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Complexes of Partial Differential Operators (*).

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dedicated to Hans Lewy

Introduction.

This research can be considered as a sort of commentary to the work of B. Malgrange on the theory of division of distributions. Our indebtness to this work is deeper than it may be superficially appearent from the context of the paper. It was from Seminars held by Malgrange in Pisa in 1962 through 1966 that we have been introduced to the theory of division of distributions developed by Ehrenpreis, Lojasiewicz and Malgrange himself for its general formulation.

Given an open set Ω in $\boldsymbol{R}^n$ and a matrix $A(\xi) = \big(a_{ij}(\xi)\big)_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant q}}$ of polynomials in n variables $\xi_1, \dots, \xi_n$, one considers the differential operator

$$A(D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^p(\Omega)$$

(where $\mathcal{E}^s(\Omega)$ is the space of C^∞ maps from Ω to $\boldsymbol{C}^s$) and the corresponding system of equations

$$(*) \qquad A(D)u = f \qquad f \in \mathcal{E}^p(\Omega)\,, \qquad u \in \mathcal{E}^q(\Omega)\,.$$

If the equation is solvable and if $Q(\xi) = \big(Q_1(\xi), \dots, Q_p(\xi)\big)$ is a vector with polynomial components such that $Q(\xi)A(\xi) \equiv 0$ then one must have $Q(D)f = 0$.

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Also if $S(\xi) = \begin{pmatrix} S_1(\xi) \\ \vdots \\ S^q(\xi) \end{pmatrix}$ is another vector with polynomial components such that $A(\xi)S(\xi) \equiv 0$, if u is a solution of $(*)$ then $u + S(D)v$, $\forall v \in \mathcal{E}(\Omega)$ is also a solution of $(*)$.

The collection of the vectors Q (integrability conditions) and the collection of the vectors S (cointegrability conditions) form finitely generated modules over the ring $\mathcal{T}$ of polynomials in the variables ξ's.

To take these into account is to insert the operator $A(D)$ into a complex

$$\mathcal{E}^s(\Omega) \xrightarrow{C(D)} \mathcal{E}^q(\Omega) \xrightarrow{A(D)} \mathcal{E}^p(\Omega) \xrightarrow{B(D)} \mathcal{E}^r(\Omega)$$

where the matrices B and C are obtained from a basis of the integrability, resp. cointegrability, conditions for $A(D)$.

Thus the study of a system of linear partial differential equations with constant coefficients leads us to the study of complexes of differential operators with constant coefficients.

Up to now only the complexes associated to the gradient operator (de Rham complex) and to the Cauchy-Riemann equations (Dolbeault complex) have received extensive study, apart from the complex arising from a single equation.

In section 1 to 5 we rehearse the fundamental theorems of the theory of division of distributions bringing in some additional remarks that will be used later.

The problem of extending the complex defined by the given operator A to the right is then related to the theorem on syzygies of Hilbert (sections 5 to 9) of which we give here a direct proof, for which we are indebted for a substantial help to E. Vesentini.

Section 9 is devoted to the problem of extending the complex to the left as much as possible, as it is an experimental truth that « the longer the complex is, the simpler is the initial operator ». This leads to the proof of a theorem that in a different form appears in Palamodov [22].

Section 10 and 11 are devoted to the study of the « generic » situation and provides us with generalized Koszul complexes. These have been introduced and studied by Auslander, Buchsbaum and Rim [2], [4], [5], [6]. The complexes we propose here are closely related to a resolution established by Eagon and Northcott [8] for the ideal generated by the top minor determinants of a matrix. The proof of the exactness of these Koszul complexes is obtained by an argument of Macaulay that here is presented for the sake of completeness.

It is clear that the theory of division of distributions transforms theorems on differential operators with constant coefficients into theorems of algebra and therefore this first chapter is a chapter of some commutative algebra.

We will postpone the theory of convexity to a second chapter.

Part of the results of Chapter I have been the subject of the J. K. Whittemore lectures given by one of the authors at Yale University in March 1974 [1].

Chapter I

COMPLEXES OF DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

1. – Notations.

a) Let Ω be an open set in R^n. We denote by $\mathcal{E}(\Omega)$ the space of all C^∞ functions on Ω with the Fréchet-Schwartz topology. This has a Hausdorff locally convex topology defined as follows. Let $K_1 \subset K_2 \subset K_3 \subset \ldots$ be an increasing sequence of compact subsets of Ω with the properties

$$K_i \subset \mathring{K}_{i+1} \quad \text{for } i = 1, 2, \ldots$$

$$\bigcup K_i = \Omega .$$

For every K_i and every integer $k \geqslant 0$ we define

$$p_{K_i,k}(f) = \sup_{x \in K_i} \sum_{|\alpha| \leqslant k} |D^\alpha f(x)| , \qquad \forall f \in \mathcal{E}(\Omega)$$

where

$$D^\alpha = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} ,$$

$x_1, \ldots, x_n$ being the coordinates in R^n.

The Fréchet-Schwartz topology on $\mathcal{E}(\Omega)$ is the topology defined by the set of seminorms $p_{K_i,k}$, for $i = 1, 2, \ldots,\ k = 0, 1, \ldots$.

This topology is independent of the choice of the sequence $\{K_i\}$ and can be defined by the sequence of seminorms

$$p_i = p_{K_i,i}$$

which has the property that, for every $f \in \mathcal{E}(\Omega)$

$$p_1(f) \leqslant p_2(f) \leqslant p_3(f) \leqslant \dots$$

For every K_i we set

$$\mathcal{D}(K_i) = \{f \in \mathcal{E}(\Omega) | \operatorname{supp} f \subset K_i\}.$$

With the induced topology this is also a Fréchet space.
We denote by

$\mathcal{D}(\Omega)$ the space of C^∞ functions in Ω with compact support in Ω

we have

$$\mathcal{D}(\Omega) = \bigcup_{i=1}^{\infty} \mathcal{D}(K_i)$$

and we consider on $\mathcal{D}(\Omega)$ the inductive limit topology

$$\mathcal{D}(\Omega) = \varinjlim_{i} \mathrm{K}(K_i).$$

We denote by

$\mathcal{D}'(\Omega) =$ the strong dual of the space $\mathcal{D}(\Omega) =$ the space of all distributions on Ω;

$\mathcal{E}'(\Omega) =$ the strong dual of the space $\mathcal{E}(\Omega) =$ the distribution in Ω with compact support.

If p is an integer $p \geqslant 1$ we denote by $\mathcal{E}^p(\Omega)$ the space

$$\mathcal{E}^p(\Omega) = \mathcal{E}(\Omega) \times \dots \times \mathcal{E}(\Omega) \qquad p \text{ times}$$

and similarly for $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$.

b) Let $\mathcal{T} = C[x_1, \dots, x_n]$ be the space of all polynomials with complex coefficients in the indeterminates $x_1, \dots, x_n$.

If $P(x) = P(x_1, \dots, x_n)$ is a polynomial we denote by $P(D)$ the differential operator obtained from $P(x)$ by the substitution

$$x_i \to \frac{\partial}{\partial x_i} = D_i.$$

For $a \in R^n$ we denote by $\phi_a = C\{x_1 - a_1, \ldots, x_n - a_n\}$ the space of formal power series centered at a, i.e. in the indeterminates $x_1 - a_1, \ldots, x_n - a_n$.

Let $\mathcal{E}_a$ be the space of germs of C^∞ functions at a point $a \in R^n$; we set for every $f \in \mathcal{E}_a$

$$\mathcal{T}_a(f) = \sum_{\alpha \in N^n} \frac{1}{\alpha!} D^\alpha f(a)(x-a)^\alpha ,$$

the Taylor expansion of f centered at a. We have thus defined a linear map

$$\mathcal{T}_a \colon \mathcal{E}_a \to \phi_a .$$

c) Given a function f which is absolutely integrable on R^n with respect to the Lebesgue measure, $f \in L^1(R^n, dx)$, we define the Fourier transform of f by

$$\hat{f}(y) = \mathcal{F}(f)(y) = \int_{R^n} f(x) \exp[-2\pi i \langle x, y \rangle]\, dx$$

where $\langle x, y \rangle = \sum x_i y_i$.

We have

$$\mathcal{F}\big(P(-2\pi i x) f\big) = P(D_y)\mathcal{F}(f)$$

$$\mathcal{F}\big(P(D_x) f\big) = P(2\pi i y)\mathcal{F}(f) .$$

Let $\mathcal{S}(R^n)$ denote the space of rapidly decreasing functions.

This can be defined as the completion of the space $\mathcal{D}(\boldsymbol{R}^n)$ under the topology defined by the set of seminorms

$$P_{\alpha,\beta}(f) = \sup_{x \in R^n} |x^\alpha D^\beta f(x)| , \qquad \alpha, \beta \in N^n .$$

Then $\mathcal{F}$ defines a topological isomorphism

$$\mathcal{F} \colon \mathcal{S}\big(R^n(x)\big) \to \mathcal{S}\big(R^n(y)\big)$$

whose inverse (left and right) is the inverse Fourier transform

$$\overline{\mathcal{F}}(g) = \int_{R^n} \exp[2\pi i \langle x, y \rangle] g(y)\, dy .$$

The strong dual of the space $\mathcal{S}$ is denoted by $\mathcal{S}'$ and is called the space of tempered distributions.

2. – Theorems of Łojasiewicz-Malgrange.

We recall four important theorems which one deduces from the theorem of division of distributions.

Let $A = \big(a_{ik}(x)\big)_{\substack{1\leq i\leq p\\ 1\leq k\leq q}}$ be a matrix of type (p, q) with polynomial entries. We denote by $A(D) = \big(a_{ik}(D)\big)$ the system of partial differential equations

$$\left\{\sum_{k=1}^{q} a_{ik}(D)\, u_k = f_i \right. \quad 1\leq i\leq p$$

defined for $u \in \mathcal{E}^q(R^n)$ and $f \in \mathcal{E}^p(R^n)$. We briefly write

$$\mathcal{E}^q(R^n) \xrightarrow{A(D)} \mathcal{E}^p(R^n)\,.$$

THEOREM A. *Let Ω be an open convex subset of R^n. The necessary and sufficient condition for the system*

$$A(D)u = f$$

to have a solution $u \in \mathcal{D}^q(\Omega)$ for a given $f \in \mathcal{D}^p(\Omega)$ is that for any point $a \in C^n$ the equation

$$\mathcal{T}_a(A)x = \mathcal{T}_a(\hat{f})$$

admits a solution $x \in \phi_a^q$ (i.e. by formal power series centered at a).

Notice that, f having compact support, the Fourier transform $\hat{f}$ is defined as en entire function on the whole space C^n.

Similarly we have a completely analogous statement for $\mathcal{E}'(\Omega)$ replacing $\mathcal{D}(\Omega)$:

THEOREM A'. *Let Ω be an open convex subset of R^n and let $f \in \mathcal{E}'^p(\Omega)$. The necessary and sufficient condition for the system*

$$A(D)u = f$$

to have a solution $u \in \mathcal{E}'^q(\Omega)$ is that, for any $a \in C^n$ the equation

$$\mathcal{T}_a(A)x = \mathcal{T}_a(\hat{f})$$

admits a solution $x \in \phi_a^q$.

The other couple of fundamental theorems is the following one.

THEOREM B. *Let Ω be an open convex subset of R^n and let $f \in \mathcal{E}^p(\Omega)$. The necessary and sufficient condition for the system*

$$A(D)u = f$$

to have a solution $u \in \mathcal{E}^q(\Omega)$ is that for any vector $v(x) = \big(v_1(x), \dots, v_p(x)\big) \in \mathcal{P}^p$ such that

$$v(x)A(x) = 0$$

one should have

$$v(D)f = 0 .$$

Replacing $\mathcal{E}(\Omega)$ by $\mathcal{D}'(\Omega)$ one has the following

THEOREM B$'$. *Let Ω be an open convex subset of R^n and let $f \in \mathcal{D}'^p(\Omega)$. If the equation*

$$A(D)u = f$$

has a solution $u \in \mathcal{D}'^q(\Omega)$ then for any vector $v(x) \in \mathcal{P}^p$ such that $v(x)A(x) = 0$ we must have also $v(D)f = 0$.

Conversely, assuming that $f \in \mathcal{D}'^p(\Omega)$ is given and that $v(D)f = 0$ for any $v(x) \in \mathcal{P}^p$ such that $v(x)A(x) = 0$, then for any compact subset $K \subset \Omega$ we can find $u \in \mathcal{D}'^q(\Omega)$ such that

$$A(D)u = f \quad \text{on } K .$$

The statements and proofs of these theorems are to be found in Malgrange [15], [16], [20].

3. – Resolutions of a differential operator with constant coefficients.

Let $\mathcal{P} = C[z_1, \dots, z_n]$ be the ring of polynomials in n indeterminates $z_1, \dots, z_n$.

Let $A_0(z)$, $A_1(z)$ be two matrices with polynomial entries of type (p, q) and (q, r) respectively and let $A_0(D)$, $A_1(D)$ the differential operator one obtains by the substitution

$$z_i \to \frac{\partial}{\partial x_i} .$$

PROPOSITION 1. *Let Ω be a convex open set in R^n. The necessary and sufficient condition for the sequence of differential operators*

$$\mathcal{D}^r(\Omega) \xrightarrow{A_1(D)} \mathcal{D}^q(\Omega) \xrightarrow{A_0(D)} \mathcal{D}^p(\Omega) \tag{1}$$

to be exact is that the sequence of $\mathfrak{T}$-homomorphism

(2) $$\mathfrak{T}^r \xrightarrow{A_1(z)} \mathfrak{T}^q \xrightarrow{A_0(z)} \mathfrak{T}^p$$

be an exact sequence.

PROOF. *Sufficiency*. Let $a \in C^n$ and let ϕ_a be the ring of formal power series centered at a; $\phi_a = C\{z_1 - a_1, \dots, z_n - a_n\}$. We can consider $\mathfrak{T}$ as a subring of ϕ_a. Then $(\mathfrak{T}, \phi_a)$ is a flat couple. Thus we get from (2) an exact sequence

(2′) $$\phi_a^r \xrightarrow{A_1(z)} \phi_a^q \xrightarrow{A_0(z)} \phi_p^a .$$

First we remark that from (2) we derive

(∗) $$A_0(z)A_1(z) \equiv 0 .$$

Setting

$$z_j = 2\pi i y_j , \qquad 1 \leqslant j \leqslant n$$

we get then for any $g \in \mathfrak{D}^r(\Omega)$

$$A_0(2\pi i y)A_1(2\pi i y)\hat{g}(y) = 0$$

thus by Fourier transform

$$A_0(D)A_1(D)g(x) = 0 , \qquad \forall g \in \mathfrak{D}^r(\Omega)$$

and this implies that

$$A_0(D)A_1(D) = 0$$

i.e. (1) is a complex (which is a directly obvious consequence of (∗) after all).
Let now $g \in \mathfrak{D}^q(\Omega)$ be such that

$$A_0(D)g(x) = 0 .$$

Then

$$A_0(2\pi i y)\hat{g}(y) = 0 .$$

Taking the Taylor series at the point $a \in C^n$ we get

$$A_0(2\pi i y)\,\mathfrak{T}_a(\hat{g}) = 0 .$$

Because of (2′) we then get for some $\varphi_a \in \phi_a^r$

$$\mathcal{C}_a(\hat{g}) = A_1(2\pi i y)\varphi_a .$$

We can then apply theorem A and we then deduce that the equation

$$A_1(D)u = g$$

admits a solution $u \in \mathfrak{D}'(\Omega)$. Thus (1) is exact.

Necessity. We prove first the following

LEMMA 1. *Let Ω be any open convex subset of R^n. Consider the space*

$$\widehat{\mathfrak{D}(\Omega)} = \{\hat{g}(y) | g \in \mathfrak{D}(\Omega)\}$$

as a linear subspace of the space $\mathcal{H}(C^n)$ of all entire functions over C^n.

(α) *Given $a \in C^n$, $\varphi \in \phi_a$ and an integer $k \geqslant 0$ we can find $g \in \mathfrak{D}(\Omega)$ such that*

$$\mathcal{C}_a(\hat{g}) \equiv \varphi \bmod \mathfrak{M}_a^k$$

where $\mathfrak{M}_a$ denotes the maximal ideal of the ring ϕ_a:

(β) *We can find a finite number $g_1, \ldots, g_l$ of functions in $\mathfrak{D}(\Omega)$ such that the entire functions $\hat{g}_1(y), \ldots, \hat{g}_l(y)$ have no common zeros in C^n.*

PROOF. (α) Let $g(x) \in \mathfrak{D}(\Omega)$ be such that $\int_\Omega g(x)\,dx = 1$.
Set

$$h_a(x) = \exp[2\pi i \langle x, a\rangle] g(x)$$

then $h_a(x) \in \mathfrak{D}(\Omega)$ and we have

$$\hat{h}_a(a) = 1 .$$

Let $P(y)$ be a polynomial of the form

$$(y_1 - a_1)^{\alpha_1} \ldots (y_n - a_n)^{\alpha_n} \qquad |\alpha| = \sum \alpha_i < k .$$

Then $P(-(1/2\pi i)D)h(x) \in \mathfrak{D}(\Omega)$ and its Fourier transform is

$$P(y)\hat{h}(y) = (y_1 - a_1)^{\alpha_1} \ldots (y_n - a_n)^{\alpha_n} + O(|\alpha| + 1) .$$

This shows that we can prescribe as we like the leading term in the Taylor series $\mathcal{T}_a(\hat{\chi})$ of a function $\chi \in \mathcal{D}(\Omega)$.

By taking linear combinations with constant complex coefficients of suitable functions in $\mathcal{D}(\Omega)$ we can prescribe ad libitum all terms of the Taylor development $\mathcal{T}_a(\hat{\chi})$ up to the order k. This proves the statement (α).

(β) First we remark that it is not restrictive to assume that the origin $0 \in R^n$ be contained in Ω by making a translation.

Let us first prove the statement in the case of one variable and let us assume $\Omega = \{x \in R \mid -\varepsilon < x < \varepsilon\}$. Let $f \in \mathcal{D}(\Omega)$, $\int f\,dx \neq 0$. Then $\hat{f}(y)$ is an entire function on C.

Set

$$g(x) = \varrho^{-1} f(x/\varrho) \qquad \text{for } 0 < \varrho \leqslant 1 .$$

Then $g(x) \in \mathcal{D}(\Omega)$ and we have

$$\hat{g}(y) = \hat{f}(\varrho y) .$$

Let $S = \{\mathfrak{z}_j\}_{j \in J}$ be the set of zeros of $\hat{f}(y)$. Since $0 \notin S$, the set $A_m = \{\varrho \mid \varrho \mathfrak{z}_m \in S\}$ is at most countable. Thus $\bigcup_{m \in J} A_m$ is also countable and thus we can select ϱ with $0 < \varrho < 1$ such that $\hat{g}(y)$ and $\hat{f}(y)$ have no common zeros.

Let now $n = 2$. We may choose $\varepsilon > 0$ so small that the square

$$Q = \{(x_1, x_2) \in R^2 \mid -\varepsilon < x_i < \varepsilon,\ 1 \leqslant i \leqslant 2\}$$

is contained in Ω.

Consider the functions in $\mathcal{D}(\Omega)$,

$$f(x_1) f(x_2) , \quad f(x_1) g(x_2) , \quad f(x_2) g(x_1) , \quad g(x_1) g(x_2)$$

whose Fourier transforms are

$$\hat{f}(y_1)\hat{f}(y_2) , \quad \hat{f}(y_1)\hat{g}(y_2) , \quad \hat{g}(y_1)\hat{f}(y_2) , \quad \hat{g}(y_1)\hat{g}(y_2)$$

these have no common zeros as $\hat{f}(y)$ and $\hat{g}(y)$ have no common zeros.

The general argument is now clear.

LEMMA 2. *The ring $\mathcal{H}(C^n)$ of entire holomorphic functions on C^n is faithfully flat over the ring $\mathcal{P} = C[z_1, \ldots, z_n]$ of polynomials in n variables.*

PROOF. We want to show that any linear system

$$(*) \qquad \sum_{j=1}^{t} a_{ij}(z) X_j = h_i(z), \quad 1 \leqslant i \leqslant l$$

with coefficients and second members in $\mathcal{T}$,

$$a_{ij} \in \mathcal{T}, \quad h_i \in \mathcal{T}$$

has the following property. Any solution $X = (X_1, \ldots, X_t) \in \mathcal{H}(C^n)^t$ is a linear finite combination of the form

$$X_i = \sum b_s z_i^{(s)} + w_i$$

where $b_s \in \mathcal{H}(C^n)$ and where

$(w_1, \ldots, w_t)$ is a solution of $(*)$ in $\mathcal{T}^n$

$(z_1^{(s)}, \ldots, z_t^{(s)})$ is a solution of the corresponding homogeneous system $(*)$ (i.e. with $h = 0$) also in $\mathcal{T}$.

Indeed this property can be taken as a definition of faithful flatness.

Suppose first that $l = 1$ and let $k = (k_1, \ldots, k_t) \in \mathcal{H}(C^n)^t$ be a solution of $(*)$ by entire functions.

We will make use of the following *theorem of M. Noether*:

Let $f, f_1, \ldots, f_s$ be polynomials in n variables and assume that

$$f = p_1 f_1 + \ldots + p_s f_s$$

where $p_1, \ldots, p_s$ are formal power series centered at 0. Then there exist polynomials $g, g_1, \ldots, g_s$ with $g(0) \neq 0$ such that

$$g f = g_1 f_1 + \ldots + g_s f_s$$

(cf. Gröbner [10], pg. 151).

