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Dirichlet Problem for Parabolic Equations with Continuous Coefficients. (*)

JULIO E. BOUILLET (**)

Summary. – *L^p-boundary value problems for strongly parabolic operators such that the coefficients of the highest order derivatives are only uniformly continuous have been studied by V. A. Solonnikov* [10]. *In the case of a cylindre $\Omega\times(0,T)$, Ω a smooth spatial domain, and initial data zero, Solonnikov assumes the Dirichlet data to belong to a trace space. More precisely, if $L\equiv\sum_{|\alpha|\leqslant 2b} a_\alpha(P,t)D_p^\alpha - D_t$ is the strongly parabolic operator, w^k, $k=0,1,\dots,b-1$, the Dirichlet data, then the problem*

$$Lu=0 \quad in\ \Omega\times(0,T)\,,\qquad D_N^k u = w^k \quad at\ \partial\Omega\times(0,T)\,,\qquad u=0 \quad on\ \Omega\ for\ t=0\,,$$

D_N indicating normal derivative to $\partial\Omega$, admits a unique solution in the space of functions whose spatial derivatives up to order $2b$, and the time derivative, belong in $L^p(\Omega\times(0,T))$, $p>1$. Solonnikov observed that this implies that w^k must have spatial derivatives up to order $2b-1-k$, and a «fractional» time derivative of order $(2b-1-k)/2k$ in $L^p(\partial\Omega\times(0,T))$. Moreover, a spatial derivative of order $2b-1-k$ of w^k will have a «fractional» derivative in the spatial direction of order $1/p'$ and in the time direction of order $1/2bp'$. Wıth this information let us denote, for right now, the space of w^k by $\mathring{L}^p_{(2b-1-k+1/p'),(2b-1-k+1/p')/2b}(\partial\Omega\times(0,T))$; in the present work we find a class of existence and uniqueness to the problem above with the assumption that $w^k\in\mathring{L}^p_{(b-1-k+\varepsilon),(b-1-k+\varepsilon)/2b}(\partial\Omega\times(0,T))$. Here $\in<1$ is an arbitrary but fixed positive number. This means that we have reduced the smoothness requirements on the data w^k by at least b derivatives in the space airection, and $b/2b=\frac{1}{2}$ in the time direction. In a subsequent paper we shall discuss the non-zero initial value case. Outline: the definitions and notation appear in I, §§ 1-5, *for the case of a half-space, and in* VIII, § 24 *for the bounded domain Ω. The problem in a half-space is treated in* I-VI, *using certain surface and volume potentials* (III-IV). *Using the half-space results, we obtain an elliptic a priori estimate* (VII) *in the half-space. The problem in a general domain is studied in* VIII *for the parabolic case (a priori parabolic estimate: Theorem,* § 25; *existence and uniqueness theorems: Theorem* 1 *and Theorem* 2, § 27). *In* IX *an a priori estimate for the strongly elliptic case is derived. This work is part of a modified version of our Dissertation, under the direction of Professor Eugene B. Fabes. We wish to thank Professor Fabes for many invaluable talks and advise.*

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I. – Definitions and notations for the problem in the half-space.

§ 1. – Points in R^n will be denoted by the letters x, z, w, while y, v, η, t, r, s will denote real numbers, the last three referring to time. $R_+^{n+1} = R^n x(0, \infty)$ will be the spatial domain (half-space), with points denoted by (x, y), (z, v). The differential operators will be defined for functions in the « cylinder » $R_+^{n+1} x(0, T)$, whose points are (x, y, t), t denoting boundary of this « cylinder » is $S_T = R^n x(0, T)$.

The following notations are standard: $f{*}g$ for the convolution of the functions f and g, $\mathcal{F}(f)(\cdot)$, occasionally also $\hat{f}(\cdot)$, the Fourier transform $\int f(\xi) \exp [i\langle\xi, \cdot\rangle]\, d\xi$, where $\langle\cdot,\cdot\rangle$ denotes scalar product of vectors. We will write, e.g. $\mathcal{F}_z(f)$ to specify the transformation in the variable z.

With $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i nonnegative integers, we set

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

$$D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}, \qquad \Delta = \Delta_x = \sum_{i=1}^n D_{x_i}^2 .$$

when there is no confusion we will use this notation to include $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, and write $D_{x,y}^\alpha$ for $D_x^{(\alpha_1,\ldots,\alpha_n)} D_y^{\alpha_{n+1}}$. This will only apply to space variables.

We will denote by $X_D(\cdot)$ the characteristic function of the set D.

§ 2. – DEFINITION. A parabolic singular integral operator is an operator of the form (cf. [3], [4])

$$Kf(x, t = \alpha(x, t) f(x, t) + \lim_{\varepsilon\to 0^+} \int_0^{t-\varepsilon} \int_{R^n} k(x, t; x - z, t - s) f(z, s)\, dz\, ds ,$$

the limit in $L^p(R^n x(0, T))$, where $a(x, t)$ is a bounded measurable function on $R^n x(0, T)$, and the variable kernel $k(x, t; z, s)$ is defined for $t \in (0, T)$, $s > 0$ as

$$k(x, t; z, s) = \frac{\Omega(x, t; z/s^{1/2b})}{s^{1+n/2b}},$$

where

(1) $\Omega(x, t; \cdot) \in \mathcal{S}(R^n)$ = space of rapidly decreasing functions.

(2) $\int_{R_n} \Omega(x, t; w)\, dw = 0$,

(3) $\sup_{(y,s)\in S_T} \left(\int_{R^n} |w^\alpha D_w^\beta \Omega(y, s; w)|^2\, dw \right)^{\frac{1}{2}} < \infty$ (see, e.g. [6]).

If in the above definition $a(x, t)$ is a constant and $k(x, t; z, s) = k(z, s)$ is independent of (x, t), the operator Kf will be said to be of convolution type.

We define the symbol of K to be the function

$$\sigma(K)(x, t; z, s) = a(x, t) + \int_0^\infty \frac{\mathcal{F}_w(\Omega)(x, t; zr^{1/2b})}{r} \exp[itr]\, dr .$$

K is known to be a continuous mapping of $L^p(R^n x(0, T))$ for $1 < p < \infty$. Its properties (cf. [2], [3], [4]) will be assumed here.

Following [3], [5] we will denote by $\mathfrak{J}_p(S_T)$, $1 < p < \infty$, the class of operators $J: L^p(S_T) \to L^p(S_T)$ satisfying for any $a \geqslant 0$

(i) $JX_{(a,\infty)} = X_{(a,\infty)} J X_{(a,\infty)}$,

(ii) $\|X_{(a,a+\varepsilon)} J(X_{(a,a+\varepsilon)} f)\|_{L^p(S_T)} \leqslant \omega(\varepsilon) \cdot \|X_{(a,a+\varepsilon)} f\|_{L^p(S_T)}$,

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in $a \geqslant 0$. We set $\mathfrak{J}(S_T) = \bigcap_p \mathfrak{J}_p(S_T)$. $\mathfrak{J}_p(R^{n+1}_+ x(0, T))$, $\mathfrak{J}(R^{n+1}_+ x(0, T))$ are defined the same way.

§ 3. – We introduce the fundamental solution of the operator $(-1)^b \Delta_x^b + D_t$ on S_T, defined by

$$F(x, t) = \mathcal{F}_z(\exp[-|z|^{2b} t])(x) \quad \text{for } t > 0,$$
$$= 0 \quad \text{for } t \leqslant 0 .$$

Definition. For β real > 0,

$$(\Lambda^{-\beta} f)(x, t) = (\Lambda^{-\beta} * f)(x, t) = \frac{1}{\Gamma(\beta/2b)} t^{\beta/2b-1} F(x, t) * f(x, t) =$$
$$= \frac{1}{t^{1+(n-\beta)/2b}} \Omega_\beta(x/t^{1/2b}) * f(x, t), \qquad \Omega_\beta(\cdot) \in \mathcal{S} .$$

For $0 < \beta \leqslant 2b$, $\Lambda^{-\beta}(x, t)$ is a tempered distribution on R^{n+1}, whose Fourier transform is $(|x|^{2b} - it)^{-\beta/2b}$. Also if $f \in L^p(S_T)$, $D_x^\alpha \Lambda^{-\beta} f \in L^p(S_T)$ for $|\alpha| \leqslant [\beta] =$ largest integer $\leqslant \beta$, and $D_t \Lambda^{-2b+\beta} f \in L^p(S_T)$. We set $\Lambda^0 f = f$ and proceed to define Λ^β.

Definition. For $0 < \beta \leqslant 2b$,

$$\Lambda^\beta f = ((-1)^b \Delta_x^b + D_t) \Lambda^{-2b+\beta} f$$

Λ^β is well defined on rapidly decreasing functions f, which vanish for $t\leqslant 0$. As tempered distributions, $\mathcal{F}(\Lambda^\beta f) = (|x|^{2b} - it)^{\beta/2b}\cdot\hat{f}$. For β integral, $(-1)^b\Delta^b\Lambda^{-2b+\beta}f$ can also be written [3] as $\sum_{|\alpha|=\beta} K^\alpha D_x^\alpha f$, with K^α a parabolic singular integral operator with symbol

$$\sigma(K_\alpha) = i^{2b-\beta}\frac{P^\alpha(x)}{(|x|^{2b}-it)^{1-\beta/2b}},$$

$P_\alpha(x)$ defined by $|x|^{2b} = \sum_{|\alpha|=\beta} P_\alpha(x)x^\alpha$: considering $(-1)^b\Delta^b\Lambda^{-2b+\beta}$ as a tempered distribution on R^{n+1} this decomposition is clear if we recall that

$$\mathcal{F}(\Lambda^{-2b+\beta}) = \frac{1}{(|x|^{2b}-it)^{1-\beta/2b}}.$$

If $f = f(z, \eta, s; x, y, v, t)$, z, η, s, y, and v being parameters, we introduce the notation $\Lambda^\beta f(z, \eta, s; x, y, v, t)$ to mean $[\Lambda^\beta f(z, \eta, s; \cdot, y, v, \cdot)](x, t)$.

