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ENRICO GIUSTI

Boundary value problems for non-parametric surfaces of prescribed mean curvature

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Boundary Value Problems for Non-Parametric Surfaces of Prescribed Mean Curvature.

ENRICO GIUSTI (*)

dedicated to Hans Lewy

0. – Introduction.

The equation of surfaces of prescribed mean curvature:

$$\operatorname{div} Tu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{\frac{\partial u}{\partial x_i}\Big/\sqrt{1+|\operatorname{grad} u|^2}\right\} = H(x, u) \tag{0.1}$$

has received considerable attention; in particular in connection with the Dirichlet problem, i.e. the problem of the existence of a solution to the equation (0.1) in an open set Ω, taking prescribed values at the boundary.

For the two-dimensional case the theory was initiated by Bernstein at the beginning of the century, and received contributions from various authors. On the contrary, the general n-dimensional problem has been successfully studied only recently; we shall mention the work of Jenkins and Serrin [17] in the case of minimal surfaces ($H = 0$), and of Serrin [25] for general H.

The method of Serrin is based on *a-priori* bounds for solutions of the Dirichlet problem for equation (0.1), in view of an application of the Leray-Schauder fixed point theorem. This allows to prove the existence of a C^2 solution to the problem, provided some conditions are satisfied, involving the function $H(x, u)$ and the mean curvature $K(x)$ of $\partial\Omega$.

In the meantime, a different approach to the Dirichlet problem for equation (0.1) was developed, starting from the observation that (0.1) is the Euler

(*) Istituto di Matematica dell'Università di Trento.
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equation for the functional

$$\int_{\Omega}\sqrt{1+|Du|^2}+\int_{\Omega}\lambda(x,u)\,dx$$

where $\lambda(x,t)=\int_0^t H(x,s)\,ds$. Euristic considerations (see [15]) suggest including the boundary datum φ in the functional, and hence looking for a minimum of

$$\mathfrak{J}_1(u)=\int_{\Omega}\sqrt{1+|Du|^2}+\int_{\Omega}\lambda(x,u)\,dx+\int_{\partial\Omega}|u-\varphi|\,dH_{n-1}$$

in the class $BV(\Omega)$ of function with bounded variation in Ω.

This variational approach to the Dirichlet problem (see [15], [11] and [23]) permits separate discussion of the assumptions on the mean curvature function $H(x,u)$ and on the boundary mean curvature $K(x)$, so that one can obtain sharp (and in many cases necessary and sufficient) conditions for the existence of a minimum for $\mathfrak{J}_1$. These conditions do not involve the mean curvature of the boundary.

A careful use of the *a-priori* estimate for the gradient (see [18], [30] and [3]) shows that the solution $u(x)$ is smooth in Ω, and is a solution of equation (0.1). If in addition φ is continuous and

$$|H(x,\varphi(x))|\leqslant(n-1)K(x) \tag{0.2}$$

at every point $x\in\partial\Omega$, then $u(x)=\varphi(x)$ at $\partial\Omega$ (see [23]) and hence is a "classical" solution to the Dirichlet problem.

The two methods outlined above have been successfully applied to the problem of capillary free surfaces. In this case one looks for a solution to (0.1), with $H(x,u)=2u$, subject to the boundary condition

$$-Tu\cdot\nu=-\sum_{i=1}^{n}\nu\,\frac{\partial u}{\partial x_i}\Big/\sqrt{1+|\operatorname{grad}u|^2}=\varkappa \qquad \text{in } \partial\Omega \tag{0.3}$$

where ν is the exterior normal, and $\varkappa$ is the cosinus of the (prescribed) angle between the surface $y=u(x)$ and the boundary of the cylinder $\Omega\times\boldsymbol{R}$.

For this problem, variational results have been obtained in [4], minimizing the functional

$$\mathfrak{J}_2(u)=\int_{\Omega}\sqrt{1+|Du|^2}+\int_{\Omega}u^2\,dx+\int_{\partial\Omega}\varkappa u\,dH_{n-1}$$

and at the same time the classical approach has been shown to work in [31] (see also [27] and [26]).

The situation is quite different when a mixed boundary value problem is considered:

$$\begin{cases} \operatorname{div} Tu = H(x, u) & \text{in } \Omega \\ u = \varphi & \text{in } \partial_1 \Omega \\ -Tu \cdot \nu = \varkappa & \text{in } \partial_2 \Omega \end{cases} \tag{0.4}$$

with $\partial_1\Omega \cup \partial_2\Omega = \partial\Omega$. In this case, when singularities at points of $\overline{\partial_1\Omega} \cap \cap \overline{\partial_2\Omega}$ can possibly occur, the classical method seems to be hardly applicable as it is; on the contrary one can show the existence of a minimum for the functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega \lambda(x, u)\, dx + \int_{\partial_1\Omega} |u - \varphi|\, dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\, dH_{n-1} .$$

The aim of this paper is to prove such existence results under sharp conditions for the functions H and $\varkappa$. As $\mathcal{F}$ reduces to $\mathfrak{J}_1$ or to $\mathfrak{J}_2$ when $\partial_2\Omega$ or $\partial_1\Omega$ is empty, we shall find the existence of a solution with pure Dirichlet or capillarity boundary conditions. We want to observe that our results are significantly new also in these situations.

The paper is divided in four sections. The first is devoted to the assumptions on H and $\varkappa$, and to the discussion of a variety of special cases. In chapter 2 we prove the existence of a minimum for the functional $\mathcal{F}$. After a brief discussion of the regularity of the solution in Ω and at $\partial_1\Omega$, we refine our method in order to treat some borderline situations, including the capillary free surfaces with $|\varkappa| = 1$ (compare [7]; see also [8] for an application of the results of ch. 4).

In conclusion, we shall get a quite general existence result for the problem (0.4). The solutions to this problem are regular in Ω, and at interior points of $\partial_1\Omega$, provided (0.2) holds. The regularity at $\partial_2\Omega$ remains still an open problem; a special case ($\varkappa = 0$) is discussed in [16].

I wish to thank R. Finn for his stimulating remarks.

1. – The variational problem.

1.*A*. Throughout this paper we shall denote by Ω a bounded connected open set in $\mathbf{R}^n$, $n \geqslant 2$, with Lipschitz-continuous boundary $\partial\Omega$. We will write $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$, where $\partial_1\Omega$ is the intersection of $\partial\Omega$ with a bounded open set A_1, such that the set

$$\Omega_1 = \Omega \cup A_1$$

is connected. We suppose that $H_{n-1}(\overline{\partial_1\Omega} \cap \partial_2\Omega) = 0$, and that $\partial_2\Omega$ coincides with the closure of its interior.

We shall discuss the existence of a minimum for the functional

$$(1.1) \qquad \mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega \lambda(x, u)\, dx + \int_{\partial_1\Omega} |u - \varphi|\, dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\, dH_{n-1}$$

in the class $BV(\Omega)$, of functions of bounded total variation in Ω.

It can be useful to recall that the symbol

$$\int_\Omega \sqrt{1 + |Du|^2}$$

means the total variation in Ω of the vector-valued measure whose components are the Lebesgue measure in $\boldsymbol{R}^n$ and the derivatives $D_i u$ of u:

$$\int_\Omega \sqrt{1 + |Du|^2} = \sup\left\{\int_\Omega \left(g_0 + \sum_{i=1}^n u D_i g_i\right) dx;\ g_i \in C_0^1(\Omega);\ \sum_{i=0}^n g_i^2 \leqslant 1\right\}.$$

The integrals on $\partial\Omega$ have sense as every function of bounded variation has a trace on $\partial\Omega$, which we denote also by u, in $L_1(\partial\Omega)$ [21].

1.*B*. Let

$$\lambda(x, t) = \int_0^t H(x, s)\, ds,$$

and let $u(x)$ be a function of class $C^2(\bar{\Omega})$, a minimum for the functional $\mathcal{F}(u)$. It is clear that $u(x)$ satisfies the equation

$$(1.2) \qquad \sum_{i=1}^n D_i\{D_i u / \sqrt{1 + |Du|^2}\} = H(x, u(x))$$

and the boundary conditions:

$$-\sum_{i=1}^n \nu_i D_i u / \sqrt{1 + |Du|^2} = \varkappa(x) \qquad \text{in } \partial_2\Omega .$$

Let B be a *Caccioppoli set*; i.e. a Borel set whose characteristic function φ_B has distributional derivatives which are measures of bounded total variation. We can integrate (1.2) over B to get:

$$\int_B H(x, u(x))\, dx = \int_{\Omega_1} (D_i u/\sqrt{1+|Du|^2}) D^i \varphi_B - \int_{\partial_2 \Omega} \varphi_B \varkappa\, dH_{n-1} .$$

Let $t_0 = \sup_\Omega |u|$ and suppose $H(x, t)$ is a non decreasing function of t. We have:

$$\int_B H(x, t_0)\, dx + \int_{\partial_2\Omega} \varkappa \varphi_B\, dH_{n-1} \geqslant -(1-\varepsilon_0) \int_{\Omega_1} |D\varphi_B|$$

and

$$\int_B H(x, -t_0)\, dx + \int_{\partial_2\Omega} \varkappa \varphi_B\, dH_{n-1} \leqslant (1-\varepsilon_0) \int_{\Omega_1} |D\varphi_B|$$

for every Caccioppoli set $B \subset \Omega$, where

$$1 - \varepsilon_0 = \sup_\Omega \{|Du|/\sqrt{1+|Du|^2}\}$$

1.*C*. We will prove the existence of a minimum for the functional $\mathcal{F}(u)$ under the following assumptions on the functions H and $\varkappa$:

(1.3) *$\varkappa(x)$ is a bounded measurable function in $\partial_2\Omega$. $H(x, t)$ is a function defined in $\Omega \times \mathbf{R}$, which is non-decreasing in t for almost every $x \in \Omega$, and belongs to $L_n(\Omega)$ for every $t \in \mathbf{R}$.*

(1.4) *There exist two positive constants ε_0 and t_0 such that for every Caccioppoli set $B \subset \Omega$ we have:*

$$\int_B H(x, t_0)\, dx + \int_{\partial_2\Omega} \varkappa \varphi_B\, dH_{n-1} \geqslant -(1-\varepsilon_0) \int_{\Omega_1} |D\varphi_B| \tag{1.4'}$$

$$\int_B H(x, -t_0)\, dx + \int_{\partial_2\Omega} \varkappa \varphi_B\, dH_{n-1} \leqslant (1-\varepsilon_0) \int_{\Omega_1} |D\varphi_B| \tag{1.4''}$$

The meaning of assumption (1.3) is clear as it implies that the functional $\mathcal{F}$ is convex. On the other hand condition (1.4), which we have shown to be necessary for the existence of a smooth minimum, can appear somewhat involved and artificial, so that it is advisable to illustrate in some detail its meaning and generality. For that we shall postpone the proof of the existence theorem to the next chapter and we will devote this section to a complete discussion of some particular hypotheses leading to (1.4).

1.*D*. Let us start from the Dirichlet problem. We have the following

PROPOSITION 1.1. *Let*

$$h(x) = \lim_{t\to\infty} H^-(x, t)$$

$$k(x) = \lim_{t\to-\infty} H^+(x, t)$$

where $H^+ = \max(H, 0)$ *and* $H^- = \min(H, 0)$.
Suppose that

$$\|h\|_{L_n(\Omega)} < n\omega_n^{1/n} \tag{1.5}$$

$$\|k\|_{L_n(\Omega)} < n\omega_n^{1/n} \tag{1.6}$$

and let $\partial_2\Omega = \emptyset$.
Then (1.4) *is satisfied.*

PROOF. Since $|H^-(x,t)|$ monotonically decreases to $|h(x)|$ we have:

$$\|h\|_{L_n(\Omega)} = \lim_{t\to+\infty} \|H^-(x,t)\|_{L_n(\Omega)}$$

and similarly

$$\|k\|_{L_n(\Omega)} = \lim_{t\to-\infty} \|H^+(x,t)\|_{L_n(\Omega)}$$

whence there exist t_0 and $\varepsilon_0 > 0$ such that

$$\|H^-(x, t_0)\|_{L_n(\Omega)} \leqslant (1-\varepsilon_0) n\omega_n^{1/n}$$

$$\|H^+(x, -t_0)\|_{L_n(\Omega)} \leqslant (1-\varepsilon_0) n\omega_n^{1/n}.$$

Let $B \subset \Omega$ be a Caccioppoli set; we have

$$\int_B H(x, t_0)\, dx \geqslant \int_B H^-(x, t_0)\, dx \geqslant -(1-\varepsilon_0) n\omega_n^{1/n} (\text{meas } B)^{1-1/n}$$

and (1.4′) follows at once from the isoperimetric inequality:

$$(\text{meas } B)^{1-1/n} \leqslant n^{-1}\omega_n^{-1/n} \int_{\Omega_1} |D\varphi_B|.$$

A similar argument lead to (1.4″). Q.E.D.