By assumption we have

$$\sum_{j=1}^{t} a_j(z) k_j = h(z).$$

Thus by application of Noether's theorem, for any point $a \in C^n$ we can find polynomials $g_a, p_{1a}, \ldots, p_{ta}$ with $g_a(a) \neq 0$ such that

$$(**) \qquad \sum p_{ja}(z) a_j(z) = g_a(z) h(z).$$

We can select a finite number $a_1, \ldots, a_s$ of points in C^n such that the polynomials

$$g_{a_1}(z), \ldots, g_{a_s}(z)$$

have no common zeros in C^n. By the Hilbert's « Nullstellensatz » we can find polynomials $r_{a_i}(z)$ such that

$$1 = \sum_{i=1}^{s} r_{a_i}(z) g_{a_i}(z) .$$

Therefore from the relations (**) we obtain

$$h(z) = \sum a_j(z) q_j(z) \quad \text{with } q_j(z) \in \mathcal{T}$$

$\left(\text{where } q_j(z) = \sum_{i=1}^{s} r_{a_i}(z) p_{j a_i}(z)\right)$.

Consequently

$$\sum_{j=1}^{t} a_j(z)(k_j - q_j) = 0 .$$

Consider the $\mathcal{T}$-homomorphism

$$\mathcal{T}^t \xrightarrow{\alpha} \mathcal{T}$$

given by

$$\alpha(p_1, \ldots, p_t) = \sum_{j=1}^{t} a_j p_j .$$

By Hilbert syzygies theorem we can find a $\mathcal{T}$-homomorphism $\beta\colon \mathcal{T}^\nu \to \mathcal{T}^t$ such that the sequence

$$\mathcal{T}^\nu \xrightarrow{\beta} \mathcal{T}^t \xrightarrow{\alpha} \mathcal{T}$$

is an exact sequence (actually only the Noetherianity of $\mathcal{T}$ is needed here).

As the ring of covergent power series is flat over the ring of polynomials, denoting by $\mathcal{O}_a$ the local ring of germs of holomorphic functions at a we deduce an exact sequence

$$\mathcal{O}^\nu \xrightarrow{\alpha} \mathcal{O}^t \xrightarrow{\beta} \mathcal{O} .$$

This shows that if $\beta^{(1)}, \ldots, \beta^{(\nu)}$ are the columns of the polynomial matrix β in a neighborhood $U(a)$ of a we must have

$$k - q = \sum_{\alpha=1}^{\nu} \lambda_\alpha^a \beta^{(\alpha)} \quad \text{with } \lambda_a^\alpha \text{ holomorphic in } U(a) .$$

We may assume that $U(a)$ is a Stein open set. Let $\mathcal{U} = \{U(a_i)\}|_{i\in I}$ be a covering of C^n by sets of the type $U(a)$.

On $U(a_i) \cap U(a_j)$ we get

$$\sum_{\alpha=1}^{\nu} (\lambda_\alpha^{a_i} - \lambda_\alpha^{a_j})\beta^{(\alpha)} = 0 .$$

Thus $\{\lambda^{a_i} - \lambda^{a_j}\}$ represents a cocycle on the covering $\mathcal{U}$ with value in the sheaf of holomorphic relations among the $\beta^{(\alpha)}$, $1 \leqslant \alpha \leqslant \nu$.

Let $\mathfrak{R}(\beta^{(1)}, \ldots, \beta^{(\nu)})$ denote this sheaf. As it is coherent, by theorem B of H. Cartan and J. P. Serre we can (making use of Leray theorem which states that $H^1(\mathcal{U}, \mathfrak{R}) = 0$) find $\sigma^{a_i} \in \Gamma(U(a_i), \mathfrak{R}(\beta^{(1)}, \ldots, \beta^{(\nu)}))$ such that

$$\lambda^{a_i} - \lambda^{a_j} = \sigma^{a_i} - \sigma^{a_j}$$

with

$$\sum_\alpha \sigma_\alpha^{a_i} \beta^{(\alpha)} = 0 , \qquad \forall i .$$

Thus setting $\Lambda = \lambda^{a_i} - \sigma^{a_i} = \lambda^{a_j} - \sigma^{a_j}$ we obtain a vector $\Lambda = (\Lambda_1, \ldots, \Lambda_\nu)$ with entire holomorphic components such that

$$k - q = \sum \Lambda_\alpha \beta^{(\alpha)} .$$

This achieves the proof of the lemma in the case $l = 1$.

We can now proceed to the general case with the inductive assumption that the lemma is proved for systems (*) of at most $l-1$ equations.

We consider the equations (*) and let $k = (k_1, \ldots, k_t) \in \mathcal{H}(C^n)^t$ be a solution.

We have in particular

$$\sum a_{1j} k_j = h_1$$

and by what we have prooved we can find a polynomial solution $p = (p_1, \ldots, p_t)$

$$\sum a_{1j} p_j = h_1 .$$

Then $k' = (k_1 - p_1, \ldots, k_t - p_t)$ satisfies the system

$$\sum a_{1j} k_j' = 0$$
$$\sum a_{2j} k_j' = h_2'$$
$$\cdots\cdots$$
$$\sum a_{lj} k_j' = h'$$

with $h'_2, \ldots, h'_l$ in $\mathcal{S}$. Moreover

$$k'_j = \sum_j \Lambda_\alpha \beta_j^{(\alpha)} \quad \text{with } \Lambda_\alpha \in \mathcal{H}(C^n) \text{ and } \beta_j^{(\alpha)} \in \mathcal{S}$$

and such that

$$\sum_j a_{1j}\beta_j^{(\alpha)} = 0 .$$

Substituting these expressions in the $(l-1)$ last equations we get a system of $l-1$ linear equations of the form

$$\sum_\alpha \Big(\sum_j a_{ji}\beta_j^{(\alpha)} \Big) \Lambda_\alpha = h'_i \qquad 2 \leqslant i \leqslant l .$$

By the inductive assumption we can find a polynomial solution $b_\alpha \in \mathcal{S}$ such that

$$\Lambda_\alpha = b_\alpha + \sum_\nu \varrho_\nu \lambda_{\alpha\nu}$$

with

$$\sum_\alpha \Big(\sum_j a_{ij}\beta_j^{(\alpha)} \Big) \lambda_{\alpha\nu} = 0 \qquad \text{and } \lambda_{\alpha\nu} \in \mathcal{S} \;\; 2 \leqslant i \leqslant l .$$

Going back to the original solution k we obtain for it the desired expression:

$$k_j = \Big(p_j + \sum_\alpha b_\alpha \beta_j^{(\alpha)} \Big) + \sum_\nu \varrho_\alpha \Big(\sum_\alpha \lambda_{\alpha\nu}\beta_j^{(\alpha)} \Big), \qquad 1 \leqslant j \leqslant t .$$

REMARK. The lemma ceases to be valid if we replace the ring of entire functions by the ring of holomorphic functions on an open Stein set Ω. For instance take $\Omega = \{z \in C \,\big|\, |z| < 1\}$ and consider the equation

$$(1-z)x = 1 .$$

This equation has no polynomial solution but it has the holomorphic solution in Ω

$$x = \sum_0^\infty z^n .$$

We can now prove the necessity of the condition given in the proposition. From

$$A_0(D)A_1(D)g = 0 , \qquad \forall g \in \mathcal{D}^r(\Omega)$$

setting $z = 2\pi i y$ we deduce

$$A_0(z)A_1(z)\hat{g} = 0 .$$

Using statement (α) of lemma 1 we deduce then that

$$A_0(z)A_1(z) = 0\,.$$

Let $h \in \mathcal{P}^q$ be such that

$$A_0(z)h = 0\,.$$

For every $g \in \mathcal{D}(\Omega)$ we thus get

$$A_0(z)h\hat{g} = 0\,.$$

But

$$h\hat{g} = \widehat{h(D)g}\,.$$

Thus

$$A_0(D)h(D)g = 0\,, \qquad \forall g \in \mathcal{D}(\Omega)\,.$$

By the hypothesis we can find $k_g \in \mathcal{D}'^r(\Omega)$ such that

$$h(D)g = A_1(D)k_g$$

or, by Fourier transform,

$$h\hat{g} = A_1(z)\hat{k}_g\,.$$

By lemma 1 (β) we can choose a finite number $g_1, \ldots, g_l$ of elements in $\mathcal{D}(\Omega)$ such that the entire functions $\hat{g}_1, \ldots, \hat{g}_l$ have no common zeros. By the « Nullstellensatz » we can find entire functions $\alpha_i \in \mathcal{H}(C^n)$ such that

$$\sum_{j=1}^{l} \alpha_i \hat{g}_i = 1\,.$$

Thus we can write

$$h = A_1(z)\tilde{k}$$

with $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_r) \in \mathcal{H}(C^n)^r$.

By lemma 2 then the system $h = A_1(z)x$ admits also a polynomial solution. This shows that the sequence (2) is an exact sequence.

With the same argument (and actually with an almost trivial proof of the analogue of lemma 1) we obtain the following

Proposition 1′. *Let Ω be an open convex set in R^n. The necessary and sufficient condition for the sequence of differential operators*

$$(1') \qquad \mathcal{E}'^r(\Omega) \xrightarrow{A_1(D)} \mathcal{E}'^q(\Omega) \xrightarrow{A_0(D)} \mathcal{E}'^p(\Omega)$$

to be exact, is that the sequence of $\mathcal{P}$-homomorphisms

$$\mathcal{P}^r \xrightarrow{A_1(z)} \mathcal{P}^q \xrightarrow{A_0(z)} \mathcal{P}^p \tag{2'}$$

be an exact sequence.

4. – We denote by ${}^tA_0(z)$, ${}^tA_1(z)$ the transposed matrices of $A_0(z)$, $A_1(z)$ respectively

PROPOSITION 2. *Let Ω be an open convex set in $\boldsymbol{R}^n$. The necessary and sufficient condition for the sequence of differential operators*

$$\mathcal{E}^p(\Omega) \xrightarrow{{}^tA_0(D)} \mathcal{E}^q(\Omega) \xrightarrow{{}^tA_1(D)} \mathcal{E}^r(\Omega) \tag{3}$$

to be exact is that the sequence of $\mathcal{P}$-homomorphisms

$$\mathcal{P}^r \xrightarrow{A_1(z)} \mathcal{P}^q \xrightarrow{A_0(z)} \mathcal{P}^p \tag{2}$$

be an exact sequence.

PROOF. *Sufficiency.* We have

$$A_0(z)A_1(z) = 0 \qquad \text{thus also } {}^tA_1(z)\,{}^tA_0(z) = 0\,.$$

Let $f \in \mathcal{E}^p(\Omega)$ and let $\{\varrho_i\}$ be a C^∞ partition of unity subordinate to an open locally finite covering $\mathcal{U} = \{U_i\}$ of Ω by relatively compact open subsets of Ω. Set $f_i = \varrho_i f$; then with the usual notation $z = 2\pi i y$ we get

$$\mathcal{F}\big({}^tA_1(D)\,{}^tA_0(D) f_i\big) = {}^tA_1(z)\,{}^tA_0(z)\hat{f}_i = 0$$

thus ${}^tA_1(D)\,{}^tA_0(D) f_i = 0$ and therefore

$${}^tA_1(D)\,{}^tA_0(D) f = \sum_i {}^tA_1(D)\,{}^tA_0(D) f_i = 0\,, \qquad \forall f \in \mathcal{E}^p(\Omega)\,.$$

Let now $f \in \mathcal{E}^q(\Omega)$ and let us assume that

$${}^tA_1(D) f = 0\,.$$

We want to show that the equation

$${}^tA_0(D) u = f$$

can be solved with $u \in \mathcal{E}^p(\Omega)$. For this we apply the criterion of theorem B of section 2; for any

$$^t v(z) = \big(v_1(z) \dots v^r(z)\big) \in \mathcal{P}^q$$

such that

$$(*) \qquad {}^t v(z)\, {}^t A_0(z) = 0$$

one should have also ${}^t v(D) f = 0$.

Now from $(*)$ we deduce $A_0(z) v(z) = 0$ thus, by assumption we can find $\eta(z) = {}^t\big(\eta_1(z), \dots, \eta_r(z)\big) \in \mathcal{P}^r$ such that

$$v(z) = A_1(z) \eta(z) .$$

Therefore

$$^t v(D) f = {}^t \eta(D)\, {}^t A_1(D) f = 0 \qquad \text{as } {}^t A_1(D) f = 0 .$$

Necessity. The proof is based on the following facts that we state as lemmas.

LEMMA 1 (Whitney's theorem). *Let M be a differentiable manifold with countable topology and let $\mathcal{E}(M)$ be the space of C^∞ functions on M with the topology of convergence of the functions and their partial derivatives (the Schwartz topology). Let $\mathfrak{J} \subset \mathcal{E}(M)$ be a* closed *ideal.*

Then the following conditions are equivalent

i) *$f \in \mathcal{E}(M)$ is an element of $\mathfrak{J}$;*

ii) *for any $a \in M$ we have*

$$\mathcal{T}_a(f) \subset \mathcal{T}_a(\mathfrak{J})$$

iii) *for any $a \in M$ and any distribution T_a with support in the point a,*

$$T_a = \sum_{|p| \leq k} C^p \delta_a^{(p)}$$

(where $\delta^{(p)} = D^p \delta_a$ and $C_p \in \mathbf{C}$) such that $T_a(\mathfrak{J}) = 0$ we also have $T_a(f) = 0$.

From this one deduces the same type of statement for closed submodules of $\mathcal{E}^p(M)$.

LEMMA 2 (division theorem for C^∞ functions). *Let Ω be an open set in $\mathbf{R}^n$ and let $A(x)$ be a $p \times q$ matrix with complex valued real analytic entries defined on Ω. Then the linear map*

$$A : \mathcal{E}^q(\Omega) \to \mathcal{E}^p(\Omega)$$

defined by

$$f(x) \to A(x)f(x)$$

has a closed image.

After these premises we can begin the proof of the necessity of the stated condition.

By assumption

$$\mathcal{E}^p(\Omega) \xrightarrow{A_0(D)} \mathcal{E}^q(\Omega) \xrightarrow{A_1(D)} \mathcal{E}^r(\Omega)$$

is an exact sequence. Thus, for any $g \in \mathcal{D}^p(\Omega)$ we have

$$^tA_1(D)\,{}^tA_0(D)g = 0$$

and by Fourier transform

$$^tA_1(z)\,{}^tA_0(z)\hat{g} = 0\ .$$

This being true for any $g \in D^p(\Omega)$ (using lemma 1 of the previous section) we deduce that $^tA_1(z)\,{}^tA_0(z) = 0$ i.e.

$$A_0(z)A_1(z) = 0\ .$$

Hence the sequence (2) is a complex.

Let now $h \in \mathfrak{I}^q$ be such that

$$A_0(z)h = 0\ .$$

We want to show that $h \in A_1(z)\mathfrak{I}^r$.

To this end we first show that

(*) $$h \in A_1(z)\,\mathcal{E}^r(C^n)\ .$$

Indeed, by lemma 2, the image of the map

$$A_1\colon \mathcal{E}^r(\boldsymbol{C}^n) \to \mathcal{E}^q(\boldsymbol{C}^n)$$

given by $f \to A_1(z)f$ is a closed submodule of $\mathcal{E}^q(\boldsymbol{C}^n)$.

Therefore we can apply the theorem of Whitney in the form of the third condition, $h \in \operatorname{Im} A_1 \Leftrightarrow$ for every point supported distribution T_a, $a \in \boldsymbol{C}^n$, such that $T_a(\operatorname{Im} A_1) = 0$ we have $T_a(h) = 0$.

A distribution of type T_a is of the form

$$r(D)\,\delta_a$$

where r is a polynomial.

Thus the inclusion (*) is verified if $\forall a \in \boldsymbol{C}^n$ we have

$$r(D)h(z)|_{z=a} = 0$$

whenever

$$(**) \qquad r(D)A_1(z)f|_{z=a} = 0\,, \qquad \forall f \in \mathcal{E}^r(\boldsymbol{C}^n)\,.$$

As the Taylor expansion of f at $z = a$ can be arbitrarily prescribed the condition (**) is equivalent to

$$(***) \qquad r(D)A_1(z) = 0 \quad \text{or} \quad r(D)A_1(z)|_{z=a} = 0 \qquad \forall a \in \boldsymbol{C}^n\,.$$

By Fourier transform this is then equivalent to the following statement. For every exponential polynomial

$$r(x)\exp[2\pi i\langle a, x\rangle]$$

satisfying

$${}^tA_1(D)\,r(x)\exp[2\pi i\langle a, x\rangle] = 0$$

we should also have

$${}^th(D)\,r(x)\exp[2\pi i\langle a, x\rangle] = 0\,.$$

Now by the assumption $A_0(z)h(z) = 0$ we get

$${}^th(D)\,{}^tA_0(D)g = 0\,, \qquad \forall g \in \mathcal{E}^p(\Omega)\,.$$

As ${}^tA_0(D)\mathcal{E}^p(\Omega) = \operatorname{Ker}{}^tA_1(D)$, this implies that ${}^th(D)$ vanishes on each $f \in \mathcal{E}^q(\Omega)$ with $f \in \operatorname{Ker}{}^tA_1(D)$ and in particular for all $f = r(x)\exp[2\pi i\langle ax\rangle] \in$ $\in \operatorname{Ker}{}^tA_1(D)$.

This proves our contention.

We have thus proved that

$$h(z) = A_1(z)k(z)$$

with $k(z) \in \mathcal{E}^r(\boldsymbol{C}^n)$. We want to show that we can replace $k(z)$ by a vector $k'(z) \in \mathcal{T}^r$.

This amounts to prove the following

LEMMA 3. *The ring $\mathcal{E}(\boldsymbol{C}^n)$ of C^∞ functions (complex valued) on C^n is faithfully flat over the ring of polynomials in n complex variables.*

This is proved in the same way as lemma 2 of the previous section taking into account the fact that the ring of germs of C^∞ functions at the origin in some $\boldsymbol{R}^N$ is (faithfully) flat over the ring of convergent power series and of formal power series (a fact which is a consequence of lemma 2) (1).

Replacing the use of theorem B with theorem B' one obtains the following

PROPOSITION 2'. *Let Ω be an open relatively compact convex set in $\boldsymbol{R}^n$ and consider the sequence of differential operators*

$$(4) \qquad \mathfrak{D}'^p(\bar{\Omega}) \xrightarrow{{}^tA_0(D)} \mathfrak{D}'^q(\bar{\Omega}) \xrightarrow{{}^tA_1(D)} \mathfrak{D}'^r(\bar{\Omega}) .$$

The necessary and sufficient condition that this sequence be exact is that the sequence

$$(2) \qquad \mathcal{P}^r \xrightarrow{A_1(z)} \mathcal{P}^q \xrightarrow{A_0(z)} \mathcal{P}^p$$

of $\mathcal{P}$-homomorphisms be an exact sequence.

Here we have denoted by $\mathfrak{D}'(\bar{\Omega})$ the space

$$\mathfrak{D}'(\bar{\Omega}) = \varinjlim_{v} \mathfrak{D}'(W_v)$$

where $\{W_v\}$ is a fundamental sequence of neighborhoods of $\bar{\Omega}$ in $\boldsymbol{R}^n$.

REMARK. In the previous statement one can replace $\bar{\Omega}$ with the open set Ω itself (cf. Palamodov [22], ch. 7, 8, th. 1 and also [23]).

5. – Given a matrix $A(D)$ of type $p \times q$ of differential operators with constant coefficients and given Ω open convex in $\boldsymbol{R}^n$ one can consider the

(1) One word should be spent over the application of M. Noether lemma. Let $f, f_1, \ldots, f_s$ be polynomials in $z_1, \ldots, z_n$. Suppose that

$$f = k_1 f_1 + \ldots + k_s f_s$$

with $k_i \in \mathcal{E}(\boldsymbol{C}^n)$. At every point $a \in \boldsymbol{C}^n$ the Taylor series of k_i is a formal power series in $(z_1 - a_1, \ldots, z_n - a_n, \bar{z}_1 - \bar{a}_1, \ldots, \bar{z}_n - \bar{a}_n)$ thus

$$f = \mathcal{T}_a(k_1) f_1 + \ldots + \mathcal{T}_a(k_s) f_s .$$

Putting in this expression $\bar{z}_1 - \bar{a}_1 = 0, \ldots, \bar{z}_n - \bar{a}_n = 0$ we obtain an expression

$$f = p_1 f_1 + \ldots + p_s f_s$$

where p_i are formal power series in $(z_i - a_i)$, $1 \leqslant i \leqslant n$, only. Thus we are still in the conditions required by M. Noether theorem.

system of partial differential equations

$$\mathfrak{D}^q(\Omega) \xrightarrow{A(D)} \mathfrak{D}^p(\Omega)$$

or

$$\mathcal{E}^q(\Omega) \xrightarrow{A(D)} \mathcal{E}^p(\Omega)\,.$$

One can then ask to construct, if possible, two other systems of differential operators with constant coefficients,

$$B(D)\colon \mathfrak{D}^p(\Omega) \to \mathfrak{D}^r(\Omega)$$

or (integrability conditions)

$$B(D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^r(\Omega)$$

and

$$C(D)\colon \mathfrak{D}^s(\Omega) \to \mathfrak{D}^q(\Omega)$$

or (cointegrability conditions)

$$C(D)\colon \mathcal{E}^s(\Omega) \to \mathcal{E}^q(\Omega)$$

such that one has, on Ω, an exact sequence

$$(\mathfrak{D}) \qquad \mathfrak{D}^s(\Omega) \xrightarrow{C(D)} \mathfrak{D}^q(\Omega) \xrightarrow{A(D)} \mathfrak{D}^p(\Omega) \xrightarrow{B(D)} \mathfrak{D}^r(\Omega)$$

$$(\mathcal{E}) \qquad \mathcal{E}^s(\Omega) \xrightarrow{C(D)} \mathcal{E}^q(\Omega) \xrightarrow{A(D)} \mathcal{E}^p(\Omega) \xrightarrow{A(D)} \mathcal{E}^r(\Omega)\,.$$

The previous propositions enable us to reduce this problem to a problem of algebra. Precisely

(a) The sequence ($\mathfrak{D}$) will be exact if and only if the sequence of $\mathfrak{P}$-homomorphisms

$$\mathfrak{P}^s \xrightarrow{C(z)} \mathfrak{P}^q \xrightarrow{A(z)} \mathfrak{P}^p \xrightarrow{B(z)} \mathfrak{P}^r$$

is an exact sequence.

