

The first part of the following lemma is contained in a more comprehensive result of D. A. Raikov ([12, p. 307]; cf. also [13, Theorem 4.7.12, pp. 235-236, and Theorem 4.7.21, pp. 238-239]).

Lemma 8.1. *For every* $h \in \mathcal{K}_{\mathcal{A}}$

$$b(h) = \sup\{f(h) : f \in \mathfrak{P}\} = \sup\{f(h) : f \in \mathfrak{Q}\},$$
$$a(h) = \inf\{f(h) : f \in \mathfrak{P}\} = \inf\{f(h) : f \in \mathfrak{Q}\}.$$

Proof. The Banach subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by h and e is a commutative $*$-subalgebra of $\mathcal{A}$. Since, by the Zorn lemma, every commutative $*$-subalgebra of $\mathcal{A}$ is contained in a maximal commutative $*$-subalgebra of $\mathcal{A}$, the involution is continuous on $\mathcal{B}$.

Since $\mathbf{C} \setminus \operatorname{Sp} h$ is connected, $\operatorname{Sp} h$ is also the spectrum of h in $\mathcal{B}$. The space $\mathfrak{M}_{\mathcal{B}}$ of maximal ideals of $\mathcal{B}$, endowed with the Gelfand topology, is canonically homeomorphic to $\operatorname{Sp} h$. This homeomorphism maps $\chi \in \mathfrak{M}_{\mathcal{B}}$

onto the real number $\hat{h}(\chi) \in \operatorname{Sp} h$, $\hat{h}$ being the Gelfand transform of h. If we identify $\mathfrak{M}_{\mathcal{B}}$ with $\operatorname{Sp} h$ via this homeomorphism, $\hat{h}$ becomes the restriction to $\operatorname{Sp} h$ of the function $\zeta \mapsto \zeta$ $(\zeta \in \boldsymbol{C})$. The algebra $\hat{\mathcal{B}}$ of all Gelfand transforms of the elements of $\mathcal{B}$ is a dense, conjugation-invariant, subalgebra of the uniform algebra $C(\operatorname{Sp} h)$ of all complex-valued continuous functions on $\operatorname{Sp} h$.

If g is any state of $\mathcal{A}$, $g|_{\mathcal{B}}$ has a continuous extension as a positive linear form on $C(\operatorname{Sp} h)$. Hence there is a finite positive Borel measure m_g on $\operatorname{Sp} h$ such that

$$g(x) = \int \hat{x}(t)\, dm_g(t)$$

for all $x \in \mathcal{B}$. In particular $\|m_g\| = g(e) = 1$, and

$$g(h) = \int t\, dm_g(t)\,,$$

so that, by (8.1) and (8.2),

$$a(h) \leqslant g(h) \leqslant b(h) = \varrho(h)\,.$$

The Dirac measures with mass 1, concentrated at the points $b(h)$ and $a(h)$, define two states on $\mathcal{B}$ which extend to two states g_1 and g_2 of $\mathcal{A}$ [13, p. 235], for which we have

$$g_1(h) = b(h)\,, \qquad g_2(h) = a(h)\,.$$

Since the pure states and 0 are all the extreme points of the set of positive linear forms on $\mathcal{A}$ with norm $\leqslant 1$, the Krein-Millman theorem completes the proof of the lemma.

Since for every $h \in \Omega$

$$\delta_\Omega(e, h) = \sup\{|\log f(h)| : f \in \mathfrak{P}\}\,.$$

By the above lemma and by (8.2) and (8.3), we have

$$\delta_\Omega(e, h) = \max\{\log \varrho(h), \log \varrho(h^{-1})\} \qquad h \in \Omega)\,. \tag{8.4}$$

Given $x, y \in \Omega$, let $h = T_x y = x^{-\frac{1}{2}} y x^{-\frac{1}{2}}$. Then

$$\begin{aligned}\delta_\Omega(x, y) &= \delta_\Omega(T_x x, T_x y) = \delta_\Omega(e, h)\\ &= \max\{\log \varrho(x^{-\frac{1}{2}} y x^{-\frac{1}{2}}), \log \varrho(x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}})\}\,,\end{aligned}$$

i.e.