§ 4. – For $\delta\geqslant 0$, $L^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))$ is the space of functions whose derivatives $D^\alpha_{x,y}u$, for $|\alpha|\leqslant 2b$, and $D_t u$ in the sense of distributions are given by functions belonging to $L^p(R^n\times(\delta,\infty)\times(0,T))$. $L^p_{2b,1}$ is a Banach space with the norm

$$\|u\|_{L^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))} = \sum_{|\alpha|\leqslant 2b}\|D^\alpha_{x,y}u\|_{L^p(R^n\times(\delta,\infty)\times(0,T))} + \|D_t u\|_{L^p(R^n\times(\delta,\infty)\times(0,T))}.$$

For $\delta = 0$ we write $L^p_{2b,1}(R^{n+1}_+\times(0,T))$.

$\mathring{L}^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))$ denotes the space of functions $u\in L^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))$ which are limits in $L^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))$ of functions $\in C_0^\infty(R^{n+1}\times(0,\infty))$.

§ 5. – A linear differential operator

$$L\equiv\sum_{|\alpha|\leqslant 2b} a_\alpha(x,y,t)D^\alpha_{x,y} - D_t$$

is said to be parabolic in the sense of Petrovski if

$$\operatorname{Re}\Big(\sum_{|\alpha|=2b} a_\alpha(x,y,t)(i\xi)^\alpha\Big) < -\pi|\xi|^{2b}, \qquad \text{for } 0\neq\xi\in R^{n+1},$$

$\pi>0$ independent of (x, y, t) in $R^{n+1}_+\times(0,T)$. Each $a_\alpha(x,y,t)$ is assumed to be measurable, bounded, and for $|\alpha| = 2b$, uniformly continuous in $R^{n+1}_+\times(0,T)$.

We will introduce a distance function: for given $\gamma, 0<\gamma<1$, and $1<p<\infty$, let γ_p be a number such that

$$1-1/p\leqslant\gamma_p<\frac{2b}{2b-\gamma}(1-1/p)\,.$$

We define $d_p(y, t) = \min(y, t^{\gamma_p/2b})$.

Throughout this work C will denote a constant, not necessarily the same at each occurrence. The connection between C and other parameters (eg. parameter of parabolicity, dimension, etc.) will be made explicit when relevant.

We will also let $\psi(r)$ denote any function of the form: constant$\cdot$ $\cdot\exp[-\text{constant}\cdot r]$, the constants and r being real and positive. When related to a solution of the operator L, these constants will depend only on the parameter of parabolicity π and on the $\max\limits_{|\alpha|\leqslant 2b}\sup|a_\alpha(x, y, t)|$.

II. – The parametrix.

§ 6. – We will construct a kernel for a generalized volume potential following [5]. We consider first a differential operator with constant coefficients

$$L_0\equiv\sum_{|\alpha|=2b}a_\alpha D^\alpha_{xy}-D_t\,.$$

Let $F(x, y, t)=\mathcal{F}_\xi\Big(\exp\sum\limits_{|\alpha|=2b}a_\alpha(i\xi)^\alpha t\Big)(x, y)$, $\xi\in R^{n+1}$, be the fundamental solution of L_0. We construct a function $G_0(x, y, v, t)$, $y, v>0$, satisfying as a function of (x, y, t),

(i) $G_0(\cdot,\cdot, v,\cdot)\in\mathring{L}^p_{2b,1}\big(R^n\times(\delta,\infty)\times(0, T)\big)$ for every $\delta>0$, $1<p<\infty$,

(ii) $L_0G_0(x, y, v, t)=0$ for $y>0$,

(iii) for $k=0, \dots, b-1$,

$$\lim_{y\to 0^+}\Lambda^{-k}(D^k_yG_0)(x, y, v, t)=[\Lambda^{-k}D^k_yF(\cdot, -v,\cdot)](x, t)\,,$$

the limit taken in $L^p(S_T)$.

It is known [3], that G_0 can be written

$$G_0(x, y, v, t)=\sum_{k,j=0}^{b-1}\int\limits_0^t\int\limits_{R^n}\Lambda^{2b-1-k}D^k_yF(x-z, y, t-s)\cdot\\ \cdot T_{k,j}[\Lambda^{-1}(D^j_yF)(\cdot, -v,\cdot)(x, t)](z, s)\,dz\,ds\,,$$

where $(T_{k,j})$ is a $b\times b$ matrix of parabolic singular integral operators (of convolution type).

Considered as a function of x, v, t for $y>0$, $G_0(x, y, v, t)$ is also a solution of the boundary value problem

(i′) $G_0(\cdot\,, y\cdot,\cdot)\in \mathring{L}^p_{2b,1}\big(R^n\times(\delta, \infty)\times(0, T)\big)$ for every $\delta>0$, $1<p<\infty$, $y>0$,

(ii′) $\Big(\sum_{|\alpha|\leqslant 2b} a_\alpha D_x^{\alpha_1\ldots\alpha_n}(-D_v)^{\alpha_{n+1}}-D_t\Big)\,G_0(x, y, v, t)=0$ for $v>0$,

(iii′) for $l=0, \ldots, b-1$,

$$\lim_{v\to 0^+} \text{ in } L^p(S_T)\Lambda^{-l}D_v^l G_0(x, y, v, t)=(-1)^l\Lambda^{-l}D_y^l F(x, y, t)\,.$$

Since for $y>0 \lim_{v\to 0^+} D_v^l G_0$ exists in $L^p(S_T)$, (iii′) implies that for $l=0, \ldots, b-1$,

$$D_v^l G_0(x, y, 0, t)=(-1)^l D_y^l F(x, y, t) \qquad \text{for every } y>0\,.$$

We now introduce the function

$$K(x, y, v, t)=F(x, y-v, t)-G_0(x, y, v, t)\,.$$

For $l=0, \ldots, b-1$,

$$D_v^l K(x, 0, v, t)=0, \quad v>0, \qquad \text{and} \qquad D_y^l K(x,y,0,t)=0, \quad y>0$$

§ 7. – The proof of the following theorem is long and technical in nature, and will not be included here.

THEOREM. For $y>0$, $v>0$, $y\neq v$, and $B>0$

(i) $$|D_x^\alpha D_y^j D_v^h \Lambda^B F(x, y-v, t)|\leqq \frac{\psi(|x|/t^{1/2b})\,\psi(|y-v|/t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}}\,,$$

(ii) $$|D_x^\alpha D_y^j D_v^h \Lambda^B G_0(x, y, v, t)\leqq \frac{\psi(|x|/t^{1/2b})\,\psi(y/t^{1/2b})\,\psi(v/t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}}\,,$$

(iii) $$|D_x^\alpha D_y^j D_v^h \Lambda^B K(x, y, v, t)|\leqq \frac{\psi(|x|/t^{1/2b})\,\psi(|y-v|t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}}\,.$$

Clearly, (iii) follows from (i) and (ii).

We shall consider the operator with constant coefficients L_{0zrs},

$$(L_{0zrs}u)(x, y, t)\equiv \sum_{|\alpha|=2b} a_\alpha(z, r, s)\,D^\alpha_{xy}u(x, y, t)-D_t u(x, y, t)\,.$$

Let $F(z, r, s; x, y, t)$, $G_0(z, r, s; x, y, v, t)$, and $K(z, r, s; x, y, v, t)$ denote the fundamental solution F, and the functions G_0 and K (introduced in § 6) which are associated with the operator L_{0zrs}, $z \in R_n$, r, s, and v being real parameters. Clearly, these functions are solutions of the equation $L_{0zrs}u = 0$.

III. – Estimates for some surface potentials.

§ 8. – For $\Phi(x, t) \in L^p(S_T)$, $1 < p < \infty$, $j = 0, 1, \ldots, b-1$, $0 < \mu < 1$, we introduce the following potentials

$$T_{\mu j}\Phi)(x, y, t) = \int_0^t \iint_{R^n} \Lambda^{b-j-\mu} D_y^j F(z, 0, s; x-z, y, t-s)\,\Phi(z, s)\,dz\,ds\,.$$

LEMMA 1. For $\Phi \in L^p(S_T)$, $1 < p < \infty$,

(i) $\|D_{x,y}^\alpha \Lambda^\mu T_{\mu j}\Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant C \cdot T^{(b-1-|\alpha|)/2b}\|\Phi\|^{L^p(S_T)}$ for $|\alpha| < b-1$,

(ii) $\|\Lambda^{b-1+\mu-k} D_y^k T_{\mu j}\Phi(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha|=b-1} \|D_{x,y}^\alpha \Lambda^\mu T_{\mu j}\Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant C \cdot$ $\cdot \|\Phi\|_{L^p(S_T)}$,

(iii) $y^{|\alpha|-(b-1)}\|D_{x,y}^\alpha \Lambda^\mu T_{\mu j}\Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant C \cdot \|\Phi\|_{L^p(S_T)}$ for $|\alpha| > b-1$.