(b) The sequence ($\mathcal{E}$) will be exact if and only if the sequence of $\mathfrak{P}$-homomorphisms

$$\mathfrak{P}^r \xrightarrow{{}^tB(z)} \mathfrak{P}^p \xrightarrow{{}^tA(z)} \mathfrak{P}^q \xrightarrow{{}^tC(z)} \mathfrak{P}^s$$

is an exact sequence.

We are thus lead to the following problem:

given a $\mathfrak{P}$-homomorphism between $\mathfrak{P}$-free modules

$$\mathfrak{P}^q \xrightarrow{A(z)} \mathfrak{P}^p$$

to find $\mathcal{S}$-homomorphisms $\mathcal{S}^s \xrightarrow{C(z)} \mathcal{S}^q$ and (if possible) $\mathcal{S}^p \xrightarrow{B(z)} \mathcal{S}^r$ so that $\operatorname{Im} C(z) = \operatorname{Ker} A(z)$ and $\operatorname{Im} A(z) = \operatorname{Ker} B(z)$.

We will discuss these questions in the next sections.

6. – Backward resolutions of a $\mathcal{S}$-homomorphism.

(*a*) We start with the following

REMARKS (*a*). *Given two $\mathcal{S}$-homomorphisms*

$$\mathcal{S}^l \xrightarrow{S} \mathcal{S}^s, \qquad \mathcal{S}^r \xrightarrow{F} \mathcal{S}^s$$

necessary and sufficient condition for S being factored through F (i.e. that there exist $A: \mathcal{S}^l \to \mathcal{S}^r$ such that $S = F \cdot A$) is that

$$\operatorname{Im} S \subset \operatorname{Im} F .$$

(*b*) *Given a diagram of $\mathcal{S}$-homomorphisms*

$$\begin{array}{ccccccccc}
\cdots \to & \mathcal{S}^{p_l} & \xrightarrow{S_l} & \mathcal{S}^{p_{l-1}} & \to \cdots \to & \mathcal{S}^{p_1} & \xrightarrow{S_1} & \mathcal{S}^{p_0} \\
 & & & & & \downarrow{\scriptstyle A_1} & & \downarrow{\scriptstyle A_0} \\
\cdots \to & \mathcal{S}^{q_l} & \xrightarrow{F_l} & \mathcal{S}^{q_{l-1}} & \to \cdots \to & \mathcal{S}^{q_1} & \xrightarrow{F_1} & \mathcal{S}^{q_0}
\end{array}$$

which is commutative with exact rows, then one can complete it in a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & \mathcal{S}^{p_l} & \xrightarrow{S_l} & \mathcal{S}^{p_{l-1}} & \to \cdots \to & \mathcal{S}^{p_1} & \xrightarrow{S_1} & \mathcal{S}^{p_0} \\
 & \downarrow{\scriptstyle A_l} & & \downarrow{\scriptstyle A_{l-1}} & & \downarrow{\scriptstyle A_1} & & \downarrow{\scriptstyle A_0} \\
\cdots \to & \mathcal{S}^{q_l} & \xrightarrow{F_l} & \mathcal{S}^{q_{l-1}} & \to \cdots \to & \mathcal{S}^{q_1} & \xrightarrow{F_1} & \mathcal{S}^{q_0}
\end{array}$$

(*c*) *Let M be a finitely generated $\mathcal{S}$-module. Any two exact sequences of $\mathcal{S}$-homomorphisms*

$$\cdots \to \mathcal{S}^{p_l} \xrightarrow{S_l} \mathcal{S}^{p_{l-1}} \to \cdots \to \mathcal{S}^{p_1} \xrightarrow{S_1} \mathcal{S}^{p_0} \xrightarrow{S_0} M \to 0$$

$$\cdots \to \mathcal{S}^{q_l} \xrightarrow{F_l} \mathcal{S}^{q_{l-1}} \to \cdots \to \mathcal{S}^{q_1} \xrightarrow{F_1} \mathcal{S}^{q_0} \xrightarrow{F_0} M \to 0$$

can be factored each one through the other. (i.e. *we can find $\mathcal{S}$-homomorphisms $A_i: \mathcal{S}^{p_i} \to \mathcal{S}^{q_i}$,* id: $M \to M$, *which complete into a commutative diagram*).

Any two factorizations $A_i, B_i\colon \mathcal{P}^{p_i} \to \mathcal{P}^{a_i}$ *of the first sequence through the second are homotopic* (i.e. *we can find* $\mathcal{P}$*-homomorphisms* $k_i\colon \mathcal{P}^{p_i} \to \mathcal{P}^{a_{i+1}}$ *such that*

$$A_0 - B_0 = F_1 k_0\,; \qquad A_i - B_i = F_{i+1} k_i + k_{i-1} S_i \qquad i \geqslant 1)\,.$$

In particular if Φ is any covariant (or contravariant) functor from the cathegory of $\mathcal{P}$-modules to the cathegory of abelian groups the homology (cohomology) groups of the complex

$$\cdots \to \Phi(\mathcal{P}^{p_l}) \xrightarrow{\Phi(S_l)} \Phi(\mathcal{P}^{p_{l-1}}) \to \cdots \to \Phi(\mathcal{P}^{p_1}) \xrightarrow{\Phi(S_1)} \Phi(\mathcal{P}^{p_0})$$

$$\left(\Phi(\mathcal{P}^{p_0}) \xrightarrow{\Phi(S_1)} \Phi(\mathcal{P}^{p_1}) \to \cdots \to \Phi(\mathcal{P}^{p_{l-1}}) \xrightarrow{\Phi(S_l)} \Phi(\mathcal{P}^{p_l}) \to \cdots\right)$$

are independent from the choice of the « resolution » and give invariants of the module M alone.

Moreover given any finitely generated $\mathcal{P}$-module M, by the fact that $\mathcal{P}$ is a Noetherian ring, it follows that we can always construct exact sequences of $\mathcal{P}$-homomorphisms (free resolution of M)

$$\cdots \to \mathcal{P}^{p_l} \xrightarrow{S_l} \mathcal{P}^{p_{l-1}} \to \cdots \to \mathcal{P}^{p_1} \xrightarrow{S_1} \mathcal{P}^{p_0} \to M \to 0\,.$$

In particular, given a $\mathcal{P}$-homomorphism $S_1\colon \mathcal{P}^{p_1} \to \mathcal{P}^{p_0}$, we can always find a backward free resolution

$$\cdots \to \mathcal{P}^{p_l} \xrightarrow{S_l} \mathcal{P}^{p_{l-1}} \to \cdots \to \mathcal{P}^{p_1} \xrightarrow{S_1} \mathcal{P}\,.$$

The proof of these statements is straightforward and thus it is omitted.

7. – Hilbert's theorem.

The following theorem is due to Hilbert:

THEOREM 1. *Let* $\mathcal{P} = k[x_1, \ldots, x_n]$ *be the ring of polynomials in* $n \geqslant 1$ *variables over an infinite field* k. *For any* $\mathcal{P}$*-homomorphism*

$$\mathcal{P}^{s_1} \xrightarrow{S_1} \mathcal{P}^{s_0}$$

we can construct a finite exact sequence

$$0 \to \mathcal{P}^{s_d} \xrightarrow{S_d} \mathcal{P}^{s_{d-1}} \to \cdots \to \mathcal{P}^{s_2} \xrightarrow{S_2} \mathcal{P}^{s_1} \xrightarrow{S_1} \mathcal{P}_{s_0}$$

of lenght $d \leqslant n + 1$.

Hilbert gave the proof of this theorem for the similar case of the graded ring of homogeneous polynomials. The proof can be easily adapted to this case and will be given below for the convenience of the reader.

PROOF. (α) We consider first the case $n=1$. Then $\mathcal{T}_1 = k[x_1]$ is a principal ideal ring. The homomorphism S_1 is represented by an $s_0\times s_1$ matrix. One can find matrices U, V of type $s_0\times s_0$, $s_1\times s_1$ respectively with $\det U = \pm 1$, $\det V = \pm 1$, with elements in $\mathcal{T}_1$ such that

$$US_1V = S' = \operatorname{diag}\langle h_1, h_2, \dots, h_l, 0, \dots, 0\rangle .$$

This means that by rechoosing the basis in $\mathcal{T}^{s_1}$ and $\mathcal{T}^{s_0}$ we can assume S_1 in diagonal form. If $S = \operatorname{diag}\langle h_1, \dots, h_l; 0, \dots, 0\rangle$ with $h_1 \dots h_l \neq 0$ then $\operatorname{Ker} S_1$ is isomorphic to $\mathcal{T}^{s_1-l}$ and we have the exact sequence (with $s_2 = s_1 - l$)

$$0 \to \mathcal{T}_1^{s_2} \to \mathcal{T}_1^{s_1} \xrightarrow{S_1} \mathcal{T} .$$

(β) The proof now proceeds by induction on the number n of variables. Let $\mathfrak{K} = k(x_1, \dots, x_n)$ be the quotient field of $\mathcal{T}_n = k[x_1, \dots, x_n]$. For a matrix S we write

$$S = (\bar{S}_1, \bar{S}_2, \dots) = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \end{pmatrix}$$

as the set of its columns or respectively of its rows.

Set

$$S_1 = (\bar{S}_1, \dots, \bar{S}_{s_1}) = \begin{pmatrix} R_1 \\ \vdots \\ R_{s_0} \end{pmatrix}$$

and let

$$\varrho = \operatorname{rank}_{\mathfrak{K}} S_1; \qquad \text{then} \qquad \varrho \leqslant \inf(s_0, s_1)$$

and we can assume that $\begin{pmatrix} R_1 \\ \vdots \\ R_\varrho \end{pmatrix}$ has rank ϱ. Therefore

$$\operatorname{Ker} S_1 = \{X \in \mathcal{T}_n^{s_1} | R_1 X = \dots = R_\varrho X = 0\}$$

and we shall have $\operatorname{Ker} S_1 \neq 0$ only if $\varrho < s_1$.

Set

$$S_1' = \begin{pmatrix} R_1 \\ \vdots \\ R_\varrho \end{pmatrix} = (\bar{S}_1', \dots, \bar{S}_{s_1}')$$

and let

$$D = \det(\bar{S}'_1, \ldots, \bar{S}'_\varrho)\,.$$

It is not restrictive to assume that

(a) $D \neq 0$;

(b) $r =$ degree of D in $x_n \geqslant$ degree of $\det(S'_{i_1}, \ldots, S'_{i_\varrho})$ in x_n for every choice of $i_1, \ldots, i_\varrho$ in $\{1, \ldots, s_1\}$;

(c) $D = ax_n^r +$ polynomial of degree less than r in x_n, and with $a \neq 0$.

To satisfy the last condition we make a « generic » linear change of coordinates x and use the fact that the field k is infinite.

Now remark that all minors of order $\varrho + 1$ of the matrix (for each $i = 1, \ldots, \varrho$)

$$\begin{pmatrix} R_i \\ R_1 \\ \vdots \\ R_\varrho \end{pmatrix}$$

have a zero determinant. This provides us with a system of vectors in $\operatorname{Ker} S_1$.

In particular those minors obtained by bordering the matrix of D by one row and column, give us $s_1 - \varrho$ vectors in $\operatorname{Ker} S_1$. Assembling them in the columns of a matrix C we get

$$C = \underbrace{\begin{pmatrix} & * & \\ D & 0 & \\ & \ddots & \\ 0 & & D \end{pmatrix}}_{s_1 - \varrho} \qquad (\text{rank } C = s_1 - \varrho)$$

where the elements in the part denoted by $*$ are of degree $\leqslant r$ in x_n.

Given any $X \in \operatorname{Ker} S_1$ we can find a vector $\begin{pmatrix} \Lambda_{\varrho+1} \\ \vdots \\ \Lambda_{s_1} \end{pmatrix}$ with $\Lambda_i \in \mathcal{T}_n$ such that

$$\Xi = X - C\Lambda = \begin{pmatrix} \Xi_1 \\ \vdots \\ \Xi_{s_1} \end{pmatrix}$$

has the property that degree in x_n of $\Xi_{\varrho+i} < r$ for $i = 1, \ldots, s_1 - \varrho$.

Indeed it is enough to choose $\Lambda_{\varrho+i}$ as the quotient of the division of X_i by D in x_n;

$$X_{\varrho+i} = D\Lambda_{\varrho+i} + \Xi_{\varrho+i}\,, \qquad i = 1, \ldots, s_1 - \varrho\,.$$

We claim that then also $\Xi_1, \ldots, \Xi_\varrho$ have a degree in x_n which is $< r$.

Indeed as $\Xi \in \operatorname{Ker} S_1$ we have $S_1 \Xi = 0$, thus

$$(S'_1, \ldots, S'_\varrho)\begin{pmatrix}\Xi_1\\ \vdots \\ \Xi_\varrho\end{pmatrix} = -(S'_{\varrho+i}, \ldots, S'_{s_1})\begin{pmatrix}\Xi_{\varrho+1}\\ \vdots \\ \Xi_{s_1}\end{pmatrix}$$

and, from Cramer's rule,

$$D\Xi_s = \Xi_{\varrho+1}A^{(s)}_{\varrho+1} + \ldots + \Xi_{s_1}A^{(s)}_{s_1}, \qquad 1 \leqslant s \leqslant \varrho$$

where $A^{(s)}_k$ are the minors determinants of order ϱ of the matrix S'_1. The degree in x_n of the right hand side is thus $< 2r$. From degree in x_n of $D\Xi_s < 2r$ we deduce, that degree in x_n of $\Xi_s < r$ for $1 \leqslant s \leqslant \varrho$.

Therefore,

every vector $X \in \operatorname{Ker} S_1$ is the sum of a linear combination $C\Lambda$ of vectors of the matrix C and a vector Ξ whose components have all a degree in x_n which is $< r$.

(γ) Set

$$\Xi = \xi_1 x_n^{r-1} + \ldots + \xi_r \qquad \text{with } \xi_i \in \mathcal{T}_{n-1}$$

$$S_1 = \sigma_1 x_n^l + \ldots + \sigma_{l+1} \qquad \sigma_i \text{ a matrix } s_0 \times s_1 \text{ with entries in } \mathcal{T}_{n-1}$$

then the condition $S_1 \Xi = 0$ is transformed into a system

$$\begin{aligned}&\sigma_1 \xi_1 = 0\\ &\sigma_2 \xi_1 + \sigma_1 \xi_2 = 0\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

in $\mathcal{T}_{n-1}$ that we will write in matrix notation

$$\varphi \xi = 0\,, \quad \text{where } \varphi = \begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ & & \sigma_1\\ & & & \ddots\\ \sigma_{l+1} & & \\ & \ddots & \\ 0 & & \sigma^{l+1}\end{pmatrix}$$

is of type $(l + r) \times r$ in the block matrices σ_j.

Let $\xi^{(1)}, \dots, \xi^{(\mu)}$ be a basis of $\operatorname{Ker}\varphi$ over $\mathfrak{T}_{n-1}$. If we set

$$\begin{aligned} \Xi^{(\alpha)} &= \xi_1^{(\alpha)} x_n^{r-1} + \dots + \xi_r^{(\alpha)}, \qquad 1 \leqslant \alpha \leqslant \mu \\ \Phi &= (\Xi^{(1)}, \dots, \Xi^{(\mu)}) \end{aligned}$$

then we get for any $X \in \operatorname{Ker} S_1$ an expression of the form

$$X = C\Lambda + \Phi\lambda \quad \text{where} \quad \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\mu \end{pmatrix} \in \mathfrak{T}_{n-1}^{\mu}$$

i.e. setting

$$S_2 = (C, \Phi)$$

we get the following commutative diagram with exact row:

$$\begin{array}{ccccc} \mathfrak{T}_n^{s_1-\varrho+\mu} & \xrightarrow{S_2} & \mathfrak{T}_n^{s_1} & \xrightarrow{S_1} & \mathfrak{T}_n^{s_0} \\ \uparrow{\scriptstyle i} & \nearrow{\scriptstyle S_2} & & & \\ \mathfrak{T}_n^{s_1-\varrho} \oplus \mathfrak{T}_{n-1}^{\mu} & & & & \end{array}$$

(δ) Now we investigate $\operatorname{Ker} S_2$.
As $x_n \Xi^{(\alpha)} \in \operatorname{Ker} S_1$ we must have

$$x_n \Xi^{(\alpha)} = C\Lambda^{(\alpha)} + \Phi\lambda^{(\alpha)} \quad \text{with } \lambda^{(\alpha)} \in \mathfrak{T}_{n-1}^{\mu},\ 1 \leqslant \alpha \leqslant \mu$$

thus

$$C\Lambda^{(\alpha)} + \Phi \begin{pmatrix} \lambda_1^{(\alpha)} \\ \vdots \\ \lambda_\alpha^{(\alpha)} - x_n \\ \vdots \\ \lambda_\mu^{(\alpha)} \end{pmatrix} = 0\,.$$

If we set

$$y^{(\alpha)} = \begin{pmatrix} \Lambda^{(\alpha)} \\ \lambda_1^{(\alpha)} \\ \vdots \\ \lambda_\alpha^{(\alpha)} - x_n \\ \vdots \\ \lambda_\mu^{(\alpha)} \end{pmatrix}$$

then

$$S_2 y^{(\alpha)} = 0\,, \quad \text{and} \quad y^{(\alpha)} \neq 0 \text{ as } \lambda_\alpha^{(\alpha)} - x_n \neq 0\,.$$

We have thus found a system of μ vectors in $\operatorname{Ker} S_2$ that we write as the columns of a matrix

$$C^{(1)} = (y^{(1)}, \dots, y^{(\mu)}) = \begin{pmatrix} \Lambda^{(1)} \dots \Lambda^{(\mu)} \\ (\lambda_\beta^{(\alpha)} - \delta_{\alpha\beta} x_n) \end{pmatrix}$$

and rank $C^{(1)} = \mu$ as $\det(\lambda_\beta^{(\alpha)} - \delta_{\alpha\beta} x_n) = (-1)^\mu x_n^\mu + \dots \neq 0$.

We want to show that *any vector* $y \in \operatorname{Ker} S_2$ *can be written as a linear combination* $C^{(1)}\Lambda$ *of the column vectors of the matrix* $C^{(1)}$ *and a vector*

$$M = \begin{pmatrix} 0 \\ \\ M' \end{pmatrix} \qquad \textit{with} \qquad M' = \begin{pmatrix} M_1 \\ \vdots \\ M_\mu \end{pmatrix} \in \mathcal{T}_{n-1}^\mu ,$$

$$y = C^{(1)}\Lambda + M .$$

Clearly one can choose Λ so that $y - C^{(1)}\Lambda$ has the last μ components independent of the variable x_n. Set $M = \begin{pmatrix} M'' \\ M' \end{pmatrix}$, then $S_1 M = 0$ implies

$$CM'' + \Phi M' = 0$$

and as $C = \begin{pmatrix} * & \\ D & 0 \\ & \ddots \\ 0 & D \end{pmatrix}$ we deduce

$DM_i'' =$ polynomial in x_n of degree $\leqslant r-1$ $(1 \leqslant i \leqslant s_1 - \varrho)$ and therefore $M_i'' = 0$.

(ε) Now

$$\Phi = \sum_{i=1}^{r} x_n^{r-i}(\xi_i^{(1)}, \dots, \xi_i^{(\mu)})$$

so that the relation $\Phi M' = 0$ and $M' \in \mathcal{T}_{n-1}^\mu$ implies

$$(\xi_i^{(1)}, \dots, \xi_i^{(\mu)}) M' = 0 \qquad \text{for } 1 \leqslant i \leqslant r .$$

Or, setting

$$\varphi_1 = (\xi^{(1)}, \dots, \xi^{(\mu)}) ,$$

we get

$$\varphi_1 M' = 0 .$$

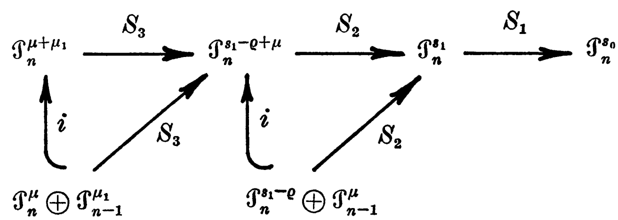
If $M'^{(1)}, \dots, M'^{(\mu_1)}$ is a basis of $\operatorname{Ker} \varphi_1$ over $\mathcal{T}_{n-1}$ and if we set

$$\varphi_2 = (M'^{(1)}, \dots, M'^{(\mu_1)})$$

we get from the last remark of (δ) that *any* $y \in \operatorname{Ker} S_1$ *has an expression of type*

$$y = C^{(1)}\Lambda + \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix}\lambda \quad \text{with } \lambda \in \mathcal{T}^{\mu_1}_{n-1}\,.$$

Therefore we conclude that we have a commutative diagram with exact row



where

$$S_3 = \begin{pmatrix} C^{(1)} & \begin{matrix} 0 \\ \varphi_2 \end{matrix} \end{pmatrix}, \quad S_2 = (C, \Phi)\,,$$

and moreover the sequence

$$\mathcal{T}^{\mu_1}_{n-1} \xrightarrow{\varphi_2} \mathcal{T}^{\mu}_{n-1} \xrightarrow{\varphi_1} \mathcal{T}^{s_1 r}_{n-1} \xrightarrow{\varphi} \mathcal{T}^{(l+r)s_0}_{n-1}$$

is an exact sequence.

