ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

C. PARENTI
F. STROCCHI
G. VELO

A local approach to some non-linear evolution equations of hyperbolic type

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 3, nº 3 (1976), p. 443-500
<http://www.numdam.org/item?id=ASNSP_1976_4_3_3_443_0>

A Local Approach to Some Non-Linear Evolution Equations of Hyperbolic Type. (*)

C. PARENTI (**) - F. STROCCHI (***) - G. VELO (*_*)

dedicated to Jean Leray

Summary. – *Existence, uniqueness and regularity theorems for a non-linear Cauchy problem of hyperbolic type in a suitable Fréchet space are proved. These results are used to treat a global in time initial value problem for systems of non-linear relativistic field equations. The initial data and the solution belong to the space of functions having locally finite kinetic energy.*

0. – Introduction.

In this paper we will be concerned with the existence and basic properties of solutions of non-linear hyperbolic partial differential equations of the form

$$\Box\varphi(x, t) - f\big(x, t; \varphi(x, t)\big) = 0 \tag{0.1}$$

where f is a possibly non-linear function of the variable φ.

The motivations for investigating this kind of equations are manifold. Equations of the type (0.1) arise in relativistic field theory and seem to play an important role in understanding the structure of elementary particles; their quantized version is the basis of relativistic quantum mechanics ([15]) and is supposed to govern the high energy physics. Recently, there has been

(*) Partially supported by C.N.R. (gruppo G.N.A.F.A.).
(**) Istituto Matematico « S. Pincherle », University of Bologna, Bologna, Italy.
(***) Scuola Normale Superiore, Pisa, Italy.
(*_*) Istituto di Fisica « A. Righi », University of Bologna, and I.N.F.N., Sez. di Bologna, Bologna, Italy.
Pervenuto alla Redazione il 31 Ottobre 1975.

great interest in a class of stable solutions (solitons) of these equations as starting points for constructing a theory of strongly interacting particles ([5]): a typical example is provided by the Sine-Gordon equation in two space-time dimensions ([4], [6]). Equations (0.1) arise also in a number of other areas of physics like non-linear optics and solid state ([14]).

The first proof of a global Cauchy problem for an interesting case of eq. (0.1) ($f = -m^2\varphi - \varphi^3$, with $m \neq 0$, in three space dimensions) was given by K. Jörgens [7]. Subsequently I. Segal ([10]) extended Jörgens' result by using a more abstract and powerful approach (see also [2], [8], [11], [3], [9]). In Segal's treatment the existence and uniqueness theorems are proved within the Hilbert space of initial data belonging to the Sobolev spaces $H^1(\boldsymbol{R}^s) \oplus L^2(\boldsymbol{R}^s)$ ($s =$ space dimensions), under suitable restrictions on the function f. This means that one identifies some « kinetic energy part » E_{kin} in the Hamiltonian (of the form $E_{\text{kin}} = \frac{1}{2}\int [(\nabla\varphi)^2 + \dot{\varphi}^2 + \varphi^2]\,dx$) and one considers only the solutions which have finite « kinetic energy ».

However there are physically interesting situations which are not covered by the discussion of ref. [10]. The function f may fail to satisfy the requirements of ref. [10], as, for example, in the constant external field problem ($f = -m^2\varphi + c$, c constant) and, in general, when spontaneous symmetry breaking solutions occur. Moreover there is a large class of solutions not having finite kinetic energy, i.e. not belonging to $H^1(\boldsymbol{R}^s)$. Of this type are the symmetry breaking solutions since they are of the form const $+ \chi$, with $\chi \in L^2(\boldsymbol{R}^s)$. Another important example is provided by the « soliton » like solutions of the Sine-Gordon equation whose behaviour at infinity is such that they do not belong to $L^2(\boldsymbol{R}^s)$.

In the approach presented in this paper we considerably enlarge the previously proposed ([10]) functional spaces in which eq. (0.1) was solved. This is obtained by looking for solutions for which the kinetic energy is locally finite but not necessarily globally finite. More precisely, the functional space we choose for discussing eq. (0.1) is

$$H^1_{\text{loc}}(\boldsymbol{R}^s) \oplus L^2_{\text{loc}}(\boldsymbol{R}^s) \equiv X$$

where the direct sum refers to the function φ and to its time derivative $\dot{\varphi}$, respectively. In this way we are able to cover the cases discussed above which were not included in Segal's treatment. The main result of this paper is in fact a uniqueness and existence proof for the Cauchy problem, for suitable functions f and for initial data in X.

It is worthwhile to mention some reasons for choosing the space X as a natural setting for our problem. From the physical point of view the local properties of the solutions are of extreme interest since any experiment and observation are necessarily localized in space. It is therefore justified to look for solutions for which the total energy is locally finite. A natural way to implement this condition is to work with the space X. The conditions we impose on the function f which guarantee the solvability of the initial value problem imply that the total energy is locally finite. The quantum field theory analog of locally finite kinetic energy is the condition of the theory being locally Fock. This condition seems to be satisfied for physically interesting interacting theories, whereas the globally Fock property does not hold.

An important feature which allows one to solve eq. (0.1) in the space X is an a priori estimate by which the function φ at a fixed time t ($t>0$) in a bounded region in space depends only on the value taken by the initial data ($t=0$) in another (larger) bounded region. As a consequence one may estimate the local norm of a solution at time t in terms of another local norm of the initial data. This phenomenon is essentially connected to the physical content of Huygens principle.

Instead of studying directly eq. (0.1) we prefer to isolate its relevant properties, and therefore discuss the problem in an abstract framework. We consider the equation

$$\frac{du}{dt} = Ku + f(t, u) \tag{0.2}$$

where

a) u belongs to a Fréchet space X which is the projective limit of a family of Banach spaces $B(\Omega)$, Ω being an arbitrary open bounded set of $\boldsymbol{R}^s$,

b) K is the generator of a continuous semigroup $W(t)$ in X ($t \geqslant 0$) with the property that

$$\|W(t)u\|_{\Omega(t)} \leqslant \text{const}\, e^{\omega t}\|u\|_{\Omega}\,, \tag{0.3}$$

Ω and $\Omega(t)$ being concentric spheres of radius R and $R-t$ respectively ($0 \leqslant t < R$),

c) $f(t, u)$ is a continuous mapping from $[0, T[\times X \to X$, $0 < T \leqslant \infty$, with the property that for any sphere Ω, any $\tau \in [0, T[$ and any $\varrho > 0$, there exists a positive constant $C(\Omega, \tau, \varrho)$ such that

$$\sup_{0 \leqslant t \leqslant \tau} \|f(t, u) - f(t, v)\|_{\Omega} \leqslant C(\Omega, \tau, \varrho)\|u - v\|_{\Omega}$$

for all $u, v \in X$ with $\|u\|_{\Omega} \leqslant \varrho$, $\|v\|_{\Omega} \leqslant \varrho$.

Within this structure, with some extra technical assumptions, we are able to prove an existence and uniqueness theorem for the Cauchy problem locally in time (Th. 1.3). By strengthening the assumptions a global theorem (Th. 1.6) is proved in subsect. (1.5).

Here we will content ourselves with listing some simple concrete cases of eq. (0.1) with a function f indipendent of space and time, to which the global existence theorem can be applied:

1) $$f(\varphi) = -\sum_{n=0}^{\infty} a_n \varphi^n$$

satisfying both the following conditions

i) $\sum_{n=1}^{\infty} |a_n| \sigma^n < \infty$ for all σ, in the case $s = 1$

$\sum_{n=1}^{\infty} |a_n| n^{n/2} \sigma^n < \infty$ for all σ, in the case $s = 2$

$a_n = 0$ for $n \geqslant 4$, in the case $s = 3$

ii) there exists real constants α, β, γ for which

$$\sum_{n=1}^{\infty} a_{n-1} \frac{\varphi^n}{n} \geqslant \alpha + \beta\varphi + \gamma\varphi^2 \qquad \forall \varphi \in \boldsymbol{R}$$

2) $f(\varphi) = a + b\varphi + c(\varphi^3 + d\varphi^2) + e \sin f\varphi$, with $c \leqslant 0$, for $s = 1, 2, 3$.

We mentioned explicitely the last example because it covers the case of Sine-Gordon type equations.

Finally the regularity properties of the solutions are discussed in detail both in the abstract setting and in the concrete cases.

Plan of the paper:

1. – Abstract formulation. Existence and uniqueness.

In this section we will discuss the integral equation corresponding to the Cauchy problem for the differential equation (0.2) in a suitable space X. The defining properties of X may be considered as an abstract version of similar properties satisfied by the usual *local Sobolev spaces*.

1.1. *Functional framework: the X space.*

Let $\mathcal{A}$ be the family of all open bounded (not empty) subsets of $\boldsymbol{R}^s$. To every $\Omega \in \mathcal{A}$ we associate a Banach space $B(\Omega)$, with norm $\|\cdot\|_{B(\Omega)}$, in such a way that the following conditions are satisfied:

i) For any $\Omega_1, \Omega_2 \in \mathcal{A}$, with $\Omega_1 \subset \Omega_2$, there is a continuous linear operator

$$r_{\Omega_1,\Omega_2}\colon B(\Omega_2) \to B(\Omega_1)\,,$$

called the « restriction » operator [1], with the properties

$$\|r_{\Omega_1,\Omega_2}(\varphi)\|_{B(\Omega_1)} \leqslant \|\varphi\|_{B(\Omega_2)}\,, \qquad \forall \varphi \in B(\Omega_2)\,,$$

$r_{\Omega_1,\Omega_1} =$ identity and $r_{\Omega_1,\Omega_2} r_{\Omega_2,\Omega_3} = r_{\Omega_1,\Omega_3} (\Omega_1 \subset \Omega_2 \subset \Omega_3)$.

ii) (Sub-additivity) For any $\Omega_1 \in \mathcal{A}$ and for any finite collection $\{\Omega_j\}_{j\in J} \subset \mathcal{A}$, with $\Omega_1 \subset \bigcup_{j\in J} \Omega_j = \Omega_2$, the following inequality holds

$$\|r_{\Omega_1,\Omega_2}(\varphi)\|_{B(\Omega_1)} \leqslant \sum_{j\in J} \|r_{\Omega_j,\Omega_2}(\varphi)\|_{B(\Omega_j)}$$

for all $\varphi \in B(\Omega_2)$.

(1) In the following such an operator will be simply denoted by r.

DEFINITION 1.1. The space X is now defined as the projective limit of the family $\{B(\Omega)\}_{\Omega\in\mathcal{A}}$ with respect to the restriction operators.

An element $u \in X$ is then identified with a family $\{u_\Omega\}_{\Omega\in\mathcal{A}}$, $u_\Omega \in B(\Omega)$, such that $u_{\Omega_1} = r_{\Omega_1,\Omega_2}(u_{\Omega_2})$, $\forall\Omega_1, \Omega_2 \in \mathcal{A}$ with $\Omega_1 \subset \Omega_2$. When equipped with the locally convex topology generated by the family of seminorms

$$X \ni u \mapsto \|u\|_\Omega \equiv \|u_\Omega\|_{B(\Omega)}, \qquad u = \{u_\Omega\}_{\Omega\in\mathcal{A}},$$

X is a complete space. It is also a Fréchet space ([1]) as a consequence of ii). Obviously we suppose X to be non trivial.

DEFINITION 1.2. $u \in X$ is said to vanish on Ω, $\Omega \in \mathcal{A}$, if $\|u\|_\Omega = 0$. The *support* of u, $\operatorname{supp} u$, is by definition the complement in $\boldsymbol{R}^s$ of the union of all $\Omega \in \mathcal{A}$ on which u vanishes.

Clearly if $\operatorname{supp} u$ is a compact set, then

$$\|u\|_{\Omega_1} = \|u\|_{\Omega_2} \qquad \forall\Omega_1, \ \Omega_2 \in \mathcal{A}, \ \operatorname{supp} u \subset \Omega_1 \cap \Omega_2 .$$

The following notation will be largely used in the sequel: if Ω is a sphere (²) of radius R, $\Omega(t)$ will denote the concentric sphere of radius $R - t$ $(0 \leqslant t < R)$, $\Omega'(t)$ the concentric sphere of radius $R + t$ $(t \geqslant 0)$ and an interval of time $[0, T]$ is called *admissible* with respect to Ω if $T \leqslant R/2$ (³).

We further require that X satisfies

CONDITION 1. For any sphere Ω of radius R and for any $u \in X$, the function $[0, R[\, \ni t \mapsto \|u\|_{\Omega(t)}$ is continuous.

CONDITION 2. (Space cut-off) For any $h \in N$ there is a linear map (cut-off map)

$$T_h : X \to X$$

with the properties

1) $\operatorname{supp} T_h\varphi \subset S_{h+1} = \{x \in \boldsymbol{R}^s \,|\, |x| \leqslant h + 1\}$, $\forall\varphi \in X$;

2) $\|\varphi - T_h\varphi\|_{\{x\in\boldsymbol{R}^s|\,|x|<h\}} = 0$, $\forall\varphi \in X$;

3) for any sphere Ω there exists a constant $\alpha(h, \Omega) > 0$ such that

$$\|T_h\varphi\|_\Omega \leqslant \alpha(h, \Omega)\|\varphi\|_\Omega \qquad \forall\varphi \in X .$$

(²) In this paper sphere will be sinonymous of open sphere.

(³) This condition of admissibility is chosen to ensure that $\inf_{0\leqslant t\leqslant T} \operatorname{diam}(\Omega(t)) > 0$. Any other upper bound on T yielding the same property would work as well.

We further require $\sup_{t\in[0,T]} \alpha(h, \Omega(t))$ to be finite if $[0, T]$ is an admissible interval of time for Ω.

A useful consequence of the subadditivity and of Condition 1 is that, given a continuous function $t \to u(t) \in X$, $t \in [0, T]$, for any sphere Ω with radius $> T$ the function $t \mapsto \|u(t)\|_{\Omega(t)}$ is continuous on $[0, T]$. This can be easily seen from the inequality

$$\Big|\|u(s')\|_{\Omega(s')} - \|u(s)\|_{\Omega(s)}\Big| \leqslant \|u(s') - u(s)\|_{\Omega(s')} + \Big|\|u(s)\|_{\Omega(s)} - \|u(s)\|_{\Omega(s')}\Big| .$$

1.2. *Statement of the problem: the integral equation.*

This subsection is devoted to the precise statement of our problem, namely the analysis of the integral equation

$$u(t) = W(t)u_0 + \int_0^t W(t-s) f(s, u(s))\, ds\,, \qquad u_0 \in X \tag{1.1}$$

under suitable assumptions on the « propagator » $W(t)$ and on the non-linear function f. Equation (1.1) may be considered as the integral version of the initial value problem for equation (0.2) if K is the « infinitesimal generator » of $W(t)$. We will look for a solution of eq. (1.1) in the space of continuous X-valued functions $t \mapsto u(t)$. For this purpose we assume that $W(t)$ and $f(t, u)$ belong to some definite classes of functions defined below.

DEFINITION 1.3. By $C(A, \omega)$ $(A \geqslant 1, \omega > 0)$ we will denote the class of all maps

$$U\colon [0, +\infty[\to \mathcal{L}(X, X)$$

such that

i) $(U(t))_{t\geqslant 0}$ is a strongly continuous semigroup ([13]);

ii) For any sphere Ω, for all $u \in X$ and for all t, $0 \leqslant t <$ radius of Ω, the following inequality

$$\|U(t)u\|_{\Omega(t)} \leqslant A e^{\omega t} \|u\|_{\Omega(0)} \tag{1.2}$$

holds.

Inequality (1.2) implies

$$\|U(\tau)u\|_{V(t)} \leqslant A e^{\omega\tau} \|u\|_{V(t-\tau)} \tag{1.2'}$$

for all $u \in X$, for any sphere V and for all t, τ with $0 \leqslant \tau \leqslant t <$ radius of V, as can be easily seen by putting $\Omega = V(t-\tau)$ in eq. (1.2).

In the following, we will call K the infinitesimal generator of a semigroup $U \in C(A, \omega)$. It is a densely defined closed operator.

DEFINITION 1.4. By $L([0, T[; X)$ $(0 < T \leqslant \infty)$ we denote the class of all maps

$$b\colon [0, T[\times X \to X$$

with the following properties

i) $b \in C^{(0)}([0, T[\times X; X)$;

ii) $b(t, 0) = 0$, $\forall t \in [0, T[$;

iii) For any sphere Ω, any $\tau \in [0, T[$ and any $\varrho > 0$, there exists a positive constant $C(\Omega, \tau, \varrho)$ such that

$$\sup_{0 \leqslant t \leqslant \tau} \|b(t, u) - b(t, v)\|_\Omega \leqslant C(\Omega, \tau, \varrho)\|u - v\|_\Omega \tag{1.3}$$

for all $u, v \in X$ with $\|u\|_\Omega$, $\|v\|_\Omega \leqslant \varrho$. We further require $\bar{C}(\Omega, \tau, \varrho) \equiv$ $\equiv \sup\limits_{0 \leqslant t \leqslant \tau} C(\Omega(t), \tau, \varrho)$ to be finite if $[0, \tau]$ is an admissible interval of time for Ω.

DEFINITION 1.5. By $L'([0, T[; X)$ $(0 < T \leqslant +\infty)$ we denote the subset of $L([0, T[; X)$ for which condition iii) holds in the stronger form

iii′) For any sphere Ω, for any $\tau \in [0, T[$, there exists a positive constant $C(\Omega, \tau)$ such that

$$\sup_{0 \leqslant t \leqslant \tau} \|b(t, u) - b(t, v)\|_\Omega \leqslant C(\Omega, \tau)\|u - v\|_\Omega$$

for all $u, v \in X$. We further require $\bar{C}(\Omega, \tau) \equiv \sup\limits_{0 \leqslant t \leqslant \tau} C(\Omega(t), \tau)$ to be finite if $[0, \tau]$ is an admissible interval of time for Ω.

REMARK. It is important to note and it will be used in the next subsection that the cut-off maps T_h, defined in Condition 2 of subsection 1.1, leave che classes $L([0, T[; X)$ and $L'([0, T[; X)$ invariant.

Inequality (1.2) is a sort of local energy estimate and implies that $U(t)$ propagate signals at finite velocity (hyperbolic character of $U(t)$). Properties iii) of Definitions 1.4 and 1.5 are a kind of local (in space) Lipschitz conditions.

In the following we will study eq. (1.1) under the assumptions that W belongs to a class $C(A, \omega)$ and f may be written as a sum

$$f = j + g \tag{1.5}$$

where $j \in C^{(0)}([0, T[; X)$ and $g \in L([0, T[; X)$. To show that under these hypotheses the right hand side of eq. (1.1) is well defined, provided $u \in C^{(0)}([0, T[; X)$, it is enough to prove that the function

$$[0, t] \ni s \mapsto W(t-s) f(s, u(s)) \in X \qquad (0 \leqslant t < T)$$

is continuous if $W \in C(A, \omega)$ and $f \in C^{(0)}([0, T[\times X; X)$ ([1]). This follows from the inequality

$$\begin{aligned} \| W(t-s) f(s, u(s)) - W(t-s') f(s', u(s')) \|_V \leqslant \\ \leqslant \| (W(t-s) - W(t-s')) f(s, u(s)) \|_V + \\ + A \exp[\omega(t-s')] \| f(s, u(s)) - f(s', u(s')) \|_{\Omega(s')} \end{aligned}$$

where $V \in \mathcal{A}$ and Ω is a sphere such that $\Omega(t) \supset V$.

1.3. *A priori estimates and removal of the space cut-off.*

A fundamental role is played by

THEOREM 1.1. *Let* $u_i \in C^{(0)}([0, T[; X)$ $(0 < T \leqslant +\infty)$, $i = 1, 2$, *be solutions of the integral equation*

$$u_i(t) = W(t) u_{0i} + \int_0^t W(t-s)[j_i(s) + g(s, u_i(s))]\, ds \tag{1.6}$$

with $W \in C(A, \omega)$, $u_{0i} \in X$, $j_i \in C^{(0)}([0, T[; X)$, $g \in L([0, T[; X)$. *Then, for any* $\tau \in [0, T[$ *and for any sphere* Ω *with radius greater than* τ *the following a priori estimate holds*

$$\begin{aligned} \| u_1(t) - u_2(t) \|_{\Omega(t)} \leqslant A \exp[(\omega + A\bar{C}(\Omega, \tau, \varrho))t] \cdot \\ \cdot \left\{ \| u_{01} - u_{02} \|_{\Omega(0)} + \int_0^t \| j_1(s) - j_2(s) \|_{\Omega(s)} ds \right\} \end{aligned} \tag{1.7}$$

for all $t \in [0, \tau]$, *with* $\varrho = \sup_{\substack{s \in [0, \tau] \\ i = 1, 2}} \| u_i(s) \|_{\Omega(s)}$.