We remark that if H does not depend on t, conditions (1.5) and (1.6) reduce to the assumptions of [1] (see also [11]):

$$\int_\Omega |H^\pm(x)|^n\,dx \leqslant n^n \omega_n .$$

Another interesting situation is

$$H(x,t) = a(x)t + b(x)$$

with a and b in $L_n(\Omega)$, and $a(x) \geqslant 0$. It is clear from the proposition that no condition has to be imposed on $b(x)$ if $a(x) > 0$ almost everywhere; if we denote by A the zero set of $a(x)$, condition (1.4) will be satisfied if

$$\|b^\pm\|_{L_n(A)} < n\omega_n^{1/n} .$$

1.*E*. We shall discuss now the general case. For that we remember the following

LEMMA 1.1 (Sobolev-Poincaré inequality). *Let Ω be a connected bounded open set with Lipschitz-continuous boundary and let $w \in BV(\Omega)$. Then*

$$\left\{\int_\Omega |w - w_\Omega|^{n/n-1}\,dx\right\}^{1-1/n} \leqslant c_1(\Omega)\int_\Omega |Dw| \tag{1.7}$$

where w_Ω is the mean value of w in Ω and c_1 is a constant independent of w.

As a corollary we get easily, taking $w = \varphi_A$, the inequality

$$(\text{meas } A)^{1-1/n} \leqslant 2c_1\int_\Omega |D\varphi_A| \tag{1.8}$$

for every Caccioppoli set A with meas $A \leqslant$ meas $\Omega/2$.

For $x \in \partial\Omega$ let $B(x,r)$ be the ball of radius r centered at x, and let

$$\Omega(x,r) = \Omega \cap B(x,r) .$$

We introduce the function

$$q(x) = \limsup_{r\to 0+}\left\{\int_{\partial_2\Omega} \varphi_A\,dH_{n-1}\Big/\int_{\Omega_1} |D\varphi_A|;\ A \subset \Omega(x,r),\ \text{meas}\,A > 0\right\} . \tag{1.9}$$

Let us start with a necessary condition.

PROPOSITION 1.2. *Let assumption* (1.4) *be satisfied and let* $\varkappa$ *be continuous on* $\partial_2\Omega$. *Then for every* $x\in\partial_2\Omega$ *we have*

$$q(x)|\varkappa(x)|\leqslant 1-\varepsilon_0\,. \tag{1.10}$$

PROOF. We can suppose $\varkappa(x)\neq 0$. Let r be a positive number such that

$$\text{meas}\ \Omega(x,r)\leqslant \text{meas}\ \Omega/2\,,$$

and let $A\subset\Omega(x,r)$.

We have from (1.8):

$$\int_A |H(x,\ \pm t_0)|dx\leqslant 2c_1 m_r\int_{\Omega_1}|D\varphi_A|$$

where

$$m_r=\max\left\{\|H(x,t_0)\|_{L_n(\Omega(x,r))},\ \|H(x,-t_0)\|_{L_n(\Omega(x,r))}\right\}.$$

We observe that m_r goes to zero with r. Recalling condition (1.4) we get

$$\left|\int_{\partial_2\Omega}\varkappa\varphi_A\,dH_{n-1}\right|\leqslant(1-\varepsilon_0+2c_1m_r)\int_{\Omega_1}|D\varphi_A|\,.$$

On the other hand

$$(|\varkappa(x)|-n_r)\int_{\partial_2\Omega}\varphi_A\,dH_{n-1}\leqslant\left|\int_{\partial_2\Omega}\varkappa\varphi_A\,dH_{n-1}\right|$$

with

$$\lim_{r\to 0+}n_r=0$$

so that in conclusion we have, for every Caccioppoli set $A\subset\Omega(x,r)$:

$$\int_{\partial_2\Omega}\varphi_A\,dH_{n-1}\leqslant\frac{1-\varepsilon_0+2c_1m_r}{|\varkappa(x)|-n_r}\int_{\Omega_1}|D\varphi_A|$$

and (1.10) follows at once. Q.E.D.

1.F. In order to obtain sufficient conditions we introduce the function

$$j(x)=\text{ess}\limsup_{y\to x}|\varkappa(y)| \tag{1.11}$$

which coincides with $|\varkappa(x)|$ whenever $\varkappa$ is continuous.

LEMMA 1.2. *Suppose that there exists a positive constant σ such that for for every $x \in \partial_2\Omega$ we have*

$$q(x)j(x) \leqslant 1 - 2\sigma .$$

Then there exists a constant c_2, depending on $\varkappa$, σ and Ω, such that for every $w \in BV(\Omega)$:

$$\Big|\int_{\partial_2\Omega} \varkappa w \, dH_{n-1}\Big| \leqslant (1-\sigma)\int_{\Omega_1} |Dw| + c_2 \int_{\Omega_1} |w| \, dx . \tag{1.12}$$

PROOF. We can suppose $w \geqslant 0$. Let $x_0 \in \partial_2\Omega$, and for $s > 0$ let r_s be such that

$$\int_{\partial_2\Omega} \varphi_B \, dH_{n-1} \leqslant (q(x_0) + s) \int_{\Omega_1} |D\varphi_B|$$

for every Caccioppoli set $B \subset \Omega(x_0, r_s)$, and

$$|\varkappa(y)| \leqslant j(x_0) + s$$

for almost all $y \in B(x_0, r_s) \cap \partial_2\Omega$.

If $\operatorname{spt} w \subset B(x_0, r_s)$ we have:

$$\Big|\int_{\partial_2\Omega} \varkappa w \, dH_{n-1}\Big| \leqslant \int_0^\infty dt \Big|\int_{\partial_2\Omega} \varkappa \varphi_{W_t} \, dH_{n-1}\Big| \leqslant (j(x_0) + s)(q(x_0) + s) \int_0^\infty dt \int_{\Omega_1} |D\varphi_{W_t}|$$

where

$$W_t = \{x \in \Omega_1 : w(x) > t\} .$$

In conclusion, choosing s small enough, we get from the coarea formula (cfr. [6], 4.3.2(2)):

$$\Big|\int_{\partial_2\Omega} \varkappa w \, dH_{n-1}\Big| \leqslant (1-\sigma) \int_{\Omega_1} |Dw| \tag{1.13}$$

and (1.12) is proved if $\operatorname{spt}(w) \subset B(x_0, r_s)$.

For general w, let $x \in \partial_2\Omega$ and let r be such that (1.13) holds. We can choose a finite covering of $\partial_2\Omega$ with balls $B(x_i, r_i)$ $(i = 1, 2, \ldots, N)$ and non-negative smooth functions f_i, with $\operatorname{spt} f_i \subset B(x_i, r_i)$, $\sum_{i=1}^N f_i \leqslant 1$ and $\sum_{i=1}^N f_i = 1$ on $\partial_2\Omega$. Writing (1.13) for each of the functions wf_i, and adding from 1 to N we obtain at once (1.12). Q.E.D.

PROPOSITION 1.3. *Let $\varkappa$ and H satisfy assumption* (1.3) *and let*

$$q(x)j(x)\leqslant 1-\sigma \tag{1.14}$$

for every $x\in\partial_2\Omega$.

Let $H(x,t)$ satisfy the assumptions of Proposition 1.1, *i.e.*

$$\begin{cases} \|h\|_{L_n(\Omega)}< n\omega_n^{1/n} \\ \|k\|_{L_n(\Omega)}< n\omega_n^{1/n} \end{cases} \tag{1.15}$$

and suppose that for almost every x in a neighborhood of $\partial_2\Omega$ we have

$$\lim_{|t|\to\infty}\operatorname{sign}(t)H(x,t)=+\infty\,. \tag{1.16}$$

Then (1.4) *is satisfied.*

PROOF. Let $A\subset\Omega_1$ be a closed set with $\partial_2\Omega\cap A=\emptyset$ and such that (1.16) is satisfied in $S=\Omega_1-A$. We can suppose that $\partial S\cap\Omega_1$ is smooth and since $\partial_2\Omega$ is compact we can assume that S has finitely many connected components.

As in the proof of Proposition 1.1, there exist positive numbers t_1 and ε_1 such that for $t>t_1$ we have

$$\|H^-(x,t)\|_{L_n(\Omega)}\leqslant(1-\varepsilon_1)n\omega_n^{1/n}$$

and

$$\|H^+(x,-t)\|_{L_n(\Omega)}\leqslant(1-\varepsilon_1)n\omega_n^{1/n}\,.$$

Let B be a Caccioppoli set in Ω; we get for $t_0>t_1$:

$$\int_B H(x,t_0)\,dx\geqslant\int_{B\cap S}H(x,t_0)\,dx-n\omega_n^{1/n}(1-\varepsilon_1)(\text{meas }(B\cap A))^{1-1/n}\geqslant$$
$$\geqslant\int_{B\cap S}H(x,t_0)\,dx-(1-\varepsilon_1)\int_A|D\varphi_B|-(1-\varepsilon_1)\int_{\partial A}\varphi_B\,dH_{n-1}$$

and hence

$$\int_{\partial_2\Omega}\varkappa\varphi_B\,dH_{n-1}+\int_B H(x,t_0)\,dx\geqslant\int_{\partial S}\hat{\varkappa}\varphi_B\,dH_{n-1}-(1-\varepsilon_1)\int_A|D\varphi_B|+\int_S H(x,t_0)\varphi_B\,dx\,,$$

where

$$\varkappa(x) = \begin{cases} \varkappa(x) & x \in \partial_2 \Omega \\ -(1-\varepsilon_1) & x \in \partial S \cap \Omega \\ 0 & \text{elsewhere in } \partial S . \end{cases}$$

Since $\partial S \cap \Omega_1$ is smooth we have $q(x) = 1$ there (see 1.G below) and therefore if $4\varepsilon_0 = \min(\sigma, \varepsilon_1)$:

$$q(x)\,\hat{\jmath}(x) \leqslant 1 - 4\varepsilon_0 \quad \text{in } \partial S .$$

Applying Lemma 1.2 we get:

$$\left| \int_{\partial S} \varkappa \varphi_B \, dH_{n-1} \right| \leqslant (1 - 2\varepsilon_0) \int_S |D\varphi_B| + c_2 \int_S \varphi_B \, dx$$

where c_2 depends on S and $\varkappa$ but not on the set B. In conclusion

$$\int_{\partial_2 \Omega} \varkappa \varphi_B \, dH_{n-1} + \int_B H(x, t_0)\, dx \geqslant -(1 - 2\varepsilon_0) \int_{\Omega_1} |D\varphi_B| + \int_S H(x, t_0) \varphi_B \, dx - c_2 \int_S \varphi_B \, dx$$

and in order to prove (1.4) we have only to show that it is possible to choose $t_0 > t_1$ in such a way that

$$\int_S (H(x, t_0) - c_2) \varphi_B \, dx + \varepsilon_0 \int_S |D\varphi_B| \geqslant 0 \tag{1.17}$$

Let Σ be a connected part of S, and for $t > t_1$ let

$$\Sigma_t = \{x \in \Sigma \colon H(x, t) < 2c_2\}$$

$$f(x, t) = \min\{H(x, t) - c_2, 0\} .$$

We have

$$\lim_{t \to \infty} \operatorname{meas} \Sigma_t = 0$$

$$\lim_{t \to \infty} \|f(x, t)\|_{L_n(\Sigma)} = 0$$

and hence we can find a number t_Σ such that for $t > t_\Sigma$:

$$\operatorname{meas} \Sigma_t < \operatorname{meas} \Sigma / 4 \tag{1.18}$$

$$\|f(x, t)\|_{L_n(\Sigma)} < \min \left\{ c_2 \left(\frac{\operatorname{meas} \Sigma}{4} \right)^{1/n} ; \; \frac{\varepsilon_0}{2c_1(\Sigma)} \right\} . \tag{1.19}$$

We discuss separately two cases:

(I) $$\text{meas}\, B \cap \Sigma \geqslant \text{meas}\, \Sigma/2 .$$

We have

$$\int_{\Sigma} \{H(x,t) - c_2\} \varphi_B \, dx \geqslant c_2 \,\text{meas}\, \Sigma/4 + \int_{\Sigma_t} f(x,t)\, dx \geqslant$$
$$\geqslant c_2 \,\text{meas}\, \Sigma/4 - \|f(x,t)\|_{L_n(\Sigma)} \left(\frac{\text{meas}\, \Sigma}{4}\right)^{1-1/n} \geqslant 0$$

(II) $$\text{meas}\, B \cap \Sigma < \text{meas}\, \Sigma/2 .$$

In this case we use (1.8) and we get

$$(\text{meas}\ B \cap \Sigma)^{1-1/n} \leqslant 2c_1(\Sigma) \int_{\Sigma} |D\varphi_B|$$

and therefore

$$\int_{\Sigma} \{H(x,t) - c_2\} \varphi_B \, dx \geqslant \int_{\Sigma} f(x,t)\, dx \geqslant - \|f(x,t)\|_{L_n(\Sigma)} 2c_1 \int_{\Sigma} |D\varphi_B|$$

whence in both cases we have for $t > t_\Sigma$.

(1.20) $$\int_{\Sigma} \{H(x,t) - c_2\} \varphi_B \, dx \geqslant - \varepsilon_0 \int_{\Sigma} |D\varphi_B| .$$

Since there are only finitely many connected open sets $\Sigma \subset S$, we get easily (1.17) with $t_0 = \max t_\Sigma$, and hence (1.4′).