(ζ) We are going to investigate $\operatorname{Ker} S_3$. We set $S_3 = \begin{pmatrix} C^{(1)}_{11} & 0 \\ C^{(1)}_{21} & \varphi_2 \end{pmatrix}$ where $C^{(1)}_{21} = (\lambda^{(\alpha)}_\beta - \delta_{\alpha\beta}x_n)$ and where $\varphi_2 = (M'^{(1)}, \dots M'^{(\mu_1)})$. Now recall that the vectors $\begin{pmatrix} 0 \\ M'^{(\alpha)} \end{pmatrix}$ are in $\operatorname{Ker} S_1$ so that we must also have

$$S_2 x_n \begin{pmatrix} 0 \\ M'^{(\alpha)} \end{pmatrix} = 0\ .$$

By the conclusion of point (ε) we must have therefore

$$\begin{pmatrix} 0 \\ x_n M'^{(\alpha)} \end{pmatrix} = C^{(1)} \begin{pmatrix} \Lambda^1_{(\alpha)} \\ \vdots \\ \Lambda^{(\alpha)}_\mu \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \begin{pmatrix} \lambda^{(\alpha)}_1 \\ \vdots \\ \lambda^{(\alpha)}_{\mu_1} \end{pmatrix}, \quad \Lambda^{(\alpha)}_i \in \mathcal{T}_n, \ \lambda^{(\alpha)}_j \in \mathcal{T}_{n-1}.$$

In particular

$$C^{(1)}_{21} \begin{pmatrix} \Lambda^{(\alpha)}_1 \\ \vdots \\ \Lambda^{(\alpha)}_\mu \end{pmatrix} + \varphi_2 \begin{pmatrix} \lambda^{(\alpha)}_1 \\ \vdots \\ \lambda^{(\alpha)}_{\mu_1} \end{pmatrix} = x_n M'^{(\alpha)}\,.$$

From Cramer's rule we deduce then

$$\det C_{21}^{(1)} \Lambda_{\beta}^{(\alpha)} = \text{polynomial in } x_n \text{ of degree } \leqslant \mu .$$

Therefore, for any $\alpha, \beta, \Lambda_{\beta}^{(\alpha)} \in \mathcal{P}_{n-1}$.

In particular the vector

$$y^{(\alpha)} = {}^t(\Lambda_1^{(\alpha)}, \dots, \Lambda_{\mu}^{(\alpha)}, \lambda_1^{(\alpha)}, \dots, \lambda_{\alpha}^{(\alpha)} - x_n, \dots, \lambda_{\mu_1}^{(\alpha)})$$

is a non zero element of $\operatorname{Ker} S_3$ for $1 \leqslant \alpha \leqslant \mu_1$. Let

$$C^{(2)} = (y^{(1)}, \dots, y^{(\mu_1)}) = \begin{pmatrix} \Lambda^{(1)} \dots \Lambda^{(\mu_1)} \\ (\lambda_{\beta}^{\alpha} - x_n) \end{pmatrix} = \begin{pmatrix} C_{11}^{(2)} \\ C_{21}^{(2)} \end{pmatrix}.$$

We have $\det C_{21}^{(2)} = (-1)^{\mu_1} x_n^{\mu_1} +$ polinomial in x_n of degree $< \mu_1$.

Given any vector $y \in \operatorname{Ker} S_3$ we can find $\Lambda \in \mathcal{P}_n^{\mu_1}$ so that

$$y = C^{(2)}\Lambda + M \quad \textit{with } M = \begin{pmatrix} 0 \\ \\ M' \end{pmatrix}, \quad M' = \begin{pmatrix} M_1' \\ \vdots \\ M_{\mu_1}' \end{pmatrix} \in \mathcal{P}_{n-1}^{\mu_1}.$$

Indeed once we have chosen Λ in such a way that the last μ_1 components of $Y - C^{(2)}\Lambda$ are independent of x_n, from the equations $S_3 M = 0$ we deduce that the first μ components of M must be zero and therefore that

$$\varphi_2 M' = 0 .$$

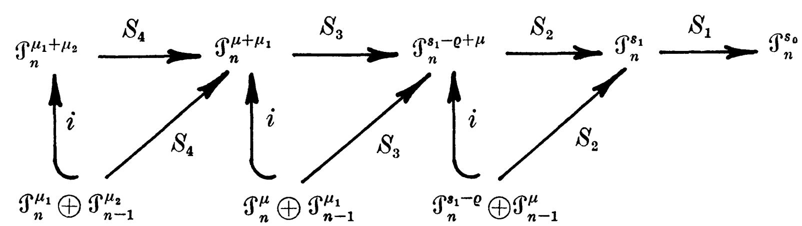
Let $M'^{(1)}, \dots, M'^{(\mu_2)}$ be a basis of $\operatorname{Ker} \varphi_2$ over $\mathcal{P}_{n-1}$ and set

$$\varphi_3 = (M'^{(1)}, \dots, M'^{(\mu_2)})$$

Then *any $Y \in \operatorname{Ker} S_3$ has an expression of the form*

$$Y = C^{(2)}\Lambda + \begin{pmatrix} 0 \\ \varphi_3 \end{pmatrix} \lambda \quad \text{with } \Lambda \in \mathcal{P}_n^{\mu_1}, \ \lambda \in \mathcal{P}_{n-1}^{\mu_2}$$

and therefore we have a commutative diagram with exact row:



where

$$S_4 = \begin{pmatrix} C^{(2)} & 0 \\ & \varphi_3 \end{pmatrix}, \quad S_3 = \begin{pmatrix} C^{(1)} & 0 \\ & \varphi_2 \end{pmatrix}, \quad S_2 = (C, \Phi)$$

and moreover we have the exact sequence

$$\mathcal{P}^{\mu_2}_{n-1} \xrightarrow{\varphi_3} \mathcal{P}^{\mu_1}_{n-1} \xrightarrow{\varphi_2} \mathcal{P}^{\mu}_{n-1} \xrightarrow{\varphi_1} \mathcal{P}^{s_1 r}_{n-1} \xrightarrow{\varphi} \mathcal{P}^{(l+r)s_0}_{n-1}$$

(η) The general procedure is now clear:

1) given the homomorphism S_1 we choose first basis and coordinates to satisfy conditions (a), (b), (c) of (β). We restrict S_1 to the subspace of $\mathcal{P}^{s_1}_n$ of those vectors whose components are polynomials of degree $< r$ in x_n. This can be considered as a free module over $\mathcal{P}_{n-1}$ of rank rs_1. The image of S_1 then falls in the subspace of $\mathcal{P}^{s_0}_n$ of vectors with components of degree less than $l + r$ in x_n, l being the maximal degree in x_n of the elements of S_1. Considering that subspace of $\mathcal{P}^{s_0}_n$ as a free module over $\mathcal{P}_{n-1}$ of rank $(l + r)s_0$, we see that S_1 determines a $\mathcal{P}_{n-1}$ homomorphism

$$\varphi : \mathcal{P}^{s_1 r}_{n-1} \to \mathcal{P}^{(1+r)s_0}_{n-1}.$$

2) We then construct a finite free resolution of φ over $\mathcal{P}_{n-1}$

$$\cdots \to \mathcal{P}^{\mu_2}_{n-1} \xrightarrow{\varphi_3} \mathcal{P}^{\mu_1}_{n-1} \xrightarrow{\varphi_2} \mathcal{P}^{\mu}_{n-1} \xrightarrow{\varphi_1} \mathcal{P}^{s_1 r}_{n-1} \xrightarrow{\varphi} \mathcal{P}^{(1+r)s_0}_{n-1}$$

which by the inductive hypothesis can be assumed to be of lenght $\leqslant n$.

3) We then determine by the previous resolution

the matrix $S_1 = (C, \Phi)$ by means of φ

the matrix $S_2 = \begin{pmatrix} C^{(1)} & 0 \\ & \varphi_2 \end{pmatrix}$ by means of φ_1 and φ_2

the matrix $S_3 = \begin{pmatrix} C^{(2)} & 0 \\ & \varphi_3 \end{pmatrix}$ by means of φ_2 and φ_3

and so on.

To show that the resolution thus obtained for S_1 has lenght $\leqslant n + 1$ we have only to remark that if φ_k is injective (thus $\varphi_{k+1} = 0$) then also $S_{k+1} = (C^{(k)})$ is injective. But this follows from the fact that rank $C^{(k)}$ = number of columns of $C^{(k)}$.

This completes the proof of the theorem.

8. – Equivalence of finitely generated $\mathcal{F}$-modules.

a) Let M and N be finitely generated $\mathcal{F}$-modules.

We say that M and N are *equivalent* if we can find two free $\mathcal{F}$-modules $\mathcal{F}^s$, $\mathcal{F}^t$ such that

$$M \oplus \mathcal{F}^s \simeq N \oplus \mathcal{F}^t$$

i.e. $M \oplus \mathcal{F}^s$ and $N \oplus \mathcal{F}^t$ are isomorphic as $\mathcal{F}$-modules.

A finitely generated $\mathcal{F}$ module M equivalent to a free module is called projective.

PROPOSITION 3. *A finitely generated $\mathcal{F}$-module P is projective if and only if every diagram of finitely generated $\mathcal{F}$-modules*

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\alpha} & B \end{array} \tag{1}$$

with $\operatorname{Im}\alpha \subset \operatorname{Im}\pi$ *can be completed into a commutative diagram*

$$\begin{array}{ccc} & & A \\ & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\alpha} & B \end{array} \tag{2}$$

PROOF. First one remarks that if P is free or if P is a direct factor of a finitely generated free module then the diagram (1) can always be completed into diagram (2).

Secondly one remarks that if (1) can be always completed in (2) then P must be a direct factor of a finitely generated free module, as we can take $B = P$, $\alpha = id$ and A any free module such that $A \xrightarrow{\pi} P$ is surjective.

Thirdly one makes use of a Hilbert resolution of P

$$0 \to L_d \to L_{d-1} \to \dots \to L_0 \to P \to 0 .$$

If P is a direct factor of a free module then by the second remark, setting $N_i = \operatorname{Ker}(L_i \to L_{i-1})$, $i = 1, 2, \dots$, $N_0 = \operatorname{Ker}(L_0 \to P)$, the sequence

$$0 \to N_0 \to L_0 \to P \to 0$$

is exact and split. Thus $L_0 = P \oplus N_0$ and also N_0 is a direct factor of a free module. Hence

$$0 \to N_1 \to L_1 \to N_0 \to 0$$

is also exact and split, so that $L_1 = N_1 \oplus N_0$ and also N_1 is a direct factor of a free module. Proceeding in this way we get

$$L_i = N_i \oplus N_{i-1} \qquad i = 1, 2, \dots \qquad (N_d = L_d,\ N_{d+1} = 0)\,.$$

Hence

$$P \oplus N_0 \oplus N_1 \oplus \dots \oplus N_d \oplus N_{d+1} = P \oplus L_1 \oplus L_3 \oplus \dots = L_0 \oplus L_2 \oplus \dots .$$

This shows that P is equivalent to a free module, finitely generated, thus P is projective.

(*b*) Let M be any finitely generated $\mathcal{T}$-module and let

$$\cdots \to L_d \xrightarrow{\alpha_{d-1}} L_{d-1} \xrightarrow{\alpha_{d-2}} \cdots L_1 \xrightarrow{\alpha_0} L_0 \to M \to 0$$

be any free resolution of M. Applying to this resolution the functor

$$\mathrm{Hom}_{\mathcal{T}}(\cdot, A) \qquad (A \text{ any finitely generated } \mathcal{T}\text{-module})$$

we get a complex

$$\mathrm{Hom}_{\mathcal{T}}(L_0, A) \xrightarrow{\alpha_0^*} \mathrm{Hom}_{\mathcal{T}}(L_1, A) \xrightarrow{\alpha_1^*} \cdots \to \mathrm{Hom}_{\mathcal{T}}(L_d, A) \to \cdots .$$

Its cohomology groups are independent of the resolution we have chosen, according to point (*c*) of 6. We set thus

$$\mathrm{Ext}^k_{\mathcal{T}}(M, A) = \frac{\mathrm{Ker}\,\alpha_k^*}{\mathrm{Im}\,\alpha_{k-1}^*} \qquad \text{for } k \geqslant 1$$

and

$$\mathrm{Ext}^0_{\mathcal{T}}(M, A) = \mathrm{Hom}_{\mathcal{T}}(M, A)\,.$$

One has the following properties:

If

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finitely generated $\mathcal{T}$-modules, then for any finitely generated $\mathcal{T}$-module A we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{T}}(A, M') \to \mathrm{Hom}_{\mathcal{T}}(A, M) \to \mathrm{Hom}_{\mathcal{T}}(A, M'') \to \\ &\to \mathrm{Ext}^1_{\mathcal{T}}(A, M') \to \mathrm{Ext}^1_{\mathcal{T}}(A, M) \to \mathrm{Ext}^1_{\mathcal{T}}(A, M'') \to \\ &\to \mathrm{Ext}^2_{\mathcal{T}}(A, M') \to \ldots \end{aligned}$$

and an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{T}}(M'', A) \to \mathrm{Hom}_{\mathcal{T}}(M, A) \to \mathrm{Hom}_{\mathcal{T}}(M', A) \to \\ &\to \mathrm{Ext}^1_{\mathcal{T}}(M'', A) \to \mathrm{Ext}^1_{\mathcal{T}}(M, A) \to \mathrm{Ext}^1_{\mathcal{T}}(M', A) \to \\ &\to \mathrm{Ext}^2_{\mathcal{T}}(M'', A) \to \ldots. \end{aligned}$$

In particular one deduces the following characterization of projective modules.

PROPOSITION 4. *If A is a finitely generated projective $\mathcal{T}$-module then for each finitely generated module M we have* $\mathrm{Ext}^i_{\mathcal{T}}(A, M) = 0$, $\forall i \geq 1$.

Conversely if for every M we have $\mathrm{Ext}^1_{\mathcal{T}}(A, M) = 0$ *then A is projective.*

It is worth noticing that the functors $\mathrm{Ext}^i_{\mathcal{T}}(\cdot, M)$ for $i \geq 1$ take the same « value » on equivalent finitely generated modules so that they represent invariants of the equivalence classes of finitely generated $\mathcal{T}$-modules.

c) Given a finitely generated $\mathcal{T}$-module A, any $\mathcal{T}$-module B appearing in a short exact sequence of the type

$$0 \to B \to \mathcal{T}^l \to A \to 0$$

is called a *module of syzygies* of the module A.

We have the following properties of easy verification

(α) equivalent finitely generated modules have equivalent modules of syzygies; in particular the modules of syzygies of any given module are all equivalent.

(β) for every finitely generated $\mathcal{T}$-module M we have

$$\mathrm{Ext}^i(B, M) = \mathrm{Ext}^{i+1}(A, M) \quad \text{for } i \geq 1$$

d) We have the following

THEOREM 2. *For any pair of $\mathcal{T}_n = k[x_1, \ldots, x_n]$-modules A and B one has* *always*

$$\mathrm{Ext}_{\mathfrak{I}}^{n+1}(A, B) = 0\,.$$

PROOF. If $n = 1$ this follows from the fact that any module of syzygies of A is free (see proof of theorem 1) i.e. we can always find a free resolution of A of type

$$0 \to L^1 \to L_0 \to A \to 0\,.$$

The general case is then treated by induction on the number of variables. A proof based on the properties of the Ext functors can be found in Northcott [21], p. 182.

COROLLARY. *Given a finitely generated $\mathfrak{I}_n$-module M we can always find a free resolution*

$$0 \to \mathfrak{I}_n^{l_d} \xrightarrow{\alpha_d} \mathfrak{I}_n^{l_{d-1}} \xrightarrow{\alpha_{d-1}} \cdots \to \mathfrak{I}_n^{l_1} \xrightarrow{\alpha_1} \mathfrak{I}_n^{l_0} \xrightarrow{\alpha_0} M \to 0$$

of lenght $d \leqslant n$.

Indeed any free resolution the module $C = \mathrm{Ker}\,\alpha_{n-1}$ is such that

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{I}}^1(C, A) &= \mathrm{Ext}_{\mathfrak{I}}^2(\mathrm{Ker}\,\alpha_{n-2}, A) = \ldots = \mathrm{Ext}^n(\mathrm{Ker}\,\alpha_0, A)\\ &= \mathrm{Ext}_{\mathfrak{I}}^{n+1}(M, A) = 0\,.\end{aligned}$$

Thus C is projective and therefore $C \oplus \mathfrak{I}^s = \mathfrak{I}^t$ for some s and t.

Modifying the considered resolution at the stage n as

$$0 \to C \oplus \mathfrak{I}^s \xrightarrow{(i\oplus id)} \mathfrak{I}^{l_{n-1}} \oplus \mathfrak{I}^s \xrightarrow{(\alpha_{n-1}\oplus 0)} \mathfrak{I}^{l_{n-2}} \to \ldots$$

where $i\colon C \to \mathfrak{I}^{l_n}$ is the natural inclusion, we obtain a free resolution of lenght n.

e) A matrix $A(x)$ of type $r \times s$ with polynomial entries can be considered as a $\mathfrak{I}$-homomorphism $\mathfrak{I}^s \xrightarrow{A(x)} \mathfrak{I}^r$ and thus we can associate to it the $\mathfrak{I}$-module

$$\alpha(A) = \mathrm{Coker}\, A(x) = \mathfrak{I}^r / A\mathfrak{I}^s\,.$$

One can define two matrices A and B with polynomial entries *equivalent* if the corresponding $\mathfrak{I}$-modules $\alpha(A)$ and $\alpha(B)$ are equivalent.

PROPOSITION 5. *Given two matrices with polynomial entries A, B the necessary and sufficient condition for their equivalence is that we can enlarge A and B*

with sets of zeros rows such that for the enlarged matrices $\begin{pmatrix} A \\ 0 \end{pmatrix}$, $\begin{pmatrix} B \\ 0 \end{pmatrix}$ *we can find polynomial matrices* c, d, Λ, M *such that*

$$d\begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix} \Lambda$$

$$c\begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix} M$$

with

$$cd \equiv id \bmod \operatorname{Im} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$dc \equiv id \bmod \operatorname{Im} \begin{pmatrix} B \\ 0 \end{pmatrix}$$

PROOF. In fact equivalence means that one can build up a commutative diagramm of $\mathfrak{P}$-homomorphisms with exact rows

$$\begin{array}{ccccccc} \mathfrak{P}^r & \xrightarrow{\binom{A}{0}} & \mathfrak{P}^N & \xrightarrow{\alpha} & N & \to & 0 \\ {\scriptstyle \Lambda}\downarrow\uparrow{\scriptstyle M} & & {\scriptstyle d}\downarrow\uparrow{\scriptstyle c} & & \| & & \\ \mathfrak{P}^s & \xrightarrow{\binom{B}{0}} & \mathfrak{P}^N & \xrightarrow{\beta} & N & \to & 0\,. \end{array}$$

Note that this notion via Fourier transform leads to a notion of equivalence for systems of differential operators with constant coefficients.

9. – Forward resolutions.

a) Given a $\mathfrak{P}$-homomorphism

$$\mathfrak{P}^{s_1} \xrightarrow{S_1} \mathfrak{P}^{s_0}$$

we can construct, by Hilbert's theorem, a backward going finite free resolution of S_1.

The problem to find the integrability conditions of the differential operator $\mathfrak{D}^{s_1}(\Omega) \xrightarrow{S_1(D)} \mathfrak{D}^{s_0}(\Omega)$ suggests the following problem:

Can the Hilbert resolution of S_1 be continued forward with an exact consequence of type

$$\mathfrak{P}^{s_1} \xrightarrow{S_1} \mathfrak{P}^{s_0} \to \mathfrak{P}^{s_{-1}} \to \mathfrak{P}^{s_{-2}} \to \cdots \to \mathfrak{P}^{s_{-k}}\,.$$

If such is the case, and $N = \operatorname{Coker} S_1$, we do have a « forward resolution » of N

$$0 \to N \to \mathcal{P}^{s_{-1}} \to \mathcal{P}^{s_{-2}} \to \cdots \to \mathcal{P}^{s_{-k}} . \tag{1}$$

b) Given a $\mathcal{P}$-module N we define the *torsion submodule*

$$\tau(N) = \{n \in N | gn = 0 \quad \text{for some } g \in \mathcal{P},\ g \neq 0\} .$$

For a submodule of a free module the torsion module must be zero.

This necessary condition is also sufficient i.e.

if N is a finitely generated $\mathcal{P}$-module without torsion $(\tau(N) = 0)$ *then one can find an injective $\mathcal{P}$-morphism of N into a free module.*

Indeed if $\mathcal{R} = k(x_1, \dots, x_n)$ is the quotient field of $\mathcal{P}$, tensoring by N the injection $\mathcal{P} \to \mathcal{R}$ we get an exact sequence $0 \to \tau(N) \to N \to N \otimes_{\mathcal{P}} \mathcal{R}$.

If $\tau(N) = 0$ then N can be considered as a $\mathcal{P}$-submodule of the vector space $N \otimes_{\mathcal{P}} \mathcal{R}$. As N is finitely generated N can be considered as a submodule of a $\mathcal{P}$-free module.

Let $N^* = \operatorname{Hom}_{\mathcal{P}}(N, \mathcal{P})$ be the dual of the finitely generated module N. It is a finitely generated $\mathcal{P}$-module. Indeed if

$$\mathcal{P}^{s_0} \xrightarrow{S_1} \mathcal{P}^{s_0} \to N \to 0$$

is a presentation of N as $\operatorname{Coker} S_1$ we have

$$N^* \cong \operatorname{Ker}(\mathcal{P}^{s_0} \xrightarrow{{}^tS_1} \mathcal{P}^{s_1}) .$$

A dual module is always without torsion.