PROOF. From the relation

$$u_1(t) - u_2(t) = W(t)(u_{01} - u_{02}) + \int_0^t W(t-s)[j_1(s) - j_2(s)]\,ds$$
$$+ \int_0^t W(t-s)[g(s, u_1(s)) - g(s, u_2(s))]\,ds$$

it follows

$$\|u_1(t) - u_2(t)\|_{\Omega(t)} \leqslant Ae^{\omega t}\Big\{\|u_{01} - u_{02}\|_{\Omega(0)} + \int_0^t e^{-\omega s}\|j_1(s) - j_2(s)\|_{\Omega(s)}\,ds\Big\} +$$
$$+ Ae^{\omega t}\int_0^t e^{-\omega s}\|g(s, u_1(s)) - g(s, u_2(s))\|_{\Omega(s)}\,ds\,.$$

The hypotheses on g yield

$$\|g(s, u_1(s)) - g(s, u_2(s))\|_{\Omega(s)} \leqslant \bar{C}(\Omega, \tau, \varrho)\|u_1(s) - u_2(s)\|_{\Omega(s)} \qquad \forall s \in [0, \tau]$$

and therefore by Gronwall's lemma one obtains the estimate (1.7).

COROLLARY 1. Let $W \in C(A, \omega)$, $j \in C^{(0)}([0, T[; X)$, $g \in L([0, T[; X)$, $u_0 \in X$ and $f = j + g$. Then the integral equation (1.1) has at most one solution $u \in C^{(0)}([0, T[; X)$.

PROOF. Trivial by Th. 1.1.

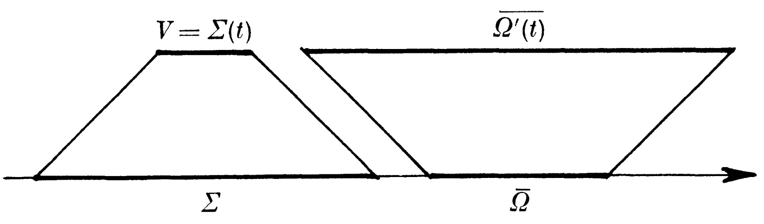
COROLLARY 2. Under the hypotheses of Cor. 1, if for a sphere Ω

$$\operatorname{supp} u_0 \subset \bar{\Omega}$$
$$\operatorname{supp} j(s) \subset \overline{\Omega'(s)}\,, \qquad 0 \leqslant s < T\,,$$

then any solution $u \in C^{(0)}([0, T[; X)$ of eq. (1.1) satisfies

$$\operatorname{supp} u(t) \subset \overline{\Omega'(t)}\,, \qquad 0 \leqslant t < T\,.$$

PROOF. Given $t \in [0, T[$, let V be any sphere with $V \cap \overline{\Omega'(t)} = \emptyset$. If Σ is the sphere such that $\Sigma(t) = V$, then obviously $\Sigma(s) \cap \overline{\Omega'(s)} = \emptyset$, $\forall s \in [0, t]$.



By applying Th 1.1 to the case $u_{02}=j_2=u_2=0$, $j_1=j$, $u_{01}=u_0$, $u_1=u$ and to the sphere Σ one obtains

$$\|u(t)\|_V=\|u(t)\|_{\Sigma(t)}\leqslant 0\,.$$

Our procedure to determine the solution of eq. (1.1) consists of two steps. First we study the theory with a space cut-off (subsects 1.4, 1.5) and then we discuss the removal of the cut-off. In the next Theorem (Th. 1.2) we will show how this removal can be performed. We need the preliminary

DEFINITION 1.6. Given $j\in C^{(0)}([0,T[;X)$, $g\in L([0,T[;X)$, $u_0\in X$ and a strictly increasing map $\ell\colon \boldsymbol{N}\to\boldsymbol{N}$, for any cut-off map T_h (see Condition 2) we define $j_h=T_h\circ j$, $g_{\ell(h)}=T_{\ell(h)}\circ g$, $u_{0h}=T_h\circ u_0$, $h\in\boldsymbol{N}$. The system $(j_h, g_{\ell(h)}, u_{0h})$ will be called the *ℓ-cut-off system of order h* corresponding to the system (j, g, u_0).

THEOREM 1.2. *Let $W\in C(A,\omega)$, $j\in C^{(0)}([0,T[;X)$, $g\in L([0,T[;X)$ $(0<T\leqslant+\infty)$, $u_0\in X$, and let $\boldsymbol{N}\ni h\to\ell(h)\in\boldsymbol{N}$ be a strictly increasing function. If, for every $h\in\boldsymbol{N}$, the integral equation*

$$u_h(t)=W(t)\,u_{0h}+\int_0^t W(t-s)\big[j_h(s)+g_{\ell(h)}(s,u_h(s))\big]\,ds \tag{1.8}$$

has a solution $u_h\in C^{(0)}([0,T[;X)$, then the integral equation (1.1), *with $f=j+g$, has a solution $u\in C^{(0)}([0,T[;X)$. Moreover to every $\tau\in[0,T[$ and every $V\in\mathcal{A}$ we can associate an integer $h(\tau,V)\in\boldsymbol{N}$ such that*

$$\sup_{0\leqslant t\leqslant\tau}\|u(t)-u_h(t)\|_V=0\,,\qquad \forall h\geqslant h(\tau,V)\,. \tag{1.9}$$

PROOF. The crucial point is to establish that to every $\tau\in[0,T[$ and every $V\in\mathcal{A}$ one can associate an integer $h(\tau,V)$ such that

$$\sup_{0\leqslant t\leqslant\tau}\|u_{h'}(t)-u_{h''}(t)\|_V=0\,,\qquad \forall h',h''\geqslant h(\tau,V)\,. \tag{1.10}$$

For this purpose we choose a sphere Ω such that $\Omega(\tau)\supset V$ and $[0,\tau]$ is an admissible interval of time for Ω. The eq. (1.8) yields

$$\begin{aligned}\|u_{h'}(t)-u_{h''}(t)\|_V&\leqslant\|u_{h'}(t)-u_{h''}(t)\|_{\Omega(t)}\leqslant\\ &\leqslant A\exp[\omega t]\Big\{\|u_{0h'}-u_{0h''}\|_\Omega+\int_0^t\|j_{h'}(s)-j_{h''}(s)\|_{\Omega(s)}\,ds+\\ &\quad+\int_0^t\exp[-\omega s]\big[\|g_{\ell(h')}(s,u_{h'}(s))-g_{\ell(h'')}(s,u_{h'}(s))\|_{\Omega(s}+\\ &\quad+\|g_{\ell(h'')}(s,u_{h'}(s))-g_{\ell(h'')}(s,u_{h''}(s))\|_{\Omega(s)}\big]\,ds\Big\}\,.\end{aligned}$$

The properties of the cut-off maps imply that

$$\|u_{0h'} - u_{0h''}\|_\Omega = 0\,, \qquad \sup_{0\le s\le \tau} \|j_{h'}(s) - j_{h''}(s)\|_{\Omega(s)} = 0$$

$$\sup_{0\le s\le \tau} \|g_{\ell(h')}(s, u_{h'}(s)) - g_{\ell(h'')}(s, u_{h'}(s))\|_{\Omega(s)} = 0$$

provided that h', h'' are greater than a suitable $h(\tau, V) \in \boldsymbol{N}$. Now for any pair $h', h'' \geq h(\tau, V)$ there is a $\varrho = \varrho(h', h'') > 0$ such that $\|u_{h'}(s)\|_{\Omega(s)}$, $\|u_{h''}(s)\|_{\Omega(s)} \leq \varrho$, $\forall s \in [0, \tau]$. Consequently

$$\|g_{\ell(h'')}(s, u_{h'}(s)) - g_{\ell(h'')}(s, u_{h''}(s))\|_{\Omega(s)} \leq$$
$$\leq \sup_{s\in[0,\tau]} \alpha(\ell(h''), \Omega(s))\bar{C}(\Omega, \tau, \varrho)\|u_{h'}(s) - u_{h''}(s)\|_{\Omega(s)}$$

and therefore, by Gronwall's Lemma, one obtains equality (1.10). The completness of $C^{(0)}([0, T[; X)$ implies now the existence of a $u \in C^{(0)}([0, T[; X)$ satisfying eq. (1.9). To show that u is actually a solution of the integral equation (1.1) it suffices to recognize that the quantity

$$\sup_{0\le t\le\tau} \|W(t)(u_{0h} - u_0) + \int_0 W(t-s)(j_h(s) - j(s))\,ds +$$
$$+ \int_0^t W(t-s)[g_{\ell(h)}(s, u_h(s)) - g(s, u(s))]\,ds\|_V$$

vanishes for h large enough. This can be seen by the same arguments as used above.

1.4. *Perturbative solution.*

In this subsection we first establish a local in time existence theorem for eq. (1.1) by a perturbative technique. (The assumptions used are satisfied by the cut-off theory). Then, in Th. 1.5, sufficient conditions are given to continue a solution defined in the time interval $[0, T[$ beyond the time T.

DEFINITION 1.7. Given $\tau, \varrho > 0$ and a family $\mathcal{F} = \{\Omega_k\}_{k\in \boldsymbol{N}}$ of spheres, with radii $> \tau$, such that $\{\Omega_k(\tau)\}_{k\in\boldsymbol{N}}$ is a locally finite covering of $\boldsymbol{R}^s$, we define the space

$$E(\mathcal{F}; \tau, \varrho) = \left\{\varphi \in C^{(0)}([0, \tau]; X) \middle|\ |\varphi|_{k,\tau} \equiv \sup_{0\le t\le\tau} \|\varphi(t)\|_{\Omega_k(t)} \leq \varrho\,,\ \forall k \in \boldsymbol{N}\right\}$$

equipped with the metric

$$d(\varphi, \psi) = \sum_{k=1}^{\infty} 2^{-k} |\varphi - \psi|_{k,\tau} .$$

It is easy to verify that the space $E(\mathcal{F}; \tau, \varrho)$ is complete.

THEOREM 1.3. *Suppose we are given* $W \in C(A, \omega)$, $j \in C^{(0)}([0, T[; X)$, $g \in C^{(0)}([0, T[\times X; X)$ *with* $g(t, 0) = 0$, $\forall t$ *and* $u_0 \in X$. *Let us assume the existence of a family* $\mathcal{F} = \{\Omega_k\}_{k\in N}$ *of spheres, with the property that* $\{\Omega_k(\tau)\}_{k\in N}$ *is a locally finite covering of* $\boldsymbol{R}^s$ *for a suitable* $\tau \in]0, T[$, *and such that there is a* $\varrho > 0$ *for which the following conditions hold*

1) $\sup_{k\in N} \|u_0\|_{\Omega_k} \leqslant \varrho/2A$;

2) $\sup_{\substack{t\in[0,\tau]\\ k\in N}} \|j(t)\|_{\Omega_k(t)} = \delta < +\infty$;

3) $\exists C(\mathcal{F}, \tau, \varrho) > 0$:

$$\|g(s, u) - g(s, v)\|_{\Omega_k(s)} \leqslant C(\mathcal{F}, \tau, \varrho) \|u - v\|_{\Omega_k(s)}$$

$\forall k \in N$, $\forall s \in [0, \tau]$ *and* $\forall u, v \in X$, *with*

$$\sup_{\substack{k\in N\\ s\in[0,\tau]}} \|u\|_{\Omega_k(s)}, \qquad \sup_{\substack{k\in N\\ s\in[0,\tau]}} \|v\|_{\Omega_k(s)} \leqslant \varrho .$$

Then there is a $\tau_0 \in]0, \tau[$ *for which the integral equation* (1.1), *with* $f = j + g$, *has a unique solution* $u \in E(\mathcal{F}; \tau_0, \varrho)$.

PROOF. One can show, by an argument analogous to the one at the end of subsetc. 1.2, that the operator S defined by

$$(1.11) \qquad (Su)(t) \equiv W(t) u_0 + \int_0^t W(t-s)\, [j(s) + g(s, u(s))]\, ds$$

maps $C^{(0)}([0, \tau]; X)$ into itself. From eq. (1.11) it follows that

$$\|(Su)(t)\|_{\Omega_k(t)} \leqslant Ae^{\omega t}\Big\{\|u_0\|_{\Omega_k} + \int_0^t \|j(s)\|_{\Omega_k(s)} + \int_0^t \|g(s, u(s))\|_{\Omega_k(s)}\, ds\Big\} \leqslant$$
$$\leqslant Ae^{\omega t}\{\varrho/2A + \delta t + C(\mathcal{F}, \tau, \varrho)\varrho t\} \equiv \zeta(t), \quad 0 \leqslant t \leqslant \tau,\ k \in N$$

provided $u \in E(\mathcal{F}; \tau, \varrho)$. Since there exists a unique $\tau_1 \in]0, \tau]$ for which $\zeta(\tau_1) = \varrho$, the operator S maps $E(\mathcal{F}; \sigma, \varrho)$ into itself for all $\sigma \in]0, \tau_1]$. Moreover, from the obvious inequality

$$|Su - Sv|_{k,\sigma} \leqslant AC(\mathcal{F}, \tau, \varrho) e^{\sigma} \sigma |u - v|_{k,\sigma}, \qquad 0 \leqslant \sigma \leqslant \tau_1, \ k \in \boldsymbol{N},$$

it is clear that we can choose a $\tau_0 \in]0, \tau_1]$ for which S is a contraction of $E(\mathcal{F}; \tau_0, \varrho)$ into itself. The result is now a consequence of the Banach theorem on contractions.

COROLLARY 3. Under the same hypotheses of Th. 1.3 there is a unique solution of eq. (1.1) ($f = j + g$) belonging to the space $C^{(0)}([0, \tau_0]; X)$.

PROOF. Obviously the solution found in Th. 1.3 belongs to $C^{(0)}([0, \tau_0]; X)$. Its uniqueness (within this space) follows by the same kind of estimates used in the proof of Th. 1.1.

THEOREM 1.4. *Let* $j \in C^{(0)}([0, T[; X)$, $g \in L([0, T[, X)$, $u_0 \in X$ *and* $h \to \ell(h)$ *be a strictly increasing map from* $\boldsymbol{N}$ *to* $\boldsymbol{N}$. *Then, for every* $h \in \boldsymbol{N}$ *the corresponding* ℓ*-cut-off system of order* h (see Def. 1.6) *satisfies the assumptions of Th.* 1.3.

PROOF. Let $\tau \in]0, T[$ and let $\mathcal{F} = \{\Omega_k\}_{k \in \boldsymbol{N}}$ be any family of spheres for which $[0, \tau]$ is an admissible interval of time and $\{\Omega_k(\tau)\}_{k \in \boldsymbol{N}}$ is a locally finite covering of $\boldsymbol{R}^s$ (the existence of such a family is obvious). For a fixed value of h, the set

$$J_h = \{k \in \boldsymbol{N} | \Omega_k \cap S_{\ell(h)+1} \neq \emptyset\}$$

is finite and by Condition 2 it follows that

$$\|u_{0h}\|_{\Omega_k} = \sup_{s \in [0,\tau]} \|j_h(s)\|_{\Omega_k} = \sup_{s \in [0,\tau]} \|g_{\ell(h)}(s, u)\|_{\Omega_k} = 0$$

for all $k \notin J_h$. It is then clear that

$$\sup_{\substack{s \in [0,\tau] \\ k \in \boldsymbol{N}}} \|j_h(s)\|_{\Omega_k} < +\infty \qquad \text{and} \qquad \sup_{k \in \boldsymbol{N}} \|u_{0h}\|_{\Omega_k} \leqslant \varrho / 2A$$

for a suitable $\varrho > 0$. Finally, if $k \in J_h$ the properties of the cut-off maps and of the function g yield

$$\|g_{\ell(h)}(s, u) - g_{\ell(h)}(s, v)\|_{\Omega_k(s)} \leqslant \Big(\sup_{s \in [0,\tau]} \alpha\big(\ell(h), \Omega_k(s)\big)\Big) \bar{C}(\Omega_k, \tau, \varrho) \|u - v\|_{\Omega_k(s)}$$

for all $s \in [0, \tau]$ and all $u, v \in X$ with $\|u\|_{\Omega_k(s)}$, $\|v\|_{\Omega_k(s)} < \varrho$ for all $k \in \boldsymbol{N}$. Consequently, condition 3 of Th. 1.3 is satisfied with

$$C(\mathcal{F}, \tau, \varrho) = \sup_{k \in J_h} \left[\bar{C}(\Omega_k, \tau, \varrho) \sup_{s \in [0,\tau]} \alpha(\ell(h), \Omega_k(s)) \right].$$

THEOREM 1.5. *Suppose we are given* $W \in C(A, \omega)$, $j \in C^0([0, T[; X)$, $g \in C^{(0)}([0, T[\times X; X)$, *with* $g(t, 0) = 0$, $\forall t$ *and* $u_0 \in X$. *Let* $u \in C^{(0)}([0, T_1[; X)$, $T_1 < T$, *be a solution of eq.* (1.1) *with* $f = j + g$. *Let us assume the existence of a family* $\mathcal{F} = \{\Omega_k\}_{k \in \boldsymbol{N}}$ *of spheres with the property that* $\{\Omega_k(2\varepsilon)\}_{k \in \boldsymbol{N}}$ *is a locally finite covering of* $\boldsymbol{R}^s$ *for a suitable* ε, $0 < \varepsilon < T - T_1$, *and such that there is a* $\varrho > 0$ *for which the following hypotheses are satisfied*

1) *There is a sequence* $t_n \uparrow T_1$ *such that*

$$\sup_{n,k \in \boldsymbol{N}} \|u(t_n)\|_{\Omega_k} \leqslant \varrho/2A$$

2) $$\sup_{\substack{s \in [T_1 - \varepsilon, T_1 + \varepsilon] \\ k \in \boldsymbol{N}}} \|j(s)\|_{\Omega_k(s - T_1 + \varepsilon)} = \delta < +\infty;$$

3) $\exists C(\mathcal{F}, \varrho) > 0$ *such that for any closed interval* $I \subset [T_1 - \varepsilon, T_1 + \varepsilon]$ *the inequality*

$$\|g(s, u) - g(s, v)\|_{\Omega_k(s - T_1 + \varepsilon)} \leqslant C(\mathcal{F}, \varrho) \|u - v\|_{\Omega_k(s - T_1 + \varepsilon)}$$

holds for every $s \in I$, $k \in \boldsymbol{N}$ *and for all,* $u, v \in X$ *with*

$$\sup_{\substack{s \in I \\ k \in \boldsymbol{N}}} \|u\|_{\Omega_k(s - T_1 + \varepsilon)}, \qquad \sup_{\substack{s \in I \\ k \in \boldsymbol{N}}} \|v\|_{\Omega_k(s - T_1 + \varepsilon)} \leqslant \varrho .$$

Then the solution $u(t)$ *can be continued beyond* T_1.