$$\delta_\Omega(x, y) = \max\{\log\varrho(x^{-1}y), \log\varrho(xy^{-1})\} \qquad (x, y \in \Omega)\,. \tag{8.5}$$

This formula proves

LEMMA 8.2. *The affine-invariant pseudo-metric δ_Ω is invariant under the map $x \mapsto x^{-1}$ of Ω onto itself.*

By (8.5), $\delta_\Omega(e, h) = 0$ for some $h \in \Omega$, if, and only if, $\operatorname{Sp} h = \{1\}$. Since any such h is the exponential of a hermitian quasi-nilpotent element, we obtain

THEOREM III. *The cone $\Omega \subset \mathcal{H}_{\mathcal{A}}$ consisting of all strictly positive hermitian elements of $\mathcal{A}$ is sharp if, and only if, $\mathcal{A}$ contains no non-trivial quasi-nilpotent hermitian element.*

9. – For any $x \in \Omega$, and $r > 0$, let $B(x, r)$ be the open ball in $\mathcal{H}_{\mathcal{A}}$

$$B(x, r) = \{y \in \mathcal{H}_{\mathcal{A}}\colon \|x - y\| < r\}\,,$$

and let

$$D(e, r) = \{h \in \Omega\colon \varrho(e - h) < r\}\,.$$

Being $\varrho(e - h) \leqslant \|e - h\|$, then $B(e, r) \cap \Omega \subset D(e, r)$ for all $r > 0$. Since the spectrum of $e - h$ is the image of $-\operatorname{Sp} h$ by the translation defined by the vector 1, then, with the notations (8.1),

$$a(e - h) = 1 - b(h)\,, \qquad b(e - h) = 1 - a(h)\,,$$

so that, for every $h \in \mathcal{H}_{\mathcal{A}}$,

$$\begin{aligned} \varrho(e - h) &= \max\big(|1 - b(h)|, |1 - a(h)|\big) \\ &= 1 - a(h) \qquad \text{if } a(h) + b(h) \leqslant 2\,, \\ &= b(h) - 1 \qquad \text{if } a(h) + b(h) > 2\,, \end{aligned}$$

and therefore

$$\begin{aligned} D(e, r) &= \{h \in \mathcal{H}_{\mathcal{A}}\colon a(h) > 0,\ 1 - b(h) > -r,\ 1 - a(h) < r\} \\ &= \{h \in \mathcal{H}_{\mathcal{A}}\colon \max(0, 1 - r) < a(h) \leqslant b(h) < 1 + r\}\,. \end{aligned}$$

Given $r > 0$, for any $R > 0$ such that $R < 1 - \exp(-r)$, we have also $\exp(-r) < 1 - R < 1 + R < \exp r$, and therefore

$$(9.1) \qquad B(e, R) \cap \Omega \subset D(e, R) \subset C(e, r) \qquad \text{for } 0 < R < 1 - \exp(-r)\,,$$

where $C(e, r)$ is defined by (4.3). Since every δ_Ω-isometry T_x defined by (6.2) is continuous for the norm-topology and since the family $\{T_x \colon x \in \Omega\}$ is transitive on Ω, (9.1) shows that—according to what has been proved at the end of n. 5—the norm topology in Ω is finer than the topology defined by δ_Ω.

Given $0 < R < 1$, for any $0 < s < \log(1 + R)$ we have $0 < 1 - R < \exp(-s) < \exp s < 1 + R$, and therefore,

$$(9.2) \qquad C(e, s) \subset D(e, R) \qquad \text{for } 0 < s < \log(1 + R)\,, \quad 0 < R < 1\,.$$