In a similar way one can prove (1.4″), thus getting the full result. Q.E.D.

1.*G*. We conclude this chapter with a computation of the function $q(x)$ in various situations.

It is easily seen that we have always $q(x) \geqslant 1$.

PROPOSITION 1.4. *Let $\partial\Omega$ be of class C^1 in a neighborhood of $x_0 \in \partial_2\Omega$. Then $q(x_0) = 1$.*

PROOF. We can suppose that $x_0 = 0$ and that $\partial\Omega$ can be represented as the graph of a function $f(x')$, $x' = (x_1, x_2, \ldots, x_{n-1})$, such that $f(0) = 0$, $Df(0) = 0$ and that

$$x_n > f(x') \quad \text{in } \Omega(0, r) .$$

Let $A \subset \Omega(0, r)$ and let $\partial_2 A = \{x \in \partial_2 \Omega: \varphi_A(x) = 1\}$ (we remember that $\varphi_A(x)$ is the trace of φ_A on $\partial\Omega$). Let π_A be the projection of $\partial_2 A$ on the hyperplane $x_n = 0$. We have:

$$\int_{\partial_2\Omega} \varphi_A \, dH_{n-1} = H_{n-1}(\partial_2 A) = \int_{\pi_A} \sqrt{1 + |Df|^2} \, dx'$$

If we set $M_r = \sup\{|Df(x')|, |x'| < r\}$ we get $\lim_{r\to 0} M_r = 0$ and

$$\int_{\partial_2\Omega} \varphi_A \, dH_{n-1} \leqslant (1 + M_r) H_{n-1}(\pi_A) .$$

On the other hand

$$\int_{\Omega_1} |D\varphi_A| \geqslant H_{n-1}(\pi_A)$$

and letting $r \to 0$ we obtain $q(x_0) = 1$. Q.E.D.

Another situation in which $q(x_0) = 1$ is when the mean curvature of $\partial\Omega$ is bounded from above in a neighborhood of x_0. More precisely we have

PROPOSITION 1.5. *Let there exist $R > 0$ and a function $K(x)$ in $L_n(\Omega_R)$ $(\Omega_R = \Omega(x_0, R))$ such that*

$$\int_{B_R} |D\varphi_{\Omega_R}| - \int_{\Omega_R} K \, dx \leqslant \int_{B_R} |D\varphi_L| - \int_L K \, dx \tag{1.21}$$

for every set $L \subset \Omega_R$, coinciding with Ω_R outside some compact set in B_R. Then $q(x_0) = 1$.

PROOF. Let $r < R$ and let $A \subset \Omega_r$. From (1.21) with $L = \Omega_R - A$ we get easily

$$\int_{\partial\Omega} \varphi_A \, dH_{n-1} - \int_{\Omega} |D\varphi_A| \leqslant \int_A K \, dx \leqslant \|K\|_{L_n(\Omega_r)} (\text{meas}\, A)^{1-1/n} .$$

If r is small enough we have meas $\Omega_r <$ meas $\Omega_R/2$ and hence from (1.8):

$$\int_{\partial\Omega} \varphi_A \, dH_{n-1} \leqslant \{1 + 2c_1(\Omega_R) \|K\|_{L_n(\Omega_r)}\} \int_{\Omega} |D\varphi_A|$$

and letting $r \to 0$ we get $q(x_0) = 1$. Q.E.D.

To conclude this section let us calculate the function $q(x)$ at the vertex of an angular region.

Let Ω be the set $\{x\in \mathbf{R}^2: x_2 > L|x_1|\}$ and let $x_0 = 0$. It is evident that the supremum in (1.9) is attained when A is the triangle

$$A = \left\{x\in \mathbf{R}^2:\ L|x_1| < x_2 < Lr/\sqrt{1+L^2}\right\}$$

For such set we have:

$$\int_{\partial\Omega} \varphi_A\, dH_{n-1} = 2r$$

$$\int_{\Omega} |D\varphi_A| = 2r/\sqrt{1+L^2}$$

and hence

$$q(0) = \sqrt{1+L^2}$$

in agreement with the results of Emmer [4].

It can be interesting to remark that if instead of Ω we consider the set $A = \mathbf{R}^2 - \Omega$, we get $q(0) = 1$.

2. – Existence of a minimum.

2.*A*. We will show in this section that conditions (1.3) and (1.4) of section 1.*C* are sufficient to guarantee the existence of a minimum for the functional

$$\mathcal{F}(u) = \int_{\Omega} \sqrt{1+|Du|^2} + \int_{\Omega} \lambda(x, u)\, dx + \int_{\partial_1\Omega} |u-\varphi|\, dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\, dH_{n-1} \tag{2.1}$$

in the class $BV(\Omega)$. To be precise we have the following

THEOREM 2.1. *Let Ω be a bounded connected open set with locally Lipschitz-continuous boundary $\partial\Omega$, and let $\varkappa$ and H be two functions satisfying conditions* (1.3) *and* (1.4) *of section* 1.*C*. *Let φ be a function in $L_1(\partial_1\Omega)$. Then the functional $\mathcal{F}(u)$ has a minimum in the class $BV(\Omega)$.*

The proof of Theorem 2.1 will take all this chapter.

The first step is quite usual in the theory of non-parametric minimal surfaces, and consists in a suitable handling of the integral involving the function φ.

Since φ is in $L_1(\partial_1\Omega)$, there exists a function $f(x)$ in the Sobolev space $H_1^1(\Omega_1)$ such that φ is the trace of f on $\partial_1\Omega$ [9]. If we denote by w the function

$$w(x) = \begin{cases} u(x) & x\in\Omega \\ f(x) & x\in\Omega_1 - \Omega \end{cases}$$

we have [21] $w \in BV(\Omega_1)$ and

$$\int_{\Omega_1} \sqrt{1+|Dw|^2} = \int_{\Omega} \sqrt{1+|Du|^2} + \int_{\Omega_1-\Omega} \sqrt{1+|Df|^2}\,dx + \int_{\partial_1\Omega} |u-\varphi|\,dH_{n-1} \tag{2.2}$$

The problem of minimizing the functional $\mathcal{F}$ in $BV(\Omega)$ is thus reduced to a minimum problem for the new functional

$$\mathcal{G}(u) = \int_{\Omega_1} \sqrt{1+|Du|^2} + \int_{\Omega} \lambda(x, u)\,dx + \int_{\partial_2\Omega} \varkappa u\,dH_{n-1} \tag{2.3}$$

in the class

$$W = \{w \in BV(\Omega_1)\colon w = f \text{ in } \Omega_1 - \Omega\}\,. \tag{2.4}$$

We remark that when $\partial_1\Omega = \emptyset$ the functionals $\mathcal{F}$ and $\mathcal{G}$ coincide, and that $W = BV(\Omega)$ in this case.

2.*B*. Let us show first that $\mathcal{G}(u)$ is bounded from below in W.

LEMMA 2.1. *Let H and $\varkappa$ satisfy* (1.3) *and* (1.4). *Then for every function $v \in BV(\Omega)$ we have*

$$\int_{\Omega} \lambda(x, v)\,dx + \int_{\partial_2\Omega} \varkappa v\,dH_{n-1} \geqslant -(1-\varepsilon_0)\Big\{\int_{\Omega} |Dv| + \int_{\partial_1\Omega} |v|\,dH_{n-1}\Big\} - c_3 \tag{2.5}$$

where c_3 is a constant independent of v.

PROOF. We extend v as zero outside Ω. Let us suppose first $v \geqslant 0$. Setting

$$V_t = \{x \in \Omega_1\colon v(x) > t\}$$

we get

$$\int_{\Omega} \lambda(x, v)\,dx + \int_{\partial_2\Omega} \varkappa v\,dH_{n-1} = \int_0^\infty dt \int_{V_t} H(x, t)\,dx + \int_0^\infty dt \int_{\partial_2\Omega} \varphi_{V_t} \varkappa\,dH_{n-1}\,.$$

On the other hand:

$$\int_0^\infty dt \int_{V_t} H(x, t)\,dx \geqslant \int_0^{t_0} dt \int_{V_t} H(x, t)\,dx + \int_{t_0}^\infty dt \int_{V_t} H(x, t_0)\,dx$$

so that from (1.4′):

$$\int_{\Omega}\lambda(x, v)\,dx + \int_{\partial_2\Omega}\varkappa v\,dH_{n-1} \geqslant -c_4 - (1-\varepsilon_0)\int_{t_0}^{\infty}dt\int_{\Omega_1}|D\varphi_{V_t}| \tag{2.6}$$

where

$$c_4 = t_0 H_{n-1}(\partial_2\Omega) + \int_0^{t_0}dt\int_{\Omega}|H(x, t)|\,dx\,.$$

In general we have (2.6) for $v_+ = \max(v, 0)$, while for $v_- = \min(v, 0)$ we get:

$$\int_{\Omega}\lambda(x, v_-)\,dx + \int_{\partial_2\Omega}\varkappa v_-\,dH_{n-1} \geqslant -c_5 - (1-\varepsilon_0)\int_{-\infty}^{-t_0}dt\int_{\Omega_1}|D\varphi_{V_t}|\,. \tag{2.7}$$

From (2.6) and (2.7) we get at once (2.5), recalling the coarea formula:

$$\int_{\Omega}|Dv| + \int_{\partial_1\Omega}|v|\,dH_{n-1} = \int_{\Omega_1}|Dv| = \int_{-\infty}^{\infty}dt\int_{\Omega_1}|D\varphi_{V_t}| \qquad \text{Q.E.D.}$$

From the preceding lemma we obtain at once the inequality

$$\mathcal{G}(u) \geqslant \varepsilon_0\int_{\Omega_1}\sqrt{1+|Du|^2} - c_6 \tag{2.8}$$

for every $u \in W$, c_6 being a constant independent of u.

Lemma 2.2. *Let $\varkappa$ and H satisfy conditions* (1.3) *and* (1.4). *Then for every $\delta > 0$ there exists a constant $c_7(\delta)$ such that for each $w \in BV(\Omega_1)$, with $w = 0$ in $\Omega_1 - \Omega$, we have:*

$$\left|\int_{\partial_2\Omega} w\varkappa\,dH_{n-1}\right| \leqslant (1-\varepsilon_0/2)\int_{S_\delta}|Dw| + c_7(\delta)\int_{S_\delta}|w|\,dx \tag{2.9}$$

where

$$S_\delta = \{x \in \Omega_1\colon \operatorname{dist}(x, \partial_2\Omega) < \delta\}\,.$$

Proof. Let us suppose that $w \geqslant 0$ and that $\operatorname{spt} w \subset S_\delta$. We have from (1.4′):

$$\int_{\partial_2\Omega}\varkappa\varphi_{W_t}\,dH_{n-1} \geqslant -(1-\varepsilon_0)\int_{\Omega_1}|D\varphi_{W_t}| - \|H(x, t_0)\|_{L_n(S_\delta)}(\text{meas } W_t)^{1-1/n}\,.$$

Suppose now that δ_0 is such that

$$\text{meas}\, S_{\delta_0} \leqslant \text{meas}\, \Omega_1/2$$

and

$$\|H(x, t_0)\|_{L_n(S_{\delta_0})} \leqslant \varepsilon_0/4c_1 .$$

If $\delta < \delta_0$ we have from (1.8):

$$\int_{\partial_2\Omega} \varkappa \varphi_{W_t}\, dH_{n-1} \geqslant -(1-\varepsilon_0/2)\int_{\Omega_1} |D\varphi_{W_t}| .$$

In a similar way, using (1.4″) instead of (1.4′), we obtain:

$$\int_{\partial_2\Omega} \varkappa \varphi_{W_t}\, dH_{n-1} \leqslant (1-\varepsilon_0/2)\int_{\Omega_1} |D\varphi_{W_t}|$$

and hence

$$\left|\int_{\partial_2\Omega} \varkappa w\, dH_{n-1}\right| \leqslant (1-\varepsilon_0/2)\int_{S_\delta} |Dw| . \tag{2.10}$$

Arguing as in Lemma 2.1 it is easy to see that (2.10) remains valid for a general w, provided $\operatorname{spt} w \subset S_\delta$, with $\delta < \delta_0$.

Let $g(x)$ be a C^∞ function with $g = 1$ on $\partial_2\Omega$, $0 \leqslant g \leqslant 1$ and $\operatorname{spt} g \subset S_\partial$. We have

$$\left|\int_{\delta_2\Omega} \varkappa w\, dH_{n-1}\right| \leqslant (1-\varepsilon_0/2)\int_{S_\delta} |D(gw)| \leqslant (1-\varepsilon_0/2)\int_{S_\delta} |Dw| + c_7\int_{S_\delta} |w|\, dx$$

where $c_7 = c_7(\delta) = \sup\limits_{S_\delta} |Dg|$ does not depend on w, so that (2.9) is proved for $\delta < \delta_0$. It is easily seen that (2.9) remains valid for every δ. Q.E.D.

We can prove now the lower semicontinuity of the functional $\mathcal{G}(u)$.