(iv) For $|\alpha| \leqslant b-1$,

$$\|D_{x,y}^\alpha \Lambda^\mu T_{\mu j}\Phi\|_{L^p(R_+^{n+1}\times(0,T))} \leqslant C \cdot T^{(b-1-|\alpha|+1/p)/2b} \cdot \|\Phi\|_{L^p(S_T)}.$$

PROOF. By the estimates on F, § 7, and Young's inequality in $dx\,dt$, we have

$$\|D_{xy}^\alpha \Lambda^\mu T_{\mu j}\Phi(\cdot, y, \cdot)\|_{(L^p S_T)} \leqslant C \cdot \left\{ y^{b-1-|\alpha|} \int_0^{T/y^{2b}} \frac{\Psi(1/s^{1/2b})}{s^{(b+1+|\alpha|)/2b}}\,ds \right\} \|\Phi\|_{L^p S_T)}\,.$$

When $|\alpha| < b-1$ the expression in brackets is bounded by $C \cdot T^{(b-1-|\alpha|)/2b}$. When $|\alpha| > b-1$, the integral in ds is finite and bounded independently of y. In both case (i) and case (iii), the constants C depend on $|\alpha|$, the parameter of parabolicity π, and the $\max_{|\alpha|=2b} \sup |a_\alpha(x, y, t)|$.

The discussion above hints that in the case (ii), for $|\alpha| = b-1$ we will find the singularity of a parabolic singular operator. In fact, (ii) is a consequence of the theory of parabolic singular interals with variable kernel (see [3]). (iv) is obtained by taking L^p-norms in dy on both sides of the estimate above.

The proof of the following Lemma is straightforward

LEMMA 2. If Φ e $L^p(S_T)$, $1<p<\infty$, and $\gamma-1/p<\mu$, then

$$\|y^{b+1-\gamma}L(T_{\mu j}\Phi)(x,y,t)\|_{L^p(R_+^{n+1}\times(0,T))}\leqslant\omega(T)\cdot\|\Phi\|_{L^p(S_T)},$$

where $\omega(T)\to 0$ as $T\to 0^+$.

IV. – Estimates for the volume potential.

§ 9. – We shall study a volume potential whose kernel is the function $K(z,\eta,s;x,y,v,t)$ (cf. § 6).

DEFINITION. For $f\in L^p(R_+^{n+1}\times(0,T))$, $1<p<\infty$,

$$V_f(x,y,t)=\int_0^t\iint_{R_+^{n+1}}K(z,v,s;x-z,y,v,t-s)f(z,y,s)\,dz\,dv\,ds.$$

THEOREM. If $f\in L^p(R_+^{n+1}\times(0,T))$, $1<p<\infty$, $0\leqslant\beta<\gamma-1/p$, $\gamma<1$, then

(i) for $k=0,\dots,b-1$, $\|\Lambda^{b-1+\beta-k}D_y^kV_f(\cdot,y,\cdot)\|_{L^p(S_T)}\to 0$ as $y\to 0^+$;

(ii)
$$\sum_{k=0}^{b-1}\|\Lambda^{b-1-k+\beta}D_y^kV_f(\cdot,y,\cdot)\|_{L^p(S_T)}+\sum_{|\alpha|\leqslant b-1}\|D_{x,v}^\alpha\Lambda^\beta V_f(\cdot,y,\cdot)\|_{L^p(S_T)}+$$
$$+\sum_{b\leqslant|\alpha|\leqslant 2b-1}\|d_p(y,\cdot)^{|\alpha|-(b-1)}D_{x,v}^\alpha\Lambda^\beta V_f(\cdot,y,\cdot)\|_{L^p(S_T)}\leqslant$$
$$\leqslant C\cdot T^{\gamma''/2b}\cdot\|d_p(\cdot,\cdot)^{b+1-\gamma}f(\cdot,\cdot,\cdot)\|_{L^p(R_+^{n+1}\times(0,T))},$$

the constant depending on π, $\max\limits_{|\alpha|=2b}\sup|a_\alpha|$, and p, and $\gamma''>0$ depending on γ,γ_p, b and a number $\gamma'<\gamma-\beta-1/p$.

$d_p(y,t)=\min(y,t^{\gamma_p/2b})$ is the distance function introduced in § 5.

PROOF. We recall that $D_v^lK(z,\eta,s;x,0,v,t)=0$ for $l\leqslant b-1$ (§ 6). We may therefore write, for $0\leqslant l\leqslant b-1$

$$K(z,\eta,s;x,y,v,t)=\frac{y^{l+1}}{l!}\int_0^1(1-\lambda)^lD_y^{l+1}K(z,\eta,s;x,\lambda y,v,t)\,d\lambda.$$

A similar remark applies to D_v^lK.

Part (i) is a direct consequence of the techniques used below, applied to

$$\Lambda^{b-1+\beta-k}D_y^kV_f=\frac{y^{b-k}}{(b-k-1)!}\int_0^t\iint_{R_+^{n+1}}\int_0^1\Lambda^{b-1+\beta-k}D_y^b\cdot$$
$$\cdot K(z,v,s;x-z,\lambda y,v,t-s)(1-\lambda)^{b-k-1}\,d\lambda f(z,v,s)\,dz\,dv\,ds.$$

PROOF OF (ii). We prove this in three Lemmas. In Lemmas A and B (§ 10) we prove the estimate for $d_p = y$. In Lemma C (§ 11) we show the estimate for $d_p = t^{\gamma_p/2b}$. For the general case we set

$$f(x, y, t) = X_{(y>t^{\gamma_p/2b})}\cdot f(x, y, t) + X_{(y\geq t^{\gamma_p/2b})}\cdot f(x, y, t) = f_1 + f_2$$

and apply to each potential V_{f_1}, V_{f_2} the corresponding estimate.

§ 10. - LEMMA A, (i). For $k = 0, \ldots, b-1$, $\beta + \gamma' < \gamma - 1/p$, $\gamma' > 0$,

$$\|\Lambda^{b-1+\beta-k} D_y^k V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C\cdot T^{\gamma'/2b}\cdot \min(1, y^{\gamma-\beta-\gamma'-1/p})\cdot$$
$$\cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1}\times(0,T))}.$$

(ii) For $b-1 < |\alpha| \leq 2b-1$,

$$y^{|\alpha|-(b-1)}\|D_{x,y}^\alpha \Lambda^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C\cdot T^{\gamma'/2b}\cdot \min(1, y^{\gamma-\beta-\gamma'-1/p})\cdot$$
$$\cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1}\times(0,T))}.$$

PROOF. Clearly,

$$\Lambda^{b-1+\beta-k} D_y^k V_f(x, y, t)| \leq \Big(\int_0^{y/2} + \int_{y/2}^{\infty}\Big)\int_0^t\int_{R^n} |\Lambda^{b-1+\beta-k} D_y^k\cdot$$
$$\cdot K(z, v, s; x-z, y, v, t-s)\cdot f(z, v, s)|\, dz\, dv\, ds.$$

We estimate first the term $\int_0^{y/2}$. It can be written

$$\int_0^{y/2}\int_0^t\int_{R^n}\left|\Lambda^{b-1+\beta-k} D_y^k \int_0^1 \frac{(1-\lambda)^b}{(b-1)!} D_v^b K(z, v, s; x-z, y, \lambda v, t-s)\, d\lambda\cdot v^b\cdot f(z, v, s)\right| dz\, dv\, ds.$$

Applying the estimates for K (§ 7) and Young's inequality, and observing that $|y - \lambda v| \geq y/2$, we get

$$\left\|\int_0^{y/2}\right\|_{L^p(S_T)} \leq C\int_0^{y/2}\int_0^T \frac{\psi(y/s^{1/2b})}{s^{1+\beta/2b}}\left(\frac{T}{s}\right)^{\gamma'/2b} ds\cdot v^{b+1-\gamma}\|f(\cdot, v, \cdot)\|_{L^p(S_T)}\frac{dv}{v^{1-\gamma}},$$

where we have introduced the factor $(T/s)^{\gamma'/2b} > 1$, $\beta + \gamma' \leq \gamma - 1/p$. Applying now Hölder inequality in dv we obtain the desired estimate for $\int_0^{y/2}$, with right-hand side

$$C\cdot T^{\gamma'/2b}\cdot y^{\gamma-\beta-\gamma'-1/p}\cdot \int_0^\infty \frac{\psi(1/s^{1/2b})}{s^{1+(\beta+\gamma')/2b}}\, ds\cdot \|v^{b+1-\gamma} f(z, v, s)\|_{(L^p(R_+^{n+1}\times(0,T))}.$$

A similar argument gives the estimate (ii) for the corresponding $\int_0^{y/2}$.