Let N^{**} be the dual of N^* i.e. the bidual of N. Every element of N can be considered as a linear function on N^* and therefore as an element of N^{**}. We thus have a natural map

$$N \xrightarrow{j_N} N^{**}$$

having the following properties:

i) *Every $\mathcal{P}$-linear map $\alpha\colon N \to \mathcal{P}$ factors uniquely through j_N:*

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & \mathcal{P} \\ & {\scriptstyle j_N}\searrow \quad \nearrow{\scriptstyle \dot\alpha} & \\ & N^{**} & \end{array} \qquad \alpha = \dot\alpha \circ j_N .$$

Indeed α can be considered as an element of N^* and therefore defining a linear map $\dot{\alpha}$ of $(N^*)^*$ into $\mathcal{S}$.

ii) $$\operatorname{Ker} j_N = \tau(N)\,.$$

Indeed $\operatorname{Ker} j_N = \{n \in N \mid l(n) = 0,\ \forall l \in N^*\}$. Thus $\operatorname{Ker} j_N \supset \tau(N)$.

But $N/\tau(N)$ has no torsion and on it $\mathcal{S}$-linear maps separate points; therefore $\operatorname{Ker} j_N \subset \tau(N)$.

One may also remark the following property

iii) *For every linear map* $\alpha\colon N \to \mathcal{S}^l$ *we get according to* i) *a factorization*

$$\begin{array}{ccc} N & \xrightarrow{\ \alpha\ } & \mathcal{S}^l \\ & {\scriptstyle j_N}\searrow \quad \nearrow{\scriptstyle \dot{\alpha}} & \\ & N^{**} & \end{array} \qquad \alpha = \dot{\alpha} \circ j_N\,.$$

If α *is injective* (*and thus* $\tau(N) = 0$) *then* $\dot{\alpha}$ *is injective.*

Indeed we first remark that $N \otimes_{\mathcal{S}} \mathfrak{R} \simeq N^{**} \otimes_{\mathcal{S}} \mathfrak{R}$ as every finite dimensional vector space is isomorphic to its bidual. Thus $T = \operatorname{Coker} j_N$ is a torsion module: $\tau(T) = T$. If α is injective we thus have an exact sequence $0 \to N \to N^{**} \to T \to 0$. Let $n^{**} \in N^{**}$ be such that $\dot{\alpha}(n^{**}) = 0$. There exists $g \in \mathcal{S}$, $g \neq 0$, such that $gn^{**} \in N$.

Thus $\alpha\ (gn^{**}) = 0$ and, as α is injective, $gn^{**} = 0$. But then $n^{**} = 0$ as N^{**} has no torsion.

c) Let N be a finitely generated $\mathcal{S}$-module without torsion and let $(f_1, \dots, f_l) = f$ be a basis of N^*. Let us consider the $\mathcal{S}$-homomorphism

$$\sigma_f\colon N \to \mathcal{S}^l$$

defined by $n \to (f_1(n), \dots, f_l(n))$.

This map has the following universal property:

Every $\mathcal{S}$*-homomorphism* $\mu\colon N \to \mathcal{S}^t$ *can be factored through* σ_f:

$$\begin{array}{ccc} N & \xrightarrow{\ \mu\ } & \mathcal{S}^t \\ & {\scriptstyle \sigma_f}\searrow \quad \nearrow{\scriptstyle \mu_f} & \\ & \mathcal{S}^l & \end{array}$$

This property follows from the fact that the t $\mathcal{S}$-homomorphisms that constitute the map μ are linear conbinations of $f_1, \dots, f_l$. In particular *the map* σ_f *is an injective map* because $\tau(N) = 0$ implies the existence of an injective μ.

LEMMA. *Let* μ *be injective and let* $N_f = \operatorname{Coker}\sigma_f$, $N_\mu = \operatorname{Coker}\mu$. *We do have then a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{\sigma_f} & \mathfrak{I}^l & \xrightarrow{\pi_f} & N_f & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \mu_f} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \to & N & \xrightarrow{\mu} & \mathfrak{I}^t & \xrightarrow{\pi_\mu} & N_\mu & \to & 0 \end{array}$$

We claim that $\lambda|\tau(N_f)$ *is injective.*

PROOF. Assume $\tau(N_f) \neq 0$ and let $\alpha \in N_f - \{0\}$, $g\alpha = 0$ some $g \in \mathfrak{I}$, $g \neq 0$. Then we can find $a \in \mathfrak{I}^l$, $a \notin \sigma_f(N)$, $ga \in \sigma_f(N)$. If $\lambda(\alpha) = 0$ then $\mu_f(a) \in \mu(N)$. Now we remark that $\operatorname{Ker}\mu_f \cap \operatorname{Im}\sigma_f = 0$ as $\mu_f\sigma_f(n) = 0$ implies $\mu(n) = 0$, thus $n = 0$. Hence $\pi_f|\operatorname{Ker}\mu_f$ is injective and therefore $\pi_f(\operatorname{Ker}\mu_f)$ is a submodule of N_f without torsion. If $\mu_f(a) = \mu(n_0)$ for some $n_0 \in N$, $\mu_f(a) = \mu_f\sigma_f(n_0)$, thus $\mu_f(a - \sigma_f(n_0)) = 0$ i.e. $a - \sigma_f(n_0) \in \operatorname{Ker}\mu_f$. But $g(a - \sigma_f(n_0)) = ga - \sigma_f(gn_0) \in \sigma_f(N)$ thus $\pi_f(a - \sigma_f(n_0))$ is a torsion element. This implies, by the previous remark, that $a = \sigma_f(n_0)$, which is a contradiction.

Given a finitely generated $\mathfrak{I}$-module N (without torsion) a $\mathfrak{I}$-linear map

$$\sigma_f\colon N \to \mathfrak{I}^l$$

given by a basis of N^* will be called a *stable* map (¹).

d) If N *is a torsion free module and if*

$$\sigma_f\colon N \to \mathfrak{I}^l, \qquad \sigma_g\colon N \to \mathfrak{I}^h$$

are two stable maps then $\operatorname{Coker}\sigma_f$ *and* $\operatorname{Coker}\sigma_g$ *are equivalent.*

Set $f = (f_1, \ldots, f_l)$, $g = (g_1, \ldots, g_h)$. Let $\sigma'\colon N \to \mathfrak{I}^{l+1}$ with $\sigma'(n) = (f_1(n), \ldots, f_l(n), g_1(n))$. As $g_1 = \sum_1^l b_j f_j$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{\sigma'} & \mathfrak{I}^{l+1} & \xrightarrow{\pi_{\sigma'}} & \operatorname{Coker}\sigma' & \to & 0 \\ & & \| & & {\scriptstyle \lambda}\downarrow\uparrow{\scriptstyle \alpha} & & {\scriptstyle \lambda'}\downarrow\uparrow{\scriptstyle \alpha'} & & \\ 0 & \to & N & \xrightarrow{\sigma_f} & \mathfrak{I}^l & \xrightarrow{\pi_\sigma} & \operatorname{Coker}\sigma & \to & 0 \end{array}$$

where $\lambda(a_1, \ldots, a_{l+1}) = (a_1, \ldots, a_l)$ and

$$\alpha(a_1, \ldots, a_l) = \left(a_1, \ldots, a_l, \sum^l b_j a_j\right).$$

(¹) Note that $\sigma\colon N \to \mathfrak{I}^l$ is stable if and only if $\operatorname{Ext}^1(\operatorname{Coker}\sigma, \mathfrak{I}) = 0$.

Now $\operatorname{Ker}\lambda \cap \operatorname{Im}\sigma' = 0$; thus $\pi_{\sigma'}|\operatorname{Ker}\lambda$ is an isomorphism. But $\operatorname{Ker}\lambda \simeq \mathfrak{I}$. Hence $\operatorname{Coker}\sigma' \simeq \operatorname{Coker}\sigma \oplus \mathfrak{I}$.

Repeated application of this remark yields the above statement.

More generally one has the following statement (of almost immediate proof).

If N and N' are torsion free equivalent $\mathfrak{I}$-modules and if $\sigma: N \to \mathfrak{I}^l$, $\sigma': N' \to \mathfrak{I}^{l'}$ are stable maps then also Coker σ *and* Coker σ' *are equivalent.*

An exact sequence

$$0 \to N \xrightarrow{\alpha_0} \mathfrak{I}^{s-1} \xrightarrow{\alpha_1} \mathfrak{I}^{s-2} \xrightarrow{\alpha_2} \cdots \to \mathfrak{I}^{s-k-1} \xrightarrow{\alpha_{k-1}} \mathfrak{I}^{s-k}$$

in which α_0 is a stable map and $\forall i \geqslant 1$, $\alpha_i: \operatorname{Coker}\alpha_{i-1} \to \mathfrak{I}^{s-i-1}$ is stable will be called a *stable forward resolution of N of lenght k.*

e) THEOREM 3. *Let N be a finitely generated $\mathfrak{I}$-module.*

i) *Necessary and sufficient condition that N be included in an exact sequence*

$$(1) \qquad 0 \to N \xrightarrow{\alpha_0} \mathfrak{I}^{s-1}$$

is that $\tau(N) = 0$ i.e. *N be torsion free.*

ii) *Necessary and sufficient condition that N be included in an exact sequence*

$$(1) \qquad 0 \to N \xrightarrow{\alpha_0} \mathfrak{I}^{s-1} \xrightarrow{\alpha_1} \mathfrak{I}^{s-2}$$

is that $\tau(N) = 0$ and $N \simeq N^{**}$ i.e. *N be reflexive (this second condition implies the first).*

iii) *Necessary and sufficient condition that N be included in a stable exact sequence*

$$(1) \qquad 0 \to N \xrightarrow{\alpha_0} \mathfrak{I}^{s-1} \xrightarrow{\alpha_1} \mathfrak{I}^{s-2} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} \mathfrak{I}^{s-k}$$

of length $k \geqslant 3$ is that N be reflexive, $N = N^{**}$; and

$$\operatorname{Ext}^1(N^*, \mathfrak{I}) = \ldots = \operatorname{Ext}^{k-2}(N^*, \mathfrak{I}) = 0\,.$$

PROOF OF i). Follows from the first remark of point *b*).

PROOF OF ii). Assume we do have an exact sequence (1) and let $N_1 = \operatorname{Ker}\alpha_1 = \operatorname{Coker}\alpha_0$. By taking biduals we get a commutative diagramm

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{\alpha_0} & \mathcal{P}^s & \xrightarrow{\beta} & N_1 & \to & 0 \\ & & \downarrow{\scriptstyle j_N} & & \| & & \downarrow{\scriptstyle j_{N_1}} & & \\ 0 & \to & N^{**} & \xrightarrow{\alpha_0^{**}} & \mathcal{P}^s & \xrightarrow{\beta^{**}} & N_1^{**} & & \end{array}$$

in which the first row is exact, the second is a complex and in it α_0^{**} is injective as α_0 is, (point b) iii)).

By the assumption, $\tau(N_1) = 0$ and therefore j_{N_1} is injective. Hence

$$N^{**} \subset \operatorname{Ker}\beta^{**} = \operatorname{Ker} j_{N_1}\beta = \operatorname{Ker}\beta = N\,.$$

This implies $N = N^{**}$ (and the bottom row of the diagram is exact).

Conversely if $N = N^{**}$ we can take a resolution of N^*

$$\mathcal{P}^{s-2} \to \mathcal{P}^{s-1} \to N^* \to 0$$

and apply to it the functor $\operatorname{Hom}_{\mathcal{P}}(\cdot, \mathcal{P})$. We then obtain an exact sequence

$$0 \to N^{**} \to \mathcal{P}^{s-1} \to \mathcal{P}^{s-2}\,.$$

As $N = N^{**}$ we conclude the proof of point ii).

PROOF OF iii). Assume we do have a stable exact sequence (1). We set $N_i = \operatorname{Ker}\alpha_i$. We then consider the commutative diagram

$$\begin{array}{cccccccccccccccccc} 0 \to N & \xrightarrow{\alpha_0} & \mathcal{P}^{s-1} & \xrightarrow{\sigma_1} & N_1 & \xrightarrow{\mu_1} & \mathcal{P}^{s-2} & \xrightarrow{\sigma_2} & N_2 & \xrightarrow{\mu_2} & \mathcal{P}^{s-3} & \to \cdots & \xrightarrow{\sigma_{k-1}} & N_{k-1} & \xrightarrow{\mu_{k-1}} & \mathcal{P}^{s-k} \\ \downarrow & & \| & & \downarrow & & \| & & \downarrow & & \| & & & \downarrow & & \| \\ 0 \to N^{**} & \xrightarrow{\alpha_0^{**}} & \mathcal{P}^{s-1} & \xrightarrow{\sigma_1^{**}} & N_1^{**} & \xrightarrow{\mu_1^{**}} & \mathcal{P}^{s-2} & \xrightarrow{\sigma_2^{**}} & N_2^{**} & \xrightarrow{\mu_2^{**}} & \mathcal{P}^{s-3} & \to \cdots & \xrightarrow{\sigma_{k-1}^{**}} & N_{k-1}^{**} & \xrightarrow{\mu_{k-1}^{**}} & \mathcal{P}^{s-k} \end{array}$$

where $\alpha_i = \mu_i \circ \sigma_i$ and where σ_i is surjective and μ_i injective.

Also σ_i^{**} is surjective for $1 \leqslant i \leqslant k-2$ and μ_i^{**} is injective for $1 \leqslant i \leqslant k-1$.

Now we remark that the resolution being stable we do have exact sequences

$$\begin{array}{l} 0 \to N_1^* \to \mathcal{P}^{s-1} \to N^* \to 0 \\ 0 \to N_2^* \to \mathcal{P}^{s-2} \to N_1^* \to 0 \\ \cdots\cdots\cdots\cdots\cdots \\ 0 \to N_{k-1}^* \to \mathcal{P}^{s-k-1} \to N_{k-2}^* \to 0 \end{array}$$

and

$$\mathcal{P}^{s-k} \to N_{k-1} \to 0$$

These provide an exact sequence

$$\mathcal{T}^{s-k} \to \mathcal{T}^{s-(k-1)} \to \mathcal{T}^{s-(k-2)} \to \cdots \to \mathcal{T}^{s-1} \to N^* \to 0 . \tag{2}$$

If we apply to (2) the functor $\mathrm{Hom}_{\mathcal{T}}(\cdot, \mathcal{T})$ we get the exact sequence (1) (except the first step). Then we must have

$$\mathrm{Ext}^i(N^*, \mathcal{T}) = \frac{\mathrm{Ker}\, \alpha_i^{**}}{\mathrm{Im}\, \alpha_{i-1}^{**}} = 0 \qquad \text{if } 1 \leqslant i \leqslant k-2 .$$

Conversely let us assume that we have the nullity of the specified invariants and let (2) be a resolution of N^*. By application of the functor $\mathrm{Hom}_{\mathcal{T}}(\cdot, \mathcal{T})$ we obtain an exact stable sequence

$$0 \to N^{**} \to \mathcal{T}^{s-1} \to \mathcal{T}^{s-2} \to \cdots \to \mathcal{T}^{s-k} .$$

As by assumption $N = N^{**}$ we get the desired conclusion.

REMARK 1. The statement of this theorem, without the assumption or condition that the resolution be stable, can be found with a different proof in Palamodov [22]. From Palamodov statement follows that if there is a resolution of type (1) at all then there is a stable resolution also of the same length.

REMARK 2. If a finitely generated $\mathcal{T}$-module N is considered up to equivalence and if $\nu(N) =$ maximal length of a forward (stable) resolution, $\delta(N) =$ minimal length of a Hilbert resolution, we must have

$$\delta(N) + \nu(N) \leqslant n .$$

10. – Koszul complexes.

a) Let $\mathcal{T} = \boldsymbol{C}[X_1, \dots, X_n]$. We denote by $\mathcal{A}_s^h$ the space of exterior forms of degree h, with coefficients in $\mathcal{T}$, in the indeterminates $dt_1, \dots, dt_s$:

$$\mathcal{A}_s^h = \Big\{ \sum_{1 \leqslant i_1 < i_2 < \dots < i_h \leqslant s} \alpha_{i_1 \dots i_h} dt_{i_1} \dots dt_{i_h} \, ; \; \alpha_{i_1 \dots i_h} \in \mathcal{T} \Big\} .$$

We do have $\mathcal{A}_s^h \simeq \mathcal{T}^{\binom{s}{h}}$ and in particular

$$\mathcal{A}_s^s \simeq \mathcal{T} ; \qquad \mathcal{A}_s^{s-1} \simeq \mathcal{T}^s .$$

Let

$$\varphi = \sum_{j=1}^{s} \varphi_j dt_j$$

be a fixed 1-form and let $\mathfrak{p}$ denote the ideal generated by $\varphi_1, \dots, \varphi_s$;

$$\mathfrak{p} = \mathcal{T}(\varphi_1, \dots, \varphi_s) .$$

We set

$$N_{\mathcal{T}} = \mathcal{T}/\mathfrak{p}$$

and we consider the following augmented complex (as $\varphi \wedge \varphi = 0$)

$$(1) \qquad 0 \to \mathcal{A}_s^0 \xrightarrow{\wedge\varphi} \mathcal{A}_s^1 \xrightarrow{\wedge\varphi} \cdots \xrightarrow{\wedge\varphi} \mathcal{A}_s^{s-1} \xrightarrow{\wedge\varphi} \mathcal{A}_s^s \xrightarrow{\varepsilon} N_{\mathcal{T}} \to 0$$

where ε denotes the natural map $\mathcal{T} \to N$ via the identification $\mathcal{A}_s^s \simeq \mathcal{T}$.

We will denote the complex (1) as the *Koszul complex associated to the sequence* $(\varphi_1, \dots, \varphi_s)$. Note that $\varphi \wedge \mathcal{A}^{s-1} = \mathfrak{p}$, so we do have exactness at $\mathcal{A}_s^s$ and ε is surjective.

b) We will say that a sequence $(a_1, \dots, a_l) \in \mathcal{T}$ is a *principal sequence* (or an a-sequence in the sense of Serre) *of length* l if for every $j = 2, 3, \dots, l$, a_j is not a zero divisor in $\mathcal{T}/\mathcal{T}(a_1, \dots, a_{j-1})$.

This means that whenever $g \in \mathcal{T}$ satisfies a relation of the form

$$g a_j = \mu_1 a_1 + \dots + \mu_{j-1} a_{j-1} \qquad \mu_i \in \mathcal{T},\ 1 \leqslant i \leqslant j-1$$

then we must also have

$$g = \nu_1 a_1 + \dots + \nu_{j-1} a_{j-1}, \qquad \nu_i \in \mathcal{T},\ 1 \leqslant i \leqslant j-1 .$$

REMARK. Given a sequence $(a_1, \dots, a_l) \in \mathcal{T}$ let us denote by $\mathfrak{p}_j = \mathcal{T}(a_1, \dots, a_j)$, the ideal generated by $a_1, \dots, a_j$. We can consider in $\boldsymbol{C}^n$ the algebraic variety

$$V_j = \{z \in \boldsymbol{C}^n | a_1(z) = \dots = a_j(z) = 0\}$$

of common zeros of the elements of $\mathfrak{p}_j$. We do have

$$V_1 \supset V_2 \supset \dots \supset V_l .$$

One has the following geometric criterion:

the sequence $(a_1, \ldots, a_l)$ *is a principal sequence if and only if either*

$$\dim_{\boldsymbol{C}} V_j = n - j \quad \text{or } V_j = \emptyset \ (j = 2, \ldots, l).$$

PROOF. If $(a_1, \ldots, a_l)$ is a principal sequence then a_j cannot be contained in any prime component of the ideal $\mathfrak{p}_{j-1}$. Hence $\dim V_j < \dim V_{j-1}$. But, if $V_j \neq \emptyset$ $\dim V_j \geq n - j$. Hence the conclusion.

Conversely assume that V_{j-1} is either empty or of dimension $n - j + 1$ and that V_j is either empty or of dimension $n - j$.

If $V_{j-1} = \emptyset$ then the implication « number j » for a principal sequence is satisfied as $\mathfrak{p}_{j-1} = \mathcal{T}$ by virtue of the Nullstellensatz.

If $V_{j-1} \neq \emptyset$ then by the assumption and the Ungemishtheitsatz (Gröbner [10], pg. 125) it follows that a_j is not contained in any prime component associated to $\mathfrak{p}_{j-1}$, thus a_j is not a zero divisor in $\mathcal{T}/\mathfrak{p}_{j-1}$. This completes the proof.

An ideal of $\mathcal{T}$ admitting a basis which is a principal sequence is called of *principal class.*

LEMMA. *Let* $(a_1, \ldots, a_l)$ *be a principal sequence. If we have a relation*

$$\sum_{i=1}^{l} X_i a_i = 0 \quad \text{with } X_i \in \mathcal{T},\ 1 \leq i \leq l,$$

then we can find polynomials $\mathfrak{h}_{ij} \in \mathcal{T}$ *with* $\mathfrak{h}_{ij} = -\mathfrak{h}_{ij}$ *such that*

$$X_i = \sum_{j=1}^{l} \mathfrak{h}_{ij} a_j \quad 1 \leq i \leq l.$$

PROOF. If $l = 1$ the lemma is trivial. We can proceed by induction on l. We have

$$-X_l a_l = + X_1 a_1 + \ldots + X_{l-1} a_{l-1}$$

thus, by the assumption there exist polynomials $\mu_1, \ldots, \mu_{l-1}$ such that

$$X_l = \mu_1 a_1 + \ldots + \mu_{l-1} a_{l-1}.$$

Then

$$(X_1 + \mu_1 a_l) a_1 + \ldots + (X_{l-1} + \mu_{l-1} a_l) a_{l-1} = 0.$$

By the inductive assumption there exist $\mathfrak{h}_{ij} \in \mathcal{T}$ with $\mathfrak{h}_{ij} = -\mathfrak{h}_{ji}$ such that

$$\begin{cases} X_1 = \mathfrak{h}_{12} a_2 + \ldots + \mathfrak{h}_{1l-1} a_{l-1} - \mu_1 a_l \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ X_{l-1} = \mathfrak{h}_{l-11} a_1 + \ldots + \mathfrak{h}_{l-1l-2} a_{l-2} - \mu_{l-1} a_l \\ X_l = \mu_1 a_1 + \ldots + \mu_{l-1} a_{l-1} . \end{cases}$$

This is the statement we wanted to prove.

c) PROPOSITION 6. *Let $(\varphi_1, \ldots, \varphi_s)$ be a sequence containing a principal sequence of lenght $l \leqslant s$. Then the corresponding Koszul complex is exact at $\mathcal{A}^0$, $\mathcal{A}^1, . \ , \mathcal{A}^{l-1}$.*

PROOF. We may assume that $(\varphi_1, \ldots, \varphi_l)$ is a principal sequence. If $l = 1$, $\varphi_1 \neq 0$ and thus $\mathcal{A}^0 \to \mathcal{A}^1$ is injective. We can thus proceed by induction assuming the proposition proved for the integers $\leqslant l-1$.