PROPOSITION 2.1. *Let $\{v_k\}$ be a sequence of functions in W, bounded in $L_{n/n-1}(\Omega_1)$, and convergent in $L_1(\Omega_1)$ to a function $v \in W$. Suppose that* (1.3) *and* (1.4) *are satisfied. Then*

$$\mathcal{G}(v) \leqslant \liminf_{k\to\infty} \mathcal{G}(v_k) \tag{2.11}$$

PROOF. Let us prove first the lower semicontinuity of the term

$$\int_{\Omega} \lambda(x, v)\, dx . \tag{2.12}$$

For that we define, for $m > 0$, the function

$$H^{(m)}(x, t) = \begin{cases} H(x, m) & \text{if } t > m \\ H(x, t) & \text{if } |t| < m \\ H(x, -m) & \text{if } t < -m \end{cases}$$

and let

$$\lambda^{(m)}(x, t) = \int_0^t H^{(m)}(x, s)\, ds .$$

We have

$$\int_\Omega \lambda(x, v(x))\, dx = \sup_{m>0} \int_\Omega \lambda^{(m)}(x, v(x))\, dx$$

and hence it is sufficient to prove the lower semicontinuity of the integral

$$\int_\Omega \lambda^{(m)}(x, v)\, dx$$

for each fixed $m > 0$.

Let $v_k \to v$ in L_1 and let

$$\varphi_k = \max(v_k - v, 0)$$
$$\psi_k = \min(v_k - v, 0) .$$

Since φ_k and ψ_k tend to zero in L_1 and are bounded in $L_{n/n-1}$, they converge to zero weakly in $L_{n/n-1}$. On the other hand

$$\int_\Omega \lambda^{(m)}(x, v_k)\, dx - \int_\Omega \lambda^{(m)}(x, v)\, dx = \int_\Omega dx \int_v^{v_k} H^{(m)}(x, t)\, dt \geqslant$$
$$\geqslant \int_\Omega H^{(m)}(x, -m)\varphi_k\, dx + \int_\Omega H^{(m)}(x, m)\, \psi_k\, dx .$$

The right-hand side of (2.13) tends to zero as $k \to \infty$, thus proving the lower semicontinuity of (2.12).

For the remaining part of $\mathcal{G}(u)$ we use Lemma 2.2 and a technique similar to [9].

Let

$$\mathcal{E}(v) = \int_{\Omega_1} \sqrt{1 + |Dv|^2} + \int_{\partial_2\Omega} \varkappa v\, dH_{n-1}$$

We have from Lemma 2.2 applied to $w = v - v_k$:

$$\mathcal{E}(v) - \mathcal{E}(v_k) \leqslant \int_{\Omega_1} \sqrt{1 + |Dv|^2} + \int_{S_\delta} |Dv| - \int_{\Omega_1 - S_\delta} \sqrt{1 + |Dv_k|^2} + c_7 \int_{S_\delta} |v - v_k| \, dx \, .$$

Letting $k \to \infty$ and taking in account the lower semicontinuity of the area integral we get

$$\mathcal{E}(v) - \liminf_{k\to\infty} \mathcal{E}(v_k) \leqslant 2 \int_{S_\delta} |Dv|$$

for every $\delta > 0$, and hence

$$\mathcal{E}(v) \leqslant \liminf_{k\to\infty} \mathcal{E}(v_k) \qquad \text{Q.E.D.}$$

2.*D*. The proof of Theorem 2.1 will be complete if we can show that there exists a minimizing sequence which is bounded in $L_1(\Omega_1)$. For, let $\{u_k\}$ be such a sequence; from (2.8) we easily see that

$$\int_{\Omega_1} \sqrt{1 + |Du_k|^2} \leqslant c_8 \tag{2.14}$$

and hence $\{u_k\}$ is bounded in $BV(\Omega_1)$.

Passing possibly to a subsequence we can suppose that u_k converges in $L_1(\Omega_1)$ to a function $u \in W$. From Lemma 1.1 it follows that $\{u_k\}$ is bounded in $L_{n/n-1}(\Omega_1)$ and hence we can apply the semicontinuity result proved above to get the conclusion of the theorem.

Depending whether $\partial_1 \Omega \neq \emptyset$ or $\partial_1 \Omega = \emptyset$ we need two different arguments. The first situation can be handled by means of the following well-known result:

LEMMA 2.3. *Let $\Omega_1 - \Omega$ be non empty and let $w(x)$ be a function in $BV(\Omega_1)$ with $w = 0$ in $\Omega_1 - \Omega$. Then*

$$\int_{\Omega_1} |w| \, dx \leqslant c_9 \int_{\Omega_1} |Dw| \tag{2.15}$$

where c_9 depends only on Ω and Ω_1.

It is easily seen that Lemma 2.3 settles the case $\partial_1 \Omega \neq \emptyset$. In fact every minimizing sequence is bounded in $L_1(\Omega_1)$ since we have:

$$\int_{\Omega_1} |u_k| dx \leqslant \int_{\Omega_1} |f| dx + c_9 \int_{\Omega_1} |Df| + c_9 \int_{\Omega_1} |Du_k|$$

and the last integral is bounded by (2.14).

2E. When $\partial_1\Omega = \emptyset$ the previous lemma does not work and we need a different argument. Let us remark first that if $\partial_1\Omega=\emptyset$ condition (1.4) implies

$$\int_\Omega H(x,t_0)\,dx \quad + \int_{\partial\Omega} \varkappa\, dH_{n-1} \geqslant 0 \tag{2.16}$$

$$-\int_\Omega H(x,-t_0)\,dx - \int_{\partial\Omega} \varkappa\, dH_{n-1} \geqslant 0\,. \tag{2.17}$$

LEMMA 2.4. *Suppose that there exists a positive number h_0 such that*

$$\int_\Omega H(x,t_0)\,dx + \int_{\partial\Omega} \varkappa\, dH_{n-1} \geqslant h_0 \tag{2.18}$$

and

$$-\int_\Omega H(x,-t_0)\,dx - \int_{\partial\Omega} \varkappa\, dH_{n-1} \geqslant h_0 \tag{2.19}$$

and let $u \in BV(\Omega)$ satisfy

$$p(u) = \int_\Omega \lambda(x,u(x))\,dx + \int_{\partial\Omega} \varkappa u\, dH_{n-1} \leqslant 1\,.$$

Then

$$\int_\Omega |u|\,dx \leqslant c_{10}\Big\{1 + \int_\Omega |Du|\Big\}\,. \tag{2.20}$$

PROOF. Let

$$v_1 = \max(u - t_0, 0)$$

$$v_2 = \max(-u - t_0, 0)\,.$$

We have

$$p(v_1 + t_0) - p(t_0) \geqslant \int_\Omega H(x,t_0)v_1\,dx + \int_{\partial\Omega} \varkappa v_1\, dH_{n-1}$$

and

$$p(-v_2 - t_0) - p(-t_0) \geqslant -\int_\Omega H(x,-t_0)v_2\,dx - \int_{\partial\Omega} \varkappa v_2\, dH_{n-1}\,.$$

Setting

$$\bar v_i = (\text{meas }\Omega)^{-1}\int_\Omega v_i\,dx \qquad (i = 1, 2)\,;$$

we get

$$h_0\bar{v}_1 \leqslant \Big[\int_\Omega H(x, t_0)\,dx + \int_{\partial\Omega} \varkappa\, dH_{n-1}\Big]\bar{v}_1 \leqslant$$

$$\leqslant \int_\Omega H(x, t_0)(\bar{v}_1 - v_1)\,dx + \int_{\partial\Omega} \varkappa(\bar{v}_1 - v_1)\,dH_{n-1} + p(v_1 + t_0) - p(t_0)\,.$$

From Lemma 1.1 we obtain:

$$\Big|\int_\Omega H(x, t_0)(\bar{v}_1 - v_1)\,dx\Big| \leqslant \|H(x, t_0)\|_{L_n}\|v_1 - \bar{v}_1\|_{L_{n/n-1}} \leqslant c_{11}\int_\Omega |Dv_1|$$

a similar estimate holding for the boundary integral. In conclusion

$$h_0\bar{v}_1 \leqslant c_{12}\int_\Omega |Dv_1| + p(v_1 + t_0) - p(t_0)\,.$$

In a similar way

$$h_0\bar{v}_2 \leqslant c_{13}\int_\Omega |Dv_2| + p(-v_2 - t_0) - p(-t_0)$$

and hence

$$h_0(\bar{v}_1 + \bar{v}_2) \leqslant c_{14}\int_\Omega |Du| + p(v_1 + t_0) + p(-v_2 - t_0) - p(t_0) - p(-t_0)\,. \tag{2.21}$$

On the other hand we have

$$(\text{meas } \Omega)^{-1}\int_\Omega |u|\,dx \leqslant \bar{v}_1 + \bar{v}_2 + t_0$$

and

$$p(v_1 + t_0) + p(-v_2 - t_0) - p(t_0) - p(-t_0) = p(u) - p(u_0)$$

where

$$u_0(x) = \max\{\min(u, t_0), -t_0\}\,.$$

Combining these relations we get

$$\int_\Omega |u|\,dx \leqslant c_{15}\int_\Omega |Du| + c_{16}$$

since

$$|p(u_0)| \leqslant t_0\{H_{n-1}(\partial\Omega) + \|H(x, t_0)\|_{L_n} + \|H(x, -t_0)\|_{L_n}\} \qquad \text{Q.E.D.}$$

The preceding lemma plays the rôle of Lemma 2.3 in the proof of Theorem 2.1. Let $\{u_k\}$ be a minimizing sequence; we can suppose that

$$\mathcal{G}(u_k) \leqslant \mathcal{G}(0) + 1 = \text{meas}\, \Omega + 1$$

and hence

$$p(u_k) \leqslant 1,$$

for every k. From Lemma 2.4 and (2.14) we can conclude that $\{u_k\}$ is bounded in $L_1(\Omega)$ and therefore we get the conclusion of Theorem 2.1.

2.*F*. It remains the case when $\partial_1 \Omega = \emptyset$ and either

$$\int_\Omega H(x, t)\, dx + \int_{\partial\Omega} \varkappa\, dH_{n-1} = 0 \tag{2.22}$$

or

$$\int_\Omega H(x, -t)\, dx + \int_{\partial\Omega} \varkappa\, dH_{n-1} = 0 \tag{2.23}$$

for every $t \geqslant t_0$.

It is evident that an *a-priori* estimate for the L_1 norms of a general minimizing sequence cannot hold, as one can realize considering the case $H = \varkappa = 0$, and hence we need a different argument.

To be definite let us suppose that (2.22) holds for every $t \geqslant t_0$. Since $H(x, t)$ is a non-decreasing function of t it follows that $H(x, t) = H(x, t_0)$ for almost every x in Ω and for each $t \geqslant t_0$.

LEMMA 2.5. *Let $\partial_1 \Omega = \emptyset$ and let* (2.22) *and condition* (1.4′) *hold. Then for every Caccioppoli set $B \subset \Omega$ we have*:

$$\left| \int_B H(x, t_0)\, dx + \int_{\partial\Omega} \varkappa \varphi_B\, dH_{n-1} \right| \leqslant (1 - \varepsilon_0) \int_\Omega |D\varphi_B| \,. \tag{2.24}$$

PROOF. Let $B \subset \Omega$ and let $A = \Omega - B$. We have

$$\int_\Omega |D\varphi_B| = \int_\Omega |D\varphi_A| \,.$$

From (1.4′) relative to the set A we get

$$\int_\Omega \varphi_A H(x, t_0)\, dx + \int_{\partial\Omega} \varphi_A \varkappa\, dH_{n-1} \geqslant -(1 - \varepsilon_0) \int_\Omega |D\varphi_B| \,.$$

Since $\varphi_A = 1 - \varphi_B$ in Ω, we obtain from (2.22):

$$\int_\Omega \varphi_B H(x, t_0)\, dx + \int_{\partial\Omega} \varkappa\varphi_B\, dH_{n-1} \leq (1-\varepsilon_0)\int_\Omega |D\varphi_B| \tag{2.25}$$

and (2.24) follows at once from (1.4′) relative to B. Q.E.D.

Let us introduce now the functional

$$\mathcal{G}_0(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega H(x, t_0)\, u(x)\, dx + \int_{\partial\Omega} \varkappa u\, dH_{n-1}\,.$$

It follows from (2.22) that

$$\mathcal{G}_0(u) = \mathcal{G}_0(u + c)$$

for every real number c and every $u \in BV(\Omega)$, so that a minimum for $\mathcal{G}_0$ in the class

$$V_0 = \{v \in BV(\Omega) \colon \int_\Omega v\, dx = 0\}$$

will be also a minimum for $\mathcal{G}_0$ in $BV(\Omega)$.

The existence of a minimum for $\mathcal{G}_0$ in V_0 follows from (2.24) with the same argument as before; the L_1 norm of a minimizing sequence being bounded from Lemma 1.1, since $v_\Omega = 0$.

2.*G*. We shall prove now an *a-priori* bound for the supremum of a function $u(x)$ minimizing the functional $\mathcal{G}(u)$. The following lemma is a simple consequence of (1.8).