For the term $\int_{y/2}^{\infty}$ in (i), we observe that

$$D_y^k K = \int_0^1 (1-\mu)^{k-1} v^k D_v^k \int_0^1 (1-\lambda)^{b-k-1} D_y^b K(\ldots; x-z, \lambda y, \mu v, t-s)\, d\lambda\, d\mu .$$

Therefore,

$$\Big\| \int_{y/2}^{\infty} \Big\|_{L^p(S_T)} \leqslant C \cdot y^{b-k} \int_{y/2}^{\infty} dv \int_0^1 d\mu \int_0^1 d\lambda \int_0^T \frac{\psi(|\lambda y - \mu v|/s^{1/2b})}{s^{1+\beta/2b}} \left(\frac{T}{s}\right)^{\gamma'/2b} ds \cdot$$

$$\cdot \frac{v^k v^{b-k+1-\gamma}}{v^{b-k+1-\gamma}} \cdot \|f(\cdot, v, \cdot)\|_{(L^p S_T)} .$$

Applying Hölder inequality in $dv\, d\mu\, d\lambda$ we get

$$\Big\| \int_{v/2}^{\infty} \Big\|_{L^p(S_T)} \leqslant C \cdot T^{\gamma'/2b} \cdot y^{b-k} \cdot \left(\int_{\gamma/2}^{\infty} \int_0^1 \int_0^1 \frac{dv\, d\lambda\, d\mu}{\{|\lambda y - \mu v|^{\beta+\gamma'} v^{b-k+1-\gamma}\}^{p/(p-1)}} \right)^{1-1/p} \cdot$$

$$\cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0,T))}$$

$$\leqslant C \cdot T^{\gamma'/2b} \frac{y^{b-k} y^{1-1/p}}{y^{\beta+\gamma'+b-k+1-\gamma}} \cdot \left\{ \int_{\frac{1}{2}}^{\infty} \int_0^1 \int_0^1 \frac{dv\, d\lambda\, d\mu}{[|\lambda - \mu v|^{\beta+\gamma'} v^{b-k+1-\gamma}]^{p/(p-1)}} \right\}^{1-1/p} \cdot$$

$$\cdot \|v^{b+1-\gamma} f(z, v, s)\| .$$

For the term $\int_{v/2}^{\infty}$ in (ii) we have

$$\Big\| \int_{v/2}^{\infty} \Big\|_{L^p(S_T)} \leqslant C \int_{v/2}^{\infty} \int_0^1 v^{2b-1-|\alpha|} \int_0^T \frac{\psi(|y - \lambda v|/s^{1/2b})}{s^{1+\beta/2b}} \left(\frac{T}{s}\right)^{\gamma'/2b} ds \cdot \frac{v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|^{L^p(S_T)} dv\, d\lambda}{v^{b+1-\gamma}}$$

$$\leqslant \frac{C \cdot T^{\gamma'/2b} y^{1-1/p}}{y^{|\alpha|-b} y^{2-\gamma+\beta+\gamma'}} \left\{ \int_{\frac{1}{2}}^{\infty} \int_1^0 \frac{dv\, d\lambda}{[|1 - \lambda v|^{\beta+\gamma'} v^{2-\gamma}]^{p/p-1}} \right\}^{1-1/p} \cdot$$

$$\cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0,T))} .$$

For $|a| = 2b - 1$, the integral $\int_0^1 d\lambda$ will not be present.

Observe that setting $\gamma' = \gamma - \beta - 1/p$ we have an estimate independent of $y^{\gamma-\beta-\gamma'-1/p}$. The proof is complete.

LEMMA B. For $|\alpha|\leqslant b-1$,

$$\|D^{\alpha}_{x,y}\Lambda^{\beta}V_f(\cdot,y,\cdot)\|_{L^p(S_T)}\leqslant C\cdot T^{\gamma'2/b}\cdot\min(1,y^{\gamma-\beta-\gamma'-1/p})\cdot \cdot\|v^{b+1-\gamma}f(z,v,s)\|_{L^p(R^{n+1}_+\times(0,T))}\cdot$$

PROOF. Set $D^{\alpha}_{x,y}\Lambda^{\beta}=D^k_yD^{\delta}_x\Lambda^{\beta}=(D^{\delta}_x\Lambda^{-(b-1-k)}D^k_y\Lambda^{b-1+\beta-k}$, $|\delta|+k=|\alpha|\leqslant\leqslant b-1$. The result follows from Lemma A and the fact that $D^{\delta}_x\Lambda^{-(b-1-k)}$ is an $L^p(S_T)$ operator.

§ 11. – LEMMA C. For $|\alpha|\leqslant b-1$, γ_p as in § 5, and

$$\gamma_p<(2b-\beta-1/p)/(2b-\gamma),$$
$$\|D^{\alpha}_{x,y}\Lambda^{\beta}V_f(\cdot,y,\cdot)\|_{L^p(S_T)}\leqslant C_1(T)\cdot\|s^{(\gamma_p/2b)(b+1-\gamma)}f(z,v,s)\|_{L(^pR^{n+1}_+\times(0,T))},$$

and for $b\leqslant|\alpha|\leqslant 2b-1$,

$$\|t^{(\gamma_p/2b)(|\alpha|-(b-1))}D^{\alpha}_{xy}\Lambda^{\beta}V_f(\cdot,y,\cdot)\|_{L^p(S_T)}\leqslant C_2(T)\cdot\|s^{(\gamma_p/2b)(b+1-\gamma)}f\|_{L^p(R^{n+1}_+\times(0,T))}\,.$$

PROOF. Case $|\alpha|\leqslant b-1$.

$$\|D^{\alpha}_{xy}\Lambda^{\beta}V_f(\cdot,y,\cdot)\|_{L^p(R^n)}\leqslant C\int\limits_0^t\frac{1}{(t-s)^{(1+|\alpha|+\beta)/2b}}\int\limits_0^{\infty}\Psi(|y-v|/(t-s)^{1/2b})\cdot\|f(\cdot,v,s)\|\,dv\,ds\,.$$

We apply Hölder's inequality in dv and observe that

$$\Big(\int\limits_0^{\infty}\Psi^{p'}(|y-v|/(t-s)^{1/2b})\,dv\Big)^{1-1/p}\leqslant C\cdot(t-s)^{(1-1/p)/2b}\,.$$

Hence

$$\|D^{\alpha}_{xy}\Lambda^{\beta}V_f(\cdot,y,t)\|_{L^p(R^n)}\leqslant C\int\limits_0^{\infty}\frac{X_{(t,(0}(s)(T/t)^{1-\{|\alpha|+\beta+1/p+\gamma_p(b+1-\gamma)\}/2b}}{(t-s)^{(|\alpha|+\beta+1/p)/2b}s^{(\gamma_p/2b)(b+1-\gamma)}}\cdot \cdot\big(s^{(\gamma_p/2b)(b+1-\gamma)}\|f[\cdot,\cdot,s)\|_{L^p(R^{n+1}_+)}\big)\,ds\,.$$

The power of (T/t) is positive, provided $\gamma_p<\big(b+1-(\beta+1/p)\big)/(b+1-\gamma)$. Now the power of $(t-s)$ is clearly integrable, and $(\gamma_p/2b)(b+1-\gamma)++1/p<1$ due to the choice of γ_p. Hence by Hardy-Schur's Lemma (cf. [8], [9]), we obtain the desired inequality.

CASE $|\alpha|\geqslant b$. We proceed as in previous case, introducing now the factor $t^{(\gamma_p/2b)(|\alpha|-(b-1))}$, to obtain

$$(*)\quad t^{(\gamma_p/2b)(|\alpha|-(b-1))}\|D^\alpha_{xy}\Lambda^\beta V_f(\cdot,y,t)\|_{L^p(R^n)}\leqslant$$

$$\leqslant C\int_0^\infty \frac{X_{(0,t)}(s)(T/t)^{1-\{|\alpha|+\beta+1/p+\gamma_p(2b-\gamma-|\alpha|)\}/2b}\,t^{(\gamma_p/2b)(|\alpha|-(b-1))}}{(t-s)^{(|\alpha|+\beta+1/p)/2b}\,s^{(\gamma_p/2b)(b+1-\gamma)}}\cdot$$

$$\cdot\left(s^{(\gamma_p/2b)(b+1-\gamma)}\|f(\cdot,\cdot,s)\|_{L^p(R^{n+1}_+)}\right)ds\,.$$

We show that the power of (T/t) is positive by showing that the quantity in brackets is $<2b$. The ratio $\big((2b-\beta-1/p)-|\alpha|\big)/\big((2b-\gamma)-|\alpha|\big)$ is an increasing function of $|\alpha|$; therefore its minimum value is attained at $|\alpha|=0$: $1<(2b-\beta-1/p)/(2b-\gamma)$. As the function $(r-\beta-1/p)/(r-\gamma)$ is decreasing, the condition $\gamma_p<(2b-\beta-1/p)/(2b-\gamma)$ also implies $\gamma_p<(b+1-\beta-1/p)/(b+1-\gamma)$ (see Case $|\alpha|\leqslant b-1$). It is clear that for $1<p<\infty$ and $0<\gamma<1$, a number γ_p satisfying

$$1-1/p\leqslant\gamma_p<\frac{2b}{2b-\gamma}(1-1/p)\qquad\text{and}\qquad \gamma_p<\frac{2b-\beta-1/p}{2b-\gamma}$$

will suit to our requirements for all $|\alpha|\leqslant 2b-1$.

Returning to the estimate, we have $\|t^{(\gamma_p/2b)(|\alpha|-(b-1))}D^\alpha_{xy}\Lambda^\beta V_f(\cdot,y,\cdot)\|_{L^p(S_T)}\leqslant$ $\leqslant L^p$-norm in t of the integral in the right hand side of $(*)$. The result follows now from Hardy's Lemma.