PROOF. For all n for which $t_n \geqslant T_1 - \varepsilon$ we define $\Sigma_k^{(n)} \equiv \Omega_k(t_n - T_1 + \varepsilon)$, $k \in \boldsymbol{N}$. It is then obvious that the family $\{\Sigma_k^{(n)}(\varepsilon)\}_{k \in \boldsymbol{N}}$ is a locally finite covering of $\boldsymbol{R}^s$. As an immediate consequence of the assumptions 1), 2), 3) one obtains

$$\sup_{k \in \boldsymbol{N}} \|u(t_n)\|_{\Sigma_k^{(n)}(\varepsilon)} \leqslant \varrho/2A ,$$

$$\sup_{\substack{\sigma \in [0,\varepsilon] \\ k \in \boldsymbol{N}}} \|j(t_n + \sigma)\|_{\Sigma_k^{(n)}(\sigma)} = \sup_{\substack{\sigma \in [0,\varepsilon] \\ k \in \boldsymbol{N}}} \|j(t_n + \sigma)\|_{\Omega_k(\sigma + t_n - T_1 + \varepsilon)} \leqslant \delta ,$$

$$\|g(t_n + \sigma, u) - g(t_n + \sigma, v)\|_{\Sigma_k^{(n)}(\sigma)} = \|g(t_n + \sigma, u) - g(t_n + \sigma, v)\|_{\Omega_k(\sigma + t_n - T_1 + \varepsilon)} \leqslant$$
$$\leqslant C(\mathcal{F}, \varrho) \|u - v\|_{\Sigma_k^{(n)}(\sigma)}$$

for all $\sigma \in [0, \varepsilon]$ and for all $u, v \in X$ with

$$\sup_{\substack{\sigma \in [0,\varepsilon] \\ k \in \boldsymbol{N}}} \|u\|_{\Sigma_k^{(n)}(\sigma)}, \qquad \sup_{\substack{\sigma \in [0,\varepsilon] \\ k \in \boldsymbol{N}}} \|v\|_{\Sigma_k^{(n)}(\sigma)} \leqslant \varrho .$$

We can now apply Th. 1.3 and Cor. 3 to the integral equation

$$\varphi_n(\sigma) = W(\sigma)u(t_n) + \int_0^\sigma W(\sigma - s) f\big(t_n + s, \varphi_n(s)\big)\, ds \tag{1.11'}$$

with respect to the family $\mathcal{F}_n = \{\Sigma_k^{(n)}\}_{k\in N}$. There exists a $\tau_0 \in \,]0, \varepsilon]$, independent of n, for which eq. (1.11′) has a unique solution $\varphi_n \in C^{(0)}([0, \tau_0]; X)$. It is then easy to see that the function

$$v(t) = \begin{cases} u(t)\,, & 0 \leqslant t \leqslant t_{\bar n} \\ \varphi_{\bar n}(t - t_{\bar n})\,, & t_{\bar n} < t \leqslant t_{\bar n} + \tau_0 \end{cases}$$

defined for a $t_{\bar n}$ with $t_{\bar n} + \tau_0 > T_1$, belongs to $C^{(0)}([0, t_{\bar n} + \tau_0]; X)$. Finally, by a change of variable in eq. (1.11′), one recognizes that v is a solution of eq. (1.1) in the interval $[0, t_{\bar n} + \tau_0]$ which continues u beyond T_1.

1.5. *Global existence theorem.*

Throughout the whole subsection we make the assumption that, for each $\Omega \in \mathcal{A}$, $B(\Omega)$ is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_\Omega$. Then we prove a global existence and uniqueness theorem for the integral equation (1.1) by using Th. 1.5, provided the following essential hypothesis is satisfied.

HYPOTHESIS A. Given $W \in C(1, \omega)$, $j \in C^{(0)}([0, T[; X)$, $g^{(1)} \in L([0, T[; X)$, $g^{(2)} \in L'([0, T[; X)$, $0 < T \leqslant +\infty$, we say that $(W, f = j + g^{(1)} + g^{(2)})$ satisfies the Hypothesis A if for any sphere Ω there exist two maps

$$V_\Omega,\ W_\Omega \colon [0, T_\Omega[\times X \to \mathbf{R}\,, \qquad T_\Omega = \min\{T, \text{radius of } \Omega\}\,,$$

with the following properties

a) For any $T_1 \in \,]0, T_\Omega[$

$$\inf_{\substack{u\in X,\ \mathrm{supp}\, u \subset \Omega \\ t\in[0,T_1]}} V_\Omega(t, u) > -\infty\,. \tag{1.12}$$

b) For any $T_1 \in \,]0, T_\Omega[$ there exist two constants $M(\Omega, T_1)$, $N(\Omega, T_1) \geqslant 0$ such that

$$W_\Omega(t, u) \leqslant M(\Omega, T_1)\big[V_\Omega(t, u) + \|u\|_\Omega^2\big] + N(\Omega, T_1) \tag{1.13}$$

for all $t \in [0, T_1]$ and for all $u \in X$ with supp $u \subset \Omega$.

c) For all ℓ–cut-off systems of any order h corresponding to $(j, g = g^{(1)} + g^{(2)}, u_0)$ and for any solution u_h of eq. (1.8) which is continuous on the interval $[0, T_1[$, $T_1 \in]0, T_\Omega[$, and such that $\operatorname{supp} u_h(t) \subset \overline{\Omega_1(t)} \subset \Omega(t)$, for a suitable sphere Ω_1, $0 \leqslant t < T_1$, the functions $t \mapsto V_\Omega(t, u_h(t))$, $W_\Omega(t, u_h(t))$ are continuous and moreover the equality

$$(1.14)\qquad \operatorname{Re}\int_0^t \langle u_h(s), g^{(1)}(s, u_h(s))\rangle_{\Omega(s)}\, ds = -\, V_\Omega(t, u_h(t)) + \\ + V_\Omega(0, u_{0h}) + \int_0^t W_\Omega(s, u_h(s))\, ds$$

holds for all $t \in [0, T_1[$.

Hypothesis A is suggested by energy considerations. The quantity $V_\Omega(t, u)$ should be identified with the content of potential energy in the sphere $\Omega(t)$, corresponding to the function $g^{(1)}$, whereas the function $W_\Omega(t, u)$ may be identified with the variation in time of the potential energy density, integrated over the sphere $\Omega(t)$. This interpretation roughly explains the meaning of eq. (1.14), which, however, (in condition *c*)), is required to be satisfied only for a rather special class of functions. This is exactly what we need to prove Th. 1.6. Moreover, in the applications we have in mind (see Sect. 3), equality (1.14) can be directly checked to hold for a class of functions much larger than the one considered in condition *c*). Concerning conditions *a*) and *b*), inequality (1.12) essentially means that the potential energy must be bounded from below, locally in space and time, and inequality (1.13) states that its rate of increase must be locally controlled by the sum of the kinetic and potential energy.

LEMMA. Suppose we are given $W \in C(1, \omega)$, $j \in C^{(0)}([0, T[; X)$, $g \in C^{(0)}([0, T[\times X; X)$ with $g(t, 0) = 0$, $\forall t$, and $u_0 \in X$. Let $u \in C^{(0)}([0, T_1[; X)$, $T_1 < T$, be a solution of eq. (1.1) with $f = j + g$, such that

$$\operatorname{supp} u(t) \subset \overline{\Omega_1(t)}\,, \qquad 0 \leqslant t < T_1\,,$$

for a suitable sphere Ω_1. Then, for any sphere $\Omega \supset \bar{\Omega}_1$ for which $[0, T_1]$ is an admissible interval of time, the following inequality holds

$$(1.15)\qquad \|u(t)\|^2_{\Omega(t)} \leqslant \|u_0\|^2_{\Omega(0)} + 2\omega\int_0^t \|u(s)\|^2_{\Omega(s)}\, ds + 2\operatorname{Re}\int_0^t \langle u(s), f(s, u(s))\rangle_{\Omega(s)}\, ds$$

for all $t \in [0, T_1[$.

PROOF. To obtain inequality (1.15) it is convenient to reduce the integral equation (1.1) to a differential equation. For this purpose, let us define the bounded operator (mollifier)

$$M_\varepsilon\colon X \to X\,, \qquad M_\varepsilon u \equiv \frac{1}{\varepsilon}\int_0^\varepsilon W(t)u\,dt\,, \qquad \varepsilon > 0\,.$$

It can be easily verified that M_ε has the following properties

a) $M_\varepsilon u \xrightarrow[\varepsilon\to 0^+]{} u$ in X, $\forall u \in X$;

b) $M_\varepsilon W(t) = W(t)M_\varepsilon$, $\forall t \geqslant 0$, $\forall \varepsilon > 0$;

c) $M_\varepsilon K \subset K M_\varepsilon$, $\forall \varepsilon > 0$, where K is the infinitesimal generator of $\big(W(t)\big)_{t\geqslant 0}$.

Let Ω be any sphere containing $\bar{\Omega}_1$ for which $[0, T_1]$ is an admissible interval of time. Then, as a consequence of Def. 1.3, there is an $\bar{\varepsilon} > 0$ for which $\operatorname{supp}\big(M_\varepsilon u(t)\big) \subset \Omega(t)$, $t \in [0, T_1[$, $\varepsilon \leqslant \bar{\varepsilon}$. Application of the operator M_ε to eq. (1.1) yields

$$M_\varepsilon u(t) = W(t)M_\varepsilon u_0 + \int_0^t W(t-s)M_\varepsilon f\big(s, u(s)\big)\,ds\,. \tag{1.16}$$

By property c) we can differentiate eq. (1.16) and obtain

$$\dot{v}_\varepsilon(t) \equiv \frac{d}{dt}v_\varepsilon(t) = Kv_\varepsilon(t) + M_\varepsilon f\big(t, u(t)\big) \tag{1.17}$$

where $v_\varepsilon(t) \equiv M_\varepsilon u(t)$, $t \in [0, T_1[$, $\varepsilon \leqslant \bar{\varepsilon}$. From eq. (1.17) it follows

$$\begin{aligned} \langle \dot{v}_\varepsilon(t), v_\varepsilon(t)\rangle_{\Omega(t)} + \langle v_\varepsilon(t), \dot{v}_\varepsilon(t)\rangle_{\Omega(t)} &= \\ = 2\operatorname{Re}\langle v_\varepsilon(t), Kv_\varepsilon(t)\rangle_{\Omega(t)} &+ 2\operatorname{Re}\langle v_\varepsilon(t), M_\varepsilon f\big(t, u(t)\big)\rangle_{\Omega(t)}\,. \end{aligned} \tag{1.18}$$

On the other hand the limit as $h \to 0$ of the expression

$$\begin{aligned} &\frac{1}{h}\big[\langle v_\varepsilon(t+h), v_\varepsilon(t+h)\rangle_{\Omega(t+h)} - \langle v_\varepsilon(t), v_\varepsilon(t)\rangle_{\Omega(t)}\big] \\ &\quad = \left\langle \frac{v_\varepsilon(t+h) - v_\varepsilon(t)}{h}, \; v_\varepsilon(t+h)\right\rangle_{\Omega(t+h)} + \left\langle v_\varepsilon(t), \frac{v_\varepsilon(t+h) - v_\varepsilon(t)}{h}\right\rangle_{\Omega(t+h)} + \\ &\quad + \frac{1}{h}\big[\langle v_\varepsilon(t), v_\varepsilon(t)\rangle_{\Omega(t+h)} - \langle v_\varepsilon(t), v_\varepsilon(t)\rangle_{\Omega(t)}\big] \equiv J_1 + J_2 + J_3 \end{aligned}$$

can be easily computed. In fact

$$J_1 \xrightarrow[h\to 0]{} \langle \dot{v}_\varepsilon(t), v_\varepsilon(t)\rangle_{\Omega(t)}, \qquad J_2 \xrightarrow[h\to 0]{} \langle v_\varepsilon(t), \dot{v}_\varepsilon(t)\rangle_{\Omega(t)}$$

and $J_3 = 0$ for h sufficiently small, because for such an h $\operatorname{supp}(v_\varepsilon(t)) \subset \subset \Omega(t+h) \cap \Omega(t)$. Then eq. (1.18) becomes

$$\text{(1.19)} \qquad \frac{d}{dt}\|v_\varepsilon(t)\|^2_{\Omega(t)} = 2\operatorname{Re}\langle v_\varepsilon(t), Kv_\varepsilon(t)\rangle_{\Omega(t)} + 2\operatorname{Re}\langle v_\varepsilon(t), M_\varepsilon f(t, u(t))\rangle_{\Omega(t)}.$$

Now, from the obvious estimate

$$|\langle v_\varepsilon(t), e^{-\omega h} W(h) v_\varepsilon(t)\rangle_{\Omega(t)}| \leqslant \|v_\varepsilon(t)\|_{\Omega(t)} \|e^{-\omega h} W(h) v_\varepsilon(t)\|_{\Omega(t)} \leqslant \|v_\varepsilon(t)\|_{\Omega(t)} \|v_\varepsilon(t)\|_{\Omega(t-h)},$$

since $\|v_\varepsilon(t)\|_{\Omega(t-h)} = \|v_\varepsilon(t)\|_{\Omega(t)}$ for h sufficiently small (see Def. 1.2), it follows

$$\operatorname{Re}\langle v_\varepsilon(t), \big(e^{-\omega h} W(h) - 1\big) v_\varepsilon(t)\rangle_{\Omega(t)} \leqslant 0$$

and therefore

$$\text{(1.20)} \qquad \operatorname{Re}\langle v_\varepsilon(t), Kv_\varepsilon(t)\rangle_{\Omega(t)} \leqslant \omega \|v_\varepsilon(t)\|^2_{\Omega(t)}.$$

Substitution of inequality (1.20) in eq. (1.19) and integration from 0 to t, $(t < T_1)$, yields

$$\text{(1.21)} \qquad \|v_\varepsilon(t)\|^2_{\Omega(t)} \leqslant \|v_\varepsilon(0)\|^2_{\Omega(0)} + 2\omega \int_0^t \|v_\varepsilon(s)\|^2_{\Omega(s)}\, ds + {} + 2\operatorname{Re}\int_0^t \langle v_\varepsilon(s), M_\varepsilon f(s, u(s))\rangle_{\Omega(s)}\, ds.$$

Now we take the limit as $\varepsilon \to 0^+$ in eq. (1.21) and obtain (1.15).

THEOREM 1.6. *Let Hypothesis A be satisfied, then for any* $u_0 \in X$ *eq.* (1.1) *has a unique solution* $u \in C^{(0)}([0, T[; X)$.

PROOF. Uniqueness is obvious by Cor. 1. Existence will be established if we show that for any τ, $0 < \tau < T$, eq. (1.1) has a solution $u \in C^{(0)}([0, \tau[; X)$. Given such a τ, let us define $\ell: \boldsymbol{N} \to \boldsymbol{N}$, $\ell(h) = h + 3 + 2([\tau] + 1)$ and consider the integral equation (see Def. 1.6)

$$\text{(1.22)} \quad u_h(t) = W(t)u_{0h} + \int_0^t W(t-s)\big[j_h(s) + g^{(1)}_{\ell(h)}(s, u_h(s)) + g^{(2)}_{\ell(h)}(s, u_h(s))\big]\, ds.$$

If we prove that, for any h, eq. (1.22) has a solution $u_h \in C^{(0)}([0, \tau[; X)$, then by Th. 1.2 also eq. (1.1) has a continuous solution in the same interval.

It remains therefore to analyze eq. (1.22) corresponding to a fixed integer h. As a consequence of Ths. 1.3 and 1.4 such an equation has always a perturbative solution. The crucial step is now to recognize that any solution u_h defined on any interval $[0, \tau_1[$, $0 < \tau_1 < \tau$, can be continued beyond τ_1. By Condition 2), $\operatorname{supp} u_{0h} \subset S_{h+1}$, $\operatorname{supp} j_h(s) \subset S_{h+1}$ and therefore by Cor. 2, $\operatorname{supp} u_h(t) \subset S_{h+1+t}$, $0 \leqslant t < \tau_1$. It is now clear that, if Ω is the sphere with center the origin and radius $h + 2 + 2([\tau] + 1)$, then $\Omega(t) \supset S_{h+1+t}$, for all $t \in [0, \tau_1[$. By the properties of the cut-off maps

$$|\langle u_h(s), g_{l(h)}(s, u_h(s)) - g(s, u_h(s))\rangle_{\Omega(s)}| \leqslant \\ \leqslant \|u_h(s)\|_{\Omega(s)} \|g_{l(h)}(s, u_h(s)) - g(s, u_h(s))\|_{\Omega(s)} = 0$$

so that

$$\langle u_h(s), g_{l(h)}(s, u_h(s))\rangle_{\Omega(s)} = \langle u_h(s), g(s, u_h(s))\rangle_{\Omega(s)} \tag{1.23}$$

for all $s \in [0, \tau_1[$. By the preceding Lemma, applied to eq. (1.22) (it is clear that there exists an Ω_1 satisfying the hypotheses of the Lemma), we obtain the inequality

$$\|u_h(t)\|^2_{\Omega(t)} \leqslant \|u_{0h}\|^2_{\Omega(0)} + 2\omega \int_0^t \|u_h(s)\|^2_{\Omega(s)}\, ds + \\ + 2 \operatorname{Re} \int_0^t \langle u_h(s), j_h(s)\rangle_{\Omega(s)}\, ds + \\ + 2 \operatorname{Re} \int_0^t \langle u_h(s), g^{(1)}(s, u_h(s))\rangle_{\Omega(s)}\, ds + \\ + 2 \operatorname{Re} \int_0^t \langle u_h(s), g^{(2)}(s, u_h(s))\rangle_{\Omega(s)}\, ds\,. \tag{1.24}$$

Moreover, by Hypothesis A, after some trivial computations, the r.h.s. of inequality (1.24) turns out to be smaller or equal than

$$(1 + 2\omega + 2\bar{C}(\Omega, \tau) + 2M(\Omega, \tau)) \int_0^t \|u_h(s)\|^2_{\Omega(s)}\, ds + \|u_{0h}\|^2_{\Omega(0)} + \\ + \int_0^t \|j_h(s)\|^2_{\Omega(s)}\, ds - 2V_\Omega(t, u_h(t)) + 2V_\Omega(0, u_{0h}) + \\ + 2M(\Omega, \tau) \int_0^t V_\Omega(s, u_h(s))\, ds + 2N(\Omega, \tau)\, \tau\,. \tag{1.25}$$

At this point application of Gronwall's lemma, which is possible by condition (1.12), yields the following a priori estimate

$$(1.26)\qquad \|u_h(t)\|^2_{\Omega(t)} + 2V_\Omega(t, u_h(t)) \leqslant \big(\|u_{0h}\|^2_{\Omega(0)} + 2V_\Omega(0, u_{0h}) + \\ + N_1(\Omega, \tau, h)\big)\exp[N_2(\Omega, \tau, h)t]$$

for all $t \in [0, \tau_1[$ and for suitable non-negative constants N_1, N_2. Obviously this implies that

$$(1.27)\qquad \sup_{t\in[0,\tau_1[} \|u_h(t)\|_{\Omega(t)} < \infty .$$

We can now verify that the hypotheses of Th. 1.5 are satisfied. Let $\mathcal{F} = \{\Omega_k\}_{k\in N}$ be a family of spheres with the property that $\{\Omega_k(2\varepsilon)\}_{k\in N}$ is a locally finite covering of $\boldsymbol{R}^s$ for a small enough $\varepsilon > 0$, such that $[0, \varepsilon]$ is an admissible interval of time for all Ω_k. Now, inequality (1.27) and Defs. 1.1 and 1.2 yield

$$\sup_{\substack{k\in N\\ t\in[0,\tau_1[}} \|u_h(t)\|_{\Omega_k} \leqslant \sup_{\substack{k\in N\\ t\in[0,\tau_1[}} \|u_h(t)\|_{\Omega_k\cup\Omega(t)} \leqslant \sup_{t\in[0,\tau_1[} \|u_h(t)\|_{\Omega(t)} < \infty .$$

This establishes the highly non trivial condition 1) in Th. 1.5. Conditions 2) and 3) are easily obtained by arguing as in the proof of Th. 1.4.

2. – Regularity in the abstract case.

In this section we will establish some regularity properties of continuous X-valued solutions of eq. (1.1), provided the initial data u_0 and the function f are suitably smooth. More precisely we want to investigate the conditions under which a function $u \in C^{(0)}([0, T[; X)$, solution of eq. (1.1), belongs to the space $\bigcap_{s=0}^{n} C^{(n-s)}([0, T[; D_{K^s})$, $n = 0, 1, 2, \ldots$, where D_{K^s} is the domain of the s-th power of the infinitesimal generator K of the semigroup $W(t)$ (equipped with the « graph topology »).

The hypotheses we are going to make here on the function f are of a different nature as compared to those used to prove Th. 1.6, and therefore the content of this section is largely independent of the general treatment of Sect. 1. As in subsect. 1.5 we suppose that, $\forall \Omega \in \mathcal{A}$, $B(\Omega)$ is a Hilbert space. We propose to discuss first the case $n = 1$ and then the general case by using induction on n.