Let $\alpha \in \mathcal{A}^{l-1}$. We write α as a polynomial in $dt_{l+1}, \ldots, dt_s$

$$\alpha = \alpha_0 + \sum_{j=l+1}^{s} \alpha_j dt_j + \sum_{j,k=l+1}^{s} \alpha_{jk} dt_j \wedge dt_k + \ldots$$
$$= \alpha_0 + \alpha_1 + \alpha_2 + \ldots .$$

Similarly we set

$$\varphi = \varphi_0 + \varphi_1$$

where $\varphi_0 = \sum_{j=1}^{l} \varphi_j dt_j$, $\varphi_1 = \sum_{j=l+1}^{s} \varphi_j dt_j$.

We have to show that if $\varphi \wedge \alpha = 0$ then there exists $\beta \in \mathcal{A}^{l-2}$ such that

$$\alpha = \varphi \wedge \beta .$$

The condition $\varphi \wedge \alpha = 0$ gives

$$(*) \qquad \begin{gathered} \varphi_0 \wedge \alpha_0 = 0 \\ \varphi_0 \wedge \alpha_1 + \varphi_1 \wedge \alpha_0 = 0 \\ \varphi_0 \wedge \alpha_2 + \varphi_1 \wedge \alpha_1 = 0 \\ \ldots\ldots\ldots \end{gathered}$$

If $\alpha_0 = \sum (-1)^j \alpha_j dt_1 \wedge \ldots \wedge \widehat{dt}_j \wedge \ldots \wedge dt_l$, from the first of these conditions we deduce

$$\sum_{1}^{l} \alpha_j \varphi_j = 0 .$$

Thus by the assumption and the lemma proved above we deduce that we must have

$$\alpha_j = \sum_{k=1}^{l} \mathfrak{h}_{jk}\varphi_k \qquad \text{with } \mathfrak{h}_{jk} \in \mathfrak{T},\ \mathfrak{h}_{jk} = -\mathfrak{h}_{kj},\ 1 \leqslant j \leqslant l\,.$$

Set

$$\beta_0 = -\sum_{j>k} \mathfrak{h}_{jk}\, dt_1 \wedge .. \wedge \widehat{dt}_j \wedge ... \wedge \widehat{dt}_k \wedge ... \wedge dt_l$$

Then

$$\alpha_0 = \varphi_0 \wedge \beta_0\,.$$

Substituting in the second relation (*) we get

$$\varphi_0 \wedge (\alpha_1 - \varphi_1 \wedge \beta_0) = 0$$

thus, by the same argument, we can write (with $\beta_1 \in \mathcal{A}^{l-2}$ of degree 1 in $dt_{l+1}, \ldots, dt_s$)

$$\alpha_1 = \varphi_1 \wedge \beta_0 + \varphi_0 \wedge \beta_1\,.$$

Substituting in the third of the relations (*) we get

$$\varphi_0 \wedge (\alpha_2 - \varphi_1 \wedge \beta_1) = 0\,.$$

Hence

$$\alpha_2 = \varphi_1 \wedge \beta_1 + \varphi_0 \wedge \beta_2\,.$$

Continuing in this way we get

$$\alpha = \varphi \wedge (\beta_0 + \beta_1 + \beta_2 + \ldots)\,.$$

COROLLARY 1. *If* $(\varphi_1, \ldots, \varphi_s)$ *is a principal sequence then the Koszul complex gives a free resolution of the* $\mathfrak{T}$*-module* $N_{\mathfrak{T}}$.

d) Consider now the isomorphism

$$*\colon \mathcal{A}^h \to \mathcal{A}^{s-h}$$

defined by

$$\alpha = \sum \alpha_{i_1 \ldots i_h} dt_{i_1} \wedge \cdots \wedge dt_{i_h} \to *\alpha = \sum \operatorname{sgn}(i_1 \ldots i_h, j_1, \ldots, j_{s-h})\ \alpha_{i_1 \ldots i_h} dt_{j_1} \wedge \cdots \wedge dt_{j_{s-h}}\,.$$

Then the operator $\varphi\wedge$ transforms into the transposed operator $e(\varphi)$ defined by

$$\alpha \to e(\varphi)\alpha$$

given by the formula

$$\alpha = \sum \alpha_J \, dt_J \to \sum \left(\sum_j \varphi_j \alpha_{jI}\right) dt_I \,,$$

where $J = (j_1, \ldots, j_h)$ denotes a block of h indices and $I = (i_1, \ldots, i_{h-1})$ denotes a block of $h-1$ indices.

Let now Ω be an open set in $\boldsymbol{R}^n$ and let $\mathcal{E}^{(k)}(\Omega)$ denote the space of C^∞ differential forms on Ω of degree k in $dt_1, \ldots, dt_s$. For any

$$\omega = \sum_{i_1 \ldots i_k} \omega_{i_1 \ldots i_k}(x) \, dt_{i_1} \wedge \ldots \wedge dt_{i_k} \in \mathcal{E}^{(k)}(\Omega)$$

we define the operator

$$\partial_\varphi \colon \mathcal{E}^{(k)}(\Omega) \to \mathcal{E}^{(k+1)}(\Omega)$$

by setting

$$\partial_\varphi \omega = \sum_{i_0 > \ldots > i_k} \left(\sum (-1)^k \varphi_{i_h}(D) \omega_{i_0 \ldots \hat{i}_h \ldots i_k}\right) dt_{i_0} \wedge \ldots \wedge dt_{i_k} .$$

From proposition 1 and 2 we deduce then the following corollaries.

Corollary 2. (α) *If $(\varphi_1, \ldots, \varphi_s)$ is a principal sequence then on any convex set we have an exact sequence of differential operators*

$$\mathcal{E}^{(0)}(\Omega) \xrightarrow{\partial_\varphi} \mathcal{E}^{(1)}(\Omega) \xrightarrow{\partial_\varphi} \cdots \xrightarrow{\partial_\varphi} \mathcal{E}^{(s-1)}(\Omega) \xrightarrow{\partial_\varphi} \mathcal{E}^{(s)}(\Omega) \to 0$$

(β) *If $(\varphi_1, \ldots, \varphi_s)$ contains a principal sequence of lenght l then the above complex is exact on* $\mathcal{E}^{(s)}(\Omega)$, $\mathcal{E}^{(s-1)}(\Omega), \ldots, \mathcal{E}^{(s-l+1)}(\Omega)$. Similarly setting

$$\mathcal{D}^{(k)}(\Omega) = \{\omega \in \mathcal{E}^{(k)}(\Omega) | \operatorname{supp} \omega \text{ compact in } \Omega\}$$

we get the following

Corollary 3 (α). *If $(\varphi_1, \ldots, \varphi_s)$ is a principal sequence then on any open convex set we have an exact sequence of differential operators*

$$0 \to \mathcal{D}^{(0)}(\Omega) \xrightarrow{\partial_\varphi} \mathcal{D}^{(1)}(\Omega) \xrightarrow{\partial_\varphi} \ldots \xrightarrow{\partial_\varphi} \mathcal{D}^{(s-1)}(\Omega) \xrightarrow{\partial_\varphi} \mathcal{D}^{(s)}(\Omega) .$$

(β) *If* $(\varphi_1, \ldots, \varphi_s)$ *contains a principal sequence of length* l *then the above sequence is exact on*

$$\mathfrak{D}^{(0)}(\Omega)\,,\ \mathfrak{D}^{(1)}(\Omega), \ldots, \mathfrak{D}^{(l-1)}(\Omega)\,.$$

11. – A generalized Koszul complex.

a) Let $\mathcal{M}_{r\times s}(\boldsymbol{C}) = \operatorname{Hom}_{\boldsymbol{C}}(\boldsymbol{C}^s, \boldsymbol{C}^r)$ denote the space of $r\times s$ matrices with elements in $\boldsymbol{C}$.

We let the group $GL(r, \boldsymbol{C})\times GL(s, \boldsymbol{C})$ operate on $\mathcal{M}_{r\times s}(\boldsymbol{C})$ by

$$(\alpha\times\beta)M = \alpha M\beta$$

where $\alpha\in GL(r, \boldsymbol{C})$, $\beta\in GL(s, \boldsymbol{C})$ and $M\in\mathcal{M}_{r\times s}(\boldsymbol{C})$. We assume that $r\leqslant s$ and we denote by

$$J_\varrho = \{M\in\mathcal{M}_{r\times s}(\boldsymbol{C})|\operatorname{rank} M = r-\varrho\}$$

for $\varrho = 0, 1, \ldots, r$.

For every $M\in J_\varrho$ we can find $\alpha\in GL(r, \boldsymbol{C})$, $\beta\in GL(s, \boldsymbol{C})$ such that

$$\alpha M\beta = \begin{pmatrix} I_{r-\varrho} & 0\\ 0 & 0\end{pmatrix}.$$

Therefore the sets J_ϱ are nothing else but the orbits of $GL(r, \boldsymbol{C})\times GL(s, \boldsymbol{C})$ on $\mathcal{M}_{r\times s}(\boldsymbol{C})$, so that

$$\mathcal{M}_{r\times s}(\boldsymbol{C}) = J_0\cup J_1\cup J_2\cup\ldots\cup J_r\,.$$

We do have the following properties

i)
$$\begin{aligned}\bar{J}_\varrho &= \{M\in J_{r\times s}(\boldsymbol{C})|\operatorname{rank} M\leqslant r-\varrho\}\\ &= J_\varrho\cup J_{\varrho+1}\cup\ldots\cup J_r\end{aligned}$$

so that $\bar{J}_\varrho$ is an algebraic irreducible variety.

ii) J_ϱ is a locally closed submanifold of $\mathcal{M}_{r\times s}(\boldsymbol{C})$ and codimension $\bar{J}_\varrho = \varrho(s-r+\varrho)$.

Indeed it is clear that $J_{\varrho+1}\cup\ldots\cup J_r$ are in the closure of J_ϱ.

Moreover

$$J_\varrho = \left\{\alpha\begin{pmatrix} I^{r-\varrho} & 0\\ 0 & 0\end{pmatrix}\beta\,,\quad (\alpha, \beta)\in GL(r, \boldsymbol{C})\times GL(s, \boldsymbol{C})\right\}$$

is irreducible as the image under a holomorphic map of the irreducible manifold $GL(r, \boldsymbol{C}) \times GL(s, \boldsymbol{C})$.

Finally near the point $\begin{pmatrix} I_{r-\varrho} & 0 \\ 0 & 0 \end{pmatrix}$ every matrix can be written as

$$\begin{pmatrix} I_{r-\varrho} & 0 \\ 0 & 0 \end{pmatrix} + (a_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}$$

and the equations of J_ϱ are of the form

$$a_{ij} + O(a^2) = 0 \qquad r - \varrho < i \leqslant r, \ s - r - \varrho \leqslant j \leqslant s$$

and this proves the contention about the codimension.

REMARK. A « generic » point of the manifold J_ϱ is given by

$$(z_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} = \begin{pmatrix} \alpha & \alpha\gamma \\ \beta & \beta\gamma \end{pmatrix}$$

where $\alpha \in \mathcal{M}_{(r-\varrho)\times(r-\varrho)}$, $\gamma \in \mathcal{M}_{(r-\varrho)\times(s-r+\varrho)}$, $\beta \in \mathcal{M}_{\varrho\times(r-\varrho)}$. Thus we get parametric equations in terms of $rs - \varrho(s - r + \varrho)$ parameters.

In particular it follows that $\bar{J}_\varrho$ is a rational variety. (The correspondence with the parameter space being generally one to one as $\det\alpha \neq 0$ in general).

b) Let us consider a matrix of type $r \times s$ with $r \leqslant s$

$$A = \begin{pmatrix} \varphi_{11}, \ldots, \varphi_{1s} \\ \varphi_{r1}, \ldots, \varphi_{rs} \end{pmatrix}$$

with entries $\varphi_{ij} \in \mathcal{T} = \boldsymbol{C}[x_1, \ldots, x_n]$. We introduce the 1-exterior forms

$$\varphi_1 = \sum_{j=1}^{s} \varphi_{1j}\, dt_j$$

$$\cdots\cdots$$

$$\varphi_r = \sum_{j=1}^{s} \varphi_{rj}\, dt_j$$

and we set

$$\omega_A = \varphi_1 \wedge \ldots \wedge \varphi_r .$$

We want to investigate the set of points in $\boldsymbol{C}^n$

$$V_A = \{z \in \boldsymbol{C}^n | \omega_A(z) = 0\} .$$

If $\mathfrak{a}_A$ denotes the ideal in $\mathcal{P}$ generated by the subdeterminants of order r of A, as these are nothing else but the coefficients of ω_A, we have that V_A is the set of common zeros of the polynomials in $\mathfrak{a}_A$ and therefore V_A is an algebraic variety.

The algebraic variety V_A can also be viewed in the following way.

The matrix A defines a map

$$\alpha_A\colon \boldsymbol{C}^n \to \mathcal{M}_{r\times s}(\boldsymbol{C})$$

by

$$x \to \big(\varphi_{ij}(x)\big)$$

then

$$V_A = \alpha_A^{-1}(\bar{J}_1)\,.$$

We will make use of the following known fact

Let $(\alpha, 0)$, $(\beta, 0)$ *be two irreducible germs of analytic subsets of* $\boldsymbol{C}^n$ *at the origin* $0 \in \boldsymbol{C}^n$. *If* $\dim_0 \alpha = a$, $\dim_0 \beta = b$ *then each irreducible component of the germ* $(\alpha \cap \beta, 0)$ *has a dimension at the origin which is* $\geqslant a + b - n$.

PROPOSITION 7. *For any choice of the matrix* A *we have that either* $V_A = \emptyset$ *or else each irreducible component of* V_A *has a dimension* $\geqslant n - (s - r + 1)$.

PROOF. We first remark that $\bar{J}_1$ is an irreducible algebraic variety in $\mathcal{M}_{r\times s}(\boldsymbol{C})$. Let us consider the graph of the map α

$$G_\alpha = \{(z, m) \in \boldsymbol{C}^n \times \mathcal{M}_{r\times s}(\boldsymbol{C}) \,|\, m = \alpha(z)\}\,.$$

We do have a natural isomorphism $\pi\colon G_\alpha \xrightarrow{\sim} \boldsymbol{C}^n$ induced by the projection on the first factor of the product $\boldsymbol{C}^n \times \mathcal{M}_{r\times s}(\boldsymbol{C})$. Also we have

$$V_A = \pi\big(G_\alpha \cap (\boldsymbol{C}^n \times \bar{J}_1)\big)\,.$$

As π is an isomorphism it is enough to show that the analytic set $G_\alpha \cap (\boldsymbol{C}^n \times \bar{J}_1)$ has every irreducible component of dimension $\geqslant n - (s - r + 1)$.

Now G_α is irreducible, thus each germ of G_α has dimension n.

Also $\boldsymbol{C}^n \times \bar{J}_1$ is irreducible and thus each one of its germs has dimension $= n + \big(rs - (s - r + 1)\big)$, $\big(rs - (s - r + 1) = \dim \bar{J}_1\big)$.

At a point $w_0 \in G_\alpha \cap (\boldsymbol{C}^n \times \bar{J}_1)$ we can apply the above remark, taking into account that the dimension of the surrounding space $\boldsymbol{C}^n \times \mathcal{M}_{r\times s}(\boldsymbol{C})$ is $n + rs$. We do get for the dimension of each irreducible germ γ at w_0 of

$G_\alpha \cap (C^n \times \bar{J}_1)$ the estimate

$$\begin{aligned}\dim_{w_0}(\gamma) &\geq \{rs + [n-(s-r+1)]\} + n - (rs+n)\\ &\geq n-(s-r+1)\end{aligned}$$

and this is what we wanted to prove.

We will say that α_A is *transversal* to the stratification of $\mathcal{M}_{r\times s}(C)$ if for every point $x_0 \in C^n$ (with $\alpha_A(x_0) \in J_\varrho$) we do have

$$T_{\alpha(x_0)}(\mathcal{M}_{r\times s}(C)) = d\alpha_A(x_0)(T_{x_0}(C^n)) + T_{\alpha(x_0)}(J_\varrho)$$

where $T_y(M)$ denote, as usual, the tangent space at y to the complex manifold M.

COROLLARY. *If α_A is transversal to the stratification of $\mathcal{M}_{r\times s}(C)$ then V_A is either empty or else each irreducible component W of V_A has dimension equal to $n-(s-r+1)$, and $W \cap \alpha^{-1}(J_1)$ is dense in W.*

PROOF. We have

$$V_A = \alpha_A^{-1}(\bar{J}_1) = \alpha_A^{-1}(J_1) \cup \alpha(J_2) \cup \ldots \cup \alpha_A^{-1}(J_r)$$

and, by the transversality assumption $\alpha_A^{-1}(J)_\varrho$, for any ϱ, is a locally closed submanifold of C^n of dimension $n-\varrho(s-r+\varrho)$.

For $\varrho \geq 2$ we have $\varrho(s-r+\varrho) > s-r+1$. This implies that each non empty irreducible component of V_A must contain a Zariski open subset all made of points of $\alpha_A^{-1}(J_1)$. From this we get the conclusion.

REMARK. In any case if $V_A \neq \emptyset$ or if $\dim_C V_A = n-(s-r+1)$ then V_A is purely dimensional i.e. each irreducible component of V_A has the same dimension than V_A itself.

DEFINITION. We say that the matrix $A = (\varphi_{ij})_{\substack{1\leq i\leq r\\ 1\leq j\leq s}}$, $r \leq s$, is a *Macaulay matrix* if for any choice of $h \leq k \leq s$, for the matrices $A_{hk} = (\varphi_{ij})_{\substack{1\leq i\leq h\\ 1\leq j\leq k}}$ we do have that either $V_{A_{hk}} = \emptyset$ or $\dim V_{A_{hk}} = n-(k-h+1)$.

PROPOSITION 8 (MACAULAY). *If A is a Macaulay matrix then*

(α) *the ideal $\mathfrak{a}_A$ is pure* (ungemisht);

(β) *for any exterior form $\alpha^{(s-r)} \in \mathcal{A}_s^{s-r}$ such that*

(*) $$\alpha \wedge \omega_A = 0$$

we can find exterior forms $\eta_i^{(s-r-1)} \in \mathcal{A}_s^{s-r-1}$ *such that*

$$\alpha^{(s-r)} = \sum_{i=1}^{r} \eta_i^{(s-r-1)} \wedge \varphi_i .$$

PROOF. (β) The theorem is true for $s = r$. The proof proceeds by a double induction assuming the theorem proved for the matrices $A_{r,s-1}$ and $A_{r-1,s-1}$.

If we write

$$\alpha = \sum_{\substack{q_1 > \ldots > q_{r-s} \\ p_1 > \ldots > p_r}} \operatorname{sgn}(q_1, \ldots, q_{r-s}, p_1, \ldots, p_r) \alpha_{p_1, \ldots, p_r} dt_{q_1} \wedge \ldots \wedge dt_{q_{r-s}}$$

then condition (*) gives

$$(**) \qquad \sum \alpha_{p_1 \ldots p_r} D_{p_1 \ldots p_r} = 0 .$$

Where $D_{p_1 \ldots p_r}$ are the minor determinants of A made with the columns of indices $p_1, \ldots, p_r$.

By the inductive assumption one of the determinants in $\mathfrak{a}_{A_{rs}}$ is prime to the ideal $\mathfrak{a}_{A_{r,s-1}}$ unless this last is trivial. We can thus assume that if

$$D_{1 \ldots r-1 s}\, g \in \mathfrak{a}_{A_{r,s-1}}$$

then also

$$g \in \mathfrak{a}_{A_{r,s-1}} .$$

Now relation (**) can be written as

$$(**) \qquad \sum_{p_1 < \ldots < p_r < s} \alpha_{p_1 \ldots p_r} D_{p_1 \ldots p_r} + \sum_{p_1 < \ldots < p_{r-1} < s} \alpha_{p_1 \ldots p_{r-1} s} D_{p_1 \ldots p_{r-1} s} = 0 .$$

Also if we denote by $D^{(j)}_{1 \ldots r-1}$ the subdeterminant of $A_{r,r-1}$ obtained by deleting the j-th row we do have

$$\sum_{j=1}^{r} D^{(j)}_{1 \ldots r-1} \varphi_{jp} = D_{1 \ldots r-1 p} .$$

Letting $p = p_1, \ldots, p_{r-1}, s$ and solving this system with respect to $D^{(r)}_{1 \ldots r-1}$ we get

$$\begin{aligned} D_{p_1 \ldots p_{r-1} s} D^{(r)}_{1 \ldots r-1} = & \pm D_{1 \ldots r-1\, p_1} D^{(r)}_{p_2 \ldots p_{r-1} s} \pm \cdots \\ & \pm D_{1 \ldots r-1\, p_{r-1}} D^{(r)}_{p_1 \ldots p_{r-2} s} \\ & \pm D_{1 \ldots r-1\, s} D^{(r)}_{p_1 \ldots p} \end{aligned}$$

thus

$$D^{(r)}_{1\dots r-1}D_{v_1\dots v_{r-1}s} \pm D^{(r)}_{v_1\dots v_{r-1}}D_{1\dots r-1\,s} \equiv 0 \bmod \mathfrak{a}_{A_{r,s-1}}\,.$$

Thus moltiplying (**) by $D^{(r)}_{1\dots r-1}$ we do get

$$D_{1\dots r-1\,s}\sum \alpha_{v_1\dots v_{r-1}s}D^{(r)}_{v_1\dots v_{r-1}} \equiv 0 \qquad \bmod \mathfrak{a}_{A_{r,s-1}}\,.$$

Therefore, by the above specified assumption, we do also have

$$\sum \alpha_{v_1\dots v_{r-1}s}D^{(r)}_{v_1\dots v_{r-1}} \equiv 0 \qquad \bmod \mathfrak{a}_{A_{r,s-1}}$$

i.e.

$$\sum \alpha_{v_1\dots v_{r-1}s}D^{(r)}_{v_1\dots v_{r-1}} = \sum_{v_1<\dots<v_r<s} \beta_{v_1\dots v_r}D_{v_1\dots v_r}\,.$$

Now

$$D_{v_1\dots v_r} = \varphi_{rv_r}D^{(r)}_{v_1\dots v_{r-1}} - \varphi_{rv_{r-1}}D^{(r)}_{v_1\dots v_{r-2}v_r} \pm \dots \pm \varphi_{rv_1}D^{(r)}_{v_2\dots v_r}\,.$$

Therefore

$$\sum \left(\alpha_{v_1\dots v_{r-1}s} - \sum \beta_{v_1\dots v_r}\varphi_{r\,v_r}\right)D^{(r)}_{v_1\dots v_{r-1}} = 0\,.$$

Now the conclusion of statement (β) can be expressed as saying that there exists for the coefficients α an expression

$$(***) \qquad \alpha_{v_1\dots v_r} = \sum \eta^i_{v_1\dots v_r v}\varphi_{iv}$$

where η^i are alternate in the lower indices.