Suppose now that $\mathcal{A}$ is a C^* algebra with an identity. The algebra $\mathcal{A}$ contains no quasi-nilpotent element $\neq 0$, and therefore Ω is sharp. Furthermore, for every hermitian element h, we have $\varrho(h) = \|h\|$, so that $D(e, r) = B(e, r) \cap \Omega$. Thus, by (9.1) and (9.2), the δ_Ω- and the $\|\,\|$-topology coincide at e, and therefore—in view of the transitivity of the group $\{T_x \colon x \in \Omega\}$—coincide throughout Ω. If $x, y \in \Omega$, both $x^{-\frac12} y x^{-\frac12}$ and $y^{-\frac12} x y^{-\frac12}$ are hermitian,

$$\|x^{-\frac12} y x^{-\frac12}\| = \varrho(x^{-\frac12} y x^{-\frac12}) = \varrho(x^{-1} y)\,,$$
$$\|y^{-\frac12} x y^{-\frac12}\| = \varrho(y^{-\frac12} x y^{-\frac12}) = \varrho(y^{-1} x)\,,$$

and (8.5) becomes

$$(9.3) \qquad \delta_\Omega(x, y) = \max\{\log \|x^{-\frac12} y x^{-\frac12}\|, \log \|y^{-\frac12} x y^{-\frac12}\|\} \qquad (x, y \in \Omega)\,.$$

Summing up the above conclusions we have

PROPOSITION 9.1. – *If $\mathcal{A}$ is a C^* algebra with identity, the cone Ω is sharp, the metric δ_Ω is defined on Ω by formula* (9.3), *and the two topologies defined on Ω by the norm and by δ_Ω coincide.*

10. – In the remainder of this paper we will further investigate the metric δ_Ω in the case in which $\mathcal{A}$ is a von Neumann algebra of operators in a complex Hilbert space $\mathcal{E}$ (and Ω is defined by (7.1)). We will prove first that the metric structures defined in Ω by δ_Ω and by the norm, although topologically equivalent, are indeed quite different.

Denoting by $(,)$ the inner product in $\mathcal{E}$, the numerical radius of $h \in \mathcal{K}_{\mathcal{A}}$ is related to $\varrho(h)$ by

$$(10.1)\qquad \sup\{(h\xi, \xi): \xi \in \mathcal{E}, (\xi, \xi) = 1\} = \\ = \sup\{\|h^{\frac{1}{2}}\xi\|^2: \xi \in \mathcal{E}, (\xi, \xi) = 1\} = \|h^{\frac{1}{2}}\|^2 = \|h\| = \varrho(h).$$

We show now that, with the notation (8.1),

$$(10.2)\qquad a(h) = \inf\{(h\xi, \xi): \xi \in \mathcal{E}, (\xi, \xi) = 1\}.$$

Since $x \mapsto (x\xi, \xi)$ is a state of $\mathcal{A}$ for every $\xi \in \mathcal{E}$, $(\xi, \xi) = 1$, the left hand side of (10.2) is less or equal than the right hand side. Suppose that it is strictly less, and let $c \in \boldsymbol{R}$ be such that

$$a(h) < c < \inf\{(h\xi, \xi): \xi \in \mathcal{E}, (\xi, \xi) = 1\}.$$

Since

$$\operatorname{Sp}(ce - h) = c - \operatorname{Sp} h,$$

by (8.1) $\operatorname{Sp}(ce - h) \cap \boldsymbol{R}_{\bullet}^{+} \neq \emptyset$, i.e. the hermitian operator $h - ce$ is *not* positive, contradicting the fact that, for all $\xi \in \mathcal{E}$,

$$\big((h - ce)\xi, \xi\big) = (h\xi, \xi) - c(\xi, \xi) \geqslant 0,$$

and thereby proving (10.2).