DEFINITION 2.1. By $BC^{(1)}([0, T[; X)$, $0 < T \leqslant \infty$, we denote the class of all functions $f: [0, T[\times X \to X$ for which there exist two maps

$$D_t f: [0, T[\times X \to X$$
$$D_u f: [0, T[\times X \to \mathfrak{L}(X, X)$$

with the following properties

i) $D_t f$ is continuous and $D_u f$ is strongly continuous;

ii) For any sphere Ω, for any $\tau \in]0, T[$ and for any $\varrho > 0$, there is a $C(\Omega, \tau, \varrho) > 0$ such that

$$\sup_{0 \leqslant t \leqslant \tau} \|D_u f(t, u) v\|_\Omega \leqslant C(\Omega, \tau, \varrho) \|v\|_\Omega$$

$\forall u, v \in X$, $\|u\|_\Omega \leqslant \varrho$. Moreover we require

$$\bar{C}(\Omega, \tau, \varrho) \equiv \sup_{0 \leqslant t \leqslant \tau} C(\Omega(t), \tau, \varrho)$$

to be finite if $[0, \tau]$ is an admissible interval of time for Ω;

iii) If we put

$$\omega(t, u; \tau, v) = f(t + \tau, u + v) - f(t, u) - D_t f(t, u) \tau - D_u f(t, u) v ,$$

then for any sphere Ω_1 there exists a sphere Ω_2 such that

$$\|\omega(t, u; \tau, v)\|_{\Omega_1} / \big(|\tau| + \|v\|_{\Omega_2}\big) \to 0$$

as $|\tau| + \|v\|_{\Omega_2} \to 0$, for all $t \in [0, T[$ and for all $u \in X$.

REMARK 1. If $f \in BC^{(1)}([0, T[; X)$ then, clearly, the function

$$[0, 1] \ni \sigma \to \psi(\sigma) = f\big(t_1 + \sigma(t_2 - t_1), v_1 + \sigma(v_2 - v_1)\big) \big(t_1, t_2 \in [0, T[, v_1, v_2 \in X\big)$$

is differentiable and

$$\psi'(\sigma) = D_t f\big(t_1 + \sigma(t_2 - t_1), v_1 + \sigma(v_2 - v_1)\big)(t_2 - t_1) + \\ + D_u f\big(t_1 + \sigma(t_2 - t_1), v_1 + \sigma(v_2 - v_1)\big)(v_2 - v_1) .$$

This yields the mean value theorem

$$f(t_2, v_2) - f(t_1, v_1) = \int_0^1 \psi'(\sigma) \, d\sigma$$

from which the estimate

$$\sup_{0\leqslant t\leqslant\tau}\|f(t,v_2)-f(t,v_1)\|_\Omega\leqslant\sup_{\substack{0\leqslant t\leqslant\tau\\0\leqslant\sigma\leqslant1}}\|D_uf(t,v_1+\sigma(v_2-v_1))(v_2-v_1)\|_\Omega\leqslant$$
$$\leqslant C(\Omega,\tau,\varrho)\|v_2-v_1\|_\Omega,\qquad\forall\tau\in]0,T[,\ \forall v_1,v_2\in X,\ \|v_1\|_\Omega\leqslant\varrho,\ \|v_2\|_\Omega\leqslant\varrho$$

follows.

REMARK 2. If $f\in BC^{(1)}([0,T[;X)$ then the function

$$[0,T[\times X\times X\ni(t,u,v)\mapsto D_uf(t,u)v\in X$$

is continuous.

This is a consequence of the inequality

$$\|D_uf(t+\Delta t,u+\Delta u)(v+\Delta v)-D_uf(t,u)v\|_\Omega\leqslant$$
$$\leqslant\|[D_uf(t+\Delta t,u+\Delta u)-D_uf(t,u)]v\|_\Omega+\|D_uf(t+\Delta t,u+\Delta u)\Delta v\|_\Omega$$

and of conditions i), ii) of Def. 2.1.

LEMMA 2.1. Given $f\in BC^{(1)}([0,T[;X)$ and $u\in C^{(0)}([0,T[;X)$ we define

$$E^{(1)}(t,v)\equiv D_tf(t,u(t))+D_uf(t,u(t))v,\qquad v\in X.$$

Then the integral equation

$$v(t)=W(t)v_0+\int_0^tW(t-s)E^{(1)}(s,v(s))\,ds\tag{2.1}$$

where $W\in C(1,\omega)$, has a unique solution $v\in C^{(0)}([0,T[;X)$ for any $v_0\in X$.

PROOF. It is clear, by Remark 2, that $E^{(1)}\in C^{(0)}([0,T[\times X;X)$. Moreover, by ii) of Def. 2.1, for any $\tau\in]0,T[$ and for any sphere Ω the following inequality

$$\sup_{0\leqslant t\leqslant\tau}\|E^{(1)}(t,v_1)-E^{(1)}(t,v_2)\|_\Omega\leqslant C(\Omega,\tau,\varrho)\|v_1-v_2\|_\Omega$$

holds $\forall v_1,v_2\in X$ and for any positive $\varrho\geqslant\sup_{0\leqslant t\leqslant\tau}\|u(t)\|_\Omega$. We can now apply Th. 1.6 with the identifications

$$j(t)=D_tf(t,u(t)),\qquad g^{(1)}=0,\qquad g^{(2)}(t,v)=D_uf(t,u(t))v.$$

THEOREM 2.1. *Let* $u \in C^{(0)}([0, T[; X)$, $0 < T \leqslant \infty$, *be a solution of eq.* (1.1) *with*

i) $W \in C(1, \omega)$;

ii) $f \in BC^{(1)}([0, T[; X)$;

iii) $u_0 \in D_K$.

Then

1) $u \in C^{(1)}([0, T[; X)$;

2) $u(t) \in D_K$, $\forall t \in [0, T[$;

3) *u is a solution of the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt}(t) = Ku(t) + f\big(t, u(t)\big), & 0 \leqslant t < T, \\ u(0) \quad = u_0 \end{cases}$$

4) $u' \equiv du/dt$ *is a solution of the integral equation*

$$u'(t) = W(t)u'(0) + \int_0^t W(t-s) E^{(1)}\big(s, u'(s)\big)\, ds\,. \tag{2.2}$$

PROOF. Let $v \in C^{(0)}([0, T[; X)$ be the solution of eq. (2.1) obtained by applying Lemma 2.1 in which $v_0 = Ku_0 + f(0, u_0)$ and $u(t)$ is the solution of eq. (1.1) considered here. The main point to be proved is that the function

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}[u(t+\varepsilon) - u(t)] - v(t)\,; \qquad \varepsilon \neq 0,\ 0 \leqslant t + \varepsilon < T, \tag{2.3}$$

tends to 0 in X as $\varepsilon \to 0$. Here we discuss only the case in which $\varepsilon > 0$ (the case $\varepsilon < 0$ can be treated in a similar way). Obviously it suffices to show that

$$\sup_{0 \leqslant t \leqslant \tau} \|\varphi_\varepsilon(t)\|_{\Omega(t)} \to 0\,, \qquad \varepsilon \to 0+\,, \tag{2.4}$$

for all compact subsets $[0, \tau] \subset [0, T[$ and any sphere Ω for which $[0, \tau]$ is an admissible interval of time.

To prove eq. (2.4) let us consider the identity

$$\varphi_\varepsilon(t) = \left(\frac{W(t+\varepsilon)-W(t)}{\varepsilon} - W(t)K\right)u_0 + \\ + \frac{1}{\varepsilon}\int_0^\varepsilon [W(t+\varepsilon-s)f(s,u(s)) - W(t)f(0,u_0)]\,ds + \\ + \frac{1}{\varepsilon}\int_\varepsilon^{t+\varepsilon} W(t+\varepsilon-s)f(s,u(s))\,ds - \frac{1}{\varepsilon}\int_0^t W(t-s)f(s,u(s))\,ds - \\ - \int_0^t W(t-s)E^{(1)}(s,v(s))\,ds$$

which, by a change of variables, becomes

$$\varphi_\varepsilon(t) = \left(\frac{W(t+\varepsilon)-W(t)}{\varepsilon} - W(t)K\right)u_0 + \\ + \frac{1}{\varepsilon}\int_0^\varepsilon [W(t+\varepsilon-s)f(s,u(s)) - W(t)f(0,u_0)]\,ds + \\ + \int_0^t W(t-s)\left[\frac{f(s+\varepsilon,u(s+\varepsilon)) - f(s,u(s))}{\varepsilon} - E^{(1)}(s,v(s))\right]ds \equiv \\ \equiv J_{1\varepsilon}(t) + J_{2\varepsilon}(t) + J_{3\varepsilon}(t)\,.$$

Now

$$\sup_{0\leqslant t\leqslant\tau} \|J_{1\varepsilon}(t)\|_{\Omega(t)} \leqslant \exp[\omega\tau] \left\|\left(\frac{W(\varepsilon)-1}{\varepsilon} - K\right)u_0\right\|_{\Omega(0)} \tag{2.5}$$

and therefore the l.h.s. of inequality (2.5) tends to 0 as $\varepsilon \to 0+$. In the same way one obtains

$$\sup_{0\leqslant t\leqslant\tau} \|J_{2\varepsilon}(t)\|_{\Omega(t)} \leqslant \exp[\omega\tau]\frac{1}{\varepsilon}\int_0^\varepsilon \|W(\varepsilon-s)f(s,u(s)) - f(0,u_0)\|_{\Omega(0)}\,ds\,. \tag{2.6}$$

Since the function $(\varepsilon, s) \to \|W(\varepsilon-s)f(s,u(s)) - f(0,u_0)\|_{\Omega(0)}$ is continuous for $0\leqslant s\leqslant\varepsilon$ and tends to 0 as $\varepsilon\to 0+$, it follows that the l.h.s. of inequality (2.6) tends to 0 as $\varepsilon\to 0+$. Concerning the term $J_{3\varepsilon}(t)$, application of Remark 1 yields

$$J_{3\varepsilon}(t) = \int_0^t W(t-s) \int_0^1 \Big\{ \Big[D_t f\big(s+\sigma\varepsilon, u(s)+\sigma(u(s+\varepsilon)-u(s))\big) - D_t f(s, u(s)) \Big] + \\ + \Big[D_u f\big(s+\sigma\varepsilon, u(s)+\sigma(u(s+\varepsilon)-u(s))\big) - D_u f(s, u(s)) \Big] v(s) + \\ + \Big[D_u f\big(s+\sigma\varepsilon, u(s)+\sigma(u(s+\varepsilon)-u(s))\big) \varphi_\varepsilon(s) \Big] \Big\} d\sigma\, ds \equiv \\ \equiv Z_{1\varepsilon}(t) + Z_{2\varepsilon}(t) + Z_{3\varepsilon}(t).$$

Now

$$\text{(2.7)} \quad \sup_{0 \leqslant t \leqslant \tau} \|Z_{1\varepsilon}(t)\|_{\Omega(t)} \leqslant \\ \leqslant \exp[\omega\tau] \sup_{\substack{0 \leqslant t \leqslant \tau \\ 0 \leqslant \sigma \leqslant 1}} \|D_t f\big(s+\sigma\varepsilon, u(s)+\sigma(u(s+\varepsilon)-u(s))\big) - D_t f(s, u(s))\|_{\Omega(0)}$$

and therefore the l.h.s. of inequality (2.7) tends to 0 as $\varepsilon \to 0+$. By similar arguments one concludes that

$$\text{(2.8)} \qquad \sup_{0 \leqslant t \leqslant \tau} \|Z_{2\varepsilon}(t)\|_{\Omega(t)} \to 0, \qquad \text{as } \varepsilon \to 0+.$$

Finally

$$\text{(2.9)} \qquad \|Z_{3\varepsilon}(t)\|_{\Omega(t)} \leqslant e^{\omega\tau} \bar{C}(\Omega, \tau, \varrho) \int_0^t \|\varphi_\varepsilon(s)\|_{\Omega(s)}\, ds, \qquad 0 \leqslant t \leqslant \tau,$$

for a suitable $\varrho > 0$. Equation (2.4) is then a trivial consequence of Gronwall's Lemma. Up to now we have established assertions 1) and 4). These results imply that the function

$$[0, T[\ni t \to \int_0^t W(t-s) f(s, u(s))\, ds$$

is differentiable, which means that the following limit exists

$$\lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon} \int^{+\varepsilon} W(t+\varepsilon-s) f(s, u(s))\, ds + \frac{W(\varepsilon)-1}{\varepsilon} \int_0^t W(t-s) f(s, u(s))\, ds \right).$$

Consequently

$$\int_0^t W(t-s) f(s, u(s))\, ds \in D_K, \qquad \forall t \in [0, T[,$$

and

$$\frac{d}{dt}\int_0^t W(t-s)f(s,u(s))\,ds = f(t,u(t)) + K\int_0^t W(t-s)f(s,u(s))\,ds\,.$$

This completes the proof of the Theorem.

COROLLARY 2.1. Let the assumptions of Th. 2.1 be satisfied. Then $u\in C^{(1)}([0,T[,X)\cap C^{(0)}([0,T[;D_K)$, if D_K is equipped with the graph topology.

PROOF. Trivial by assertion 3) of Th. 2.1.

REMARK 3. The graph topology on D_K is, by definition, the weakest topology for which the maps $D_K\ni u\to u, Ku$ are continuous.

We may now proceed to analyze the general case with $n\geqslant 1$. First it is convenient to fix some notation. For any $n\in\mathbf{N}$ we shall denote by Λ_n the class of all pairs $\alpha=(\alpha_1,\dots,\alpha_k)$, $\beta=(\beta_1,\dots,\beta_h)$, with $h,k\in\mathbf{N}$, $|h-k|\leqslant 1$, such that

i) $\alpha_1,\beta_1\in\{0,1,\dots,n\}$, $\alpha_j,\beta_i\in\{1,\dots,n\}$, $2\leqslant j\leqslant k$, $2\leqslant i\leqslant h$;

ii) $|\alpha|+|\beta|\equiv\sum_{j=1}^{k}\alpha_j+\sum_{i=1}^{h}\beta_i=n$;

iii) $\alpha_1+\beta_1\neq 0$, $\alpha_1\beta_1=0$; $\alpha_1=0\Rightarrow h\geqslant k$; $\beta_1=0\Rightarrow k\geqslant h$.

Moreover, if $(\alpha=(\alpha_1,\dots,\alpha_k),\ \beta=(\beta_1,\dots,\beta_h))\in\Lambda_n$ we shall denote by (α',β'), (α'',β'') the following pairs of Λ_{n+1}

$$\alpha'=\begin{cases}(\alpha_1+1,\alpha_2,\dots,\alpha_k) & \text{if } \alpha_1\neq 0\\ (1,\alpha_2,\dots,\alpha_k) & \text{if } \alpha_1=0\end{cases}$$

$$\beta'=\begin{cases}\beta & \text{if } \alpha_1\neq 0\\ (0,\beta_1,\dots,\beta_h) & \text{if } \alpha_1=0\end{cases}$$

$$\alpha''=\begin{cases}(0,\alpha_1,\dots,\alpha_k) & \text{if } \beta_1=0\\ \alpha & \text{if } \beta_1\neq 0\end{cases}$$

$$\beta''=\begin{cases}(1,\beta_2,\dots,\beta_h) & \text{if } \beta_1=0\\ (\beta_1+1,\beta_2,\dots,\beta_h) & \text{if } \beta_1\neq 0\,.\end{cases}$$

We are now in a position to state

DEFINITION 2.2. By $BC^{(n)}([0, T[; X)$, $0 < T \leqslant \infty$, $n \in \boldsymbol{N}$, $n \geqslant 2$, we denote the class of all functions $f\colon [0, T[\times X \to X$ such that

i) $f \in BC^{(n-1)}([0, T[; X)$;

ii) For any $(\alpha, \beta) \in \Lambda_n$ there exists a map

$$D^{(\alpha,\beta)}f\colon [0, T[\times X \to \mathfrak{L}^{|\beta|}(X) \ (^4)$$

satisfying the following properties

a) $D^{(\alpha,\beta)}f$ is continuous, if $h = 1$ and $|\beta| = 0$, and $D^{(\alpha,\beta)}f$ is strongly continuous otherwise;

b) For any sphere Ω, for any $\tau \in]0, T[$ and for any $\varrho > 0$ there is a $C_{(\alpha,\beta)}(\Omega, \tau, \varrho) > 0$ such that

$$\sup_{0 \leqslant t \leqslant \tau} \|D^{(\alpha,\beta)}f(t, u)(v_1, v_2, \ldots, v_{|\beta|})\|_\Omega \leqslant C_{(\alpha,\beta)}(\Omega, \tau, \varrho) \prod_{j=1}^{|\beta|} \|v_j\|_\Omega ,$$
$$\forall v_1, \ldots, v_{|\beta|},\ u \in X,\ \|u\|_\Omega \leqslant \varrho ;$$

c) If, for any $(\alpha, \beta) \in \Lambda_{n-1}$, we put

$$\begin{aligned}\omega_{(\alpha,\beta)}(t, u, v_1, \ldots, v_{|\beta|}; \tau, w) = D^{(\alpha,\beta)}f(t+\tau, u+w)(v_1, \ldots, v_{|\beta|}) - \\ - D^{(\alpha,\beta)}f(t, u)(v_1, \ldots, v_{|\beta|}) - \tau D^{(\alpha',\beta')}f(t, u)(v_1, \ldots, v_{|\beta|}) - \\ - D^{(\alpha'',\beta'')}f(t, u)(v_1, \ldots, v_{|\beta|}, w)\end{aligned}$$

then for any sphere Ω_1 there exists a sphere Ω_2 such that

$$\|\omega_{(\alpha,\beta)}(t, u, v_1, \ldots, v_{|\beta|}; \tau, w)\|_{\Omega_1} / (|\tau| + \|w\|_{\Omega_2}) \to 0$$

as $|\tau| + \|w\|_{\Omega_2} \to 0$, for all $t \in [0, T[$ and for all $v_1, \ldots, v_{|\beta|}$, $u \in X$.

For convenience, if $f \in BC^{(n)}([0, T[; X)$, $n \geqslant 2$, and $(\alpha, \beta) \in \Lambda_k$, $k < n$, we will use the notation $D_t D^{(\alpha,\beta)}f$ for $D^{(\alpha',\beta')}f$ and $D_u D^{(\alpha,\beta)}f$ for $D^{(\alpha'',\beta'')}f$.

To establish the main result of this section (Th. 2.2) it is convenient to start with some preparatory lemmas.

LEMMA 2.2. If $f \in BC^{(n)}([0, T[; X)$ then, for any $(\alpha, \beta) \in \Lambda_n$, $|\beta| > 0$, the map

$$[0, T[\times X \times \underbrace{(X \times \ldots \times X)}_{|\beta|} \ni (t, u, v_1, \ldots, v_{|\beta|}) \mapsto D^{(\alpha,\beta)}f(t, u)(v_1, \ldots, v_{|\beta|}) \in X$$

is continuous.

(4) $\mathfrak{L}^{(0)}(X) \equiv X$ and $\mathfrak{L}^{|\beta|}(X)$, with $|\beta| > 0$, is the space of all $|\beta|$-linear continuous maps from $\underbrace{X \times \ldots \times X}_{|\beta|}$ to X.