LEMMA 2.6. *Let $w \in BV(\Omega)$ and let*

$$\text{meas}\,\text{spt}(w) \leq \text{meas}\,\Omega/2\,.$$

Then

$$\Big(\int_\Omega |w|^{n/n-1}\, dx\Big)^{1-1/n} \leq 2c_1 \int_\Omega |Dw|\,. \tag{2.26}$$

PROOF. We can suppose $w \geq 0$. We have

$$(\text{meas}\, W_t)^{1-1/n} \leq 2c_1 \int_\Omega |D\varphi_{W_t}|$$

where as usual

$$W_t = \{x \in \Omega \colon w(x) > t\}\,.$$

and (2.26) follows as in [2], Lemma 1. Q.E.D.

The *a-priori* bound for a solution can now be proved using the method of [28] (see [10]).

THEOREM 2.2. *Let conditions* (1.3) *and* (1.4) *be satisfied and let* $u(x)$ *be a minimum for the functional* $\mathcal{G}(u)$. *We have*:

$$\sup_{\Omega} |u| \leqslant c_{17} \tag{2.27}$$

where c_{17} *is a constant depending on* ε_0, t_0, $\|u\|_{L_1}$ *and on* $\sup\limits_{\partial_1\Omega} |\varphi|$.

PROOF. Let $m_0 = \sup\limits_{\partial_1\Omega} |\varphi|$; we can suppose that $\sup\limits_{\Omega_1} |f| = m_0$. Let $k \geqslant \max(m_0, t_0)$ and let

$$v = \min(u, k)$$
$$w = \max(u - k, 0) = u - v .$$

We have as in [10]:

$$\int_{\Omega_1} |Dw| - \text{meas}\, U_k \leqslant \int_{\Omega_1} \sqrt{1 + |Du|^2} - \int_{\Omega_1} \sqrt{1 + |Dv|^2}$$

where

$$U_k = \{x \in \Omega : u(x) > k\}$$

and therefore, since $\mathcal{G}(u) \leqslant \mathcal{G}(v)$:

$$\int_{\Omega_1} |Dw| + \int_{\Omega} dx \int_v^u H(x, t)\, dt + \int_{\partial_2\Omega} \varkappa w\, dH_{n-1} \leqslant \text{meas}\, U_k .$$

From the very definition of v and w we get

$$\int_{\Omega} dx \int_v^u H(x, t)\, dt \geqslant \int_{\Omega} H(x, t_0) w\, dx$$

and from (1.4′):

$$\int_{\Omega} H(x, t_0)\, w dx + \int_{\partial_2\Omega} \varkappa w\, dH_{n-1} \geqslant -(1 - \varepsilon_0) \int_{\Omega_1} |Dw|$$

so that in conclusion:

$$\varepsilon_0 \int_{\Omega_1} |Dw| \leqslant \text{meas}\, U_k . \tag{2.28}$$

On the other hand we have

$$k \operatorname{meas} U_k \leqslant \|u\|_{L_1(\Omega)}$$

and hence if

$$k > 2\|u\|_{L_1(\Omega)}/\operatorname{meas} \Omega$$

we get

$$\operatorname{meas} \operatorname{spt}(w) \leqslant \operatorname{meas} \Omega/2 .$$

From Lemma 2.6 we conclude that

$$\|w\|_{L_{n/n-1}} \leqslant 2c_1 \int_\Omega |Dw|$$

and therefore

(2.29) $$\int_{U_k} (u-k)\, dx \leqslant 2c_1 \varepsilon_0^{-1} (\operatorname{meas} U_k)^{1+1/n}$$

for every $k \geqslant k_0 = \max\{m_0, t_0, 2\|u\|_{L_1(\Omega)}/\operatorname{meas}\Omega\}$.

Using a well known result of Stampacchia [28] we get the estimate

$$\sup_\Omega u \leqslant k_0 + 2(n+1)\, c_1 \varepsilon_0^{-1} (\operatorname{meas}\Omega)^{1/n}.$$

A similar computation gives the estimate for the infimum of u in Ω. Q.E.D.

2.*H*. We are now ready to prove the existence of a minimum for the functional $\mathcal{G}(u)$, under the condition

(2.22) $$\int_\Omega H(x,t)\, dx + \int_{\partial\Omega} \varkappa\, dH_{n-1} = 0 \qquad \text{for every } t \geqslant t_0 .$$

We observe that (2.22) implies $H(x,t) = H(x,t_0)$ for every $t \geqslant t_0$, and hence $H(x,t) \leqslant H(x,t_0)$ for each t. If we set

$$q_0 = \int_\Omega dx \int_0^{t_0} \big(H(x,t_0) - H(x,t)\big)\, dt$$

we have, for every function $u \in BV(\Omega)$:

$$\mathcal{G}_0(u) \leqslant \mathcal{G}(u) + q_0$$

the equality holding if $u(x) \geqslant t_0$ a.e. in Ω.

Let now $v(x)$ be a minimum for the functional $\mathcal{G}_0$ in V_0; it follows from Theorem 2.2 that $|v(x)|$ is bounded by some constant c_{18} depending only on t_0, ε_0 and Ω. If $u(x)$ is a function in $BV(\Omega)$ we have

$$\mathcal{G}(u) \geqslant \mathcal{G}_0(u) - q_0 \geqslant \mathcal{G}_0(v + c_{18} + t_0) - q_0 = \mathcal{G}(v + c_{18} + t_0)\,.$$

In conclusion, the function

$$v_0 = v + c_{18} + t_0$$

gives the required minimum for the functional.

The proof of Theorem 2.1 is thus complete. We can summarize the results of this chapter as follows:

THEOREM 2.3. *Let $\varkappa$ and H be two functions satisfying conditions* (1.3) *and* (1.4) *of section* 1.*C. and let φ be in $L_1(\partial_1\Omega)$.*

The functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega dx \int_0^{u(x)} H(x, t)\, dt + \int_{\partial_1\Omega} |u - \varphi|\, dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\, dH_{n-1}$$

has a minimum in $BV(\Omega)$.

Moreover, if φ is bounded, every minimum u of $\mathcal{F}$ is bounded by a constant depending on ε_0, t_0, $\sup_{\partial_1\Omega}|\varphi|$ and $\|u\|_{L_1(\Omega)}$.

In particular if $\partial_1\Omega \neq \emptyset$ and φ is bounded, or if $\partial_1\Omega = \emptyset$ and the functions $\varkappa$ and H satisfy (2.18) and (2.19) then every minimum of $\mathcal{F}$ is bounded by a constant depending only on ε_0, t_0, $\sup|\varphi|$ and possibly on h_0.

3. – Regularity of the solution.

3.*A*. The problem of the regularity of the solutions to our variational problem is still open in what concerns the regularity at the boundary $\partial_2\Omega$. On the contrary, for interior smoothness, as well as for the regularity at $\partial_1\Omega$, the situation is quite satisfactory and, for instance, one can get complete results for the Dirichlet problem.

In this chapter we shall sketch briefly the ideas involved in the proof of these results, referring to [15], [13] and [23] for details.

3.*B*. We begin with interior regularity, and we suppose that the mean curvature function $H(x, t)$ is Lipschitz-continuous in $\Omega \times \boldsymbol{R}$.

The first step consists in the observation that if the function $u(x)$ gives a minimum for the functional $\mathcal{F}(u)$ in $BV(\Omega)$, then the set

$$U = \{(x, t) \in \Omega \times \mathbf{R} : t < u(x)\}$$

minimize the functional

$$F_K(U) = \int_K |D\varphi_U| + \int_K H(x, t)\varphi_U \, dx \, dt$$

in every compact set $K \subset \Omega \times \mathbf{R}$.

In other words we have

$$F_K(U) \leqslant F_K(V) \tag{3.1}$$

for every Caccioppoli set $V \subset \Omega \times \mathbf{R}$ and such that $\varphi_U - \varphi_V = 0$ outside the compact set K (see [20]).

We can apply the results of [19] and conclude that the boundary ∂U of U is a regular hypersurface, except possibly for a locally compact set Σ, whose Hausdorff dimension does not exceed $n - 7$.

The argument used in [15] and [13] applies to this case also (we use again the fact that $H(x, t)$ is non-decreasing in t) and we conclude that the function $u(x)$ is regular (say $C^{1+\alpha}$), except for the set $N = \operatorname{proj} \Sigma$. In addition the function u belongs to the Sobolev space $H^{1,1}(\Omega)$.

In order to get the complete regularity of the function $u(x)$ one must use the *a-priori* inequality for the gradient (see [18] and [30]), and an approximation procedure, for which we refer to [15], [12] and [22]. The final result is the following

THEOREM 3.1. *Let $H(x, t)$ be a Lipschitz-continuous function in $\Omega \times \mathbf{R}$ and let $u(x)$ be a minimum for the functional*

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega \lambda(x, u)\, dx + \int_{\partial_1 \Omega} |u - \varphi| \, dH_{n-1} + \int_{\partial_2 \Omega} \varkappa u \, dH_{n-1} \tag{3.2}$$

Then u belongs to $C^{2+\alpha}(\Omega)$, for every $\alpha < 1$.

In addition for every $x_0 \in \Omega$ and for every $R < \operatorname{dist}(x_0, \partial\Omega)$ we have

$$|Du(x_0)| \leqslant c_{19} \exp\{c_{20}\,\omega(R)/R\} \tag{3.3}$$

where $\omega(R)$ is the oscillation of u in the ball $B(x_0, R)$ and c_{19} and c_{20} depend on H, DH and $\sup |u|$.

3.*C*. It has been proved in [3] that the inequality (3.3) does not hold in general if the function $H(x, t)$ is not Lipschitz-continuous. To be precise, if H does not depend on t it is not sufficient to assume that $H(x)$ belongs to the Sobolev space $H^{1,p}$, for every $p < \infty$.

In this section we shall give an example showing that if H is not Lipschitz-continuous the function $u(x)$ does not belong in general to the space $H^{1,1}(\Omega)$. The example will concern the one-dimensional problem, but it is easily seen that it works in any dimension.

Let

$$f(x) = \begin{cases} \exp\left(-(x-1)^{-1}\right) & x > 1 \\ 0 & -1 \leqslant x \leqslant 1 \\ \exp\left((x+1)^{-1}\right) & x < -1 \end{cases}$$

If we set

$$h(t) = \begin{cases} 0 & t = 0 \\ t \log^3 |t| (2 + \log |t|)(1 + t^2 \log^4 |t|)^{-\frac{3}{2}} & t \neq 0 \end{cases}$$

we have

$$\frac{d}{dx}\left(f'(1+f'^2)^{-\frac{1}{2}}\right) = h(f(x)) .$$

The function $h(t)$ is increasing for $|t| < T = e^{-6}$; if we set

$$Q = \{x \in \mathbf{R} \colon |x| < 1 + \tfrac{1}{6}\}$$

we get $|f(x)| < T$ in Q and hence

$$\frac{d}{dx}\left(f'(1+f'^2)^{-\frac{1}{2}}\right) = H(f(x))$$

where

$$H(t) = \begin{cases} h(t) & |t| < T \\ h(T) + h'(T)(t - T) & t \geqslant T \\ h(-T) + h'(-T)(t + T) & t \leqslant -T \end{cases}$$

The function $H(t)$ is increasing in $\mathbf{R}$ and therefore the set

$$F = \{(x, y) \in Q \times \mathbf{R} \colon f(x) < y < T\}$$

minimizes the functional

$$\int_K |D\varphi_F| + \int_K \varphi_F H(y)\,dx\,dy$$

in every compact set $K \subset \Lambda = Q \times (-T, T)$.

Let

$$v(y) = \operatorname{sign}(y)(1 - (\log|y|)^{-1})\ ;$$

we have $v \in BV(-T, T)$ and

$$F = \{(x, y) \in \Lambda\colon |y| < T, x < v(y)\}\,.$$

From the minimum properties of F it follows that

$$\int_{-T}^{T}\sqrt{1+|Dv|^2} + \int_{-T}^{T} H(y)\,v(y)\,dy \leqslant \int_{-T}^{T}\sqrt{1+|Dw|^2} + \int_{-T}^{T} H(y)\,w(y)\,dy \tag{3.4}$$

for every $w \in BV(-T, T)$, such that $\operatorname{spt}(v - w) \subset (-T, T)$ and such that the graph of w is contained in Λ, i.e. $|w| < 1 + \frac{1}{6}$.

From the convexity of the functional it follows at once that the function $v(y)$ satisfies (3.4) for every $w \in BV(-T, T)$ with $\operatorname{spt}(v - w) \subset (-T, T)$.

It is easily seen that $H(y)$ is in $H^{1,p}(-T, T)$ for every $p < +\infty$ and $v(y)$ does not belong to $H^{1,1}$.

3.*D*. For what concerns the regularity of the solution at points of $\partial_1\Omega$ we refer to [23] and [13]. We have the following

THEOREM 3.2. *Let $\partial_1\Omega$ be of class C^3 and let $\varphi(x)$ be a continuous function on $\partial_1\Omega$. Let $x_0 \in \partial_1\Omega$ be such that the sum of the principal curvatures of $\partial\Omega$ at x_0 is greater than $|H(x_0, \varphi(x_0))|$. Let $u(x)$ minimize the functional* (3.2) *and let $H(x, t)$ be continuous. Then*

$$\lim_{x \to x_0} u(x) = \varphi(x_0)\,.$$

If in addition $\varphi(x)$ is of class $C^{1+\alpha}$ in a neighborhood of x_0 and $H(x, t)$ is Lipschitz-continuous, then the gradient of $u(x)$ is bounded in a neighborhood of x_0.