Thus by (8.2), (8.3) and (8.4),

$$\delta_{\Omega}(e, h) = \sup\{|\log(h\xi, \xi)|: \xi \in \mathcal{E}, (\xi, \xi) = 1\}.$$

Since, for any $\xi \in \mathcal{E}$, $x \mapsto (x\xi, \xi)$ $(x \in \mathcal{A})$ is a normal positive linear form on $\mathcal{A}$, then we have also

$$\delta_{\Omega}(e, h) = \sup\{|\log f(h)|: f \text{ normal state on } \mathcal{A}\}.$$

In conclusion we have

LEMMA 10.1. *If $\mathcal{A}$ is a von Neumann algebra, the invariant metric δ_{Ω} is expressed for all $x, y \in \Omega$ by* ((8.5), (9.3) *and by*)

$$\begin{aligned}\delta_{\Omega}(x, y) &= \sup\{\sigma\big((x\xi, \xi), (y\xi, \xi)\big): \xi \in \mathcal{E}, (\xi, \xi) = 1\} \\ &= \sup\{\sigma\big(f(x), f(y)\big): f \text{ normal positive linear form on } \mathcal{A}\} \\ &= \sup\{\sigma\big(f(x), f(y)\big): f \text{ normal state on } \mathcal{A}\}.\end{aligned}$$

THEOREM IV. *If $\mathcal{A}$ is a von Neumann algebra, the metric δ_Ω is complete on Ω.*

PROOF. *a*) Let $\{x_\nu\}$ be a Cauchy sequence in Ω for the metric δ_Ω. The distances $\{\delta_\Omega(e, x_\nu)\}$ are bounded by a finite constant $k > 0$. Hence, by (8.1), (8.2), (8.3) and (8.4), we have

$$\exp(-k) < a(x_\nu) \leqslant b(x_\nu) < \exp k \qquad (\nu = 1, 2 \ldots). \tag{10.3}$$

Furthermore

$$\|x_\nu\| = \varrho(x_\nu) < \exp k .$$

Denoting by $\mathcal{A}_*$ the predual of $\mathcal{A}$, x_ν converges to some hermitian element $x \in \mathcal{A}$ for the weak topology defined by $\mathcal{A}_*$ on $\mathcal{A}$. By Lemma 10.1 and by (10.3), we have $x \in \Omega$ and $\delta_\Omega(e, x) \leqslant k$. We will prove that $\delta_\Omega(x, x_\nu) \to 0$ as $\nu \to \infty$. By Proposition 9.1 this is equivalent to showing that

$$\|x - x_\nu\| \to 0 \quad \text{ as } \nu \to \infty .$$

If this fact does not hold, there is a subsequence $\{x_{\nu_i}\}$ of $\{x_\nu\}$ and a positive constant $r > 0$ such that

$$\|x_{\nu_{i+1}} - x_{\nu_i}\| > 2r \quad \text{ for } i = 1, 2, \ldots .$$

We may assume $0 < r < \exp k$.
Since for all $y \in \mathcal{A}$

$$\|y\| = \sup\{|f(y)| : f \in \mathcal{A}_*, \|f\| \leqslant 1\},$$

then there is some $f_i \in \mathcal{A}_*$, $\|f_i\| \leqslant 1$ such that

$$|f_i(x_{\nu_{i+1}}) - f_i(x_{\nu_i})| > 2r \quad \text{ for } i = 1, 2, \ldots . \tag{10.4}$$

By considering the orthogonal decomposition $f_i = f_i^+ - f_i^-$ of f_i [15, p. 31] we see that (10.4) must be satisfied—with r in place of $2r$—when we substitute to f_i at least one of the two positive normal forms f_i^+ or f_i^-. In other words, there is some normal state f_i on $\mathcal{A}$, such that

$$|f_i(x_{\nu_{i+1}}) - f_i(x_{\nu_i})| > r \quad \text{ for } i = 1, 2, \ldots . \tag{10.5}$$

b) By Lemma 10.1, for any $\varepsilon > 0$ there exists an index ν_0 such that, whenever $\nu, \mu > \nu_0$, then

$$\sigma(f(x_\nu), f(x_\mu)) < \varepsilon$$

for every positive normal form f on $\mathcal{A}$. Thus being $\exp(-k) \leqslant f(x_\mu) \leqslant \exp k$, for all μ and for every normal state f, we have

$$(10.6) \qquad (\exp(-\varepsilon) - 1)\exp k < f(x_\nu) - f(x_\mu) < (\exp\varepsilon - 1)\exp k \quad \text{for } \mu, \nu > \nu_0 .$$

Choosing $0 < \varepsilon < \log(1 + r\exp(-k))$, then,

$$-r < (\exp(-\varepsilon) - 1)\exp k\,, \qquad (\exp\varepsilon - 1)\exp k < r\,.$$

Thus (10.6) contradicts the conclusion of a) and thereby proves the theorem.