V. – The operator *J*.

§ 12. – We shall study a commutator for the operator

$$(Jf)(x,y,t)=\lim_{\varepsilon\to0^+}\int_0^{t-\varepsilon}\iint_{R^{n+1}_+}(L_{xyt}K)(z,v,s;x-z,y,v,t-s)f(z,v,s)\,dz\,dv\,ds\,,$$

Consider

$$[y^{b+1-\gamma}J(f)-J\big(v^{b+1-\gamma}f(z,v,s)\big)](x,y,t)=$$

$$=\sum_{|\alpha|=2b}T^0_\alpha\big([a_\alpha(x,y,t)-a_\alpha(z,v,s)]f(z,v,s)\big)(x,y,t)+$$

$$+\sum_{|\alpha|<2b}a_\alpha(x,y,t)\,T^0_\alpha\big(f(z,v,s)\big((x,y,t)\,,$$

where we have set

$$T_\alpha^0(f)(x, y, t) = \int_0^t \iint_{R_+^{n+1}} (y^{b+1-\gamma} - v^{b+1-\gamma}) D_{x,y}^\alpha K(z, v, s; x-z, y, v, t-s) \cdot$$
$$\cdot f(z, v, s)\, dz\, dv\, ds .$$

We also set

$$T_\alpha^1(f)(x, y, t) = \int_0^t \iint_{R_+^{n+1}} (t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)})(D_{x,y}^\alpha K) f(z, v, s)\, dz\, dv\, ds ,$$

T_α^1 being the corresponding operators for the commutator

$$\left[t^{(\gamma_p/2b)(b+1-\gamma)} J(f) - J\left(s^{(\gamma_p/2b)(b+1-\gamma)} f(z, v, s)\right)\right](x, y, t) .$$

LEMMA. Let $f \in L^p(R_+^{n+1} \times (0, T))$. For $0 \leqslant |\alpha| \leqslant 2b$, $1/p < \gamma < 1$, and $1 < p < \infty$,

(i) $\|T_\alpha^0 f\|_{L^p(R_+^{n+1} \times (0,T))} \leqslant C \cdot T^{1-(|\alpha|/2b)} \|y^{b+1-\gamma} f(x, y, t)\|_{L^p(R_+^{n+1} \times (0,T))}$,

(ii) $\|T_\alpha^1 f\|_{L^p(R_+^{n+1} \times (0,T)} \leqslant C \cdot T^{1-(|\alpha|/2b)} \|t^{(\gamma_p/2b)(b+1-\gamma)} f(x, y, t)\|_{L(^p R_+^{n+1} \times (0,T))}$.

PROOF OF (i). We set as usual

$$|(T_\alpha^0 f)(x, y, t)| \leqslant \left(\int_0^{y/2} + \int_{y/2}^\infty\right) \int_0^t \int_{R^n} |y^{b+1-\gamma} - v^{b+1-\gamma}| \cdot |D_{x,y}^\alpha K| \cdot |f|\, dz\, dv\, ds .$$

To estimate the first term, we use the known properties of K together with Young's inequality in the variables z and s, and the fact that $v \leqslant y/2$ to obtain

$$\left\| \int_0^{y/2} \right\|_{L^p(S_T)} \leqslant C \cdot T^{1-(|\alpha|/2b)} \int_0^\infty \frac{X_{(0,y/2)}(v)}{y^\gamma \cdot v^{1-\gamma}} \left(v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)}\right) dv .$$

The desired inequality follows from this one, applying Hardy's lemma ([8], [9]) and recalling that $\gamma > 1/p$.

For the second term, we apply the known estimate directly to K to get

$$\left\| \int_{y/2}^{2y} \right\|_{L^p(S_T)} \leqslant C \cdot T^{1-(|\alpha|/2b)} \int_0^\infty \frac{X_{(y/2,2y)}(v) |y^{b+1-\gamma} - v^{b+1-\gamma}|}{v^{b+1-\gamma} |y - v|} \cdot \left(v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)}\right) dv .$$

We now apply Hardy's lemma to obtain the desired inequality.

PROOF OF (ii). It is clear that

$$|T^1_\alpha(f)(x, y, t)| \leq \int_0^t \iint_{R_u^{n+1}} |t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}||D^\alpha_{x,y} K||f| dz\, dv\, ds .$$

By applying Young's inequality in the variables z, v to the known estimates for K, and observing that $(t-s)^{1-(|\alpha|/2b)} \leq T^{1-(|\alpha|/2b)}$, it is easy to see that

$$\|T^1_\alpha f\|_{L^p(R_+^{n+1}\times(0,T))} \leq C\cdot T^{1-(|\alpha|/2b)} \left\| \int_0^\infty k(t, s)\left(s^{(\gamma_p/2b)(b+1-\gamma)}\|f(\cdot,\cdot, s)\|_{L^p(R^{n+1})}\right) ds\right\|_{L^p(0,\infty)},$$

where

$$k(t, s) = \frac{X_{(0,t)}(s)}{s^{(\gamma_p/2b)(b+1-\gamma)}} \frac{t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}}{t-s}.$$

Due to the conditions on γ_p and the integrability of the second factor $k(t, s)$ satisfies the hypotheses in Hardy's lemma, from which part (ii) follows. This completes the proof of the lemma.

§ 13. – LEMMA. For $f \in L^p(R_+^{n+1}\times(0, T))$, $1/p < \gamma < 1$, $1 < p < \infty$,

(i) $\|y^{b+1-\gamma}J(f) - J(v^{b+1-\gamma}f)\|_{L^p(R_+^{n+1}\times(0,T))} \leq \omega(T)\|y^{b+1-\gamma}f\|_{L^p(R_+^{n+1}\times(0,T))}$

(ii) $\|t^{(\gamma_p/2b)(b+1-\gamma)}J(f) - J(s^{(\gamma_p/2b)(b+1-\gamma)}f)\|_{L^p(R_+^{n+1}\times(0,T))} \leq$
$\leq \omega(T)\|t^{(\gamma_p/2b)(b+1-\gamma)}f(x, y, t)\|_{L^p(R_+^{n+1}\times(0,T))}$,

with $\omega(T)\to 0$ as $T\to 0^+$, ω depending on the moduli of continuity of the a_α, $|\alpha| = 2b$.

PROOF. Let $\varphi\in C_0^\infty(R^{n+2})$, $\varphi \geq 0$, with $\iiint_{R^{n+2}} \varphi = 1$ and support φ contained in the set $|x|^2 + y^2 + t^2 < 1$. For $|\alpha| = 2b$, we extend $a_\alpha(x, y, t)$ to all R^{n+2}, preserving uniform continuity and define, for $\lambda > 0$,

$$a^\lambda_\alpha(x, y, t) = (1/\lambda^{n+2})\iiint_{R^{n+2}} a_\alpha(w, u, r)\,\varphi\left(\frac{x-w}{\lambda}, \frac{y-u}{\lambda}, \frac{t-r}{\lambda}\right) dw\, du\, dr .$$

Then $a^\lambda_\alpha \in C^\infty(R^{n+2})$, $|a_\alpha(x, y, t) - a^\lambda_\alpha(x, y, t)| \leq \omega(\lambda) =$ maximum of moduli of continuity of a_α, $|\alpha| = 2b$, $|Da_\alpha(x, y, t)| \leq (C/\lambda)\omega(\lambda)$, D being any derivative, and therefore $|a^\lambda_\alpha(x, y, t) - a^\lambda_\alpha(z, v, s)| \leq (C/\lambda)\omega(\lambda)(|x-z| + |y-v| + |t-s|)$.

According to the decomposition and the estimates in previous §, it will be sufficient to consider the case $|\alpha| = 2b$, $j = 0, 1$. We have

$$T_\alpha^j([a_\alpha(x, y, t) - a_\alpha(z, v, s)]f) = T_\alpha^j([a_\alpha(x, y, t) - a_\alpha^\lambda(x, y, t)]f) + \\ + T_\alpha^j([a_\alpha^\lambda(z, v, s) - a_\alpha(z, v, s)]f) + T_\alpha^j([a_\alpha^\lambda(x, y, t) - a_\alpha^\lambda(z, v, s)]f) .$$

The first two terms are in absolute value less than or equal to $\omega(\lambda)$ times (bounds for $|T_\alpha^j f|$ in proof (i) and (ii) of lemma, § 12). For the third, we set $|y^{b+1-\gamma} - v^{b+1-\gamma}| = E^0$, $|t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}| = E^1$ and ovserve that it is in absolute value

$$\leqslant \int_0^t \int_0^\infty \int_{R^n} |D_{x\,y}^\alpha K| \cdot E^j \cdot (C/\lambda)\omega(\lambda)(|x-z| + |y-v| + |t-s|) \cdot |f(z, v, s)|\, dz\, dv\, ds .$$

The usual procedure applies to each term above. For the third, $t - s < T$, so we set $\lambda = T^{1/2b}$ and obtain with a new $\omega(T)$ that tends to zero as $T \to 0+$, the same type of bound for $|T_\alpha^j f|$ used in Lemma, § 12. The proofs of § 12 apply again.

§ 14. – Lemma. Let $f \in L^p(R_+^{n+1} \times (0, T))$. Then the operator (Jf) maps $L^p(R_+^{n+1} \times (0, T))$ continuously into itself, and belongs to $\mathfrak{J}(R_+^{n+1} \times (0, T))$ (cf. § 2).