The first assertion of the Theorem is a special case of [23], Theorem 6; the last part can be easily proved with the method of [13] using inequality (3.3) and the bound for $\sup |u|$.

4. – Existence revisited.

4.*A*. In this chapter we shall come back to the existence of a minimum for the functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \lambda(x, u)\,dx + \int_{\partial_1\Omega} |u-\varphi|\,dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\,dH_{n-1}$$

with the purpose of generalizing the results of ch. 2.

We shall make the following assumptions:

(4.1) *The boundary of Ω, $\partial\Omega$, is a hypersurface of class C^3; and $\varphi(x)$ and $\varkappa(x)$ are bounded measurable functions in $\partial_1\Omega = \Omega_1 \cap \partial\Omega$ and $\partial_2\Omega$, respectively, with $|\varkappa(x)| \leqslant 1$.*
$H(x, t)$ is a Lipschitz-continuous function in $\bar{\Omega}\times \mathbf{R}$, non-decreasing in t for every $x \in \bar{\Omega}$.

(4.2) *There exist two positive constants t_0 and α_0 such that for every Caccioppoli set $B \subset \Omega$:*

$$(4.2') \qquad \int_B H(x, t_0)\,dx + \int_{\partial_2\Omega} \varkappa\varphi_B\,dH_{n-1} \geqslant -\int_{\Omega_1} |D\varphi_B| + 2\alpha_0 \min\{|B|, |\Omega - B|\}$$

$$(4.2'') \qquad \int_B H(x, -t_0)\,dx + \int_{\partial_2\Omega} \varkappa\varphi_B\,dH_{n-1} \leqslant \int_{\Omega_1} |D\varphi_B| - 2\alpha_0 \min\{|B|, |\Omega - B|\}$$

where we have denoted by $|B|$ the measure of B.

It is clear from (1.8) that condition (4.2) is more general than the corresponding assumption (1.4); in particular we are able to treat the case where $\varkappa(x)$ takes the values ± 1, formerly forbidden by the proposition 1.2. From the results of [14] it is apparent that one must have quantitative conditions on the mean curvature of that part of the boundary where $\varkappa(x)$ takes the values $+1$ or -1. For $x \in \Omega$ let $d(x) = \operatorname{dist}(x, \partial\Omega)$ and let

$$\Gamma_{+1} = \{x \in \partial_2\Omega : \varkappa(x) = \ \ 1\}.$$

$$\Gamma_{-1} = \{x \in \partial_2\Omega : \varkappa(x) = -1\}.$$

We have:

PROPOSITION 4.1. *Let x_0 be an interior point of Γ_{+1} and let (4.2″) hold. Then*

$$(4.3) \qquad H(x_0, -t_0) \leqslant \Delta d(x_0) - 2\alpha_0 .$$

If instead x_0 is an interior point of Γ_{-1} and if (4.2′) holds, we have:

$$H(x_0, t_0) \geqslant -\Delta d(x_0) + 2\alpha_0 . \tag{4.4}$$

PROOF. Let B_r be a ball centered at x_0 such that $|B_r| \leqslant |\Omega|/2$ and $\partial\Omega \cap B_r \subset \Gamma_{+1}$, and let $B \subset B_r$. We have from (4.2″):

$$\int_B \{H(x, -t_0) + 2\alpha_0\}\, dx + \int_{\partial\Omega} \varphi_B\, dH_{n-1} \leqslant \int_\Omega |D\varphi_B| .$$

Setting

$$H_0 = \inf_{B_r} H(x, -t_0) + 2\alpha_0$$

we obtain

$$\int_{\partial\Omega} \varphi_B\, dH_{n-1} \leqslant \int_\Omega |D\varphi_B| - H_0 |B| .$$

Let $Q = \mathbf{R}^n - \Omega$, and let $A = Q \cup B$. We have

$$\int_{\bar{B}_r} |D\varphi_Q| = \int_{\bar{B}_r} |D\varphi_Q| + \int_\Omega |D\varphi_B| - \int_{\partial\Omega} \varphi_B\, dH_{n-1}$$

and hence

$$\int_{\bar{B}_r} |D\varphi_Q| - H_0 |Q \cap B_r| \leqslant \int_{\bar{B}_r} |D\varphi_A| - H_0 |A \cap B_r|$$

for every set $A \supset Q$, and coinciding with Q outside B_r.

The last relation implies that the sum of the principal curvatures of $\partial\Omega \cap B_r$ does not exceed $-H_0$; from Lemma 1.2 of [25] we conclude

$$-\Delta d(x_0) \leqslant -H_0 .$$

Since r is arbitrary, we can let $r \to 0$, getting (4.3). With the same argument one can prove (4.4) and hence the Proposition. Q.E.D.

The preceding result justifies the additional assumption

(4.5) SUPPLEMENTARY CONDITION. *There exist two positive constants $\varkappa_1 < 1$ and α_1, and two open sets L_{+1} and L_{-1} such that*

$$\varkappa(x) \leqslant \varkappa_1 \quad \text{in } \partial_2\Omega - L_{+1}, \quad \text{and} \quad H(x, -t_0) \leqslant \Delta d(x) - 2\alpha_1 \text{ in } L_{+1} , \tag{4.5′}$$

$$\varkappa(x) \geqslant -\varkappa_1 \text{ in } \partial_2\Omega - L_{-1}, \quad \text{and} \quad H(x, t_0) \geqslant -\Delta d(x) + 2\alpha_1 \text{ in } L_{-1} . \tag{4.5″}$$

We remark that if $\varkappa(x)$ is a continuous function and if the sets Γ_{+1} and Γ_{-1} coincide with the closure of their interior, the supplementary condition (with $\alpha_1 = \frac{1}{2}\alpha_0$) follows from (4.2) and Proposition 4.1.

4.*B*. As in ch. 1 we shall begin with a short discussion of condition (4.2).

Let us consider first the Dirichlet problem (i.e. $\partial_2\Omega = \emptyset$) for constant mean curvature H, in the borderline case [3]:

$$|H| = n\omega_n^{1/n}|\Omega|^{-1/n}. \tag{4.6}$$

We have

$$\Big|\int_B H\,dx\Big| = |H||B|^{1-1/n}|\Omega|^{1/n}(1 - |\Omega - B|/|\Omega|)^{1/n}$$

and hence

$$\Big|\int_B H\,dx\Big| \leqslant \int_{\Omega_1}|D\varphi_B| - n^{-1}|H|(|B|/|\Omega|)^{1-1/n}|\Omega - B| \leqslant \int_{\Omega_1}|D\varphi_B| - n^{-1}|H||B||\Omega - B||\Omega|^{-1}$$

which gives at once (4.2).

In general, in the case of Dirichlet problem, condition (4.2) is satisfied if for some t_0 and some p, $n < p \leqslant +\infty$,

$$\|H^{\pm}(x, \mp t_0)\|_{L_p(\Omega)} \leqslant n\omega_n^{1/n}|\Omega|^{1/p-1/n}.$$

where, as usual, $H^+ = \max(H, 0)$ and $H^- = \min(H, 0)$.

The situation is more involved in the case of mixed boundary conditions. We need the following

LEMMA 4.1. *Let L be a set with C^3 boundary and let $w \in BV(L)$. Then*

$$\int_{\partial L}|w|\,dH_{n-1} \leqslant \int_L |Dw| + c_{21}(L)\int_L |w|\,dx. \tag{4.7}$$

PROOF (see [10], Lemma 1). Let d_0 be such that the distance function $d(x) = \operatorname{dist}(x, \partial L)$ is of class C^2 in the strip

$$S = \{x \in L : d(x) < d_0\}.$$

Arguing as in [10], we get (4.7) with

$$c_{21}(L) = d_0^{-1} + \sup_S \max(-\Delta d, 0).$$

Q.E.D.

PROPOSITION 4.2. *Let Ω, $\varkappa$ and H satisfy assumptions* (4.1) *and let*

$$h(x)=\lim_{t\to+\infty} H^-(x,t)$$

$$k(x)=\lim_{t\to-\infty} H^+(x,t)$$

Suppose that

$$\|h\|_{L_n(\Omega)}< n\omega_n^{1/n}$$

$$\|k\|_{L_n(\Omega)}< n\omega_n^{1/n}$$

and

$$\lim_{|t|\leftarrow+\infty} \operatorname{sgn}(t) H(x,t)=+\infty \tag{4.8}$$

uniformly for x in some neighborhood L of $\partial_2\Omega$. Then (4.2) *is satisfied.*

PROOF. We remark that since H is continuous and non-decreasing in t, uniform convergence in (4.8) is equivalent to pointwise convergence. Moreover, we can suppose that ∂L is smooth. Arguing as in Proposition 1.3 we get the inequality:

$$\int_B H(x,t_0)\,dx+\int_{\partial_2\Omega}\varphi_B\,dH_{n-1}\geqslant\int_{\partial L}\tilde{\varkappa}\varphi_B\,dH_{n-1}-(1-\varepsilon_1)\int_{\Omega_1-L}|D\varphi_B|+ \\ +\int_L H(x,t_0)\varphi_B\,dx+\varepsilon_1\int_{\partial L\cap\Omega_1}\varphi_B\,dH_{n-1}$$

for every $t_0>t_1$, where

$$\tilde{\varkappa}=\begin{cases}\varkappa & \text{in } \partial\Omega\\ -1 & \text{in } \partial L\cap\Omega_1.\end{cases}$$

From Lemma 4.1 we obtain

$$\int_B H(x,t_0)\,dx+\int_{\partial_2\Omega}\varkappa\varphi_B\,dH_{n-1}\geqslant-\int_{\Omega_1}|D\varphi_B|+\int_L (H(x,t_0)-c_{21})\varphi_B\,dx+ \\ +\varepsilon_1\Big\{\int_{\Omega_1-L}|D\varphi_B|+\int_{\partial L\cap\Omega_1}\varphi_B\,dH_{n-1}\Big\}.$$

If we choose t_0 in such a way that $H(x,t_0)\geqslant c_{21}+1$ in L, we get immediately (4.2′) observing that

$$\int_{\Omega_1-L}|D\varphi_B|+\int_{\partial L\cap\Omega_1}\varphi_B\,dH_{n-1}\geqslant n\omega_n^{1/n}|\Omega|^{-1/n}|B\cap(\Omega-L)|.$$

The proof of (4.2″) is similar and will be omitted. Q.E.D.

4.*C*. The existence of a minimum for the functional $\mathcal{F}(u)$ with the assumptions of section 4.*A*. cannot be proved directly as before, because in general we don't have an *a-priori* bound for the area of minimizing sequences. However, for a suitable minimizing sequence we shall prove an estimate in BV, from which we will derive the existence of a minimum.

The idea consists in approximating the functions H and $\varkappa$ by means of the functions

$$H_\varepsilon(x, t) = (1-\varepsilon)H(x, t)$$

and

$$\varkappa_\varepsilon(x) \quad = (1-\varepsilon)\varkappa(x)\,.$$

For small values of $\varepsilon > 0$ these new functions satisfy the hypotheses of section 4.*A*. and moreover conditions (1.4) of section 1.*C* with $\varepsilon_0 = \varepsilon$, so that the corresponding functional $\mathcal{F}_\varepsilon$ has a minimum u_ε in $BV(\Omega)$ which, according to Theorems 2.3 and 3.1, belongs to the space $C^{2,\alpha}(\Omega) \cap L_\infty(\Omega)$. Due to the convexity of the functional, the minimizing function is unique up to an additive constant.

Our goal is to get a bound for $\sup\limits_\Omega |u_\varepsilon|$, independent of ε. To this purpose we shall consider an auxiliary obstacle problem.

Let

$$\hat{\varkappa}_\varepsilon = \begin{cases} (1-\varepsilon)\varkappa & \text{in } \partial_2\Omega \\ 1-\varepsilon & \text{in } \partial_1\Omega \end{cases}$$

and let

$$\mathcal{G}_\varepsilon(v) = \int_\Omega \sqrt{1+|Dv|^2} + \int_\Omega H_\varepsilon(x, t_0)\, v\, dx + \int_{\partial\Omega} \hat{\varkappa}_\varepsilon v\, dH_{n-1}\,.$$

We have from (4.2′)

$$\int_B H_\varepsilon(x, t_0)\, dx + \int_{\partial\Omega} \hat{\varkappa}_\varepsilon \varphi_B\, dH_{n-1} \geqslant -(1-\varepsilon)\int_\Omega |D\varphi_B| + 2\alpha_0(1-\varepsilon)\min\big(|B|, |\Omega - B|\big)$$

and therefore arguing as in chapter 2 (see also [5]) we can conclude that the functional $\mathcal{G}_\varepsilon$ has a minimum v_ε in the class

$$K_T = \{v \in BV(\Omega) : v \geqslant T\}\,.$$

The minimum v_ε is actually of class $C^{1,\alpha}$ in Ω.