By the inductive assumption for $\mathfrak{a}_{A_{r-1,s-1}}$ we thus get

$$\alpha_{v_1\dots v_{r-1}s} = \sum \gamma^i_{v_1\dots v_{r-1}s\,v}\varphi_{iv}$$

thus the relation (***) for $r = s$.

From this it follows that replacing α with

$$\alpha' = \alpha - \sum \gamma_i \wedge \varphi_i$$

we get for α' a new form which satisfies $\alpha' \wedge \omega_A = 0$ and in which the coefficients $\alpha'_{v_1\dots v_{r-1}s} = 0$.

Thus the relation (**) and the inductive assumption for $\mathfrak{a}_{r,s-1}$ give the conclusion (β).

(α) It remains to prove that $\mathfrak{a}_A$ is unmixed. For this one applies the following criterion. The ideal $\mathfrak{a}_A$ is pure if and only if adding to the basis of $\mathfrak{a}_A$ $t \leqslant n-(s-r+1)-1$ generic linear polynomials, $x_1-a_1, \ldots, x_t-a_t$, the ideal thus obtained should not have any 0-dimensional primary component. If this is not the case, then we can find $F \in \mathfrak{I}$ such that

$$(x_n-a_n)F \equiv 0 \qquad \operatorname{mod}(\mathfrak{a}_A, x_1-a_1, \ldots, x_t-a_t)$$

but $F \notin (\mathfrak{a}_A, x_1-a_1, \ldots, x_t-a_t)$.

Now as $a_1, \ldots, a_t$ are generic the matrix obtained from A setting $x_i = a_i$, $1 \leqslant i \leqslant t$, $i = n$, is again a Macaulay matrix.

The above congruence gives

$$(x_n-a_n)F = \sum \alpha_{p_1 \ldots p_r} D_{p_1 \ldots p_r} \qquad \operatorname{mod}(x_1-a_1, \ldots, x_t-a_t)\,.$$

Setting $x_n = a_n$ we get by the result (β)

$$\alpha_{p_1 \ldots p_r} = \sum \gamma^i_{p_1 \ldots p_r p} \varphi_{ip} + (x_n-a_n) y_{p_1 \ldots p_r} \qquad \operatorname{mod}(x_1-a_1, \ldots, x_t-a_t)$$

Hence

$$(x_n-a_n)\big(F - \sum y_{p_1 \ldots p_r} D_{p_1 \ldots p_r}\big) \equiv 0 \qquad \operatorname{mod}(x_1-a_1, \ldots, x_t-a_t)\,.$$

But the ideal $(x_1-a_1, \ldots, x_t-a_t)$ is relatively prime to x_n-a_n so that

$$F - \sum y_{p_1 \ldots p_r} D_{p_1 \ldots p_r} \equiv 0 \qquad \operatorname{mod}(x_1-a_1, \ldots, x_t-a_t)$$

i.e.

$$F \in (\mathfrak{a}_A, x_1-a_1, \ldots, x_t-a_t)$$

COROLLARY 1. *If A is a Macaulay matrix and if $\vartheta^{(l)} \in \mathcal{A}^l_s$ with $l < s$; if*

$$\vartheta^{(l)} \wedge \varphi_i = 0 \qquad \textit{for } 1 \leqslant i \leqslant r$$

then there exist $\beta \in \mathcal{A}^{l-r}_s$ such that

$$\vartheta^{(l)} = \beta \wedge \varphi_1 \wedge \ldots \wedge \varphi_r\,.$$

PROOF. (α) First we remark that if $l \leqslant s-r$ and if

$$\vartheta \wedge \varphi_r = \sum_1^{r-1} \beta_j \varphi_j\,.$$

We do also have

$$\vartheta = \sum_1^s \alpha_j \varphi_j\,.$$

Indeed let us introduce $((s-r)-l)\times s$ new variables z_{ij}, $1\leqslant i\leqslant s-r-l$, $1\leqslant j\leqslant s$ and set

$$\eta_h=\sum_1^s z_{hj}\,dt_j$$

the extended matrix $\hat{A}=\begin{pmatrix}A\\ z_{ij}\end{pmatrix}$ is again a Macaulay matrix.

Moreover we have

$$\vartheta\wedge\varphi_1\wedge\ldots\wedge\varphi_r\wedge\eta_1\ldots\wedge\eta_\varrho=0\qquad(\varrho=s-r-l)$$

thus

$$\vartheta=\sum\sigma_j\varphi_j+\sum\tau_j\eta_j\,.$$

Setting $z_{ij}=0$ we get an expression

$$\vartheta=\sum\alpha_j\varphi_j\,.$$

(β) Now from $\vartheta\wedge\varphi_1=0$ and $l\leqslant s-1$ we deduce

$$\vartheta=\sigma^{(l-1)}\wedge\varphi_1\,.$$

Also $\sigma^{(l-1)}\wedge\varphi_1\wedge\varphi_2=0$ and since $l-1\leqslant s-2$ we do get

$$\sigma^{(l-1)}=\alpha^{(l-1)}\wedge\varphi_1+\beta^{(l-2)}\wedge\varphi_2\,.$$

Hence

$$\vartheta=\beta\wedge\varphi_2\wedge\varphi_1\,.$$

Also $\beta^{(l-2)}\wedge\varphi_1\wedge\varphi_2\wedge\varphi_3=0$ and since $(l-2)\leqslant s-3$ we get

$$\beta^{(l-2)}=\alpha_1\wedge\varphi_1+\alpha_2\wedge\varphi_2+\gamma^{(l-3)}\wedge\varphi_3\,.$$

Hence

$$\vartheta=\gamma\wedge\varphi_3\wedge\varphi_2\wedge\varphi_1\,.$$

Proceeding in this way we get the conclusion.

COROLLARY 2. *Same assumption; if* $u\in\mathcal{A}_s^l$ *and if* $l\leqslant s-r$ *and if moreover*

$$u\wedge\varphi_1\ldots\wedge\varphi_r=0\,.$$

Then we can find $\beta_i \in \mathcal{A}_s^{l-1}$ *such that*

$$u = \sum_1^r \beta_i \wedge \varphi_i .$$

PROOF. If $r = 1$ the theorem is true and was already proved with the Koszul complex. We can proceed by induction on r. Set

$$v = u \wedge \varphi_1 \wedge \ldots \wedge \varphi_{r-1}$$

then

$$v \wedge \varphi_i = 0 \qquad \text{for } 1 \leqslant i \leqslant r .$$

Hence by Corollary 1 we get

$$v = \beta \wedge \varphi_1 \wedge \ldots \wedge \varphi_r$$

therefore

$$(\pm \beta \wedge \varphi_r + u) \wedge \varphi_1 \wedge \ldots \wedge \varphi_{r-1} = 0 .$$

By the inductive assumption

$$u = \mp \beta \wedge \varphi_r + \sum_1^{r-1} \beta_j \wedge \varphi_j .$$

COROLLARY 3. *Same assumption; if* $u_i \in \mathcal{A}_s^l$ *and* $l \leqslant s - r$ *and if moreover*

$$\sum_1^r u_i \wedge \varphi_i = 0$$

then one can find $h_{ij} \in \mathcal{A}_s^{l-1}$ *with* $h_{ij} = h_{ji}$ *such that*

$$u_i = \sum_j h_{ij} \varphi_j \qquad 1 \leqslant i \leqslant r .$$

PROOF. If $r = 1$ the statement has already been prooved. By induction on r: We have

$$- u_r \wedge \varphi_r = \sum_{j=1}^{r-1} u_j \wedge \varphi_j .$$

Hence

$$u_r \wedge \varphi_r \wedge \varphi_{r-1} \wedge \ldots \wedge \varphi_1 = 0$$

and therefore (Corollary 2)

$$(*)\qquad u_r = \sum_{i=1}^{r} \lambda_i \wedge \varphi_i .$$

Now

$$0 = \sum u_i \wedge \varphi_i = (u_1 - \lambda_1 \varphi_r)\wedge \varphi_1 + \dots + (u_{r-1} - \lambda_{r-1}\varphi_r)\wedge \varphi_{r-1} = 0 .$$

By the inductive hypothesis

$$(**)\qquad u_i - \lambda_i \varphi_r = \sum \sigma_{ij} \wedge \varphi_j \qquad 1 \leqslant i \leqslant r-1$$

with $\sigma_{ij} = \sigma_{ji}$. The relations (**) and (*) are equivalent to the statement of the corollary.

REMARK. Corollary 1, 2 and 3 are all consequences of the following statement (that we have deduced from the theorem of Macaulay)

(P_r) *If* $\vartheta \in \mathcal{A}_s^l$ *and if we have*

$$\vartheta \wedge \varphi_r = \sum_{j=1}^{r-1} \beta_j \varphi_j \qquad \textit{for some } \beta_j \in \mathcal{A}_s^l,\ \deg \vartheta = l \leqslant s - r$$

then we do have

$$\vartheta = \sum_1^r \alpha_j \wedge \varphi_j \qquad \textit{for some } \alpha_j \in \mathcal{A}_s^{l-1} .$$

A matrix $A = (\varphi_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}$ with $r \leqslant s$ will be called of the *principal type* if setting

$$\varphi_i = \sum_{j=1}^{s} \varphi_{ij} dt_j$$

we have that φ_1 verifies (P_1), φ_1, φ_2 verify (P_2), ..., $\varphi_1, \dots, \varphi_r$ verify (P_r).

c) Let us denote by $\mathcal{B}_h^k$ the space of forms of degree k in the indeterminates $y_1, \dots, y_r$ and with coefficients in $\mathcal{A}_s^h$. Thus $\beta \in \mathcal{B}_h^k$ is an expression of the form

$$\beta = \sum_{|\alpha| = k} \beta_{\alpha_1} \dots \beta_{\alpha_r} y_1^{\alpha_1} \dots y_r^{\alpha_r} \qquad \text{with } \beta_{\alpha_1} \dots \beta_{\alpha_r} \in \mathcal{A}^h .$$

We consider β as a polynomial in one, y_i, of the variables:

$$\beta = \beta_0 + \beta_1 y_i + \dots + \beta_t y_i^t$$

where β_i are polynomials in $y_1, \ldots, \hat{y}_i, \ldots, y_r$. We define

$$\nabla_i \beta = \varphi_i \wedge \sum_1^t \beta_l y_i^{l-1}$$

where $\varphi_1, \ldots, \varphi_r$ are r 1-forms given once for all.

One verifies easily that

$$\nabla_i \nabla_j = -\nabla_j \nabla_i \quad \text{if } i \neq j$$

$$\nabla_i^2 = 0 .$$

Therefore if we set

$$\nabla = \sum_{i=1}^r \nabla_i$$

we define a map

$$\nabla : \mathcal{B}_h^k \to \mathcal{B}_{k-1}^{h+1}$$

with the property

$$\nabla \circ \nabla = 0 .$$

LEMMA. *Let $h < s - r$, $k \geqslant 1$ and let $\varphi_1, \ldots, \varphi_r$ be the 1-forms associated to a matrix of principal type.*

Let $\beta \in \mathcal{B}_k^h$ with $\nabla \beta = 0$; then if $h = 0$ we have $\beta = 0$ and, if $h > 0$, we can find $\gamma \in \mathcal{B}_{k-1}^{h+1}$ with

$$\beta = \nabla \gamma .$$

PROOF. If $\beta = \sum \beta_{\alpha_1 \ldots \alpha_r} y_1^{\alpha_1} \ldots y_r^{\alpha_r}$ then

$$\nabla \beta = \sum \beta_{\alpha_1 \ldots \alpha_i + 1 \ldots \alpha_r} \varphi_i y_1^{\alpha_1} \ldots y_i^{\alpha_i} \ldots y_r^{\alpha_r}$$

thus if $h = 0$ and $\nabla \beta = 0$ we have

$$\sum \beta_{\alpha_1 \ldots \alpha_i + 1 \ldots \alpha_r} \varphi_i = 0 .$$

Hence all $\beta_{\alpha_1 \ldots \alpha_i + 1 \ldots \alpha_r} = 0$. But this imples $\beta = 0$ as $k \geqslant 1$.

Assume now $h > 0$. If $r = 1$ then the lemma says that if $\beta \wedge \varphi_1 = 0$ then $\beta = \varphi_1 \wedge \gamma$. This has already been proved. Therefore we can proceed by induction on r.

Set

$$\beta = \beta_0(y_1 \cdots y_{r-1}) + \beta_1(y_1 \cdots y_{r-1}) y_r + \ldots + \beta_l(y_1 \cdots y_{r-1}) y_r^l .$$

Denoting $\nabla' = \nabla_1 + \dots + \nabla_{r-1}$ the condition $\nabla\beta = 0$ translates into the conditions

$$\begin{aligned} \nabla'\beta_0 \quad &+ \varphi_r\beta_1 = 0 \\ \nabla'\beta_1 \quad &+ \varphi_r\beta_2 = 0 \\ &\dots\dots \\ \nabla'\beta_{l-1} &+ \varphi_r\beta_l = 0 \\ &\nabla'\beta_l = 0\,. \end{aligned}$$

From these, by the inductive assumption we derive

$$\begin{aligned} &\beta_l = \nabla'\sigma_l \\ \nabla'(\beta_{l-1} - \varphi_r\wedge\sigma_l) &= 0 \quad \text{thus } \beta_{l-1} = \varphi_r\wedge\sigma_l + \nabla'\sigma_{l-1} \\ \nabla'(\beta_{l-2} - \varphi_r\wedge\sigma_{l-1}) &= 0 \quad \text{thus } \beta_{l-2} = \varphi_r\wedge\sigma_{l-1} + \nabla'\sigma_{l-2} \\ &\dots\dots \\ \nabla'(\beta_0 - \varphi_r\wedge\sigma_1) &= 0 \quad \text{thus } \beta_0 = \varphi_r\wedge\sigma_1 + \nabla'\sigma_0\,. \end{aligned}$$

Set

$$\gamma = \sigma_0 + \sigma_1 y_r + \dots + \sigma_l y_r^l$$

we get

$$\nabla\gamma = (\nabla'\sigma_0 + \varphi_r\wedge\sigma_1) + (\nabla'\sigma_1 + \varphi_r\wedge\sigma_2)y_r + \dots + \nabla'\sigma_l y_r^l\,.$$

Hence the lemma is proved, as one verifies that the degrees of the σ_j in the $y_1, \dots, y_{r-1}$ are the right ones.

d) Given a principal matrix A we can consider the $\mathfrak{I}$-homomorphism

$$\mathfrak{I}^s \xrightarrow{A} \mathfrak{I}^r \to N_A \to 0$$

whose cokernel is denoted by N_A.

We identify

$$\begin{aligned} \mathfrak{I}^s &\simeq \mathcal{A}_s^{s-1} \\ \mathfrak{I}^r &\simeq (\mathcal{A}_s^s)^r \end{aligned}$$

so that the above homomorphism can be described as follows:

$$\mathcal{A}^{s-1} \xrightarrow{\begin{smallmatrix}\wedge\varphi_1 \\ \vdots \\ \wedge\varphi_r\end{smallmatrix}} (\mathcal{A}^s)^r\,.$$

Combining the above lemma with Corollaries 1, 2, 3 which are valid for a principal matrix we obtain the following

PROPOSITION 9. *Given a principal matrix $A = (\varphi_{ij})_{\substack{1\leq i\leq r\\ 1\leq j\leq s}}$ for the cokernel N_A of the $\mathcal{S}$-homomorphism defined by A one has the following exact sequence which provides a free resolution of N_A:*

$$0 \to \mathcal{B}^0_{s-r-1} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{B}^{s-r-3}_2 \xrightarrow{\nabla} \mathcal{B}^{s-r-2}_1 \xrightarrow{\nabla} \mathcal{A}^{s-r-1} \xrightarrow{\wedge\varphi_1\wedge\ldots\varphi_r} \mathcal{A}^{s-1} \xrightarrow{\substack{\wedge\varphi_1\\ \vdots\\ \wedge\varphi_r}} (\mathcal{A}^s_s)^r \to N_A \to 0\,.$$

REMARK. 1) The length of the resolution is $s - r + 1$.

2) As rank $A = r$ we have that N_A is a torsion module

$$\tau(N_A) = N_A\,.$$

Exercise. Note that the image of the map

$$\mathcal{A}^{s-r} \xrightarrow{\wedge\varphi_1\ldots\wedge\varphi_r} \mathcal{A}^s$$

is the ideal $\mathfrak{a}_A$. Denoting by $N_\mathfrak{a}$ its cokernel, we do get with the same argument the following resolution

$$0 \to \mathcal{B}^0_{s-r} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{B}^{s-r-2}_2 \xrightarrow{\nabla} \mathcal{B}^{s-r-1}_1 \xrightarrow{\nabla} \mathcal{A}^{s-r} \xrightarrow{\varphi_1\wedge\ldots\wedge\varphi_r} \mathcal{A}^s \to N_\mathfrak{a} \to 0$$

of length $s - r + 1$. Since the dimension of $\mathfrak{a}$ is $n - (s - r + 1)$ it follows that $\mathfrak{a}$ is a perfect ideal.

Applying proposition 1 and 2 we do get resolution for the matrix of differential operators ${}^tA(D)$ (forward going on C^∞ functions) or for the matrix $A(D)$ (backward going on C^∞ functions with compact support).

REMARK. The resolution of the exercise can be found in Eagon and Northcott [8].

e) As an example of a free resolution which (it seems to us) cannot be derived from the Koszul complexes one can consider the following situation.

Let (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) be two principal sequences of length 4 in $\mathcal{S}$. We set

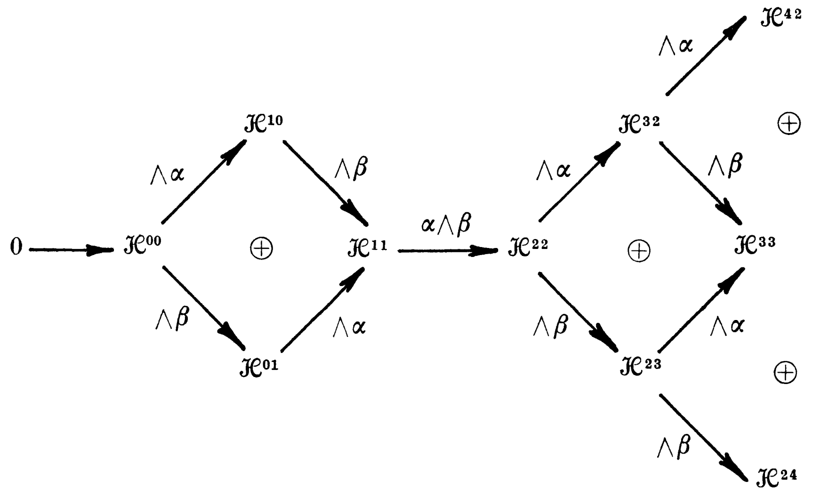
$$\alpha = a_1 dt_1 + a_2 dt_2 + a_3 dt_3 + a_4 dt_4$$

$$\beta = b_1 d\theta_1 + b_2 d\theta_2 + b_3 d\theta_3 + b_4 d\theta_4$$

where dt_i and $d\theta_j$ are indeterminates.

Let $\mathcal{H}^{r,s}$ denote the space of exterior forms with coefficients in $\mathcal{T}$ of degree r in the dt_i's and of degree s in the $d\theta_j$'s.

One can then verify, by making use of the previous considerations, that the following sequence is exact



This example can be generalized in many ways (cf. Bigolin [3]).

f) By the theory of division of distributions we can transform the generalized Koszul complex into a complex of differential operators.

We set for any open set $\Omega\subset \boldsymbol{R}^n$

$\mathcal{E}^{(k)}(\Omega)$ = space of exterior forms of degree k in the indeterminates $dt_1, \dots, dt_s$ with C^∞ coefficients;

$\mathcal{E}^{(k)}_h(\Omega)$ = space of homogeneous forms of degree h in $y_1, \dots, y_r$ with coefficients in $\mathcal{E}^{(k)}(\Omega)$

$\mathcal{D}^{(k)}_h(\Omega) = \{\alpha\in \mathcal{E}^{(k)}_n(\Omega) | \text{support of } \alpha \text{ compact in } \Omega\}$.