11. – Let Ω be an open convex cone in a real Banach space $\mathfrak{R}$. It is easily seen that Ω is sharp if, and only if, there is no non-trivial translation of $\mathfrak{R}$ such that the image of Ω is a cone (with vertex 0).

Let $\mathfrak{L}(\mathfrak{R})$ be the real Banach algebra consisting of all bounded linear operators in $\mathfrak{R}$. Suppose that Ω is sharp and let $\mathfrak{L}(\Omega)$ and $\mathfrak{L}(\bar{\Omega})$ be the cones in $\mathfrak{L}(\mathfrak{R})$ consisting of all bounded linear operators mapping, respectively, Ω into itself and the closure $\bar{\Omega}$ of Ω into itself. Let $\mathfrak{L}(\mathfrak{R})^{-1}$ be the open group of all invertible elements in the Banach algebra $\mathfrak{L}(\mathfrak{R})$, and let $G(\Omega) = \mathfrak{L}(\Omega) \cap \mathfrak{L}(\mathfrak{R})^{-1}$.

By Banach's homomorphism theorem, a bounded linear operator in $\mathfrak{R}$ belongs to $G(\Omega)$ if, and only if, its restriction to Ω is a bijective map of Ω onto Ω.

The cone Ω being open and convex, $\mathring{\bar{\Omega}} = \Omega$. Hence any element of $\mathfrak{L}(\mathfrak{R})^{-1}$ mapping $\bar{\Omega}$ onto $\bar{\Omega}$ belongs to $G(\Omega)$, i.e. $G(\Omega) = \mathfrak{L}(\bar{\Omega}) \cap \mathfrak{L}(\mathfrak{R})^{-1}$. Thus *$G(\Omega)$ is a closed subgroup of $\mathfrak{L}(\mathfrak{R})^{-1}$ (for the norm topology).*

$G(\Omega)$ acts continuously on Ω, that is, the natural map $G(\Omega) \times \Omega \to \Omega$ defined by $(A, x) \mapsto Ax$ is continuous.

THEOREM V. *Let $\mathcal{A}$ be a von Neumann algebra and let Ω be the open convex cone in $\mathfrak{R} = \mathcal{H}_{\mathcal{A}}$ consisting of all strictly positive hermitian elements of $\mathcal{A}$. Let $\mathfrak{L}(\Omega) \subset \mathfrak{L}(\mathcal{H}_{\mathcal{A}})$ be the set of all bounded linear operators on $\mathcal{H}_{\mathcal{A}}$ mapping Ω into itself. If V_1 and V_2 are any two bounded sets in Ω for the metric δ_Ω, the set*

$$(11.1) \qquad \{A \in \mathfrak{L}(\Omega) : A(V_1) \cap V_2 \neq \emptyset\}$$

is bounded in norm in $\mathfrak{L}(\mathcal{H}_{\mathcal{A}})$.

PROOF. There is no restriction in assuming V_1 and V_2 to be two open balls $C(x, r)$ and $C(y, r)$ with centers $x, y \in \Omega$ and radius $r > 0$, for the metric δ_Ω. According to Proposition 9.1, there is some $R > 0$ such that

$$B(x, R) = \{z \in \mathcal{H}_{\mathcal{A}}: \|z - x\| < R\} \subset C(x, r),$$
$$B(y, R) = \{z \in \mathcal{H}_{\mathcal{A}}: \|z - y\| < R\} \subset C(y, r).$$

The norm $\|A\|$ of any A in (11.1) is given by

$$\|A\| = \frac{1}{R} \sup\{\|A(z - x)\|: z \in B(x, R)\}. \tag{11.2}$$

Denoting again by $\mathcal{A}_*$ the predual of $\mathcal{A}$,

$$\|A(z - x)\| = \sup\{|f(A(z - x))|: f \in \mathcal{A}_*, \|f\| \leqslant 1\}.$$