LEMMA 4.2. *Let v_ε be a minimum for the functional $\mathcal{G}_\varepsilon$ in K_T, and let u_ε be a minimum for $\mathcal{F}_\varepsilon$ in $BV(\Omega)$. Suppose that*

$$T \geqslant \max\left\{t_0, \sup_{\partial_1\Omega} |\varphi|\right\}.$$

Then either $u_\varepsilon \leqslant v_\varepsilon$ in Ω or $\mathcal{F}_\varepsilon(v_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon)$.

PROOF. Let

$$w = \min(u_\varepsilon, v_\varepsilon)$$

and

$$A = \{x \in \Omega : v_\varepsilon(x) < u_\varepsilon(x)\}.$$

We have

$$\mathcal{F}_\varepsilon(w) = \mathcal{F}_\varepsilon(u_\varepsilon) + \int_A \left(\sqrt{1+|Dv_\varepsilon|^2} - \sqrt{1+|Du_\varepsilon|^2}\right) dx + \\ + \int_A \left(\lambda_\varepsilon(x, v_\varepsilon) - \lambda_\varepsilon(x, u_\varepsilon)\right) dx + \int_{\partial_1\Omega} \left(|w-\varphi| - |u_\varepsilon - \varphi|\right) dH_{n-1} + \int_{\partial_2\Omega} \varkappa_\varepsilon(w - u_\varepsilon)\, dH_{n-1} \tag{4.9}$$

Recalling that for x in A we have $u_\varepsilon > v_\varepsilon \geqslant T \geqslant t_0$, we get

$$\int_A \left(\lambda_\varepsilon(x, v_\varepsilon) - \lambda_\varepsilon(x, u_\varepsilon)\right) dx \leqslant \int_A H_\varepsilon(x, t_0)(v_\varepsilon - u_\varepsilon)\, dx . \tag{4.10}$$

On the other hand the function $y = \max(u_\varepsilon, v_\varepsilon)$ is in K_T, and therefore $\mathcal{G}_\varepsilon(v_\varepsilon) \leqslant \mathcal{G}_\varepsilon(y)$. This implies

$$\int_A \left(\sqrt{1+|Dv_\varepsilon|^2} - \sqrt{1+|Du_\varepsilon|^2}\right) dx + \int_A H_\varepsilon(x, t_0)(v_\varepsilon - u_\varepsilon)\, dx \leqslant \int_{\partial\Omega} \hat{\varkappa}_\varepsilon(y - v_\varepsilon)\, dH_{n-1}.$$

Comparing with (4.9) and (4.10):

$$\mathcal{F}_\varepsilon(w) \leqslant \mathcal{F}_\varepsilon(u_\varepsilon) + \int_{\partial_2\Omega} (w - u_\varepsilon + y - v_\varepsilon)\varkappa_\varepsilon\, dH_{n-1} + \\ + \int_{\partial_1\Omega} \{(1-\varepsilon)(y - v_\varepsilon) + |w - \varphi| - |u_\varepsilon - \varphi|\}\, dH_{n-1} \leqslant \mathcal{F}_\varepsilon(u_\varepsilon) .$$

From the uniqueness of the minimum we get

$$w = u_\varepsilon + c, \qquad c \leqslant 0$$

and hence either $w = u_\varepsilon$, so that $u_\varepsilon \leqslant v_\varepsilon$, or $w = v_\varepsilon$, and then

$$\mathcal{F}_\varepsilon(v_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon)\,. \qquad \text{Q.E.D.}$$

In any case there exists a function u_ε minimizing the functional $\mathcal{F}_\varepsilon$, satisfying $u_\varepsilon \leqslant v_\varepsilon$, so that an upper bound for v_ε will give a corresponding upper bound for u_ε.

In a similar way we can find a lower bound for u_ε by means of the solution w_ε to the problem

$$\int_\Omega \sqrt{1 + |Dw|^2} + \int_\Omega H_\varepsilon(x, -t_0)\, w\, dx + \int_{\partial_2\Omega} \varkappa_\varepsilon w\, dH_{n-1} - \int_{\partial_1\Omega} (1-\varepsilon)\, w\, dH_{n-1} \to \min$$

in the class

$$\{w \in BV(\Omega) \colon w \leqslant -T\}\,.$$

We are then reduced to the problem of finding a uniform upper bound for v_ε (and a lower bound for w_ε). Since the arguments are perfectly symmetrical we shall consider in detail only the first problem.

4.*D*. In order to avoid unnecessary complications we shall omit the suffix ε; we will derive an upper estimate for a function $v(x)$, minimizing the functional

$$\mathcal{G}(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \int_\Omega H(x)\, v\, dx + \int_{\partial\Omega} \hat{\varkappa} v\, dH_{n-1}$$

in the class K_T; the functions H and $\hat{\varkappa}$ satisfy the relations

$$\int_B H\, dx + \int_{\partial\Omega} \hat{\varkappa}\varphi_B\, dH_{n-1} \geqslant -\int_\Omega |D\varphi_B| + \alpha_0 \min\{|B|, |\Omega - B|\} \tag{4.11}$$

for every set $B \subset \Omega$;

$$\hat{\varkappa}(x) \geqslant -\varkappa_1 \qquad \forall x \in \partial\Omega - L_{-1} \tag{4.12}$$

$$H(x) \geqslant -\Delta d(x) + \alpha_1 \qquad \forall x \in L_{-3} \tag{4.13}$$

where L_{-1} and L_{-3} are open sets, with $L_{-1} \subset L_{-1} \subset L_{-3}$.

It is clear that $H_\varepsilon(x, t_0)$ and $\hat{\varkappa}_\varepsilon$ satisfy the preceding relations uniformly for $\varepsilon > 0$ in a neighborhood of 0.

LEMMA 4.2. *For every* $v \in K_T$ *we have*

$$\int_\Omega |v - v_\Omega| dx \leqslant 2\mathcal{G}(v)/\alpha_0 \,, \tag{4.14}$$

where v_Ω *denotes the mean value of* v *in* Ω.

PROOF. Let

$$V_t = \{x \in \Omega : v(x) > t\} \,;$$

we have from (4.11):

$$\int_0^\infty \left\{ \int_\Omega H\varphi_{V_t} dx + \int_{\partial\Omega} \hat{\varkappa}\varphi_{V_t} dH_{n-1} + \int_\Omega |D\varphi_{V_t}| \right\} dt \geqslant \alpha_0 \int_0^\infty \min\{|V_t|, |\Omega - V_t|\} dt$$

and hence

$$\mathcal{G}(v) \geqslant \int_\Omega |Dv| + \int_\Omega Hv\, dx + \int_{\partial\Omega} \hat{\varkappa} v\, dH_{n-1} \geqslant \alpha_0 \int_0^\infty \min\{|V_t|, |\Omega - V_t|\} dt \,.$$

Let τ be such that

$$|V_t| \geqslant |\Omega|/2 \quad \text{for } t < \tau$$

$$|V_t| \leqslant |\Omega|/2 \quad \text{for } t > \tau \,;$$

then

$$\int_0^\infty \min\{|V_t|, |\Omega - V_t|\} dt = \int_\Omega |v - \tau| dx$$

and (4.14) follows from the simple inequality

$$\int_\Omega |v - v_\Omega| dx \leqslant 2 \int_\Omega |v - \tau| dx \,. \qquad \text{Q.E.D.}$$

In particular, if v minimizes $\mathcal{G}$ in K_T, we have the uniform estimate:

$$\int_\Omega |v - v_\Omega| dx \leqslant 2\mathcal{G}(T)/\alpha_0 \leqslant c_{22} \,. \tag{4.15}$$

The next step consists in getting an estimate for the area of v in compact subsets of Ω. For $s > 0$ let

$$\Omega_s = \{x \in \Omega : d(x) > s\} \,.$$

LEMMA 4.3. *For every $s>0$ there exists a constant $c_{23}(s)$ such that if $v(x)$ minimizes $\mathcal{G}$ in K_T we have:*

$$\int_{\Omega_s}\sqrt{1+|Dv|^2}\,dx\leqslant c_{23}(s). \tag{4.16}$$

PROOF. Let $g(x)$ be a smooth function in Ω, with $0\leqslant g\leqslant 1$, $g=1$ in Ω_s and $g=0$ near $\partial\Omega$.

Let

$$w=\max(0,v-v_\Omega)$$

and let

$$h=v-gw.$$

We have $h\in K_T$ and therefore $\mathcal{G}(v)\leqslant\mathcal{G}(h)$. If we observe that $h=v$ near $\partial\Omega$, we get

$$\int_\Omega\left(\sqrt{1+|Dv|^2}-\sqrt{1+|Dh|^2}\right)dx\leqslant-\int_\Omega Hgw\,dx.$$

On the other hand

$$\sqrt{1+|Dv|^2}-\sqrt{1+|Dh|^2}=\sqrt{1+|Dw|^2}-\sqrt{1+|D(w-gw)|^2}\geqslant$$
$$\geqslant\sqrt{1+|Dw|^2}-\sqrt{1+(1-g)^2|Dw|^2}-w|Dg|$$

and whence

$$\int_{\Omega_s}\sqrt{1+|Dw|^2}\,dx\leqslant|\Omega|+\int_\Omega w(|H|+|Dg|)\,dx.$$

In a similar way, if $z=\min(0,v-v_\Omega)$ and $k=v-gz$, we get

$$\int_{\Omega_s}\sqrt{1+|Dz|^2}\,dx\leqslant|\Omega|+\int_\Omega|z|(|H|+|Dg|)\,dx.$$

Combining the last two inequalities:

$$\int_{\Omega_s}\sqrt{1+|Dv|^2}\,dx\leqslant 2|\Omega|+c_{24}(s)\int_\Omega(w+|z|)\,dx=2|\Omega|+c_{24}\int_\Omega|v-v_\Omega|\,dx$$

and (4.16) follows at once from (4.15). Q.E.D.

We conclude this section with an estimate of the oscillation of v.

LEMMA 4.4. *For every $s > 0$ there exists a constant $c_{25}(s)$ such that*

$$\operatorname*{osc}_{\Omega_{2s}}(v) \leq c_{25} \int_{\Omega_s} \sqrt{1 + |Dv|^2}\, dx \tag{4.17}$$

PROOF. Let

$$V = \{(x, t) \in \Omega \times \mathbf{R} : t < v(x)\} .$$

If B is a $(n+1)$-ball centered on $\partial V = \operatorname{graph}(v)$ and contained in the set $\Omega_s \times (T, +\infty)$, we have ([20]):

$$\int_B |D\varphi| + \int_B H\varphi_V\, dx\, dt \leq \int_B |D\varphi_Q| + \int_B H\varphi_Q\, dx\, dt$$

for every set Q coinciding with V in a neighborhood of ∂B.

From [19] we get the estimate

$$\int_B |D\varphi_V| \geq \omega_n r^n - n\omega_{n+1} \sup |H| r^{n+1}$$

where r is the radius of B.

If $n\omega_{n+1} \sup |H| r < \omega_n/2$ we have

$$\int_B |D\varphi_V| \geq \omega_n r^n/2 . \tag{4.18}$$

Let now

$$\lambda = \operatorname*{osc}_{\Omega_{2s}}(v);$$

if $r < s$ there are at least $[\lambda/2r]$ disjoint balls contained in $\Omega_s \times (T, +\infty)$ for which (4.18) holds. We have therefore

$$\int_{\Omega_s} \sqrt{1 + |Dv|^2}\, dx = \int_{\Omega_s \times \mathbf{R}} |D\varphi_V| \geq [\lambda/2r]\, \omega_n r^n/2$$

and (4.17) follows at once. Q.E.D.

4.*E*. The results of the preceding section give the uniform estimate

$$\operatorname*{osc}_{\Omega_s}(v) \leq c_{26}(s). \tag{4.19}$$

Using the supplementary conditions (4.12) and (4.13) we shall get a bound for the supremum of v in Ω.

Let $d_0 > 0$ be such that the distance function $d(x)$ is twice differentiable outside Ω_{d_0}, and let

$$A = 1 + \sup_{\Omega - \Omega_{d_0}} \{|\Delta d| + |H|\}. \tag{4.20}$$

The function

$$g(s) = \frac{1}{A}\{1 - (As + \varkappa_1)^2\}^{\frac{1}{2}}$$

satisfies the equation

$$g'' = -A(1 + g'^2)^{\frac{3}{2}}$$

in the interval $0 < s < (1 - \varkappa_1)/2A = s_0$.

If we set $s_1 = \min(d_0, s_0)$ and

$$z(x) = g(d(x)) - g(0)$$

we have, for every $x \in \Omega - \Omega_{s_1}$:

$$\mathfrak{L}(z) = -A - H(x) + \Delta d g'(1 + g'^2)^{-\frac{1}{2}} \tag{4.21}$$

where $\mathfrak{L}$ is the Euler operator relative to the functional $\mathfrak{G}$:

$$\mathfrak{L}(z) = (1 + |Dz|^2)^{-\frac{3}{2}}\{(1 + |Dz|^2)\,\Delta z - z_{x_i} z_{x_j} z_{x_i x_j}\} - H(x)\,.$$

From (4.20) it follows immediately

$$\mathfrak{L}(z) \leqslant -1 \tag{4.22}$$

so that z is a strict supersolution in the strip $\Omega - \Omega_{s_1}$.