Let $A = (\varphi_{ij}(x))_{\substack{1\leq i\leq r\\ 1\leq j\leq s}}$ with $r<s$ be a principal matrix. For

$$\omega = \sum_{i_1>\dots>i_k} \omega_{i_1\dots i_k}(x)\, dt_{i_1}\wedge\dots\wedge dt_{i_k}$$

we set

$$\partial_{\varphi_i}\omega = \sum_{i_0<\dots<i_k}\Big(\sum_{h=0}^{k}(-1)^h\varphi_{i\,i_h}(D)\,\omega_{i_0\dots\hat{i}_h\dots i_k}\Big)\, dt_{i_0}\wedge\dots\wedge dt_{i_k}$$

$$\nabla = \sum y_i\,\partial_{\varphi_i} \qquad \text{and} \qquad D = \sum \partial/y_i\,\partial_{\varphi_i}.$$

Then on any convex open set $\Omega \subset \boldsymbol{R}^n$ we get the following exact sequences

$$(\mathcal{E}^{(0)}(\Omega))^r \xrightarrow{\Sigma \partial_{\varphi_i}} \mathcal{E}^{(1)}(\Omega) \xrightarrow{\partial_{\varphi_1} \dots \partial_{\varphi_r}} \mathcal{E}^{(r+1)}(\Omega) \xrightarrow{\nabla} \mathcal{E}_1^{(r+2)}(\Omega) \xrightarrow{\nabla} \cdots$$
$$\cdots \to \mathcal{E}^{(s-1)}_{s-r-2}(\Omega) \xrightarrow{\nabla} \mathcal{E}^{(s)}_{s-r-1}(\Omega) \to 0$$

where the first map is defined by $(u_1, \dots, u_r) \to \Sigma \partial_{\varphi_i} u_i$, and

$$0 \to \mathcal{D}^{(0)}_{s-r-1}(\Omega) \xrightarrow{D} \mathcal{D}^{(1)}_{s-r-2}(\Omega) \xrightarrow{D} \cdots$$
$$\cdots \xrightarrow{D} \mathcal{D}_1^{(s-r-2)}(\Omega) \xrightarrow{D} \mathcal{D}^{(s-r-1)}(\Omega) \xrightarrow{\partial_{\varphi_1} \dots \partial_{\varphi_r}} \mathcal{D}^{(s-1)}(\Omega) \xrightarrow{\partial_{\varphi_1} \vdots \partial_{\varphi_r}} (\mathcal{D}^{(s)}(\Omega))^r$$

the last map being defined by $u \to {}^t(\partial_{\varphi_1} u, \dots, \partial_{\varphi_r} u)$.

g) If instead of the matrix $A = (\varphi_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}$, $r \leqslant s$, we consider the transposed matrix ${}^tA$ as a $\mathcal{P}$-homomorphism

$$\mathcal{P}^r \xrightarrow{{}^tA} \mathcal{P}^s$$

identifying $\mathcal{P}$ with $\mathcal{A}^0$ and $\mathcal{P}^s$ with $\mathcal{A}^1$ we get the map

$$(\mathcal{A}^0)^r \xrightarrow{\alpha} \mathcal{A}^1$$

given by

$$(u_1, \dots, u_r) \to \sum u_i \varphi_i .$$

Setting

$$D\sigma = (\sum y_i \varphi_i) \sigma$$

we then obtain the following

PROPOSITION ^{t}9. *Given a principal matrix* $A = (\varphi_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}$, $r \leqslant s$ *for the* $\mathcal{P}$*-homomorphism* α *defined by* ${}^tA$ *one obtains the following exact sequence*

$$0 \to (\mathcal{A}^0)^r \xrightarrow{\alpha} \mathcal{A}^1 \xrightarrow{\varphi_1 \wedge \dots \wedge \varphi_r} \mathcal{A}^{r+1} \xrightarrow{D} \mathcal{B}_1^{r+2} \xrightarrow{D} \mathcal{B}^{s-1}_{s-r-2} \xrightarrow{D} \mathcal{B}^s_{s-r-1} \to N \to 0$$

where N *is the cokernel of the last map* D. *The module* $N_{{}^tA}$ *is a torsion module* $\tau(N_{{}^tA}) = N_{{}^tA}$.

(Generalized coKoszul complex)

We can formulate similar remarks to those stated in point *f*) of this section.

NOTE: this complex is obtained from the generalized Koszul complex applying the functor

$$\mathrm{Hom}_{\mathcal{T}}(\cdot, \mathcal{T}), \quad \text{thus} \quad \mathrm{Ext}^i(N_A, \mathcal{T}) = 0 \qquad \text{for } 1 \leqslant i \leqslant s-r.$$

Similarly $\mathrm{Ext}^i(N_{{}^tA}, \mathcal{T}) = 0$ for $1 \leqslant i \leqslant s-r$.

PROOF OF THE PROPOSITION ${}^t9$. We first prove the exactness of the sequence.

(α) On $(\mathcal{A}^0)^r$. If $\sum u_i\varphi_i = 0$ then $u_i = 0$ as the u_i's are of degree 0.

(β) If $\beta\wedge\varphi_1\wedge\ldots\wedge\varphi_r = 0$ then $\beta = \sum u_i\varphi_i$ and we have exactness on $\mathcal{A}^1$.

(γ) If $\beta \in \mathcal{A}^{r+1}$, $D\beta = 0 \Leftrightarrow \beta\wedge\varphi_i = 0$, $\forall i$, thus there exists $\lambda \in \mathcal{A}^1$ such that $\beta = \lambda\wedge\varphi_1\wedge\ldots\wedge\varphi_r$ and we have exactness at $\mathcal{A}^{r+1}$.

(δ) Let us prove exactness at $\mathcal{B}_k^h$, $r+2 \leqslant h \leqslant s$, $1 \leqslant k \leqslant s-r-1$. Note that $D\cdot D = 0$.

Let $\beta \in \mathcal{B}_h^k$ and let us write

$$\beta = \beta_0 + \beta_1 y_r + \ldots + \beta_k y_r^k$$

where β_k will not contain any of the y's.

Also set $\alpha_0 = \sum_1^{r-1} y_i\varphi_i$. We get from $D\beta = 0$

$$(\beta_0 + \beta_1 y_r + \ldots + \beta_k y_r^k)(\alpha_0 + \varphi_r y_r) = 0$$

thus

$$\begin{aligned} \beta_0\alpha_0 &= 0 \\ \beta_0\varphi_r + \beta_1\alpha_0 &= 0 \\ \beta_1\varphi_r + \beta_2\alpha_0 &= 0 \\ \cdots\cdots & \\ \beta_{k-1}\varphi_r + \beta_k\alpha_0 &= 0 \\ \beta_k\varphi_r &= 0\,. \end{aligned}$$

Note that if $r = 1$ the theorem has already been proved; thus we can proceed by induction on r, we get therefore

$$\beta_0 = \gamma_0\alpha_0$$

hence $(\beta_1 - \gamma_0\varphi_r)\alpha_0 = 0$ thus

$$\beta_1 = \gamma_0\varphi_r + \gamma_1\alpha_0$$

hence $(\beta_2 - \gamma_1\varphi_r)\alpha_0 = 0$ thus

$$\beta_2 = \gamma_1\varphi_r + \gamma_2\alpha_0 .$$

Proceeding in this way we get thus

$$\begin{aligned}
\beta_0 &= \gamma_0\alpha_0 \\
\beta_1 &= \gamma_0\varphi_r + \gamma_1\alpha_0 \\
\beta_2 &= \gamma_1\varphi_r + \gamma_2\alpha_0 \\
&\cdots\cdots\cdots \\
\beta_{k-1} &= \gamma_{k-2}\varphi_r + \gamma_{k-1}\alpha_0 \\
\beta_k &= \gamma_{k-1}\varphi_r + \gamma_k\alpha_0 .
\end{aligned}$$

But β_k is indipendent from the y's, thus we must have $\gamma_k = 0$, i.e.

$$\beta_k = \gamma_{k-1}\varphi_r .$$

Therefore if

$$\gamma = \gamma_0 + \gamma_1 y_r + \ldots + \gamma_{k-1} y_r^{k-1}$$

we must have $\beta = D\gamma$. This achieves the proof.

It remains to prove that $\tau(N) = N$. We remark the following.

If $r + 1 = s$ then over the field of rational functions every s-form can be written as

$$\alpha = \beta \wedge \varphi_1 \wedge \ldots \wedge \varphi_r$$

thus chasing denominators

$$p\alpha = b \wedge \varphi_1 \wedge \ldots \wedge \varphi_r$$

with $p \in \mathcal{T}$, $p \neq 0$ and $b \in \mathcal{A}^1$.

If $s \geqslant r + 2$ then $\mathcal{B}^s_{s-r-1} \simeq$ space of homogeneous polynomials in $y_1, \ldots, y_r$ of degree $s - r - 1$ with coefficients in $\mathcal{T} \simeq \mathcal{A}^s$.

Let $a = a_{\alpha_1 \ldots \alpha_r} y_1^{\alpha_1} \ldots y_r^{\alpha_r} \in \mathcal{B}^s_{s-r-1}$ $\left(\sum \alpha_i = s - r - 1\right)$ be a monomial.

Again over the field of rational functions

$$a_{\alpha_1 \ldots \alpha_r} = \beta \varphi_1 \wedge \ldots \wedge \varphi_r$$

i.e.

$$p a_{\alpha_1 \ldots \alpha_r} = b \wedge \varphi_1 \wedge \ldots \wedge \varphi_r$$

with $p \in \mathcal{T} \simeq \mathcal{A}^s$, $p \neq 0$ and $b \in \mathcal{A}^{s-r}$. If, say $\alpha_1 > 1$, we can then write

$$p a_{\alpha_1 \ldots \alpha_r} y_1^{\alpha_1} \ldots y_r^{\alpha_r} = \pm \left(\sum y_i \varphi_i\right) \wedge \left(b \wedge \varphi_2 \wedge \ldots \wedge \varphi_r y_1^{\alpha_1 - 1} y_2^{\alpha_2} \ldots y_r^{\alpha_r}\right).$$

Thus $pa \in \operatorname{Im} D$ and therefore every element of N is a torsion element.

12. – Symbol sequences and elliptic complexes.

a) Let us consider a complex of differential operators with constant coefficients

$$(1) \qquad \mathcal{E}_0(\boldsymbol{R}^n) \xrightarrow{D^0} \mathcal{E}^1(\boldsymbol{R}^n) \xrightarrow{D^1} \mathcal{E}^2(\boldsymbol{R}^n) \xrightarrow{D^2} \ldots$$

where $\mathcal{E}^j(\boldsymbol{R}^n) \simeq \mathcal{E}(\boldsymbol{R}^n)^{p_j}$ for some integer p_j and where

$$D^j = \sum_{|\alpha| \leqslant k_j} a_\alpha^{(j)} D^\alpha$$

where a_α^j are matrices $p_{j+1} \times p_j$ with elements in $\boldsymbol{C}$. We assume that, for some α with $|\alpha| = k_j$, $a_\alpha^{(j)} \neq 0$, so that k_j is the true order of the operator D^j.

We define the total symbol of D^j

$$\Sigma_\xi(D^j) = \sum_{|\alpha| \leqslant k_j} a_\alpha^{(j)} (i)^{|\alpha|} \xi^\alpha$$

and the

principal symbol of D^j

$$\sigma_\xi(D^j) = (i)^{k_j} \sum_{|\alpha| = k_j} a^{(j)} \xi^\alpha .$$

For every $\xi \in \boldsymbol{R}^n - \{0\}$ these give linear maps

$$\Sigma_\xi(D^j)\colon \boldsymbol{C}^{p_j} \to \boldsymbol{C}^{p_{j-1}}$$

$$\sigma_\xi(D^j)\colon \boldsymbol{C}^{p_j} \to \boldsymbol{C}^{p_{j-1}}$$

where we can consider $\boldsymbol{C}^{p_j}$ as the vector space $E^j_{x_0}$ obtained for every $x^0 \in \boldsymbol{R}^n$ by tensoring $\mathcal{E}^j(\boldsymbol{R}^n)$ with $\mathcal{E}(\boldsymbol{R}^n)/\mathcal{M}_{x^0}$ where $\mathcal{M}_{x^0} = \mathcal{E}(\boldsymbol{R}^n)(x_1 - x_1^0, \ldots, x_n - x_n^0)$:

$$\boldsymbol{C}^{p_j} \simeq E^j_{x^0} = \mathcal{E}^j(\boldsymbol{R}^n) \underset{\mathcal{E}(\boldsymbol{R}^n)}{\otimes} \mathcal{E}(\boldsymbol{R}^n)/\mathcal{M}_{x^0} .$$

As

$$D^{j+1} \circ D^j = 0$$

we must have for every ξ

$$\Sigma_\xi(D^{j+1})\circ\Sigma_\xi(D^j)=0$$

and therefore (replacing ξ by $\lambda\xi$ and considering the coefficient of $\lambda^{k_j+k_{j+1}}$)

$$\sigma_\xi(D^{j+1})\circ\sigma_\xi(D^j)=0\,.$$

Therefore we obtain two complexes for every $\xi\in\boldsymbol{R}^n-\{0\}$.

$$0\to E^0_{x_0}\xrightarrow{\Sigma_\xi(D^0)}E^1_{x_0}\xrightarrow{\Sigma_\xi(D^1)}E^2_{x_0}\xrightarrow{\Sigma_\xi(D^2)}\dots \tag{2}$$

$$0\to E^0_{x_0}\xrightarrow{\sigma_\xi(D^0)}E^1_{x_0}\xrightarrow{\sigma_\xi(D^1)}E^2_{x_0}\xrightarrow{\sigma_\xi(D^2)}\dots \tag{2_σ}$$

the total and the principal symbol sequences.

b) If the complex (1) is obtained from a Hilbert resolution

$$0\to\mathcal{T}^{p_d}\xrightarrow{{}^tA_{d-1}(x)}\cdots\to\mathcal{T}^{p_2}\xrightarrow{{}^tA_1(x)}\mathcal{T}^{p_1}\xrightarrow{{}^tA_0(x)}\mathcal{T}^{p_0}\to N\to 0 \tag{3}$$

with

$$D_j=A_j(x)=\sum_{|\alpha|\leqslant k_j}a_\alpha(i)^{|\alpha|}x^\alpha$$

then the total symbol sequence for $\xi=\xi_0$ is obtained from the complex (3) with the following operations

α) Consider the complex obtained from (3) by applying the functor $\mathrm{Hom}_{\mathcal{T}}(*,\mathcal{T})$:

$$0\to N^*\to\mathcal{T}^{p_0}\xrightarrow{A_0(x)}\mathcal{T}^{p_1}\xrightarrow{A_1(x)}\cdots\xrightarrow{A_{d-1}(x)}\mathcal{T}^{p_d}\to 0\,. \tag{4}$$

β) Setting $\mathcal{M}_{\xi_0}=\mathcal{T}(x_1-\xi_1^0,\dots,x_n-\xi_n^0)$, tensoring (4) with $\boldsymbol{C}=\mathcal{T}/\mathcal{M}_{\xi_0}$

c) DEFINITION. *We say that the complex* (1) *at the place* $\mathcal{E}^j$ *is determined in the direction* $\xi_0\in\boldsymbol{R}^n-\{0\}$ *if, at* ξ_0 *the total symbol sequence is exact at* E^j_x, *elliptic in the direction* $\xi_0\in\boldsymbol{R}^n-\{0\}$ *if at* ξ_0 *the principal symbol sequence is exact at* $E^j_{x_0}$.

A complex which is determined (elliptic) at every place and for every direction is called determined (elliptic).

On each space $\boldsymbol{C}^{p_j}=E^j_{x_0}$ we introduce a hermitian product

$$\langle u,w\rangle={}^t\bar{u}w\,.$$

For every $\xi_0 \in \boldsymbol{R}^n - \{0\}$ we can then consider the hermitian forms

$$\boldsymbol{\Delta}_j(\xi_0)(v) = \langle {}^t\Sigma_{\xi_0}(D^{j-1})v, {}^t\Sigma_{\xi_0}(D^{j-1})v\rangle + \langle \Sigma_{\xi_0}(D^j)v, \Sigma_{\xi_0}(D^j)v\rangle$$
$$\Delta_j(\xi_0)(v) = \langle {}^t\sigma_{\xi_0}(D^{j-1})v, {}^t\sigma_{\xi_0}(D^{j-1})v\rangle + \langle \sigma_{\xi_0}(D^j)v, \sigma_{\xi_0}(D^j)v\rangle .$$

In matrix notations

$$\boldsymbol{\Delta}_j(\xi_0) = A_{j-1}(\xi_0)\, {}^t\overline{A_{j-1}(\xi_0)} + {}^t\overline{A_j(\xi_0)}A_j(\xi_0)$$

and symilarly for $\Delta_j(\xi_0)$ replacing A^{j-1} and A_j by the corresponding principal parts.

LEMMA 1. *The complex* (1) *is determined* (*elliptic*) *on* E^j *in the direction* ξ_0 *if and only if* $\boldsymbol{\Delta}(\xi_0)$ $(\Delta(\xi_0))$ *is positive definite.*

PROOF. If $\boldsymbol{\Delta}(\xi_0)(v) = 0$ then

$${}^t\Sigma_{\xi_0}(D^{j+1})v = 0 \qquad \text{and} \qquad \Sigma_{\xi_0}(D^j)v = 0 .$$

Now note that for any linear map $\alpha\colon \boldsymbol{C}^r \to \boldsymbol{C}^s$ we have

$$\langle \alpha u, w\rangle_{\boldsymbol{C}^s} = \langle u, {}^t\alpha w\rangle_{\boldsymbol{C}^r}, \qquad \forall u \in \boldsymbol{C}^r,\ \forall w \in \boldsymbol{C}^s$$

therefore

$$\operatorname{Ker} {}^t\alpha = (\operatorname{Im} \alpha)^\perp .$$

Since the symbol sequence is a complex we have

$$\operatorname{Im} \Sigma_{\xi_0}(D^{j-1}) \subset \operatorname{Ker} \Sigma_{\xi_0}(D^j) .$$

If $v \neq 0$ and $\boldsymbol{\Delta}(\xi_0)(v) = 0$ then there exists

$$(*) \qquad 0 \neq v \in \operatorname{Ker} \Sigma_{\xi_0}(D^j) \cap \big(\operatorname{Im} \Sigma_{\xi_0}(D^{j-1})\big)^\perp$$

and thus the symbol sequence cannot be exact. Conversely if at ξ_0 and E^j the symbol sequence is not exact there must exist $v \neq 0$ verifying (*) and thus such that $\boldsymbol{\Delta}(\xi_0)(v) = 0$. For the ellipticity case the proof is the same.

LEMMA 2. *A complex* (1) *coming from a Hilbert resolution* (3) *is determined at the place* $j > 0$ *and in the direction* $\xi_0 \in \boldsymbol{R}^n - \{0\}$ *if and only if* $\operatorname{Tor}^j(N, \mathcal{T}/\mathcal{M}_{\xi_0}) = 0$.

PROOF. For $j \geqslant 1$ the exacteness of

$$C^{p_{j-1}} \xrightarrow{\Sigma_{\xi_0}(D^{j-1})} C^{p_j} \xrightarrow{\Sigma_{\xi_0}(D^j)} C^{p_{j+1}}$$

i.e. of

$$\mathcal{T}^{p_{j-1}} \underset{\mathcal{T}}{\otimes} \mathcal{T}/\mathcal{M}_{\xi_0} \xrightarrow{A_{j-1}(\xi_0)} \mathcal{T}^{p_j} \underset{\mathcal{T}}{\otimes} \mathcal{T}/\mathcal{M}_{\xi_0} \xrightarrow{{}^tA_j(\xi_0)} \mathcal{T}^{p_{j+1}} \underset{\mathcal{T}}{\otimes} \mathcal{T}/\mathcal{M}_{\xi_0}$$

is equivalent to the exactness of the transposed sequence (as we are dealing with finite dimensional vector spaces and linear maps)

$$\mathcal{T}^{p_{j+1}} \otimes \mathcal{T}/\mathcal{M}_{\xi_0} \xrightarrow{{}^tA_j(\xi_0)} \mathcal{T}^{p_j}\mathcal{T}/\otimes\mathcal{M}_{\xi_0} \xrightarrow{{}^tA_{j-1}(\xi_0)} \mathcal{T}^{p_{j-1}} \otimes \mathcal{T}/\mathcal{M}_{\xi_0} .$$

The cohomology of this complex is, by definition, equal to

$$\operatorname{Tor}^j_{\mathcal{T}}(N, \mathcal{T}/\mathcal{M}_{\xi_0}) .$$

d) Let us assume that the given complex is finite

$$\text{(1)} \qquad \mathcal{E}^0(\boldsymbol{R}^n) \xrightarrow{D_0} \mathcal{E}^1(\boldsymbol{R}^n) \xrightarrow{D^1} \cdots \xrightarrow{D^{d-1}} \mathcal{E}^d(\boldsymbol{R}^n) \to 0 .$$

One can then consider the adjoint operators (formal adjoint)

$$*D^j\colon \mathcal{E}^{j+1}(\boldsymbol{R}^n) \to \mathcal{E}^j(\boldsymbol{R}^n)$$

with total symbol

$$\Sigma_\xi(*D^j) = {}^t\overline{\Sigma_\xi(D^j)}$$

and the adjoint complex:

$$\text{(1)*} \qquad \mathcal{E}^d(\boldsymbol{R}^n) \xrightarrow{*D^{d-1}} \mathcal{E}^{d-1}(\boldsymbol{R}^n) \xrightarrow{*D^{d-2}} \cdots \xrightarrow{*D^0} \mathcal{E}^0(\boldsymbol{R}^n) \to 0 .$$

If (1) is a determined (elliptic) complex so is (1)* and conversely, i.e.

If determinateness (*ellipticity*) *if satisfied by a finite complex then it is also satisfied by the adjoint complex.*

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