PROOF. This is an immediate consequence of Def. 2.2 and of the identity

$$D^{(\alpha,\beta)}f(t+\Delta t, u+\Delta u)(v_1+\Delta v_1, \dots, v_{|\beta|}+\Delta v_{|\beta|}) - D^{(\alpha,\beta)}f(t,u)(v_1,\dots,v_{|\beta|}) =$$
$$= [D^{(\alpha,\beta)}f(t+\Delta t, u+\Delta u) - D^{(\alpha,\beta)}f(t,u)](v_1,\dots,v_{|\beta|}) +$$
$$+ \sum_{j=1}^{|\beta|} D^{(\alpha,\beta)}f(t+\Delta t, u+\Delta u)(v_1+\Delta v_1, \dots, v_{j-1}+\Delta v_{j-1}, \Delta v_j, v_{j+1}, \dots, v_{|\beta|}).$$

LEMMA 2.3. If $f \in BC^{(n)}([0,T[;X)$ then, for any $(\alpha,\beta)\in\Lambda_{n-1}$ and for any $u, v_1, \dots, v_{|\beta|} \in C^{(1)}([0,T[;X)$, the function

$$\Phi(t) \equiv D^{(\alpha,\beta)}f(t,u(t))(v_1(t),\dots,v_{|\beta|}(t)), \qquad t\in[0,T[,$$

is continuously differentiable and

$$\frac{d\Phi}{dt}(t) = D_t D^{(\alpha,\beta)}f(t,u(t))(v_1(t),\dots,v_{|\beta|}(t)) + \tag{2.10}$$
$$+ D_u D^{(\alpha,\beta)}f(t,u(t))(v_1(t),\dots,v_{|\beta|}(t), u'(t)) +$$
$$+ \sum_{j=1}^{|\beta|} D^{(\alpha,\beta)}f(t,u(t))(v_1(t),\dots,v_{j-1}(t), v_j'(t), v_{j+1}(t),\dots,v_{|\beta|}(t)).$$

PROOF. This follows easily from Def. 2.2, the previous Lemma and the identity

$$\frac{\Phi(t+\tau)-\Phi(t)}{\tau} = D_t D^{(\alpha,\beta)}f(t,u(t))(v_1(t),\dots,v_{|\beta|}(t)) +$$
$$+ D_u D^{(\alpha,\beta)}f(t,u(t))\left(v_1(t),\dots,v_{|\beta|}(t), \frac{u(t+\tau)-u(t)}{\tau}\right) +$$
$$+ \frac{1}{\tau}\,\omega_{(\alpha,\beta)}(t,u(t),v_1(t),\dots,v_{|\beta|}(t);\tau, u(t+\tau)-u(t)) +$$
$$+ \sum_{j=1}^{|\beta|} D^{(\alpha,\beta)}f(t+\tau, u(t+\tau))(v_1(t+\tau),\dots,v_{j-1}(t+\tau), v_j'(t), v_{j+1}(t),\dots,v_{|\beta|}(t)) +$$
$$+ \sum_{j=1}^{|\beta|} D^{(\alpha,\beta)}f(t+\tau,u(t+\tau))\cdot$$
$$\cdot\left(v_1(t+\tau),\dots,v_{j-1}(t+\tau), \frac{v_j(t+\tau)-v_j(t)}{\tau} - v_j'(t), v_{j+1}(t),\dots,v_{|\beta|}(t)\right).$$

LEMMA 2.4. For any $f\in BC^{(n)}([0,T[;X)$, $n\geq 2$, and for any $u\in C^{(1)}([0,T[;X)$ the map

$$\Psi\colon [0,T[\times X \to X,\ \Psi(t,w) \equiv D_u f(t,u(t))\,w$$

belongs to $BC^{(1)}([0, T[; X)$ and

$$\left\{\begin{aligned} & D_t\Psi(t, w) = D_tD_uf(t, u(t))w + D_u^2f(t, u(t))(w, u'(t)) \\ & D_w\Psi(t, w)v = D_uf(t, u(t))v = \Psi(t, v)\,. \end{aligned}\right. \tag{2.11}$$

PROOF. The assertion follows from Def. 2.2 and the obvious identity

$$\begin{aligned} \Psi(t+\Delta t, w+\Delta w) - \Psi(t, w) = {} & D_tD_uf(t, u(t))(\Delta t w) + \\ & + D_u^2f(t, u(t))(w, \Delta t u'(t)) + D_u^2f(t, u(t))\left(w, \Delta t\left(\frac{u(t+\Delta t)-u(t)}{\Delta t} - u'(t)\right)\right) \\ & + \omega_1(t, u(t), w; \Delta t, u(t+\Delta t) - u(t)) + D_uf(t, u(t))\Delta w + \\ & + D_tD_uf(t, u(t))(\Delta t\,\Delta w) + D_u^2f(t, u(t))(\Delta w, u(t+\Delta t) - u(t)) + \\ & + \omega_2(t, u(t), \Delta w; \Delta t, u(t+\Delta t) - u(t))\,. \end{aligned}$$

DEFINITION 2.3. Given $f \in BC^{(n)}([0, T[; X)$ and $u \in C^{(n-1)}([0, T[; X)$, $n \geqslant 1$, we define, for $k = 0, 1, \ldots, n$, the maps

$$E^{(k)}: [0, T[\times X \to X$$

$$E^{(k)}(t, w) = \left\{\begin{aligned} & f(t, w)\,, \quad \text{if } k = 0 \\ & \frac{d}{dt}E^{(k-1)}(t, v)\Big|_{v=u^{(k-1)}(t)} + D_uf(t, u(t))w\,, \quad \text{if } k = 1, \ldots, n\,. \end{aligned}\right. \tag{2.12}$$

The next Lemma guarantees that Def. 2.3 makes sense and states some useful properties of the maps (2.12).

LEMMA 2.5. The $E^{(k)}$ of formula (2.12) are well defined, $E^{(k)} \in BC^{(1)}([0, T[; X)$ for $0 \leqslant k \leqslant n-1$ and $E^{(n)} \in C^{(0)}([0, T[\times X; X)$.

PROOF. The Lemma is a consequence of Lemma 2.3 and of the following representation formula

$$\begin{aligned} E^{(k)}(t, w) = \sum_{\alpha,\beta,\sigma\in\Gamma_k} C^{(k)}_{\alpha,\beta,\sigma} D^{(\alpha,\beta)}f(t, u(t))\left(u^{(\sigma_1)}(t), \ldots, u^{(\sigma_{|\beta|})}(t)\right) + \\ + D_uf(t, u(t))w\,, \quad k \geqslant 1\,, \end{aligned} \tag{2.13}$$

where Γ_k is the set of all multiindices α, β, σ such that $|\alpha| + |\beta| \leqslant k$; $1 \leqslant \sigma_j < k$, $j = 1, \ldots, |\beta|$; $|\alpha| + |\sigma| = k$ and the $C^{(k)}_{\alpha,\beta,\sigma}$ are suitable non negative constants. Formula (2.13) is obviously verified for $k = 1$. To prove it for a general k one proceeds by induction making again use of Lemma 2.3.

DEFINITION 2.4. Given $f \in BC^{(n)}([0, T[; X)$, we define the maps

$$(2.14)\qquad S^{(k)}(t, z) = \begin{cases} f(t, z) \quad \text{if } k = 0,\ (t, z) \in [0, T[\times X \\ \sum_{\alpha,\beta,\sigma \in \Gamma_k} C^{(k)}_{\alpha,\beta,\sigma} D^{(\alpha,\beta)} f(t, z_0)(z_{\sigma_1}, \ldots, z_{\sigma_k}) + D_u f(t, z_0) z_k \\ \text{if } \ k = 1, \ldots, n, \quad (t, z = (z_0, \ldots, z_k)) \in [0, T[\times \underbrace{X \times \ldots \times X}_{k+1}. \end{cases}$$

Here the set Γ_k and the $C^{(k)}_{\alpha,\beta,\sigma}$ are the same as in formula (2.13).

We are now in a position to state the main result of this section.

THEOREM 2.2. *Let $u \in C^{(0)}([0, T[; X)$, $0 \leq T < \infty$, be a solution of eq. (1.1) with*

i) $W \in C(1, \omega)$;

ii) $f \in BC^{(n)}([0, T[; X)$;

iii)
$$\begin{cases} u_0^{(0)} \equiv u_0 \in D_K \\ u_0^{(1)} \equiv K u_0^{(0)} + S^{(0)}(0, u_0^{(0)}) \in D_K \\ u_0^{(2)} \equiv K u_0^{(1)} + S^{(1)}(0, (u_0^{(0)}, u_0^{(1)})) \in D_K \\ \cdots\cdots\cdots\cdots \\ u_0^{(n-1)} \equiv K u_0^{(n-2)} + S^{(n-2)}(0, (u_0^{(0)}, \ldots, u_0^{(n-2)})) \in D_K \end{cases}$$

Then

1) $u \in C^{(n)}([0, T[; X)$;

2) $u^{(j)}(t) \equiv (d^j/dt^j) u(t) \in D_K$, $0 \leq j \leq n-1$, $t \in [0, T[$;

3) $u^{(j)}(t)$, $0 \leq j \leq n-1$, *is a solution of the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt} u^{(j)}(t) = K u^{(j)}(t) + E^{(j)}(t, u^{(j)}(t)) \\ u^{(j)}(t)|_{t=0} = u_0^{(j)} \end{cases}$$

4) $u^{(j)}(t)$, $0 \leq j \leq n$, *is a solution of the integral equation*

$$(2.15)\qquad u^{(j)}(t) = W(t) u^{(j)}(0) + \int_0^t W(t-s) E^{(j)}(s, u^{(j)}(s))\, ds .$$

PROOF. If $n = 1$ this is nothing else but Th. 2.1. To prove it for $n > 1$ we proceed by induction. Let us suppose that it has been already proved that $u \in C^{(r)}([0, T[; X)$, that 2) and 3) are satisfied for $j \leq r-1$ and that 4) holds for $j \leq r$, for some r, $1 \leq r < n$. Since $u_0^{(r)} \in D_K$ by hypothesis and

$E^{(r)} \in BC^{(1)}([0, T[; X)$ by Lemma 2.4, we can apply Th. 2.1 to the integral equation (2.15) with $j = r$.

COROLLARY 2.2. Let $u \in C^{(0)}([0, T[; X)$, $0 < T \leqslant \infty$, be a solution of eq. (1.1) with

i) $W \in C(1, \omega)$;

ii) $f \in BC^{(n)}([0, T[; X)$;

iii) $u_0 \in D_{K^n}$;

iv) For all j, $0 \leqslant j < n-2$, if $\big(t, z = (z_0, \dots, z_j)\big) \in [0, T[\times \bigoplus_{j+1} D_{K^{n-j-1}}$ the value of the function $S^{(j)}(t, z)$ belongs to $D_{K^{n-j-1}}$.

Then the hypotheses of Th. 2.2 are satisfied and, moreover,

$$u^{(j)}(t) \in D_{K^{n-j}}, \quad t \in [0, T[, \quad 0 \leqslant j \leqslant n \ (D_{K^0} = X). \tag{2.16}$$

PROOF. Concerning Th. 2.2 it remains only to check hypotheses iii). Actually we are going to show more, i.e. that $u_0^{(j)} \in D_{K^{n-j}}$, $0 \leqslant j \leqslant n-1$. The first step of the usual induction argument is obviously true because $u_0^{(0)} \in D_{K^n}$. If we suppose that $u_0^{(r)} \in D_{K^{n-r}}$, $r = 0, 1, \dots, j$, then, by our hypothesis, $Ku_0^{(j)} + S^{(j)}(0, (u_0^{(0)}, \dots, u_0^{(j)})) \in D_{K^{n-j-1}}$ and therefore $u_0^{(j+1)} \in D_{K^{n-j-1}}$. Equation (2.16) is now easily proved by using part 3) of Th. 2.2 and the same induction argument as above.

The next Corollary contains some useful results concerning the regularity of solutions of eq. (1.1). To state it we need the following convenient

DEFINITION 2.5. If D_{K^r}, $r = 1, 2, \dots$ is the domain of the r-th power of the infinitesimal generator of a semigroup $W \in C(1, \omega)$, by the graph topology on D_{K^r} we denote the weakest topology for which all the maps $D_{K^r} \ni u \mapsto u, Ku, K^2u, \dots, K^r u$ are continuous [5].

COROLLARY 2.3. Let the hypothesis of Corollary 2.2 be satisfied. Let us suppose, moreover, that for all j, $0 \leqslant j \leqslant n-2$, and for all $t \in [0, T[$ $S^{(j)}(t, \cdot)$ is a continuous map from $\bigoplus_{j+1} D_{K^{n-j-1}}$ to $D_{K^{n-j-1}}$. Then

$$u \in \bigcap_{r=0}^{n} C^{(n-r)}([0, T[; D_{K^r}) \ [6]. \tag{2.17}$$

[5] These domains are dense in X.

[6] Here the domains D_{K^r} are equipped with the graph topology.

PROOF. Equation (2.17) is equivalent to prove that $u^{(j)} \in C^{(0)}([0, T[; D_{K^{n-j}})$, $0 \leqslant j \leqslant n$. This can be seen by arguing as in the proof of Corollary 2.2.

3. – Applications.

3.1. *Position of the problem and free theory.*

In this section we want to use the previously developed abstract theory to treat a particularly interesting class of applications, namely the Cauchy problem for the system

$$\left\{\begin{array}{l}\dfrac{\partial^2\varphi_k}{\partial t^2}(x, t) - \sum_{j=1}^{s}\dfrac{\partial^2\varphi_k}{\partial x_j^2}(x, t) - f_k\big(x, t; \varphi_1(x, t), \dots, \varphi_n(x, t)\big) = 0 \\ k = 1, \dots, n;\ (x, t) \in \boldsymbol{R}^s \times [0, T[\,.\end{array}\right. \tag{3.1}$$

The system (3.1) can be more conveniently rewritten in the first order formalism as

$$\frac{d}{dt}\begin{bmatrix}\varphi_1\\ \psi_1\\ \cdot\\ \cdot\\ \cdot\\ \varphi_n\\ \psi_n\end{bmatrix} = \begin{bmatrix}0 & 1 & & & & \\ \Delta & 0 & & & & \\ & & 0 & 1 & & \\ & & \Delta & 0 & & \\ & & & & \ddots & & \\ & & & & & 0 & 1\\ & & & & & \Delta & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \psi_1\\ \cdot\\ \cdot\\ \cdot\\ \varphi_n\\ \psi_n\end{bmatrix} + \begin{bmatrix}0\\ f_1\\ 0\\ f_2\\ \vdots\\ 0\\ f_n\end{bmatrix} \tag{3.2}$$

or, more concisely,

$$\frac{d}{dt}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = K\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + f\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) \tag{3.2'}$$

where

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{bmatrix}\varphi_1\\ \psi_1\\ \vdots\\ \varphi_n\\ \psi_n\end{bmatrix}, \qquad \varphi = (\varphi_1, \dots, \varphi_n), \qquad \psi = (\psi_1, \dots, \psi_n),$$

and

$$f\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) = \begin{bmatrix} 0 \\ f_1(t, \varphi) \\ 0 \\ \vdots \\ 0 \\ f_n(t, \varphi) \end{bmatrix}.$$

What we are going to investigate is actually the corresponding integral equation

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(t) = W(t)\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} + \int_0^t W(t-s) f\left(s, \begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}\right) ds \tag{3.3}$$

where $W(t)$ is the semigroup having K as infinitesimal generator.

To introduce the functional framework, in which eq. (3.3) will be studied we recall the definition of a class of localized Sobolev spaces. For any open bounded subset Ω of $\boldsymbol{R}^s$ we denote by $H^r(\Omega)$, $r = 0, 1, \ldots,$ the space of all real functions $\varphi(x)$, $x \in \Omega$, whose distributional derivatives $(\partial/\partial x)^\alpha \varphi(x)$, $|\alpha| \leqslant r$, are square integrable in Ω. $H^r(\Omega)$ equipped with the norm

3.4) $$\|\varphi; H^r(\Omega)\| = \left(\sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right|^2 dx\right)^{\frac{1}{2}}$$

is a Hilbert space. By $H^r_{\text{loc}}(\boldsymbol{R}^s)$ we denote the projective limit of the family of spaces $H^r(\Omega)$ as Ω runs over the open bounded subsets of $\boldsymbol{R}^s$. Since we are interested in vector valued functions, it is useful to define

$$X_r \equiv \underset{n}{\oplus} H^r_{\text{loc}}(\boldsymbol{R}^s) \equiv \underbrace{H^r_{\text{loc}}(\boldsymbol{R}^s) \oplus \ldots \oplus H^r_{\text{loc}}(\boldsymbol{R}^s)}_{n}$$

with its natural topology: if $\varphi = (\varphi_1, \ldots, \varphi_n) \in X_r$ then

$$\|\varphi; X_r(\Omega)\| \equiv \left(\sum_{j=1}^n \|\varphi_j; H^r(\Omega)\|^2\right)^{\frac{1}{2}}.$$

We propose to study eq. (3.3) in the space

$$X = X_1 \oplus X_0 . \tag{3.5}$$

In physical terms this means that we are looking for solutions of eq. (3.3) having finite kinetic energy in all open bounded regions of the space $\boldsymbol{R}^s$. In this functional space it is possible to introduce a family of cut-off maps T_h, $h \in \boldsymbol{N}$, by

$$T_h \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \chi_h \varphi \\ \chi_h \psi \end{pmatrix}, \quad \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X, \tag{3.6}$$

where χ_h is a real $C_0^\infty(\boldsymbol{R}^s)$ function such that $0 \leqslant \chi_h(x) \leqslant 1$ and $\chi_h(x) = 1$ if $|x| \leqslant h$, $\chi_h(x) = 0$ if $|x| \geqslant h+1$. It is then immediate to verify that the space X satisfies all properties listed in subsect. 1.1. We can now proceed to give an exact meaning to the semi-group $W(t)$ contained in eq. (3.3). A useful intermediate notion is that of the group $G(t)$ defined by

$$G(t) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv (2\pi)^{-s} \begin{pmatrix} \displaystyle\int \exp[i\langle x, \xi\rangle] \left[\hat{\varphi}(\xi) \cos|\xi|t + \hat{\psi}(\xi) \frac{\sin|\xi|t}{|\xi|}\right] d\xi \\ \displaystyle\int \exp[i\langle x, \xi\rangle] \left[-|\xi|\hat{\varphi}(\xi) \sin|\xi|t + \hat{\psi}(\xi) \cos|\xi|t\right] d\xi \end{pmatrix} \tag{3.7}$$

where $\varphi, \psi \in \mathcal{S}(\boldsymbol{R}^s)$ and $t \in \boldsymbol{R}$ (7). The properties of $G(t)$ we list below are well known; however, for the convenience of the reader, a proof of them is given in Appendix A.

a) For any sphere Ω and for any $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathcal{S}(\boldsymbol{R}^s) \oplus \mathcal{S}(\boldsymbol{R}^s)$ the following energy estimate holds

$$\left\| G(t) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} ;\ H^1(\Omega(|t|)) \oplus L^2(\Omega(|t|)) \right\| \leqslant \exp[|t|/2] \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} ;\ H^1(\Omega) \oplus L^2(\Omega) \right\| \tag{3.8}$$

for all t such that $0 \leqslant |t| < $ radius of Ω.

Inequality (3.8) implies that $G(t)$ defines a strongly continuous group of linear operators in $H^1_{\text{loc}}(\boldsymbol{R}^s) \oplus L^2_{\text{loc}}(\boldsymbol{R}^s)$.

b) The infinitesimal generator A of the group $G(t)$ is the unbounded linear operator

$$A = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \tag{3.9}$$

with domain

$$D_A = H^2_{\text{loc}}(\boldsymbol{R}^s) \oplus H^1_{\text{loc}}(\boldsymbol{R}^s). \tag{3.10}$$

(7) $\hat{\ell}(\xi) \equiv \int \exp[-i\langle \xi, x\rangle] \ell(x)\, dx$, if $\ell \in \mathcal{S}(\boldsymbol{R}^s)$.

In general

$$D_{A^j} = H^{1+j}_{\mathrm{loc}}(\boldsymbol{R}^s) \oplus H^j_{\mathrm{loc}}(\boldsymbol{R}^s), \quad j = 0, 1, \ldots, (A^0 = \text{identity}). \tag{3.11}$$

From eq. (3.11) it follows that the graph topology of D_{A^j} of Def. 2.5 is equivalent to the topology of $H^{1+j}_{\mathrm{loc}}(\boldsymbol{R}^s) \oplus H^j_{\mathrm{loc}}(\boldsymbol{R}^s)$.

Now we define

$$W(t) = G(t) \otimes I_{C^n}. \tag{3.12}$$

By property *a*), $W(t)$ defines a group of linear continuous operators on X of class $C(1, \frac{1}{2})$ (see Def. 1.3), and by property *b*) its infinitesimal generator is the operator

$$K = A \otimes I_{C^n}. \tag{3.13}$$

Obviously $D_{K^j} = X_{1+j} \oplus X_j$, $j = 0, 1, \ldots$, and the graph topology of D_{K^j} (see Def. 2.5) is equivalent to the topology of $X_{1+j} \oplus X_j$.

3.2. *Global existence and uniqueness.*

In this subsection we apply the general Th. 1.6 to establish the existence and uniqueness, in $C^{(0)}([0, T[; X)$, of solutions of the integral eq. (3.3). To satisfy the hypothesis of the theorem it is natural to introduce some special class of functions.