Considering the orthogonal decomposition $f = f^+ - f^-$ of any normal linear form f on $\mathcal{A}$, we have

$$|f(A(z - x))| \leqslant f^+(A(z + x)) + f^-(A(z + x)) \qquad (z \in B(x, R)).$$

Hence

$$\|A(z - x)\| \leqslant 2 \sup f(A(z + x)) \leqslant 2 \left\{\sup \left(\frac{f(Az)}{f(y)}\right) \sup f(y) + \sup \left(\frac{f(Ax)}{f(y)}\right) \sup f(y)\right\},$$

where all the supremums are taken over the family of all normal positive linear forms f with $\|f\| \leqslant 1$.

By Lemma 10.1, for all $z \in C(x, r)$,

$$\begin{aligned} \|A(z - x)\| &\leqslant 2\{\exp \delta_\Omega(Az, y) + \exp \delta_\Omega(Ax, y)\} \exp \delta_\Omega(y, e) \\ &\leqslant 2\{\exp (\delta_\Omega(Az, Ax) + \delta_\Omega(Ax, y)) + \exp \delta_\Omega(Ax, y)\} \exp \delta_\Omega(y, e) \\ &\leqslant 2 \exp \delta_\Omega(Ax, y)(1 + \exp r) \exp \delta_\Omega(y, e). \end{aligned}$$

Since A belongs to the set (11.1), then $\delta_\Omega(Ax, y) < 2r$.

Therefore

$$\|A(z - x)\| < 2 \exp (2r)(1 + \exp r) \exp \delta_\Omega(y, e).$$

The theorem follows then from (11.2).

Appendix.

12. – We prove now Proposition 3.2. Re-arranging the indices, if necessary, we may assume $x_1 < y_1$.

a) We consider first the case where the affine line passing through x and y has a half-line in common with $\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$. Since $x_1 < y_1$, this implies $x_2 \leqslant y_2$.

If $\sigma(x_1, y_1) \leqslant \sigma(x_2, y_2)$, then $x_2 < y_2$, and, by (3.1) and Proposition 2.4, we have

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) \geqslant \sigma(x_2, y_2) . \tag{12.1}$$

The affine function $f\colon \boldsymbol{R} \to \boldsymbol{R} \times \boldsymbol{R}$ defined by

$$f(t) = \left(x_1 + \frac{y_1 - x_1}{y_2 - x_2}(t - x_2), t\right) \qquad (t \in \boldsymbol{R}) ,$$

maps $\boldsymbol{R}_*^+$ into $\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$; furthermore

$$f(x_2) = (x_1, x_2) , \qquad f(y_2) = (y_1, y_2) .$$

Proposition 2.2 and (12.1) yield

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) = \sigma(x_2, y_2) .$$

If $\sigma(x_1, y_1) \geqslant \sigma(x_2, y_2)$, replacing the affine function f by the function $g\colon \boldsymbol{R} \to \boldsymbol{R} \times \boldsymbol{R}$ defined by

$$g(t) = \left(t, x_2 + \frac{y_2 - x_2}{y_1 - x_1}(t - x_1)\right) \qquad (t \in \boldsymbol{R}) ,$$

and repeating the above considerations, we obtain

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) = \sigma(x_1, y_1) .$$

Note that, if the affine line through x, y intersects $\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$ along a half-line, for any point z belonging to the segment $[x, y]$, we have

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(x, y) = \gamma(x, z) + \gamma(z, y)$$

b) Let us consider now the case where the affine line through x and y has a finite segment in common with $\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+$.