Proof. Condition (i), § 2 is clear. To prove condition (ii), we set $J = J_1 - J_2$, where

$$J_1 f(x, y, t) = \lim_{\varepsilon \to 0^+} \text{ in } L^p \int_0^{t-\varepsilon} \iint_{R_+^{n+1}} L_{xvt} F(z, v, s; x - z, y - v, t - s) f(z, v, s)\, dz\, dv\, ds ,$$

$$J_2 f(x, y, t) = \int_0^t \iint_{R_+^{n+1}} L_{xvt} G_0(z, v, s; x - z, y, v, t - s) f(z, v, s)\, dz\, dv\, ds .$$

J_1 is known to belong in $\mathfrak{J}$ (cf. [5]). Furthermore,

$$J_1 = \sum_{|\alpha| = 2b} (a_\alpha \bar{K}_\alpha - \bar{K}_\alpha a_\alpha) + \sum_{|\alpha| < 2b} a_\alpha J_\alpha ,$$

$\bar{K}_\alpha$ being a variable kernel, and the J_α, L^p-operators in $\mathfrak{J}$. For J_2, we see that it can be decomposed in two sums, setting

$$N_\alpha(f) = \int_0^t \iint_{R_+^{n+1}} D^\alpha_{x,y} G_0(z, v, s; x-y, v, t-s) f(z, v, s)\, dz\, dv\, ds$$

$$J_2 f = \sum_{|\alpha|=2b} N_\alpha\big([a_\alpha(x, y, t) - a_\alpha(z, v, s)] f\big) + \sum_{|\alpha|<2b} a_\alpha N_\alpha(f) .$$

Consider first the case $|\alpha| = 2b$. Replacing f by $X_{(a,a+\varepsilon)}(s) f(z, v, s)$ se see that each term in the first sum above can be written as

$$N_\alpha\big([a_\alpha(x, y\ t) - a^\lambda_\alpha(x\ y, t)] X_{(a,a+\varepsilon)} f\big) + N_\alpha\big([a^\lambda_\alpha(z, v, s) - a_\alpha(z, v, s)] X f\big) + \\ + N_\alpha\big([a^\lambda_\alpha(x, y, t) - a^\lambda_\alpha(z, v, s)] X_{(a,a+\varepsilon)} f\big) ,$$

where each term is in absolute value

$$\leqslant \omega(\varepsilon^{1/2b}) \int_0^t \iint_{R_+^{n+1}} (\text{bounds for } |D^\alpha_{x,y} G_0| \text{ in } \S\, 7) \cdot X_{(a,a+\varepsilon)}(s) |f(z\ v\ s)|\, dz\, dv\, ds ,$$

if $\lambda = \varepsilon^{1/2b}$. (We observe that for $a \leqslant t \leqslant a + \varepsilon$ and $a \leqslant s \leqslant t$, $t - s \leqslant \varepsilon$ and $X_{(0,\varepsilon)}(t - s) = 1$).

By Young's inequality in $dx\, dt$, the $L^p(R^n \times (a, a + \varepsilon))$-norm of the above expression is bounded by

$$C\omega(\varepsilon^{1/2b}) \int_0^\infty \frac{1}{|y + v|} \int_0^{\varepsilon/|y+v|^{2b}} \frac{\Psi(1/s^{1/2b})}{s^{1+1/2b}}\, ds\, \|f(\cdot, v, \cdot)\|_{L^p(S_{(a,a+\epsilon)})}\, dv .$$

(Here we have set $\Psi(y/s^{1/2b}) \Psi(v/s^{1/2b}) \leqslant \Psi(|y + v|/s^{1/2b})$, the Ψ being of exponential type (cf. § 5). The integral in ds is bounded independently of y, v).

We now apply Hardy's lemma to obtain the inequality

$$\sum_{|\alpha|=2b} \|N_\alpha\big([a_\alpha(x, y, t) - a_\alpha(z, v, s)] X_{(a,a+\varepsilon)} f\big)\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} \leqslant \\ \leqslant C\omega(\varepsilon^{1/2b}) \|f\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} .$$

For $|\alpha| < 2b$, using the fact that $\Psi(y)\Psi(v) \leqslant \Psi(|y - v|)$, $y, v > 0$, and applying the known estimates for G_0 and the remarks on $X_{(0,\varepsilon)}(t - s)$, we easily get

$$\|N_\alpha(X_{(a,a+\varepsilon)} f)\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} \leqslant C\varepsilon^{1-(|\alpha|/2b)} \|f\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} ,$$

C depending on $\max_{|\alpha|<2b} \sup |a_\alpha|$, completing the proof of the lemma.

REMARK. We have essentially shown that

$$L_{xvt}\Big(\int_0^t \iint_{R_+^{n+1}} G_0(z, v, s; x-z, y, v, t-s) f(z, v, s)\, dz\, dv\, ds\Big) = J_2 f$$

belongs to $L^p(R_+^{n+1}\times(0, T))$ provided $f\in L^p(R_+^{n+1}\times(0, T))$, and its norm is $\leqslant C\|f\|_{L^p(R_+^{n+1}\times(0,T))}$. We observe that the boundary term in the computation of the D_t is zero, due to the estimates for G_0 (§ 7); we also see that

$$\int_0^t \iint_{R_+^{n+1}} G_0(z, v, s; x-z, y, v, t-s) f(z, v, s)\, dz\, dv\, ds \in \mathring{L}^p_{2b,1}(R_+^{n+1}\times(0, T)) .$$

Estimate for J. As a consequence of the results above, we have the following

LEMMA. For $f\in L^p(R_+^{n+1}\times(0, T))$, $1/p<\gamma<1$, $1<p<\infty$,

$$\|d_p(y, t)^{b+1-\gamma} Jf(x, y, t)\|_{L(^pR_+^{n+1}\times(0,T))} \leqslant \omega(T)\|d_p(y, t)^{b+1-\gamma} f(x, y, t)\|_{L^p(R_+^{n+1}\times(0,T))} ,$$

where $\omega\to 0$ as $T\to 0^+$.

§ 15. – The operator J having small norm as an operator on $L^p(R_+^{n+1}\times(a, a+\varepsilon))$ for ε suitably small, it follows that $I-J$ is invertible over $L^p(R_+^{n+1}\times(0, T))$. In fact, choosing m large enough,

$$\|J(X_{(a,a+(T/m))}g\|_{L^p(R_+^{n+1}\times(a,a+(T/m)))} \leqslant (\tfrac{1}{2})\|g\|_{L^p(R_+^{n+1}\times(a,a+(T/m)))}) ,$$

provided $a+(T/m)\leqslant T$. Let g_1 be a function with support in $R_+^{n+1}\times(0, T/m)$ such that $(I-J)g_1 = f$ on $R_+^{n+1}\times(0, T/m)$, and in general let g_k have support in $R_+^{n+1}\times((k-1)T/m, k(T/m))$, and satisfy $(I-J)g_k = f - \sum_{h=1}^{k-1}(I-J)g_h$ on $R_+^{n+1}\times((k-1)T/m, k(T/m))$. Set $g=\sum_{k=1}^{n} g_k$: $g\in L^p(R_+^{n+1}\times(0, T)$, being a sum of functions in that space, and $(I-J)g=f$.

With the construction above it is easy to prove the following

LEMMA. If $f\in L^p(R_+^{n+1}\times(0, T))$, $1<p<\infty$, $1/p<\gamma<1$, then

$$\|d_p(y, t)^{b+1-\gamma}(I-J)^{-1}f\|_{L^p(R_+^{n+1}\times(0,T))} \leqslant C\|d_p(y, t)^{b+1-\gamma}f\|_{L^p(R_+^{n+1}\times(0,T))} .$$

($d_p(y, t)$ can be replaced by y or $t^{(\gamma p/2b)}$).

REMARK. If the function f and the coefficients of L_{xyt} are differentiable, so is the function $(I-J)^{-1}f = g$. In fact, the only possible points of discontinuity of g or its derivatives with respect to time are those in the partition kT/m, $k \leqslant m$. To see that g is smooth at those points, we only have to construct $(I-J)^{-1}f$ with a different partition. The uniqueness of $(I-J)^{-1}f$ shows the differentiability of g at the points kT/m.

The differentiability of $(I-J)^{-1} = \sum J^k$ for small t can be seen from the fact that $DJ(f) = J^{\sim}(f) + J(Df)$, D denoting any derivative, and $J^{\sim}$ being a $\mathfrak{J}$-operator that depends on the derivatives of the coefficients. From this identity we derive the recursion inequality

$$\|DJ^n(f)\| \leqslant \omega(r)\big(\|J^{n-1}(f)\| + \|DJ^{n-1}(f)\|\big), \qquad \omega \to 0 \text{ as } r \to 0^+,$$

which shows convergence in norm of the series $\sum DJ^n(f)$ for r sufficiently small.

VI. – The main results in the half-space.

§ 16. – THEOREM. Let $f \in L^p\big(R_+^{n+1} \times (0, T)\big)$ for some p, $1 < p < \infty$. Set

$$\begin{aligned} V(x, y, t) &= -\int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s)(I-J)^{-1} f(z, v, s)\, dz\, dv\, ds \\ &= - V_{(I-J)^{-1}f}(x, y, t)\,. \end{aligned}$$

Then for $0 \leqslant \beta < \gamma - 1/p$,

(i) $V(x, y, t) \in \mathring{L}^p_{2b,1}\big(R_+^{n+1} \times (0, T)\big)$, and $LV = f$ for $y > 0$,

(ii) for $k = 0, \ldots, b-1$,

$$\lim_{y \to 0^+} \text{ in } L^p(S_T) \Lambda^{b-1+\beta-k} D_y^k V(\cdot, y, \cdot) = 0\,.$$

(iii)
$$\begin{aligned} &\sum_{k=0}^{b-1} \|\Lambda^{b-1-k+\beta} D_y^k V(\cdot, y, \cdot)\|_{L^p(S_T)} \\ &+ \sum_{|\alpha| \leqslant b-1} \|D_{x,y}^\alpha \Lambda^\beta V(\cdot, y, \cdot)\|_{L^p(S_T)} \\ &+ \sum_{b \leqslant |\alpha| \leqslant 2b-1} \|d_a(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^a \Lambda^\beta V(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant C(T) \|d_p^{b+1-\gamma} f\|_{L^p}\,, \end{aligned}$$

where $C(T) \to 0$ as $T \to 0^+$.