DEFINITION 3.1. By $P([0, T[; n)$, $0 < T \leqslant \infty$, we denote the class of all real vector valued functions

$$b(x, t; z) = \begin{bmatrix} 0 \\ b_1(x, t; z) \\ 0 \\ b_2(x, t; z) \\ 0 \\ \vdots \\ 0 \\ b_n(x, t; z) \end{bmatrix}, \qquad (x, t; z) \in \boldsymbol{R}^s \times [0, T[\times \boldsymbol{R}^n,$$

with the properties

i) $b_k(x, t; 0) = 0$, $\forall (x, t) \in \boldsymbol{R}^s \times [0, T[$, $k = 1, \ldots, n$;

ii) $\forall (t, \varphi) \in [0, T[\times X_1$, $\forall k = 1, \ldots, n$, the functions $b_k(x, t; \varphi) \in L^2_{\mathrm{loc}}(\boldsymbol{R}^s)$ and are continuous in the t variable, at φ fixed, in the $L^2_{\mathrm{loc}}(\boldsymbol{R}^s)$ topology;

iii) The functions b_k have partial derivatives $\partial b_k/\partial z_j$, $j=1,\dots,n$, continuous in the z variables and for any sphere Ω, for any $\tau\in\,]0,T[$ and for any $\varrho>0$, there exists a positive constant $C(\Omega,\tau,\varrho)$ such that

$$(3.14)\qquad \sup_{\substack{0\leqslant t\leqslant\tau\\ k=1,\dots,n}}\left\|\sum_{j=1}^{n}\frac{\partial b_k}{\partial z_j}(x,t;\varphi^{(1)})\varphi_j^{(2)};\ L^2(\Omega)\right\|\leqslant C(\Omega,\tau,\varrho)\|\varphi^{(2)};X_1(\Omega)\|$$

$\forall\varphi^{(1)},\varphi^{(2)}\in X_1$, $\|\varphi^{(1)};X_1(\Omega)\|\leqslant\varrho$. We further require

$$\bar{C}(\Omega,\tau,\varrho)=\sup_{0\leqslant t\leqslant\tau}C(\Omega(t),\tau,\varrho)$$

to be finite if $[0,\tau]$ is an admissible interval of time for Ω.

DEFINITION 3.2. By $P'([0,T[;n)$ we denote the subset of $P([0,T[;n)$ for which the inequality (3.14) holds in the stronger form

$$(3.14')\qquad \sup_{\substack{0\leqslant t\leqslant\tau\\ k=1,\dots,n}}\left\|\sum_{j=1}^{n}\frac{\partial b_k}{\partial z_j}(x,t;\varphi^{(1)})\varphi_j^{(2)};\ L^2(\Omega)\right\|\leqslant C(\Omega,\tau)\|\varphi^{(2)};X_1(\Omega)\|$$

$\forall\varphi^{(1)},\varphi^{(2)}\in X_1$. We further require $\bar{C}(\Omega,\tau)\equiv\sup_{0\leqslant t\leqslant\tau}C(\Omega(t),\tau)$ to be finite if $[0,\tau]$ is an admissible interval of time for Ω.

THEOREM 3.1. *If $b\in P([0,T[;n)$ $(b\in P'([0,T[;n))$ then the map*

$$(3.15)\qquad \begin{cases}\beta\colon[0,T[\times X\to X\\ \beta\left(t,\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)\equiv b(x,t;\varphi)\end{cases}$$

belongs to the class $L([0,T[;X)$ $(L'([0,T[;X))$.

PROOF. It is clear, from ii) of Def. 3.1, that β is well defined. Application of the mean value theorem and of Minkowski integral inequality yields, $\forall k=1,\dots,n$,

$$(3.16)\qquad \left(\int_\Omega|b_k(x,t;\varphi^{(2)})-b_k(x,t;\varphi^{(1)})|^2dx\right)^{\frac12}=$$

$$=\left\{\int_\Omega\left|\int_0^1\sum_{j=1}^{n}\frac{\partial b_k}{\partial z_j}(x,t;\varphi^{(1)}+\sigma(\varphi^{(2)}-\varphi^{(1)}))(\varphi_j^{(2)}-\varphi_j^{(1)})\,d\sigma\right|^2dx\right\}^{\frac12}\leqslant$$

$$\leqslant\int_0^1\left\{\int_\Omega\left|\sum_{j=1}^{n}\frac{\partial b_k}{\partial z_j}(x,t;\varphi^{(1)}+\sigma(\varphi^{(2)}-\varphi^{(1)}))(\varphi_j^{(2)}-\varphi_j^{(1)})\right|^2dx\right\}^{\frac12}d\sigma\leqslant$$

$$\leqslant C(\Omega,\tau,\varrho)\|\varphi^{(2)}-\varphi^{(1)};X_1(\Omega)\|,$$

if $t \in [0, \tau] \subset [0, T[$, Ω is any sphere, and $\|\varphi^{(i)}; X_1(\Omega)\| \leqslant \varrho$, $i = 1, 2$. Therefore condition iii) of Def. 1.4 is satisfied. Moreover, $\forall\ k = 1, \ldots, n$,

$$\begin{aligned}\|b_k(x, t + \Delta t; \varphi + \Delta\varphi) - b_k(x, t; \varphi); L^2(\Omega)\| \leqslant \\ \leqslant \|b_k(x, t + \Delta t; \varphi + \Delta\varphi) - b_k(x, t + \Delta t; \varphi); L^2(\Omega)\| + \\ + \|b_k(x, t + \Delta t; \varphi) - b_k(x, t; \varphi); L^2(\Omega)\| \equiv J_1 + J_2 .\end{aligned}$$

Now $J_2 \to 0$, as $\Delta t \to 0$, by ii) of Def. 3.1, and $J_1 \to 0$, as $\Delta\varphi \to 0$ in X_1, uniformly on compact intervals of time by inequality (3.16). This establishes i) of Def. 1.4. Similarly one proves the corresponding statement for P' and L'.

DEFINITION 3.3. By $J([0, T[; n)$, $0 < T \leqslant \infty$, we denote the class of all real vector valued functions

$$c(x, t) = \begin{bmatrix} 0 \\ c_1(x, t) \\ 0 \\ c_2(x, t) \\ 0 \\ \vdots \\ 0 \\ c_n(x, t) \end{bmatrix}, \qquad (x, t) \in \boldsymbol{R}^s \times [0, T[,$$

with the property that $c_k(x, t) \in L^2_{\text{loc}}(\boldsymbol{R}^s)$ and that the map $t \mapsto c_k(x, t)$ is continuous in the $L^2_{\text{loc}}(\boldsymbol{R}^s)$ topology, $k = 1, \ldots, n$.

It is obvious that, if $c \in J([0, T[; n)$, the map

$$(3.17) \qquad \gamma : [0, T[\to X , \quad \gamma(t) \equiv c(x, t) ,$$

belongs to $C^{(0)}([0, T[; X)$.

In the following we will study eq. (3.3) under the assumption that $\begin{pmatrix}\varphi_0 \\ \psi_0\end{pmatrix} \in X$ and that

$$(3.18) \qquad f\left(t, \begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right) = g(x, t; \varphi) + j(x, t) ; \qquad g \in P([0, T[; n), j \in J([0, T[; n) .$$

It is clear from Th. 3.1 and the discussion in subsect. 1.1 that the integral equation (3.3) makes sense.

LEMMA 3.1. Let $\begin{pmatrix}\varphi \\ \psi\end{pmatrix} \in C^{(0)}([0, T[; X)$, $0 < T \leqslant \infty$, be a solution of eq. (3.3) with $\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} \in X$ and f as in (3.18). Then the map

$$[0, T[\ni t \mapsto \varphi(t)$$

is X_0-differentiable and $d\varphi/dt = \psi$.

PROOF. By applying the mollifier

$$M_\varepsilon \equiv \frac{1}{\varepsilon}\int_0^\varepsilon W(t)\cdot dt$$

to eq. (3.3) and differentiating, one obtains

$$\frac{d}{dt} M_\varepsilon \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix} = K M_\varepsilon \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix} + M_\varepsilon f\left(t, \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix}\right). \tag{3.19}$$

Integration of eq. (3.19) yields

$$M_\varepsilon \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix} = M_\varepsilon \begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} + \int_0^t \left[K M_\varepsilon \begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix} + M_\varepsilon f\left(s, \begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}\right)\right] ds. \tag{3.20}$$

Since the operator $E \equiv \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \otimes I_{C^n}$ is bounded from X to X and $EK = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \otimes I_{C^n}$, the application of E to both sides of eq. (3.20) gives, in the limit as $\varepsilon \to 0+$,

$$\begin{pmatrix}0\\ \varphi(t)\end{pmatrix} = \begin{pmatrix}0\\ \varphi_0\end{pmatrix} + \int_0^t \begin{pmatrix}0\\ \psi(s)\end{pmatrix} ds.$$

DEFINITION 3.4. By $Q([0, T[; n)$ we denote the class of functions $b \in P([0, T[; n)$ for which the following properties are satisfied

1) There exists a real function $G(x, t; z)$ differentiable in the z variables with $G(x, t; 0) = 0$, $\forall (x, t)$, such that

   i) $b_k = -\dfrac{\partial G}{\partial z_k}$, $\quad k = 1, \ldots, n$;

   ii) $\inf\limits_{\substack{x\in B\\ t\in[0,\tau]\\ z\in \mathbf{R}^n}} G(x, t; z) > -\infty$, $\forall \tau \in]0, T[$ and for all compact subsets $B \subset \mathbf{R}^s$.

2) There exists a real function $\dot{G}(x, t; z)$, $(x, t; z) \in \boldsymbol{R}^s \times [0, T[\times \boldsymbol{R}^n$, such that

i) $\forall \varphi \in X_1$, $\dot{G}(x, t; \varphi) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^s)$, $t \in [0, T[$;

ii) $\frac{1}{\Delta t}[G(x, t + \Delta t; \varphi) - G(x, t; \varphi)] \to \dot{G}(x, t; \varphi)$,

as $\Delta t \to 0$, in $L^1_{\mathrm{loc}}(\boldsymbol{R}^s)$, $\forall \varphi \in X_1$;

iii) If $\varphi \in C^{(0)}([0, \tau[; X)$, $\tau \leqslant T$, then $\dot{G}(x, t; \varphi(t)) \in C^{(0)}([0, T[; L^1_{\mathrm{loc}}(\boldsymbol{R}^s))$.

3) For any sphere Ω, for any $\tau \in]0, T[$, there exist two non negative constants $M(\Omega, \tau)$, $N(\Omega, \tau)$ such that

$$\dot{G}(x, t; z) \leqslant M(\Omega, \tau) G(x, t; z) + N(\Omega, \tau)$$

for all $(x, t; z) \in \Omega \times [0, \tau] \times \boldsymbol{R}^n$.

REMARK 1. Condition 2) ii) of Def. 3.4 requires some explanation. More precisely, it has to be checked that, $\forall \varphi \in X_1$, the function $G(x, t; \varphi) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^s)$ for all $t \in [0, T[$. In fact, from the identity

$$G(x, t; z) = -\sum_{j=1}^{n} \int_0^1 b_j(x, t; \sigma z) z_j d\sigma$$

one can deduce that

$$(3.20') \qquad \int_U |G(x, t; \varphi(x))| dx \leqslant \sum_{j=1}^{n} \left(\int_U \left| \int_0^1 |b_j(x, t; \sigma\varphi(x))| d\sigma \right|^2 dx \right)^{\frac{1}{2}} \|\varphi_j; L^2(U)\| \leqslant$$
$$\leqslant \sup_{j=1,\ldots,n} \left(\int_0^1 \|b_j(x, t; \sigma\varphi(x)); L^2(U)\|^2 d\sigma \right)^{\frac{1}{2}} \sum_{j=1}^{n} \|\varphi_j; L^2(U)\| < \infty .$$

LEMMA 3.2. Let $b \in Q([0, T[; n)$ and let $\varphi \in C^{(0)}([0, \tau[; X_1) \cap C^{(1)}([0, \tau[; X_0)$, $\tau \in]0, T[$. If

$$\operatorname{supp} \varphi(t) \subset \overline{\Omega_1(t)} \subset \Omega(t), \qquad 0 \leqslant t < \tau,$$

for some spheres Ω_1, Ω, then the function

$$[0, \tau[\ni t \mapsto E(t) \equiv \int_{\Omega(t)} G(x, t; \varphi(t)) dx$$

is differentiable and

$$\frac{dE}{dt}(t)=\int_{\Omega(t)}\dot{G}(x,t;\varphi(t))\,dx-\sum_{j=1}^{n}\int_{\Omega(t)}b_j(x,t;\varphi(t))\,\psi_j(t)\,dx \tag{3.21}$$

where $\psi = d\varphi/dt$.

PROOF. It is clear from Remark 1 that the function $E(t)$ is well defined. If $|\Delta t|$ is sufficiently small

$$\frac{E(t+\Delta t)-E(t)}{\Delta t}=\int_{\Omega(t)}\frac{G(x,t+\Delta t;\varphi(t+\Delta t))-G(x,t;\varphi(t))}{\Delta t}\,dx \tag{3.22}$$

as a consequence of the estimate (3.20′) and of the fact that

$$\operatorname{supp}\varphi(t+\Delta t)\cap\left[\left(\Omega(t+\Delta t)\setminus\overline{\Omega(t)}\right)\cup\left(\Omega(t)\setminus\overline{\Omega(t+\Delta t)}\right)\right]\quad\text{is empty}.$$

Eq. (3.22) can be rewritten

$$\frac{E(t+\Delta t)-E(t)}{\Delta t}=\int_{\Omega(t)}\frac{G(x,t+\Delta t;\varphi(t))-G(x,t;\varphi(t))}{\Delta t}+$$
$$+\int_{\Omega(t)}\frac{G(x,t+\Delta t;\varphi(t+\Delta t))-G(x,t+\Delta t;\varphi(t))}{\Delta t}\equiv J_1+J_2\,.$$

By Def. 3.4 $J_1\to\int_{\Omega(t)}\dot{G}(x,t;\varphi(t))\,dx$, as $\Delta t\to 0$, while J_2 can be conveniently reexpressed as

$$J_2=\left(J_2+\sum_{j=1}^{n}\int_{\Omega(t)}b_j(x,t;\varphi(t))\left[\frac{\varphi_j(t+\Delta t)-\varphi_j(t)}{\Delta t}\right]dx\right)-$$
$$-\sum_{j=1}^{n}\int_{\Omega(t)}b_j(x,t;\varphi(t))\left[\frac{\varphi_j(t+\Delta t)-\varphi_j(t)}{\Delta t}-\psi_j(t)\right]dx-$$
$$-\sum_{j=1}^{n}\int_{\Omega(t)}b_j(x,t;\varphi(t))\,\psi_j(t)\,dx\equiv R_1-R_2-R_3\,.$$

The term $R_2\to 0$, as $\Delta t\to 0$. Application of the mean value theorem yields

$$R_1=\int_{\Omega(t)}\left(\int_0^1\sum_{j=1}^{n}\left[b_j(x,t;\varphi(t))-b_j\big(x,t+\Delta t;\varphi(t)+\sigma(\varphi(t+\Delta t)-\varphi(t))\big)\right]\cdot\right.$$
$$\left.\cdot\frac{\varphi_j(t+\Delta t)-\varphi_j(t)}{\Delta t}\,d\sigma\right)dx$$

and therefore

$$|R_1| \leqslant \sum_{j=1}^{n} \left(\left\| \frac{\varphi_j(t+\Delta t) - \varphi_j(t)}{\Delta t} ;\ L^2(\Omega(t)) \right\| \cdot \right.$$
$$\left. \cdot \sup_{0 \leqslant \sigma \leqslant 1} \left\| b_j(x, t; \varphi(t)) - b_j\big(x, t + \Delta t; \varphi(t) + \sigma(\varphi(t+\Delta t) - \varphi(t))\big);\ L^2(\Omega(t)) \right\| \right).$$

By the continuity of the b_j, the r.h.s. tends to 0 as $\Delta t \to 0$.

REMARK 2. The proof of Lemma 3.2 does not require properties 1) ii), 2) iii) and 3) of Def. 3.4.

REMARK 3. It follows from Lemma 3.1 that if $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is any continuous solution of eq. (3.3) (with f as in eq. (3.18)) with compact support, then Lemma 3.2 can be applied to the function φ with a suitable Ω.

DEFINITION 3.5. Given $b \in Q([0, T[; n)$, for any sphere Ω we define

$$(3.23) \quad \begin{cases} \tilde{V}_\Omega(t; \varphi; b) \equiv \int_{\Omega(t)} G(x, t; \varphi)\, dx, \\ \qquad\qquad \varphi \in X_1,\ t \in [0, T_\Omega[,\ T_\Omega = \min\{T, \text{radius } \Omega\}. \\ \tilde{W}_\Omega(t; \varphi; b) \equiv \int_{\Omega(t)} \dot{G}(x, t; \varphi)\, dx. \end{cases}$$

These functions are well defined as a consequence of Def. 3.4 and Remark 1.

LEMMA 3.3 Let $b \in Q([0, T[; n)$ and let $\varphi \in C^{(0)}([0, \tau[; X_1)$, $\tau \in]0, T[$. Then, for any sphere Ω with radius greater than τ, the functions

$$[0, \tau[\ni t \mapsto \tilde{V}_\Omega(t; \varphi(t); b), \qquad \tilde{W}_\Omega(t; \varphi(t); b)$$

are continuous.

PROOF. By property 2) iii) of Def. 3.4 it follows that $t \mapsto \dot{G}(x, t; \varphi(t))$ is $L^1_{\text{loc}}(\boldsymbol{R}^s)$ continuous. By the same kind of argument as in Remark 1, the function $t \mapsto G(x, t; \varphi(t))$ is $L^1_{\text{loc}}(\boldsymbol{R}^s)$ continuous. The continuity of $\tilde{V}_\Omega$ and $\tilde{W}_\Omega$ is now immediate.

REMARK 4. The proof of Lemma 3.3 does not require properties 1) ii), 2) i) and 3) of Def. 3.4.

At this point we can prove the main result of this section.

THEOREM 3.2. *Let*

$$j \in J([0, T[; n), \quad g^{(1)} \in Q([0, T[; n), \quad g^{(2)} \in P'([0, T[; n), \quad \begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} \in X .$$

Then the integral equation (3.3), *with*

$$f\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) = j(x, t) + g^{(1)}(x, t; \varphi) + g^{(2)}(x, t; \varphi),$$

has a unique solution $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C^{(0)}([0, T[; X)$.

PROOF. The proof consists in showing that hypothesis A of Th. 1.6 is satisfied, with the identifications

$$V_\Omega\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) \equiv \tilde{V}_\Omega(t; \varphi; g^{(1)}), \qquad W_\Omega\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) \equiv \tilde{W}_\Omega(t; \varphi; g^{(1)}).$$

The only non trivial point that remains to be checked is condition c). By Lemma 3.1 it will be enough to verify condition c) for functions

$$\begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix}$$

with

$$\varphi \in C^{(0)}([0, \tau[; X_1) \cap C^{(1)}([0, \tau[; X_0), \qquad \psi = \frac{d\varphi}{dt},$$

and

$$\operatorname{supp} \varphi(t) \subset \overline{\Omega_1(t)} \subset \Omega(t), \qquad 0 \leqslant t < \tau,$$

for some spheres Ω_1, Ω. The required continuity of

$$V_\Omega\left(t, \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix}\right) \quad \text{and} \quad W_\Omega\left(t, \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix}\right)$$

is a consequence of Lemma 3.3. On the other hand, eq. (3.21), integrated from 0 to t $(t < \tau)$, becomes

$$\sum_{j=1}^{n} \int_0^t \int_{\Omega(s)} g_j^{(1)}(x, s; \varphi(s))\, \psi_j(s)\, dx\, ds = \int_0^t W_\Omega\left(s, \begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}\right) ds + \\ + V_\Omega\left(0, \begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix}\right) - V_\Omega\left(t, \begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix}\right),$$

which is exactly eq. (1.14).

Theorem 3.2 covers the case of the forward Cauchy problem. However $W(t)$ is a group and the local energy estimate (3.8) holds also in the backward direction. For this reason Th. 3.2 can be used also to prove existence and uniqueness for the backward Cauchy problem, once the obvious modifications are performed on the classes of functions defined in subsect. 3.2.

3.3. *Concrete cases.*

We propose to list some concrete and interesting examples of functions belonging to the classes defined in subsect. 3.2.

1) *Functions belonging to the space* $J([0, T[; n)$. If $c_k(x, t)$, $k = 1, \ldots, n$, are n real-valued continuous functions defined in $\boldsymbol{R}^s \times [0, T[$, it is clear that the vector valued function

$$c(x, t) = \begin{bmatrix} 0 \\ c_1(x, t) \\ 0 \\ \vdots \\ 0 \\ c_n(x, t) \end{bmatrix}$$

belongs to $J([0, T[; n)$.