Let now L_{-1} and L_{-3} be as in (4.12), (4.13), and let L_{-2} be an open set such that

$$L_{-1} \subset \bar{L}_{-1} \subset L_{-2} \subset \bar{L}_{-2} \subset L_{-3}\,.$$

Let $M(x)$ be a function of class C^2 in $\bar{\Omega}$, such that

$$M(x) = -1 \quad \text{in } L_{-1}$$

$$M(x) = 2/\alpha_1 \quad \text{outside } L_{-2}\,;$$

let $s_2 = \min(d_0, 2/\alpha_1)$, and for x in $\Omega - \Omega_{s_2}$ let

$$y(x) = M(x) - \frac{2}{\alpha_1}\{1 - (\alpha_1 d(x)/2 - 1)^2\}^{\frac{1}{2}} = M(x) + k(d(x))\,.$$

If we observe that the function $k(s)$ satisfies

$$k'' = \frac{\alpha_1}{2}(1 + k'^2)^{\frac{3}{2}}$$

we get easily:

$$\mathfrak{L}(y) \leqslant (1 + |Dy|^2)^{-\frac{3}{2}}\left\{m_1 + m_2 k'^2 + \frac{\alpha_1}{2}(1 + k'^2)^{\frac{3}{2}} + k'^3 \Delta d\right\} - H(x)$$

where m_1 and m_2 depend only on the C^2 norms of M and d.

Let now $x \in L_{-3} \cap (\Omega - \Omega_{s_2})$; from (4.13) we get

$$\mathfrak{L}(y) \leqslant (1 + |Dy|^2)^{-\frac{3}{2}}(m_1 + m_2 k'^2) + (\Delta d - \alpha_1/2)(1 + k'^3(1 + |Dy|^2)^{-\frac{3}{2}}) - \alpha_1/2 = \\ = -\alpha_1/2 + R\,.$$

As $s \to 0^+$, we have $|Dy| \to +\infty$ and $k'(1 + |Dy|^2)^{-\frac{1}{2}} \to -1$, and therefore $R \to 0$. We can conclude that there exists a positive number s_3 such that

$$\mathfrak{L}(y) \leqslant -\alpha_1/4 \qquad \text{in } L_{-3} \cap (\Omega - \Omega_{s_3})\,.$$

In addition we have

$$-1 < -1/A \leqslant z(x) \leqslant 0$$

and

$$-2/\alpha_1 \leqslant k(s) \leqslant 0$$

whence

$$y(x) = -1 + k(d) \leqslant -1 < z(x) \qquad x \in L_{-1}$$

$$y(x) = 2/\alpha_1 + k(d) \geqslant 0 > z(x) \qquad x \notin L_{-2}\,.$$

If we set

$$Z(x) = \min\{z(x), y(x)\}$$

the function Z is a strict supersolution for the functional $\mathcal{G}$ in the strip $\Omega - \Omega_{s_4}$ ($s_4 = \min(s_1, s_3)$), coinciding with y in L_{-1} and with z outside L_{-2}.

More precisely, if η is a non-negative function with support in the strip $\Omega - \Omega_{s_4}$, we have:

$$\int \frac{D_i Z D_i \eta}{\sqrt{1 + |DZ|^2}} dx + \int H\eta \, dx \geqslant 0 \tag{4.23}$$

the equality sign holding only for $\eta = 0$.

LEMMA 4.5. *For every $g \geqslant 0$, with $g = 0$ in Ω_{s_4}, we have:*

$$\int_\Omega D_i g D_i Z (1 + |DZ|^2)^{-\frac{1}{2}} dx + \int_\Omega Hg \, dx + \int_{\partial\Omega} \hat{\varkappa} g \, dH_{n-1} \geqslant 0 , \tag{4.24}$$

the equality sign holding only for $g = 0$.

PROOF. Let $0 < \varrho < s_4$

$$\varphi(x) = \min(d(x)/\varrho, 1)$$

and let

$$\eta(x) = g(x)\varphi(x) .$$

From (4.23) we get

$$\int_\Omega \varphi D_i g D_i Z (1 + |DZ|^2)^{-\frac{1}{2}} dx + \frac{1}{\varrho} \int_{\Omega - \Omega_\varrho} g D_i d D_i Z (1 + |DZ|^2)^{-\frac{1}{2}} dx + \int_\Omega \varphi H g \, dx \geqslant 0 . \tag{4.25}$$

As $\varrho \to 0$ we have

$$\frac{1}{\varrho} \int_{\Omega - \Omega_\varrho} g D_i d D_i Z (1 + |DZ|^2)^{-\frac{1}{2}} dx \to \int_{\partial\Omega} g\beta \, dH_{n-1}$$

where β satisfies

$$\begin{cases} \beta \leqslant -\varkappa_1 & \text{in } \partial\Omega \\ \beta = -1 & \text{in } \bar{L}_{-1} \cap \partial\Omega . \end{cases} \tag{4.26}$$

As $\beta \leqslant \hat{\varkappa}$ the conclusion of the lemma follows immediately passing to the limit in (4.25). Q.E.D.

It is clear that the conclusion of the lemma holds if Z is replaced by $Z + \text{const.}$

PROPOSITION 4.3. *Let v be a minimizing function for the functional $\mathcal{G}$ in the class K_T and let $p > 0$ be such that the function $w = Z + p$ satisfies*

$$w \geqslant T \quad \text{in } \Omega - \Omega_s$$
$$w \geqslant v \quad \text{in } \partial\Omega_s,\ s < s_4 .$$

Then

$$w \geqslant v \quad \text{in } \Omega - \Omega_s .$$

PROOF. Let

$$g = \begin{cases} v - \min(v, w) & \text{in } \Omega - \Omega_s \\ 0 & \text{in } \Omega_s . \end{cases}$$

The function g is Lipschitz-continuous and non-negative in Ω; we want to prove that $g = 0$.

Suppose this is not true, and let

$$m(t) = \mathcal{G}(\min(v, w) + tg) .$$

We have

$$m'(0) = \int_\Omega D_i w D_i g (1 + |Dw|^2)^{-\frac{1}{2}} dx + \int_\Omega Hg\, dx + \int_{\partial\Omega} \varkappa g\, dH_{n-1} > 0$$

and from the convexity of $\mathcal{G}$:

$$\mathcal{G}(v) = m(1) > m(0) = \mathcal{G}(\min(v, w))$$

contradicting the minimality of v. Q.E.D.

4.*F*. From Proposition 4.3 we get immediately the inequality

$$\sup_\Omega v(x) \leqslant \sup_{\Omega_s} v(x) + c_{27} , \qquad s < s_4 . \tag{4.27}$$

In wiew of (4.19) we need only an estimate for the quantity

$$m_s = \inf_{\Omega_s} v(x)$$

As in ch. 2 we discuss first the case where

$$\int_\Omega H\, dx + \int_{\partial\Omega} \varkappa\, dH_{n-1} \geqslant h_0 > 0 \tag{4.28}$$

(we remark that if (4.28) holds for $\varepsilon = 0$, it holds uniformly for $0 \leqslant \varepsilon \leqslant \frac{1}{2}$).

Let

$$B_s = \{x \in \Omega : v(x) < m_s\}$$

and let

$$v_s = \max(v, m_s) .$$

We have

$$\mathcal{G}(v_s) = \mathcal{G}(v) - \int_{B_s} \sqrt{1 + |Dv|^2}\, dx + |B_s| + \int_{\Omega} H w_s\, dx + \int_{\partial\Omega} \hat{\varkappa} w_s\, dH_{n-1}$$

where

$$w_s = \max(m_s - v, 0) .$$

Using lemma 4.1 we get easily:

$$\mathcal{G}(v_s) \leqslant \mathcal{G}(v) + |B_s|\{1 + (m_s - T)[c_{21}(\Omega) + \sup_{\Omega} |H|]\} =$$

$$= \mathcal{G}(v) + |B_s|[1 + c_{28}(m_s - T)] .$$

On the other hand we deduce from (4.28):

$$\mathcal{G}(v_s - m_s + T) = \mathcal{G}(v_s) - (m_s - T)\left\{ \int_{\Omega} H\, dx + \int_{\partial\Omega} \hat{\varkappa}\, dH_{n-1}\right\} \leqslant \mathcal{G}(v_s) - h_0(m_s - T)$$

and hence

$$\mathcal{G}(v_s - m_s + T) \leqslant \mathcal{G}(v) + |B_s| - (m_s - T)(h_0 - c_{28}|B_s|) .$$

Since $v_s - m_s + T \geqslant T$, we have $\mathcal{G}(v) \leqslant \mathcal{G}(v_s - m_s + T)$ and therefore

$$m_s \leqslant T + 2|\Omega|/h_0$$

provided s is so small that

$$c_{28}|B_s| \leqslant c_{28}|\Omega - \Omega_s| \leqslant h_0/2 .$$

In conclusion, if s is small enough we have the inequality

$$\inf_{\Omega_s} v(x) \leqslant c_{29} , \tag{4.29}$$

which eventually, together with (4.19) and (4.27), gives the required bound for the function $v(x)$:

$$\sup_{\Omega} v(x) \leq c_{30} . \tag{4.30}$$

4.*G*. It remains the case when

$$\int_{\Omega} H(x, t_0)\, dx + \int_{\partial_2\Omega} \varkappa\, dH_{n-1} + \int_{\partial_1\Omega} dH_{n-1} = 0 . \tag{4.31}$$

This is the typical situation when the curvature function H does not depend on t and $\partial_1\Omega = \emptyset$; actually (4.31) becomes a necessary condition in this case, as one can easily see from (4.2′) and (4.2″) (or else integrating equation (0.2) in Ω with the boundary conditions (0.3)).

As in 2.*F*, Lemma 2.5, we deduce from (4.11) and (4.31) the inequality

$$\left| \int_{B} H\, dx + \int_{\partial\Omega} \hat{\varkappa} \varphi_B\, dH_{n-1} \right| \leq \int_{\Omega} |D\varphi_B| - \alpha_0 \min\{|B|, |\Omega - B|\} \tag{4.32}$$

for every set $B \subset \Omega$, where

$$\hat{\varkappa}(x) = \begin{cases} \varkappa(x) & x \in \partial_2\Omega \\ 1 & x \in \partial_1\Omega \end{cases}$$

It follows from (4.32) and Proposition 4.1 (remember that $\partial_1\Omega$ is open in $\partial\Omega$) that the new function $\hat{\varkappa}$ satisfies the supplementary condition (4.5).

Let now $\varepsilon > 0$ and let v_ε be a minimizing function for the functional $\mathcal{G}_\varepsilon$ in K_T. We have as above the estimates

$$\operatorname*{osc}_{\Omega_s} (v_\varepsilon) \leq c_{26}(s) \tag{4.33}$$

and

$$\sup_{\Omega} v_\varepsilon \leq \sup_{\Omega_s} v_\varepsilon + c_{27} \tag{4.34}$$

with c_{26} and c_{27} independent of ε.

On the other hand from (4.31) we get $\mathcal{G}_\varepsilon(v) = \mathcal{G}_\varepsilon(v + \text{const})$, and hence v_ε minimizes $\mathcal{G}_\varepsilon$ in $BV(\Omega)$. From (4.32) and the supplementary condition (4.5) we conclude with the same argument as before

$$\inf_{\Omega} v_\varepsilon \geq \inf_{\Omega_s} v_\varepsilon - c_{31} \tag{4.35}$$

which, together with (4.33) and (4.34) gives

$$\operatorname*{osc}_{\Omega} (v_\varepsilon) \leq c_{32} .$$

Adding possibly a constant to the function v_ε we can suppose that $\inf_\Omega v_\varepsilon = T$, getting the required bound (4.30).

4.*H*. The estimate (4.30) and the lemma 4.2 give an *a-priori* bound for the supremum of the function u_ε, minimizing the functional $\mathcal{F}_\varepsilon$. This supremum is actually independent of ε.

In a similar way one can show that u_ε is bounded from below in Ω, so that we have the estimate

$$\sup_\Omega |u_\varepsilon(x)| \leqslant c_{33}\,. \tag{4.36}$$

Since

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leqslant \mathcal{F}_\varepsilon(0) = |\Omega| + \int_{\partial_1\Omega} |\varphi|\, dH_{n-1}$$

we have from (4.36) the inequality

$$\int_{\Omega_1} \sqrt{1+|Du_\varepsilon|^2} \leqslant c_{34} \tag{4.37}$$

where we have set, as usual, $u = f$ in $\Omega_1 - \Omega$.

Let now ε_j be a sequence converging to zero; from $\{u_{\varepsilon_j}\}$ we can extract a subsequence converging in $L_1(\Omega_1)$ to a function $u(x)$. It is easily seen that u gives a minimum for the functional $\mathcal{F}$. We have in conclusion:

THEOREM 4.1. *With the assumptions of section* 4.*A the functional*

$$\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \lambda(x, u)\, dx + \int_{\partial_1\Omega} |u - \varphi|\, dH_{n-1} + \int_{\partial_2\Omega} \varkappa u\, dH_{n-1}$$

has a minimum in $BV(\Omega)$.

It is clear that the results of ch. 3 apply to this case; in particular every minimizing function has Hölder-continuous second derivatives in Ω.

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