(A similar estimate holds with d_p replaced by y, and with d_p replaced by $t^{\gamma_p/2b}$).

PROOF (i). Recall that

$$K(z, v, x; y, v, t) = F(z, v, s; x, y - v, t) - G_0(z, v, s; x, y, v, t),$$

and that $(I-J)^{-1}f \in L^p(R_+^{n+1}\times(0, T))$. Set $V = -V_1 + V_2$, where

$$V_1(x, y, t) = \int_0^t \iint_{R_+^{n+1}} F(z, v, s; x - z, y - v, t - s)(I - J)^{-1}f(z, v, s)\, dz\, dv\, ds,$$

$$V_2(x, y, t) = \int_0^t \iint_{R_+^{n+1}} G_0(z, v, s; x - z, y, v, t - s)(I - J)^{-1}f(z, v, s)\, dz\, dv\, ds.$$

We showed (cf. Remark to Lemma, § 14) that $V_2 \in \mathring{L}^p_{2b,1}(R_+^{n+1}\times(0, T))$. It is known [2] that $V_1 \in \mathring{L}^p_{2b,1}(R_+^{n+1}\times(0, T))$, hence $V \in \mathring{L}^p_{2b,1}(R_+^{n+1}\times(0, T))$. Also

$$LV_1 = -(I-J)^{-1}f + J_1\big((I-J)^{-1}f\big),$$

and by same Remark, $LV_2 = J_2\big((I-J)^{-1}f\big)$. Therefore

$$\begin{aligned} LV &= (I-J)^{-1}f - (J_1 - J_2)(I-J)^{-1}f \\ &= (I-J)^{-1}f - J(I-J)^{-1}f = f. \end{aligned}$$

Part (ii) is an immediate consequence of Theorem § 9, (i), and the fact that $(I-J)^{-1}f$ belongs in $L^p(R_+^{n+1}\times(0, T))$.

Part (iii) follows from the estimates for the volume potential (Theorem, § 9, cf. lemmas A, B, C, §§ 10-11), and from estimate in Lemma, § 15.

§ 17. - For $j = 0, \ldots, b-1$, we introduce the functions

$$u_j(x, y, t) = (T_{\mu j}\Phi)(x, y, t) + \\ + \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s)(I - J)^{-1}(LT_{\mu j}\Phi)(z, v, s)\, dz\, dv\, ds \ (*).$$

(cf. definitions in § 6(K), § 8($T_{\mu j}$), and in § 9 (volume potential)).

THEOREM. If $\Phi \in L^p(S_T)$, then for $j = 0, \ldots, b-1$,

(i) $u_j(x, y, t) \in \mathring{L}^p_{2b,1}(R^n \times (\delta, \infty)\times(0, T))$ for every $\delta > 0$,

(ii) $L\mu_j = 0$ in $R_+^{n+1}\times(0, T)$.

PROOF. Is included in §§ 17-19. We introduce the following notation.

For $|\alpha| = 2b$, let $a_\alpha^\lambda(z, v, s)$ denote the regularization of the coefficient $a_\alpha(z, v, s)$ in L(cf. § 13). We will denote by $F^\lambda(z, v, s; x, y, t)$ the $\mathcal{F}_\xi\Big[\exp \sum_{|\alpha|=2b} a_\alpha^\lambda(z, v, s)(i\xi)^\alpha t\Big](x, y)$, by $T_{\mu j}^\lambda \Phi$ the surface potential

$$(T_{\mu j}^\lambda \Phi)(x, y, t) = \int_0^t \int_{R^n} \Lambda^{b-j-\mu} D_y^j F^\lambda(z, 0, s; x-z, y, t-s)\, \Phi(z, s)\, dz\, ds\,,$$

and by $u_j^\lambda(x, y, t)$ the corresponding functions (*).

If $\Phi \in C_0^\infty(R^n \times (0, \infty))$, it can be seen that $u_j^\lambda(x, y, t) \in \mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$, and that $Lu_j^\lambda = 0$ for $y > 0$. The second statement will follow from the theorem in § 16; the first is a consequence of the same theorem in § 16 and of the fact that $(I - J)^{-1}$ is a L^p-mapping, if we observe that $T_{\mu j}^\lambda \Phi \in \mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$.

When the coefficients a_α, $|\alpha| = 2b$, are only bounded and uniformly continuous, and $\Phi \in C_0^\infty(R^n \times (0, \infty))$, we will show that the expression in (*), § 17, belongs in $\mathring{L}_{2b,1}^p(R_E \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. We first state two lemmas.

LEMMA 1. Let $\lambda, \lambda' > 0$. For $j = 0, \dots, b-1$,

$$\|(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'})\, \Phi\|_{L^p(R_+^{n+1} \times (0,T))} \leqslant$$
$$\leqslant C \cdot T^{(\mu + 1/p)/2b} \max_{|\alpha|=2b} \|a_\alpha^\lambda(z, 0, s) - a_\alpha^{\lambda'}(z, 0, s)\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}\,.$$

LEMMA 2. If $\Phi \in L^p(S_T)$, $1 < p < \infty$, $\gamma - 1/p < \mu$, then

$$\|y^{b+1-\gamma} L(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0,T))} \leqslant C\omega(T) \max_{|\alpha|=2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}\,.$$

These lemmas are L^p versions of similar results in Pogorzelski [12], and can be proved by similar arguments.

If now $\theta(y) \in C^\infty(0, \infty)$, $\theta = 0$ for $y < \delta$, $\theta = 1$ for $y \geqslant 2\delta$, and we set $\theta^{(l)}(y) = D_y^l \theta(y)$, we have the

COROLLARY. For $|\alpha| + l \leqslant 2b$,

$$\|\theta^{(l)}(y) D_{xy}^\alpha (T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0,T))} \leqslant C_\delta \omega(T) \max_{|\alpha|2=2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}\,.$$

We now continue with the proof of the Theorem, § 17.

§ 18. – We recall that for $\Phi \in C_0^\infty(R^n \times (0, \infty))$, $u_j^\lambda(x, y, t) = (T_{\mu j}^\lambda \Phi) + V_{(\Phi)}^\lambda$ belongs to $\mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$ (we have set $V_{(\Phi)}^\lambda = V_{(I-J)^{-1}(LT_j^\lambda \Phi)}$).

The product $\theta(y)(u_j^\lambda - u_j^{\lambda'})$ vanishes near $y = 0$ and $t = 0$. It is known [4] that

$$\|\theta(u_j^\lambda - u_j^{\lambda'})\|_{L_{2b,1}^p(R_+^{n+1} \times (0,T))} \leqslant C \cdot \|L(\theta(u_j^\lambda - u_j^{\lambda'}))\|_{L^p(R_+^{n+1} \times (0,T))} \leqslant$$

$$\leqslant C \sum_{|\alpha| + l \leqslant 2b} \|\theta^{(l)} D_{x,y}^\alpha (T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi\|_{L^p(R_+^{n+1} \times (0,T))} +$$

$$+ C\|\theta L(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi\|_{L^p(R_+^{n+1} \times (0,T))} +$$

$$+ C\|\sum_{l>0} \theta^{(l)}(y) D_{x,y}^\alpha V_{(\Phi)}^{\lambda,\lambda'}(x, y, t)\|_{L^p(R_+^{n+1} \times (0,T))},$$

where the meaning of $V_{(\Phi)}^{\lambda,\lambda'}(x, y, t)$ is clear.

Now

$$\|\sum_{l>0} \theta^{(l)} D_{x,y}^\alpha V_{(\Phi)}^{\lambda,\lambda'}\|_{L^p(R_+^{n+1} \times (0,T))}$$

$$\leqslant \sum_{l>0} \|\theta^{(l)}(y) \|D_{x,y}^\alpha V_{(\Phi)}^{\lambda,\lambda'}(\cdot, y, \cdot)\|_{L^p(S_T)}\|_{L^p(0,\infty)}$$

$$\leqslant \sum_{l>0} \left\|\theta^{(l)}(y) \cdot \frac{1}{\delta^{|\alpha|}} \cdot y^{|\alpha|}\right\| D_{x,y}^\alpha V_{(\Phi)}^{\lambda,\lambda'}(\cdot, y, \cdot)\|_{L^p(S_T)}\|_{L^p(0,\infty)}$$

$$\leqslant C(T) \sum_{l>0} \left\|\frac{\theta(l)}{\delta^{|\alpha|}}\right\|_{L^\infty(0,\infty)} \|y^{b+1-\gamma}(I - J)^{-1} L(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{(L^p(R_+^{n+1} \times (0,T))}$$

$$\leqslant C_\delta(T) \|y^{b+1-\gamma} L(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0,T))}.$$

(We have used the estimates for the volume potential (§ 9, and lemmas A and B, § 10) with $\beta = 0$, and the Lemma in § 15)

By Lemmas 1 and 2, § 17, we conclude that

$$\theta(u_j^\lambda - u_j^{\lambda'})\|_{L_{2b,1}^p(R_+^{u+1} \times (0,T))} \leqslant C(T), \qquad \max_{|\alpha| = 2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \cdot \|\Phi\|_{L^p(S_T)}.$$

This shows that $\{u_j^\lambda\}_\lambda$ is a Cauchy sequence in $\mathring{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. Let $\bar{u}_j \in \mathring{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$ be its limit. By Lemmas 1 and 2, with $T_{\mu j}^{\lambda'}$ replaced by $T_{\mu j}$, it follows that the potentials $T_{\mu j}^\lambda \Phi$ and $V_{(\Phi)}^\lambda$ converge to $T_{\mu j} \Phi$,

$$V_{(\Phi)} = \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s)(I - J)^{-1}(LT_{\mu j} \Phi)(z, v, s)\, dz\, dv\, ds.$$

Hence $\bar{u}_j$ admits the representation (*) for $\Phi \in C_0^\infty(R^n \times (0, \infty))$, i.e.,

$$\bar{u}_j(x, y, t) = (T_{\mu j}\Phi)(x, y, t) +$$
$$+ \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s)(I-J)^{-1}(LT^{\mu j}\Phi)(z, v, s)\, dz\, dv\, ds\,, \quad (*)$$

and $\bar{u}_j \in \mathring{L}^p_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$.