2) *Functions belonging to the space* $P'([0, T[; n)$. Let $b_k(x, t; z)$, $k = 1, \ldots, n$, be n real-valued continuous functions defined in $\boldsymbol{R}^s \times [0, T[\times \boldsymbol{R}^n$ and such that

i) $b_k(x, t; 0) = 0$, $\forall (x, t)$, $\forall k$;

ii) The partial derivatives $\partial b_k / \partial z_j$ $(k, j = 1, \ldots, n)$ exist, are continuous and satisfy the estimate

$$\sup_{\substack{j,k=1,\ldots,n \\ x \in B \\ t \in [0,\tau] \\ z \in \boldsymbol{R}^n}} \left| \frac{\partial b_k}{\partial z_j}(x, t; z) \right| = L(B, \tau) < \infty$$

for all $[0, \tau] \subset [0, T[$ and for all compact subsets $B \subset \boldsymbol{R}^s$. Then, by direct inspection, one checks that the vector valued function

$$b(x, t; z) = \begin{bmatrix} 0 \\ b_1(x, t; z) \\ 0 \\ \vdots \\ 0 \\ b_n(x, t; z) \end{bmatrix}$$

belongs to $P'([0, T[; n)$.

An explicit case which falls in the class $P'([0, T[; 1)$ is given by

$$b(x, t; z) = \alpha(x, t) \sin \beta(x, t) z + \gamma(x, t) z$$

with α, β, γ continuous functions of (x, t). We mention this example because a semplified form of it (α, β, γ constants) has been recently used in elementary particle physics ([4], [5]).

3) *Functions belonging to the space* $Q([0, T[; n)$. Here we restrict ourselves to the cases in which the number of space dimensions s is equal to 1, 2, 3. Apart from the physical case ($s = 3$) which is important for obvious reasons, the cases $s = 1, 2$ are usefully studied as interesting prototypes, simulating some aspects of the actual physical theories.

In what follows an important role is played by some well-known Sobolev inequalities ([12]) that we are reporting here, but whose discussion is deferred to Appendix B. Precisely, given any sphere $\Omega_R \subset \mathbf{R}^s$ of radius $R > 0$, these inequalities can be written as

$$(3.24) \qquad \begin{cases} s = 1: & \|\varphi; L^p(\Omega_R)\| \leqslant C_1(R) \|\varphi; H^1(\Omega_R)\|, & 2 \leqslant p \leqslant \infty \\ s = 2: & \|\varphi; L^p(\Omega_R)\| \leqslant \sqrt{p}\, C_2(R) \|\varphi; H^1(\Omega_R)\|, & 2 \leqslant p < \infty \\ s = 3: & \|\varphi; L^p(\Omega_R)\| \leqslant C_3(R) \|\varphi; H^1(\Omega_R)\|, & 2 \leqslant p \leqslant 6 \end{cases}$$

with $\varphi \in H^1(\Omega_R)$ and $C_j(R) = c_j R$ if $R \geqslant 1$, $C_j(R) = c_j/R$ if $R \leqslant 1$.

It is convenient to give a different treatment of the three cases $s = 1, 2, 3$.

Case $s = 1$.

Let $b_k(x; z) = \sum_{|\alpha| \geqslant 1} a_{k\alpha}(x) z^\alpha$, $k = 1, \ldots, n$, with the properties

i) $a_{k\alpha}$ are real-valued functions belonging to $L^\infty_{\text{loc}}(\mathbf{R}^1)$, $\forall k, \alpha$;

ii) $\sum_{|\alpha| \geqslant 1} \sup_{x \in B} |a_{k\alpha}(x)| \sigma^{|\alpha|} < \infty$, $k = 1, \ldots, n$, for any compact $B \subset \mathbf{R}^1$ and for all $\sigma > 0$;

iii) There exists a real function $G(x, z)$ continuously differentiable in $\mathbf{R}^1 \times \mathbf{R}^n$ for which

$$\begin{cases} G(x; 0) = 0, \quad x \in \mathbf{R}^1 \\ b_k(x; z) = -\dfrac{\partial G}{\partial z_k}(x; z), \qquad (x; z) \in \mathbf{R}^1 \times \mathbf{R}^n, \qquad k = 1, \ldots, n \\ \inf\limits_{\substack{x \in B \\ z \in \mathbf{R}^n}} G(x; z) > -\infty, \qquad \text{for all compact } B \subset \mathbf{R}^1 . \end{cases}$$

Then the vector valued function

$$b(x, t; z) = \begin{bmatrix} 0 \\ b_1(x; z) \\ 0 \\ \vdots \\ 0 \\ b_n(x; z) \end{bmatrix}$$

belongs to $Q([0, +\infty[; n)$.

The only non trivial facts to be verified are properties ii) and iii) of Def. 3.1. Given $\varphi \in X_1$ and a sphere Ω_R of radius R, one has

$$\Big\| \sum_{|\alpha|\geq 1} a_{k\alpha}(x)\varphi_1^{\alpha_1}\varphi_2^{\alpha_2}\dots\varphi_n^{\alpha_n}; L^2(\Omega_R)\Big\| \leq \sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)|\prod_{j=1}^{n}\|\varphi_j; L^{2|\alpha|}(\Omega_R)\|^{\alpha_j} \leq$$
$$\leq \sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)|C_1(R)^{|\alpha|}\|\varphi; X_1(\Omega_R)\|^{|\alpha|} < \infty\,.$$

This proves property ii). Moreover the estimates

$$\Big\| \sum_{j=1}^{n}\sum_{|\alpha|\geq 1} a_{k\alpha}(x)\alpha_j\prod_{\ell\neq j}\varphi_\ell^{(1)\alpha_\ell}\varphi_j^{(1)(\alpha_j-1)}\varphi_j^{(2)}; L^2(\Omega_R)\Big\| \leq$$
$$\leq \sum_{j=1}^{n}\sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)|\alpha_j\prod_{\ell\neq j}\|\varphi_\ell^{(1)}; L^{2|\alpha|}(\Omega)_R\|^{\alpha_\ell}\|\varphi_j^{(1)}; L^{2|\alpha|}(\Omega_R)\|^{\alpha_j-1}\|\varphi_j^{(2)}; L^{2|\alpha|}(\Omega_R)\| \leq$$
$$\leq\Big(\sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)|C_1(R)^{|\alpha|}|\alpha|\|\varphi^{(1)}; X_1(\Omega_R)\|^{|\alpha|-1}\Big)\|\varphi^{(2)}; X_1(\Omega_R)\|$$

yield inequality (3.14), with $C(\Omega_R, \tau, \varrho) = \sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)|C_1(R)^{|\alpha|}|\alpha|\varrho^{|\alpha|-1}$, because the series $\sum_{|\alpha|\geq 1}\sup_{x\in\Omega_R}|a_{k\alpha}(x)||\alpha|\sigma^{|\alpha|-1}$ is convergent for all $\sigma > 0$. Taking into account the structure of $C_1(R)$, it is easily recognized that $\sup_{0\leq t\leq\tau} C(\Omega_R(t), \tau, \varrho)$ is finite if $[0, \tau]\subset[0, T[$ is an admissible interval of time for Ω_R.

Case $s = 2$.

Let $b_k(x; z) = \sum_{|\alpha|\geq 1} a_{k\alpha}(x)z^\alpha$, $k = 1, \dots, n$, with the properties

i) $a_{k\alpha}$ are real valued functions belonging to $L^\infty_{\text{loc}}(\boldsymbol{R}^2)$, $\forall k, \alpha$;

ii) $\sum_{|\alpha|\geq 1}\sup_{x\in B}|a_{k\alpha}(x)||\alpha|^{|\alpha|/2}\sigma^{|\alpha|} < \infty$; $k = 1, \dots, n$, for any compact $B\subset\boldsymbol{R}^2$ and for all $\sigma > 0$;

iii) There exists a real function $G(x; z)$ continuously differentiable in $\boldsymbol{R}^2 \times \boldsymbol{R}^n$ for which

$$\begin{cases} G(x; 0) = 0\,, \quad x \in \boldsymbol{R}^2 \\ b_k(x; z) = -\dfrac{\partial G}{\partial z_k}(x; z)\,, \qquad (x; z) \in \boldsymbol{R}^2 \times \boldsymbol{R}^n, \qquad k = 1, \ldots, n \\ \inf\limits_{\substack{x \in B \\ z \in \boldsymbol{R}^n}} G(x; z) > -\infty\,, \quad \text{for all compact } B \subset \boldsymbol{R}^2\,. \end{cases}$$

Then the vector valued function

$$b(x, t; z) = \begin{bmatrix} 0 \\ b_1(x; z) \\ 0 \\ \vdots \\ 0 \\ b_n(x; z) \end{bmatrix}$$

belongs to $Q([0, +\infty[; n)$.

As in the case $s = 1$, the only non trivial facts to be verified are properties ii) and iii) of Def. 3.1. Given $\varphi \in X_1$ and a sphere Ω_R of radius R, one has

$$\begin{aligned} \Big\| \sum_{|\alpha| \geq 1} a_{k\alpha}(x) \varphi_1^{\alpha_1} \varphi_2^{\alpha_2} \ldots \varphi_n^{\alpha_n}; L^2(\Omega_R) \Big\| &\leq \\ &\leq \sum_{|\alpha| \geq 1} \sup_{x \in \Omega_R} |a_{k\alpha}(x)| \prod_{j=1}^{n} \|\varphi_j; L^{2|\alpha|}(\Omega_R)\|^{\alpha_j} \leq \\ &\leq \sum_{|\alpha| \geq 1} \sup_{x \in \Omega_R} |a_{k\alpha}(x)| (2|\alpha|)^{|\alpha|/2} C_2(R)^{|\alpha|} \|\varphi; X_1(\Omega_R)\|^{|\alpha|} < \infty\,. \end{aligned}$$

This proves property ii). Property iii) may be verified by arguing as in the case $s = 1$.

For example, the function $b(x; z) = e^z - 1$ belongs to $Q([0, +\infty[; 1)$ while the function $b(x; z) = e^{z^2} - 1$ does not verify condition ii).

Case $s = 3$.

Let $b_k(x; z) = \sum\limits_{1 \leq |\alpha| \leq 3} a_{k\alpha}(x) z^\alpha$, $k = 1, \ldots, n$, with the properties

i) $a_{k\alpha}$ are real functions belonging to $L^\infty_{\mathrm{loc}}(\boldsymbol{R}^3)$, $\forall k, \alpha$;

ii) There exists a real function $G(x; z)$ continuously differentiable in

$R^3 \times R^n$ for which

$$\begin{cases} G(x;0) = 0\,, \quad x \in R^3 \\ b_k(x;z) = -\dfrac{\partial G}{\partial z_k}(x;z)\,, \quad (x;z) \in R^3 \times R^n, \quad k = 1, \dots, n \\ \inf\limits_{\substack{x \in B \\ z \in R^n}} G(x;z) > -\infty\,, \quad \text{for all compact } B \subset R^3\,. \end{cases}$$

Then the vector valued function

$$b(x,t;z) = \begin{bmatrix} 0 \\ b_1(x;z) \\ 0 \\ \vdots \\ 0 \\ b_n(x;z) \end{bmatrix}$$

belongs to $Q([0, +\infty[; n)$. This can be easily proved by using inequality (3.24) (for $s = 3$) and the same kind of arguments as in the cases $s = 1, 2$.

Finally we want to discuss some examples of functions $b(x, t; z)$ belonging to the class $Q([0, T[; n)$ and depending explicitely on t.

Let us be given a function

$$G(x,t;z) = h(x) \sum_{j=2}^{2p} a_j(x,t) z^j\,, \quad (x,t;z) \in R^s \times [0, T[\times R^1\,,$$

with the properties

i) $p \in N$ if $s = 1, 2$; $p < 2$ if $s = 3$;

ii) h is a continuous non-negative function in R^s;

iii) The functions a_j are real-valued and continuously differentiable in $R^s \times [0, T[$. Moreover, $a_{2p}(x,t) > 0$, $\forall (x,t) \in R^s \times [0, T[$.

Then the function

$$b(x,t;z) = -\frac{\partial G}{\partial z}(x,t;z)$$

belongs to $Q([0, T[; 1)$ if we define

$$\dot{G}(x,t;z) = h(x) \sum_{j=2}^{2p} \frac{\partial a_j}{\partial t}(x,t) z^j\,.$$

Since the proof that $b \in P([0, T[; 1)$ goes as in the preceding cases, the only non trivial facts that remain to be checked are properties 1) ii) and 3) of Def. 3.4. Let Ω be any sphere and $\tau \in]0, T[$. It is clear that for any $\varepsilon \in]0, \min_{\substack{x\in\bar{\Omega}\\ t\in[0,\tau]}} a_{2p}(x, t)[$ there exists a constant C_ε such that

$$\inf_{\substack{x\in\Omega\\ t\in[0,\tau]}} \sum_{j=2}^{2p} a_j(x, t) z^j \geqslant \Big(\min_{\substack{x\in\bar{\Omega}\\ t\in[0,\tau]}} a_{2p}(x, t) - \varepsilon \Big) z^{2p} - C_\varepsilon . \tag{3.25}$$

Thus property 1) ii) follows from inequality (3.25). On the other hand

$$\sup_{\substack{x\in\Omega\\ t\in[0,\tau]}} \sum_{j=2}^{2p} \frac{\partial a_j}{\partial t}(x, t) z^j \leqslant A(\Omega, \tau) z^{2p} + B(\Omega, \tau) \tag{3.26}$$

for some suitable non negative constants A, B. Combination of inequalities (3.25) and (3.26) then yields property 3) of Def. 3.4.

Another example is worth mentioning explicitely because it covers the case of adiabatic switching. Suppose we are given n real valued functions $b_k(x; z)$, $k = 1, \dots, n$, $(x; z) \in \mathbf{R}^s \times \mathbf{R}^n$, with the property that the corresponding vector valued function $b(x, t; z)$ belongs to $Q([0, +\infty[; n)$. If $t \mapsto \lambda(t)$ is a continuously differentiable real valued function on $[0, +\infty[$, we ask for conditions ensuring that $\lambda(t) b(x, t; z)$ still belongs to $Q([0, +\infty[; n)$. If we denote by $H(x; z)$ the function satisfying condition 1) of Def. 3.4 with respect to b, then we have only to check properties 1) ii) and 3) of Def. 3.4, with the identifications $G = \lambda H$, $\dot{G} = (d\lambda/dt) H$. These are obviously satisfied if

$$\sup_{\substack{x\in\Omega\\ z\in\mathbf{R}^n}} |H(x; z)| < \infty \tag{3.27}$$

for all spheres Ω. In case inequality (3.27) does not hold, it is not difficult to show that the following are necessary and sufficient conditions for λb to belong to $Q([0, +\infty[; n)$

α) $\lambda(t) \geqslant 0$, $\forall t$;

β) $\forall \tau > 0$ there exists a constant $M(\tau) > 0$ such that

$$\frac{d\lambda}{dt}(t) \leqslant M(\tau)\, \lambda(t) , \qquad t \in [0, \tau] .$$

Clearly α) and β) are verified if $\lambda(t) > 0$, $\forall t$.

3.4. *Regularity.*

In this subsection we will apply the abstract results of Sect. 2 to study the regularity of the solutions of eq. (3.3). For this purpose it is convenient to define some classes of functions satisfying conditions which guarantee the applicability of Th. 2.2 and Cor. 2.3. These conditions are certainly not the best ones, they are however sufficient to cover a large number of interesting applications.

DEFINITION 3.6. By $Y_{(r)}([0, T[; n)$, $0 < T \leqslant \infty$, we denote the class of all real vector valued functions

$$a(x, t; z) = \begin{bmatrix} 0 \\ a_1(x, t; z) \\ 0 \\ a_2(x, t; z) \\ \vdots \\ 0 \\ a_n(x, t; z) \end{bmatrix}, \quad (x, t; z) \in \boldsymbol{R}^s \times [0, T[\times \boldsymbol{R}^n,$$

with the properties

i) For almost all $x \in \boldsymbol{R}^s$, $\forall k$, a_k is $(r+1)$-times continuously differentiable in the variables t and z and the partial derivatives $\partial_t^\alpha \partial_z^\beta a_k$ are measurable functions of x, for all t and z;

ii) For any sphere Ω, for any $\tau \in]0, T[$, for any $\varrho > 0$ and for any multiindices α, β with $\alpha + |\beta| \leqslant r + 1$, there exists a positive constant $C_{\alpha,\beta}(\Omega, \tau, \varrho)$ such that

$$(3.28) \qquad \sup_{\substack{0 \leqslant t \leqslant \tau \\ k=1,\dots,n}} \|\partial_t^\alpha \partial_z^\beta a_k(x, t; \varphi) \xi_1 \xi_2 \dots \xi_{|\beta|}; L^2(\Omega)\| \leqslant C_{\alpha,\beta}(\Omega, \tau, \varrho) \prod_{j=1}^{|\beta|} \|\xi_j; H^1(\Omega)\|$$

for all $\xi_1, \dots, \xi_{|\beta|} \in H^1(\Omega)$, for all $\varphi \in X_1(\Omega)$ with $\|\varphi; X_1(\Omega)\| \leqslant \varrho$. We further require $\bar{C}(\Omega, \tau, \varrho) = \sup\limits_{\substack{0 \leqslant t \leqslant \tau \\ |\beta|=1}} C_{0,\beta}(\Omega(t), \tau, \varrho)$ to be finite if $[0, \tau]$ is an admissible interval of time for Ω.

DEFINITION 3.7. By $Y'_{(r)}([0, T[; n)$, $0 < T \leqslant \infty$, we denote the class of all real vector valued functions

$$a(x, t; z) = \begin{bmatrix} 0 \\ a_1(x, t; z) \\ 0 \\ a_2(x, t; z) \\ \vdots \\ 0 \\ a_n(x, t; z) \end{bmatrix}, \qquad (x, t; z) \in \boldsymbol{R}^s \times [0, T[\times \boldsymbol{R}^n,$$

with the properties

i) The derivatives $\partial_x^\gamma \partial_t^\alpha \partial_z^\beta a_k$ exist for all α, β, γ with $\alpha + |\beta| \leqslant r + 1$, $|\gamma| \leqslant r - 1$, $k = 1, \dots, n$, and are continuous;

ii) For any sphere Ω, for any $\tau \in]0, T[$, for any $\varrho > 0$ and for any multiindices α, β, γ, as above, there exists a positive constant $C_{\gamma,\alpha,\beta}(\Omega, \tau, \varrho)$ such that

$$(3.29) \qquad \sup_{\substack{0 \leqslant t \leqslant \tau \\ k=1,\dots,n}} \|\partial_x^\gamma \partial_t^\alpha \partial_z^\beta a_k(x, t; \varphi)\xi_1\xi_2 \dots \xi_{|\beta|}; L^2(\Omega)\| \leqslant \\ \leqslant C_{\gamma,\alpha,\beta}(\Omega, \tau, \varrho) \prod_{j=1}^{|\beta|} \|\xi_j; H^1(\Omega)\|$$

for all $\xi_1, \dots, \xi_{|\beta|} \in H^1(\Omega)$, for all $\varphi \in X_1(\Omega)$ with $\|\varphi; X_1(\Omega)\| \leqslant \varrho$. We further require $\bar{C}(\Omega, \tau, \varrho) = \sup\limits_{\substack{0 \leqslant t \leqslant \tau \\ |\beta|=1}} C_{0,0,\beta}(\Omega(t), \tau, \varrho)$ to be finite if $[0, \tau]$ is an admissible interval of time for Ω.

REMARK 5. Clearly $Y'_{(r)}([0, T[; n) \subset Y_{(r)}([0, T[, n)$.

LEMMA 3.4. If $a \in Y_{(r)}([0, T[; n)$, the vector valued map

$$\begin{cases} f: [0, T[\times X \to X \\ f\left(t, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right) \equiv a(x, t; \varphi) \end{cases}$$

belongs to $BC^{(r)}([0, T[; X)$ (see Defs. 2.1, 2.2).