Since $x_1 < y_1$, then $x_2 > y_2$. With the same notations as in the definition of $\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}$ (n. 2), we have

$$\gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}((x_1, x_2), (y_1, y_2)) \leqslant \sigma(a^1, b^1) + \ldots + \sigma(a^n, b^n) .$$

For any $\varepsilon > 0$, we can select $n \geqslant 2$, a^j, b^j, p^j, f_j $(j = 1, \ldots, n)$ in such a way that

$$\gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}((x_1, x_2), (y_1, y_2)) \geqslant \sigma(a^1, b^1) + \ldots + \sigma(a^n, b^n) - \varepsilon .$$

By Proposition 2.2 and by (3.1) we have then

$$\text{(12.2)} \qquad \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}((x_1, x_2), (y_1, y_2)) \geqslant \sum_{j=1}^{n} \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(f_j(a^j), f_j(b^j)) - \varepsilon = \sum_{j=1}^{n} \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(p^{j-1}, p^j) - \varepsilon .$$

We prove now that for every choice of n, a^j, b^j, p^j, f^j $(j = 1, \ldots, n)$ as in n. 2, we have

$$\text{(12.3)} \qquad \sum_{j=1}^{n} \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(p^{j-1}, p^j) \geqslant \sigma(x_1, y_1) + \sigma(x_2, y_2) .$$

c) If $n = 2$, the point p^1 has coordinates $p_1^1 \geqslant y_1$, $p_2^1 \geqslant x_2$ or $p_1^1 \leqslant x_1$, $p_2^1 \leqslant y_2$. In the first case, let $u = (y_1, u_2)$ and $v = (v_1, x_2)$ be the intersections of the lines $\{(y_1, t) : t \in \mathbf{R}\}$ and $\{(t, x_2) : t \in \mathbf{R}\}$ with the segments $[x, p^1]$ and $[y, p^1]$ respectively. Then $u_2 \geqslant x_2$ and $v_1 \geqslant y_1$. By the final remark of *a*)

$$\gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(x, p^1) + \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(y, p^1) \geqslant \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(x, u) + \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(y, v) \geqslant \sigma(x_1, y_1) + \sigma(x_2, y_2) .$$

An entirely similar argument, in the second case, leads to the same conclusion.

d) We proceed by induction on n, assuming (12.3) to hold for $n = n_0 (\geqslant 2)$ and proving it for $n = n_0 + 1$. If at least one, p^l say, of the vertices $p^1, \ldots, p^{n-1}$ has coordinates $p_1^l \geqslant y_1$, $p_2^l \geqslant x_2$ or $p_1^l \leqslant x_1$, $p_2^l \leqslant y_2$, then, by the triangle inequality, the left hand side of (12.3) is larger or equal than

$$\gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(x, p^l) + \gamma_{\mathbf{R}^+_* \times \mathbf{R}^+_*}(p^l, y) ,$$

which, by *c*), is at least equal to

$$\sigma(x_1, y_1) + \sigma(x_2, y_2) .$$

e) Suppose now that no such vertex like p^l exists among $p^1, \ldots, p^{n-1}$. At least one of the four half-lines $\{(t, y_2) : t < y_1\}$, $\{(x_1, t) : t < x_2\}$, $\{(t, x_2) : t > x_1\}$,

$\{(y_1, t): t > y_2\}$ has a non-empty intersection with some side, $[p^{i-1}, p^i]$ ($1 < < i < n$), of the polygonal $\{[p^1, p^2], \ldots, [p^{n-2}, p^{n-1}]\}$. Suppose that this happens to the half-line $\{(t, y_2): t < y_1\}$. (An entirely similar argument holds for the other three half-lines.)

Let $w = (w_1, y_2)$ be one of the points of intersection. By the final remark of a) and by the triangle inequality,

$$
\begin{aligned}
(12.4)\qquad \sum_{j=1}^{n} \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{j-1}, p^j) &= \sum_{j=1}^{i-1} \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{j-1}, p^j) + \\
&\quad + \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{i-1}, w) + \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(w, p^i) + \sum_{j=1+1}^{n} (p^{j-1}, p^j) \\
&\geqslant \sum_{j=1}^{i-1} \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{j-1}, p^j) + \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{i-1}, w) + \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(w, y) \\
&= \sum_{j=1}^{i-1} \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{j-1}, p^j) + \gamma_{\boldsymbol{R}_*^+ \times \boldsymbol{R}_*^+}(p^{i-1}, w) + \sigma(w_1, y_1)\,.
\end{aligned}
$$

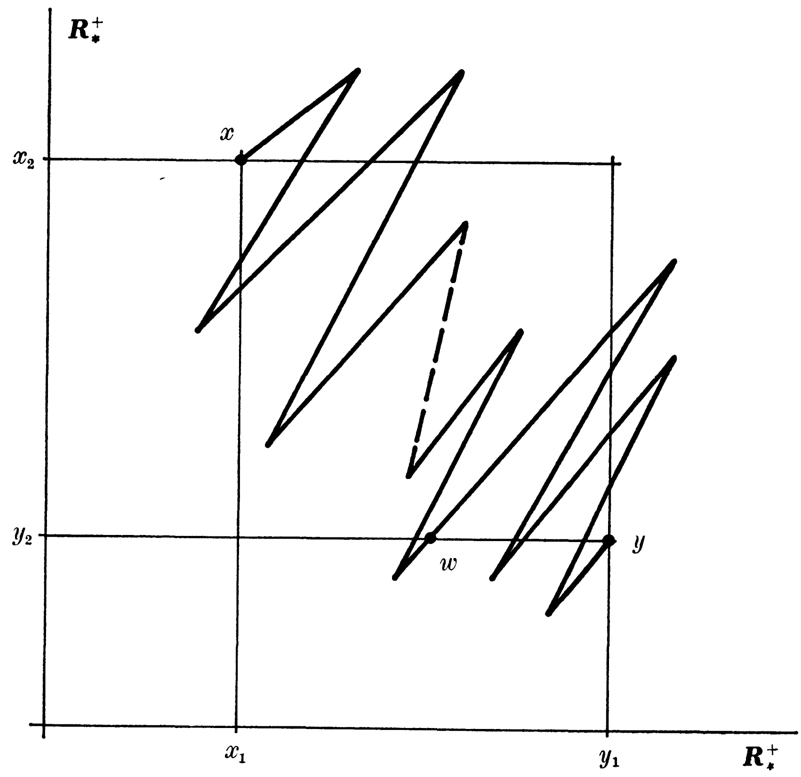

Figure 1

By d) we may assume $w_1 > x_1$: Then the affine line determined by x and w intersects $R_{\bullet}^{+} \times R_{\bullet}^{+}$ on a finite segment. Since $i < n$, the polygonal $\{[x, p^1], [p^1, p^2], \dots, [p^{i-1}, w]\}$ has $i \leqslant n_0$ sides. Then, by the inductive hypothesis

$$\sum_{j=1}^{i-1} \gamma_{R_{\bullet}^{+} \times R_{\bullet}^{+}}(p^{j-1}, p^j) + \gamma_{R_{\bullet}^{+} \times R_{\bullet}^{+}}(p^{i-1}, w) \geqslant \sigma(x_1, w_1) + \sigma(x_2, y_2),$$

so that (12.4) yields

$$\sum_{j=1}^{n} \gamma_{R_{\bullet}^{+} \times R_{\bullet}^{+}}(p^{j-1}, p^j) \geqslant \sigma(x_1, w_1) + \sigma(w_1, y_1) + \sigma(x_2, y_2) = \sigma(x_1, y_1) + \sigma(x_2, y_2),$$

thereby proving (12.3).

f) As a consequence of (12.3) we have, by (12.2),

$$\gamma_{R_{\bullet}^{+} \times R_{\bullet}^{+}}\big((x_1, x_2), (y_1, y_2)\big) \geqslant \sigma(x_1, y_1) + \sigma(x_2, y_2) - \varepsilon$$

for every $\varepsilon > 0$. Hence

$$\gamma_{R_{\bullet}^{+} \times R_{\bullet}^{+}}\big((x_1, x_2), (y_1, y_2)\big) \geqslant \sigma(x_1, y_1) + \sigma(x_2, y_2).$$

By (3.1) and Proposition 2.4 this inequality implies (3.3). Q.E.D.

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