PROOF. For all $(\alpha, \beta) \in \Lambda_j$, $j \leqslant r$ (see notations before Def. 2.2), let

$$D^{(\alpha,\beta)} f\left(t, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right) \equiv \begin{bmatrix} 0 \\ \partial_t^{|\alpha|} \partial_z^\beta a_1(x, t; \varphi) \\ 0 \\ \partial_t^{|\alpha|} \partial_z^\beta a_2(x, t; \varphi) \\ \vdots \\ 0 \\ \partial_t^{|\alpha|} \partial_z^\beta a_n(x, t; \varphi) \end{bmatrix}.$$

It is clear, by ii) of Def. 3.6, that

$$D^{(\alpha,\beta)} f: [0, T[\times X \to \mathfrak{L}^{|\beta|}(X).$$

The strong continuity of $D^{(\alpha,\beta)}f$ relies on the following estimate

$$(3.30)\quad \|[\partial_t^{|\alpha|}\partial_z^{\beta}a_k(t+\Delta t;\varphi+\Delta\varphi)-\partial_t^{|\alpha|}\partial_z^{\beta}a_k(t;\varphi)]\cdot$$
$$\cdot\prod_{j=1}^{|\beta|}\xi_j(x);L^2(\Omega)\|\leqslant\sup_{0\leqslant\sigma\leqslant 1}\left\{\left\|\sum_{j=1}^{n}\partial_t^{|\alpha|}\partial_z^{\beta+e_j}a_k(t+\Delta t;\varphi+\sigma\Delta\varphi)(\Delta\varphi)_j\prod_{h=1}^{|\beta|}\xi_h(x);L^2(\Omega)\right\|+\right.$$
$$\left.+\|\partial_t^{|\alpha|+1}\partial_z^{\beta}a_k(t+\sigma\Delta t;\varphi)\prod_{h=1}^{|\beta|}\xi_h(x);L^2(\Omega)\|\,|\Delta t|\right\}$$

because the r.h.s. of inequality (3.30) tends to 0 as $\Delta t\to 0$, $\Delta\varphi\to 0$ in X, by ii) of Def. 3.6.

Since condition b) of Def. 2.2 is obviously satisfied, what remains to be checked is condition c). It will be enough to show that, for $|\alpha|+|\beta|\leqslant r-1$, the quantities

$$\omega_{\alpha\beta}^{(k)}\equiv\Big[\partial_t^{|\alpha|}\partial_z^{\beta}a_k(t+\Delta t;\varphi+\Delta\varphi)-\partial_t^{|\alpha|}\partial_z^{\beta}a_k(t;\varphi)-$$
$$-\Delta t\,\partial_t^{|\alpha|+1}\partial_z^{\beta}a_k(t;\varphi)-\sum_{j=1}^{n}\partial_t^{|\alpha|}\partial_k^{\beta+e_j}a_k(t;\varphi)(\Delta\varphi)_j\Big]\prod_{h=1}^{|\beta|}\xi_h,\qquad k=1,\dots,n$$

satisfy the following condition: for any sphere Ω_1 there exists a sphere Ω_2 such that

$$\frac{\|\omega_{\alpha\beta}^{(k)};L^2(\Omega_1)\|}{|\Delta t|+\|\Delta\varphi;X_1(\Omega_2)\|}$$

tends to zero as $|\Delta t|+\|\Delta\varphi;X_1(\Omega_2)\|\to 0$. This is an easy consequence of the estimate

$$\|\omega_{\alpha\beta}^{(k)};L^2(\Omega_1)\|\leqslant$$
$$\leqslant\sup_{0\leqslant\sigma\leqslant 1}\|[\partial_t^{|\alpha|+1}\partial_z^{\beta}a_k(t+\sigma\Delta t;\varphi+\sigma\Delta\varphi)-\partial_t^{|\alpha|+1}\partial_z^{\beta}a_k(t;\varphi)]\prod_{j=1}^{|\beta|}\xi_j;L^2(\Omega_1)\|\,|\Delta t|+$$
$$+\sup_{0\leqslant\sigma,\eta\leqslant 1}\|\sum_{i,j=1}^{n}\partial_t^{|\alpha|}\partial_z^{\beta+e_j+e_i}a_k(t+\sigma\eta\Delta t;\varphi+\sigma\eta\Delta\varphi)(\Delta\varphi)_i(\Delta\varphi)_j\prod_{h=1}^{|\beta|}\xi_h;L^2(\Omega_1)\|\,.$$

LEMMA 3.5. Let $a\in Y'_{(r)}([0,T[;n)$ and let us define the vector valued map

$$f\colon[0,T[\times X\to X$$

$$f\left(t,\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)\equiv\begin{bmatrix}0\\ a_1(x,t;\varphi)\\ 0\\ \vdots\\ 0\\ a_n(x,t;\varphi)\end{bmatrix}.$$

Then, for any j, $0 \leqslant j \leqslant r-2$, and for all $t \in [0, T[$ the maps $S^{(j)}(t, \cdot)$ (see Def 2.4) are continuous from $\bigoplus_{j+1} D_{K^{r-j-1}}$ to $D_{K^{r-j-1}}$ in the graph topology (see Def. 2.5), with K given by eq. (3.13).

PROOF. As remarked at the end of subsect. 3.1 the graph topology of $D_{K^{r-j-1}}$ is equivalent to the topology of $X_{r-j} \oplus X_{r-j-1}$. Therefore, from the structure of the $S^{(j)}$ (see eq. (2.14)), it will be enough to estimate in $H_{\text{loc}}^{r-j-1}(\boldsymbol{R}^s)$ expressions of the type $\partial_t^{|\alpha|}\partial_z^\beta a_k(x, t; \varphi(x)) \prod_{h=1}^{|\beta|} \xi_h(x)$ with $|\alpha| + |\beta| \leqslant j$, $0 \leqslant j \leqslant r-2$, $\varphi \in X_{r-j}$, $\xi_1, \ldots, \xi_{|\beta|} \in H_{\text{loc}}^{r-j}(\boldsymbol{R}^s)$.

If γ is a multiindex with $|\gamma| \leqslant r-j-1$ we have

$$\partial_x^\gamma\Big[\partial_t^{|\alpha|}\partial_z^\beta a_k(x, t; \varphi(x)) \prod_{j=1}^{|\beta|} \xi_j(x)\Big] =$$

$$= \sum_{\gamma_0+\ldots+\gamma_{|\beta|}=\gamma} c_{\gamma_0,\ldots,\gamma_{|\beta|}} \partial_x^{\gamma_0}[\partial_t^{|\alpha|}\partial_z^\beta a_k(x, t; \varphi(x))] \prod_{j=1}^{|\beta|} \partial_x^{\gamma_j}\xi_j(x) =$$

$$= \sum_{\substack{\gamma_0+\ldots+\gamma_{|\beta|}=\gamma \\ \sigma,\mu \\ |\sigma|+|\mu|\leqslant|\gamma_0|}} d_{\gamma_0,\ldots,\gamma_{|\beta|},\sigma,\mu} \Big(\partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi(x))\Big) P_{\sigma\mu}(\varphi_1(x), \ldots, \varphi_n(x)) \prod_{j=1}^{|\beta|} \partial_x^{\gamma_j}\xi_j(x)$$

where the c's and the d's are constants, the $P_{\sigma\mu}$ are finite linear combinations of $|\mu|$-linear forms in the derivatives of the φ's up to order $|\gamma_0|$ and $|\alpha| + |\beta| + |\mu| \leqslant j + (r-j-1) = r-1$. Since by inequality (3.29)

$$\begin{aligned}(3.31)\qquad &\Big\| \partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi) P_{\sigma\mu}(\varphi_1, \ldots, \varphi_n) \prod_{j=1}^{|\beta|} \partial_x^{\gamma_j}\xi_j; L^2(\Omega)\Big\| \leqslant \\ &\qquad \leqslant C'_{\sigma,\alpha,\beta+\mu}(\Omega, \tau, \varrho) \|\varphi; X_{r-j}(\Omega)\|^{|\mu|} \prod_{h=1}^{|\beta|} \|\xi_h; H^{r-j}(\Omega)\|,\end{aligned}$$

for all $t \in [0, \tau] \subset [0, T[$ and $k = 1, \ldots, n$, if $\|\varphi; X_1(\Omega)\| \leqslant \varrho$, it follows that $S^{(j)}(t, \cdot)$ maps $\bigoplus_{j+1} D_{K^{r-j-1}}$ into $D_{K^{r-j-1}}$, for all j, $0 \leqslant j \leqslant r-2$. To establish the required continuity one has to evaluate in $L_{\text{loc}}^2(\boldsymbol{R}^s)$ the differences

$$\begin{aligned}(3.32)\qquad &\partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi + \Delta\varphi) P_{\sigma\mu}(\varphi_1 + (\Delta\varphi)_1, \ldots, \varphi_n + (\Delta\varphi)_n) \cdot \\ &\cdot \prod_{h=1}^{|\beta|} \partial_x^{\gamma_h}(\xi_h + (\Delta\xi)_h) - \partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi) P_{\sigma\mu}(\varphi_1, \ldots, \varphi_n) \prod_{h=1}^{|\beta|} \partial_x^{\gamma_h}\xi_h = \\ &= [\partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi + \Delta\varphi) - \partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi)] P_{\sigma\mu}(\varphi_1, \ldots, \varphi_n) \cdot \\ &\cdot \prod_{h=1}^{|\beta|} \partial_x^{\gamma_h}\xi_h + \partial_x^\sigma \partial_t^{|\alpha|}\partial_z^{\beta+\mu} a_k(x, t; \varphi + \Delta\varphi) \Big[P_{\sigma\mu}(\varphi_1 + (\Delta\varphi)_1, \ldots, \varphi_n + (\Delta\varphi)_n) \cdot \\ &\cdot \prod_{h=1}^{|\beta|} \partial_x^{\gamma_h}(\xi_h + (\Delta\xi)_h) - P_{\sigma\mu}(\varphi_1, \ldots, \varphi_n) \prod_{h=1}^{|\beta|} \partial_x^{\gamma_h}\xi_h\Big].\end{aligned}$$

By arguing as in the proof of Lemma 3.4 and by using the estimate (3.31), the $L^2(\Omega)$ norm of the r.h.s. of eq. (3.32) tends to 0 as $\Delta\varphi$ tends to 0 in X_{r-j} and the $(\Delta\xi_h)$'s tend to 0 in $H^{r-j}_{\text{loc}}(\boldsymbol{R}^s)$.

As a consequence of the two preceeding Lemmas one can apply Th. 2.2 and Cor. 2.3 to eq. (3.3). In particular we want to mention explicitely the following

THEOREM 3.3. *Let* $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in C^{(0)}([0, T[; X)$, $0 < T \leqslant \infty$, *be a solution of the integral eq.* (3.3). *If*

i) $\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} \in X_{r+1} \oplus X_r$;

ii) $$f\left(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) = \begin{bmatrix} 0 \\ a_1(x, t; \varphi) \\ 0 \\ \vdots \\ 0 \\ a_n(x, t; \varphi)\end{bmatrix} = a(x, t; \varphi) \qquad \text{with } a \in Y'_{(r)}([0, T[; n)\,,$$

then

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in \bigcap_{h=0}^{r} C^{(r-h)}([0, T[; X_{h+1} \oplus X_h)\,. \tag{3.33}$$

PROOF. The result follows immediately from Lemmas 3.4, 3.5 and Cor. 2.3.

REMARK 6. If the hypothesis of Th. 3.3 are verified for all r then, as a consequence of a well known embedding Sobolev theorem, $\varphi(x, t) \in C^\infty(\boldsymbol{R}^s \times [0, T[)$.

Finally we want to list here some interesting cases of vector valued functions for which it is easy to recognize that they belong to $\bigcap_r Y'_{(r)}([0, +\infty[; n)$.

A first example of this kind is given by any real vector valued function a whose components a_k do not depend on z and are C^∞ in $\boldsymbol{R}^s \times [0, +\infty[$.

Examples which are more significant from the point of view of the non-linearity are given by any real vector valued function a whose components a_k are of the form

$$a_k(x, t; z) = \sum_{|\alpha| \geqslant 1} a_{k\alpha}(x)\, z^\alpha\,, \qquad k = 1, \ldots, n\,,$$

and satisfy the following properties

i) $a_{k\alpha}$ are real C^∞ functions on $\boldsymbol{R}^s$, $\forall k, \alpha$;

ii) $\sum_{|\alpha|\geqslant 1} \sup_{x\in B} |\partial_x^\gamma a_{k\alpha}(x)|\sigma^{|\alpha|} < \infty$ if $s=1$, $r=1, \dots, n$;

$\sum_{|\alpha|\geqslant 1} \sup_{x\in B} |\partial_x^\gamma a_{k\alpha}(x)||\alpha|^{|\alpha|/2}\sigma^{|\alpha|} < \infty$ if $s=2$, $k=1, \dots, n$

for any multiindex γ, for any compact $B\subset \boldsymbol{R}^s$ and for any $\sigma > 0$, $a_{k\alpha}(x) = 0$, $\forall \alpha$, $|\alpha| > 3$ if $s=3$, $k=1, \dots, n$.

The proof goes along the same lines as in subsect. 3.3.

Appendix A.

It is convenient to put

$$G(t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \equiv$$

$$\equiv (2\pi)_{-s}\begin{pmatrix}\displaystyle\int \exp[i\langle x, \xi\rangle]\left[\hat{\varphi}(\xi)\cos|\xi|t + \hat{\psi}(\xi)\frac{\sin|\xi|t}{|\xi|}\right]d\xi \\ \displaystyle\int \exp[i\langle x, \xi\rangle]\left[-|\xi|\hat{\varphi}(\xi)\sin|\xi|t + \hat{\psi}(\xi)\cos|\xi|t\right]d\xi\end{pmatrix} \equiv \begin{pmatrix}\alpha(x,t)\\ \\ \beta(x,t)\end{pmatrix}$$

where $\varphi, \psi \in \mathcal{S}(\boldsymbol{R}^s)$ and $t\in\boldsymbol{R}$. It is clear that $\alpha, \beta \in C^\infty(\boldsymbol{R}; \mathcal{S}(\boldsymbol{R}^s))$ and that they satisfy the following identity

$$\frac{d}{dt}\left[\frac{\alpha^2 + \beta^2 + (\nabla\alpha)^2}{2}\right] - \nabla(\beta\nabla\alpha) = \alpha\beta\,. \tag{A.1}$$

Integration of eq. (A.1) on the volume $\bigcup_{0\leqslant\tau\leqslant t}\Omega(\tau)$, where Ω is a sphere and $0 < t <$ radius of Ω, application of Gauss theorem and of Gronwall's lemma yields eq. (3.8) for $\varphi, \psi\in\mathcal{S}(\boldsymbol{R}^s)$ and $t\geqslant 0$. Obviously the same argument can be used to show inequality (3.8) for $t<0$. Therefore $G(t)$ defines a group of linear continuous operators in $H^1_{\text{loc}}(\boldsymbol{R}^s)\oplus L^2_{\text{loc}}(\boldsymbol{R}^s)$. It is easy to recognize directly that

$$\left\|(G(t)-1)\begin{pmatrix}\varphi\\ \psi\end{pmatrix};\ H^1(\Omega)\oplus L^2(\Omega)\right\| \tag{A.2}$$

tends to 0 as $t\to 0$ when $\varphi, \psi\in\mathcal{S}(\boldsymbol{R}^s)$. This, coupled with a density argument and inequality (3.8), gives the strong continuity of the group $t\mapsto G(t)$.

Let B be the infinitesimal generator of $G(t)$. By definition $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in D_B$ iff the limit

$$\lim_{t\to 0+}\frac{G(t)-1}{t}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

exists in $H^1_{\text{loc}}(\boldsymbol{R}^s) \oplus L^2_{\text{loc}}(\boldsymbol{R}^s)$. Now $H^1_{\text{loc}}(\boldsymbol{R}^s) \oplus L^2_{\text{loc}}(\boldsymbol{R}^s)$ is continuously embedded in $\mathfrak{D}'(\boldsymbol{R}^s) \oplus \mathfrak{D}'(\boldsymbol{R}^s)$, so the above limit must exist also in $\mathfrak{D}' \oplus \mathfrak{D}'$. On the other hand

$$\left\langle \frac{G(t)-1}{t} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle_{\mathfrak{D}',\mathfrak{D}} = \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \frac{G(t)^*-1}{t} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle_{\mathfrak{D}',\mathfrak{D}}$$

for any $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathfrak{D}(\boldsymbol{R}^s) \oplus \mathfrak{D}(\boldsymbol{R}^s)$. It is easy to recognize by direct inspection that

$$\frac{G(t)^*-1}{t} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \Delta\eta \\ \xi \end{pmatrix}$$

in $\mathfrak{D} \oplus \mathfrak{D}$ as $t \to 0+$, and therefore that

$$\frac{G(t)-1}{t} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \to \begin{pmatrix} \psi \\ \Delta\varphi \end{pmatrix}$$

in $\mathfrak{D}' \oplus \mathfrak{D}'$, as $t \to 0+$. Consequently

$$B \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

in $\mathfrak{D}' \oplus \mathfrak{D}'$ and, by well known properties of elliptic operators ([13]), we have that $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in H^2_{\text{loc}}(\boldsymbol{R}^s) \oplus H^1_{\text{loc}}(\boldsymbol{R}^s)$. This proves that B is a restriction of the operator A of eq. (3.9). Since $\mathfrak{D} \oplus \mathfrak{D}$ is dense in $H^2_{\text{loc}}(\boldsymbol{R}^s) \oplus H^1_{\text{loc}}(\boldsymbol{R}^s)$ and it is contained in D_B, from the fact that B is closed, it follows that $A = B$.

Appendix B.

Inequalities (3.24) are well known if $\Omega_R = \boldsymbol{R}^s$ (they hold with the constants c_j independent of R) ([12]). To obtain them in the local case we proceed as follows. By translation invariance we can suppose that Ω_R is a sphere of radius R centered in the origin. Suppose now π is a continuous linear operator

$$\pi\colon H^1(\Omega_1) \to H^1(\boldsymbol{R}^s)$$

such that $\pi(\varphi)|_{\Omega_1} = \varphi$, $\varphi \in H^1(\Omega_1)$. Let us define

$$T: H^1(\Omega_R) \to H^1(\Omega_1)$$

$$(T\varphi)(x) \equiv \varphi(Rx)$$

$$S: H^1(\boldsymbol{R}^s) \to H^1(\boldsymbol{R}^s)$$

$$(S\psi)(x) \equiv \psi\left(\frac{x}{R}\right).$$

One immediately verifies that $\pi_R \equiv S\pi T$ is a linear continuous extension operator from $H^1(\Omega_R)$ into $H^1(\boldsymbol{R}^s)$ and that the following inequality holds

$$\|\pi_R(\varphi); H^1(\boldsymbol{R}^s)\| \leqslant L(R)\|\varphi; H^1(\Omega_R)\|$$

with

$$L(R) = \begin{cases} CR, & R \geqslant 1 \\ C/R, & 0 < R < 1. \end{cases}$$

The constant C is the norm of the map π.

REFERENCES

[1] N. Bourbaki, *Eléments de Mathématique. Espaces vectoriels topologiques*, Hermann, Paris (1955).

[2] F. Browder, *On non-linear wave equations*, Math. Zeit., **80** (1962), pp. 249-264.

[3] J. M. Chadam, *The classical equations in quantum field theory*, Marseille preprint 75/P. 691.

[4] S. Coleman, *Quantum Sine-Gordon equation as the massive Thirring model*, Phys. Rev., D **11** (1975), p. 2088.

[5] L. D. Faddeev, *Quantization of Solitons*, Lectures at the Institute for Advanced Study, Princeton, N. J., April 1975, and references therein.

[6] J. Frölich, *Quantized « Sine-Gordon » equation with a non-vanishing mass term in two space-time dimensions*, Phys. Rev. Letters, **34** (1975), p. 833.

[7] K. Jörgens, *Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen*, Math. Zeit., **77** (1961), pp. 295-308.

[8] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris (1969).

[9] M. Reed, *Higher order estimates and smoothness of solutions of non-linear wave equations*, to appear in Proc. A.M.S.

[10] I. Segal, *Non-linear semi-groups*, Ann. Math., **78** (1963), pp. 339-364.

[11] W. A. STRAUSS, *Non-linear scattering theory*, in *Scattering Theory in Mathematical Physics*, Proceedings of the NATO Advanced Study Institute, Denver, 1973, J. A. LAVITA and J.-P. MARCHAND (Editors), D. Reidel, Dordrecht-Holland (1974), pp. 53-78.

[12] L. R. VOLEVIC - B. P. PANEYAKH, *Certain spaces of generalized functions and embedding theorems*, Russian Math. Surveys, **20** (1965), pp. 1-73.

[13] K. YOSIDA, *Functional Analysis*, Springer, Berlin (1966).

[14] G. B. WHITHAM, *Linear and Non-Linear Waves*, J. Wiley, New York (1974).

[15] A. S. WIGHTMAN, *Partial differential equations and relativistic quantum field theory*, Lectures in Differential Equations, vol. II, A. K. AZIZ (Editor), Van Nostrand, Princeton, N. J. (1969), pp. 1-52.