Clearly $L\bar{u}_j = 0$ for $y > 0$.

§ 19. – The same representation holds for $\Phi \in L^p(S_T)$. To see this, select $\Phi^\lambda \in C_0^\infty(R^n \times (0, \infty))$, $\|\Phi - \Phi^\lambda\|_{L^p(S_T)} \to 0$ as $\lambda \to 0$, and let $\bar{u}_j^\lambda$ be the function (*) above with Φ^λ in place of Φ. Again let $\theta(y) \in C^\infty(0, \infty)$, $\theta(y) = 0$ for $y < \delta$, $\theta(y) = 1$ for $y > 2\delta$. We have

$$\|\theta(\bar{u}_j - \bar{u}_j^\lambda)\|_{L^p_{2b,1}(R_+^{n+1}\times(0,T))} \leqslant \|L[\theta(\bar{u}_j - \bar{u}_j^\lambda)]\|_{L^p(R_+^{n+1}\times(0,T))}$$
$$\leqslant C \sum_{(|\alpha|+l\leqslant 2b} \|\theta^{(l)} D^\alpha_{\mu j} T_{\mu j}(\Phi - \Phi)\|_{L^p(R_+^{n+1}\times(0,T)} +$$
$$+ \|\theta L T_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1}\times(0,T))} +$$
$$+ C\Big\| \sum_{l>0} \theta^{(l)}(y) D^\alpha_{x,y} V_{(\Phi-\Phi^\lambda)(x,y,t)}\Big\|_{L^p(R_+^{n+1}\times(0,T))}$$

where $V_{(\Phi-\Phi)^\lambda}$ denotes the volume potential in (*) corresponding to $\Phi - \Phi^\lambda$.

A slight modification of the arguments in § 18 shows that these terms are majorized by $C\|\Phi - \Phi^\lambda\|_{L^p(S_T)}$: Therefore $\bar{u}_j^\lambda \to \bar{\bar{u}}_j$ in $\mathring{L}^p_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. Also, by Lemmas 1 and 2, § 8, both

$$\|T_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1}\times(0,T))} \quad \text{and} \quad \|y^{b+1-\gamma} L T_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1}\times(0,T))}$$

tend to zero as $\lambda \to 0$. This shows that

$$\bar{\bar{u}}_j \in \mathring{L}^p_{2b,1}(R^n \times (\delta, \infty) \times (0, T)) \qquad \text{for every } \delta > 0\,,$$

and admits the representation (*) with $\Phi \in L^p(S_T)$. From this we see that $\bar{\bar{u}}_j \equiv u_j$ and $Lu_j = 0$ for $y > 0$.

This concludes the proof of Theorem, § 17.

§ 20. – It is known (cf. [3]) that

$$\Lambda^{b-1+\mu-k} D^k_y T_{\mu j}\Phi(x, y, t)$$
$$= \int_0^t \int_{R^n} \Lambda^{2b-1-(k+j)} D^{k+j}_y F(z, 0, s; x-z, y, v, t-s)\, \Phi(z, s)\, dz\, ds$$

converges in $L^p(S_T)$ as $y \to 0^+$, to a limit of the form $(S_{k,j}+J_{k,j})\Phi$, where $(S_{k,j})$ is a matrix of parabolic singular integral operators whose matrix of symbols admits an inverse $[\sigma(S_{k,j})(z, s; x, t)]^{-1} = \big(\sigma(S^*_{k,j})(z, s; x, t)\big)$ provided $(x, t) \neq (0, 0)$, $S^*_{k,j}$ being a parabolic singular integral operator such that $(S_{kj})\cdot(S^*_{kj}) = I + (J^0_{kj})$, with $J^0_{kj} \in \mathfrak{J}(S_T)$ (cf. [4]). $J_{k,j}$ belongs to $\mathfrak{J}(S_T)$, and is a limit as $y \to 0$ of a series of commutators in the variables (x, y, t) which belong to $\mathfrak{J}(S_T)$ uniformly in y [3].

From the estimates for the volume potential (lemma A, § 10) and the estimates for $y^{b+1-\gamma}(I-J)^{-1}(\cdot)$ and $y^{b+1-\gamma}LT_{\mu j}\Phi$ (§ 15 and Lemma 2, § 8), it follows that for $\beta < \mu$, with $\beta < \gamma - 1/p < \mu$,

$$\Lambda^{b-1+\beta-k} D^k_y \int_0^t \iint_{R_+^{1+u}} K(z, v, s; x-z, y, v, t-s)(I-J)^{-1}(LT_{\mu j}\Phi)(z, v, s)\, dz\, dv\, ds \to 0$$

in $L^p(S_T)$ as $y \to 0^+$.

Also, for $f \in L^p\big(R_+^{n+1} \times (0, T)\big)$ (Theorem § 16, (ii)),

$$\Lambda^{b-1+\beta-k} D^k_y \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s)(I-J)^{-1} f(z, v, s)\, dz\, dv\, ds \to 0$$

in $L^p(S_T)$ as $y \to 0^+$.

We observe now that $(J_{kj})\cdot(S^*_{kj})$ (the dot indicates matrix multiplication) is a matrix of $\mathfrak{J}$-operators, due to the fact that if $J \in \mathfrak{J}(S_T)$, and S is a singular integral operator on S_T, $JS \in \mathfrak{J}(S_T)$.

Therefore $I + [(J^0_{kj}) + (J_{kj})\cdot(S^*_{kj})]$ has an inverse.

Set

$$(\mathfrak{V}_{kj}) = (S^*_{kj})\cdot[I + (J^0_{kj}) + (J_{kj})\cdot(S^*_{kj})]^{-1}\,.$$

THEOREM. Let there be given $f \in L^p\big(R_+^{n+1} \times (0, T)\big)$, and for $k = 0, \ldots, b-1$, functions $w^k \in L^p(S_T)$ such that $\Lambda^{b-1-k+\mu} w^k \in L^p(S_T)$ for some μ, $0 < \mu < 1$.

Then there exists a function $u \in \mathring{L}^p_{2b,1}\big(R^n \times (\delta, \infty) \times (0, T)\big)$ for every $\delta > 0$ such that

(i) $Lu = f$ in $R_+^{n+1} \times (0, T)\big)$,

(ii) $\|\Lambda^{b-1-k+\beta}\big(D^k_y u(\cdot, y, \cdot) - w^k(\cdot, \cdot)\big)\|_{L^p(S_T)} \to 0$ as $y \to 0^+$, for every β, $0 \leqslant \beta < \mu$,

(iii) For $0 \leqslant \beta < \gamma - 1/p < \mu$,

$$\sum_{|\alpha| \leqslant b-1} \|D^{\alpha}_{x,y} \Lambda^{\beta} u(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k} D^k_y u(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{b \leqslant |\alpha| \leqslant 2b-1} \|d_p(y, \cdot)^{|\alpha|-(b-1)} D^{\alpha}_{x,y} \Lambda^{\beta} u(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant$$

$$\leqslant C \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k} w^k\|_{L^p(S_T)} + C(T) \|d_p^{b+1-\gamma_f} f\|_{L^p(R^{n+1}_+ \times (0,T))}$$

(d_p can be replaced with y or $t^{\gamma_p/2b}$).

PROOF. We set

$$g_{\mu l}(x, y, t) = \sum_{j=0}^{b-1} (\mho_{lj} \Lambda^{b-1+\mu-j} w)(x, y, t)$$

and define

$$u(x, y, t) = \sum_{l=0}^{b-1} (T_{\mu l} g_{\mu l})(x, y, t) + \sum_{l=0}^{b-1} \int_0^t \iint_{R^{n+1}_+} K(z, v, s; x-z, y, v, t-s) \cdot (I-J)^{-1} (L T_{\mu l} g_{\mu l})(z, v, s)\, dz\, dv\, ds -$$

$$- \int_0^t \iint_{R^{n+1}_+} K(z, v, s; x-z, y, v, t-s) (I-J)^{-1} f(z, v, s)\, dz\, dv\, ds .$$

The fact that $u \in \mathring{L}^p_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$, and Part (i) follow from the theorems, §§ 16 and 17. To prove (ii), it is enough to consider the terms $\sum_{l=0}^{b-1} (T_{\mu l} g_{\mu l})(x, y, t)$ and to recall the definition of $g_{\mu l}$ and the fact that $\Lambda^{\beta-\mu}$ is an integrable function.

§ 21. – THEOREM. Suppose $u \in \mathring{L}^p_{2b,1}(R^{n+1}_+ \times (0, T))$, $1 < p < \infty$. Then $u(x; y, t)$ admits the representation $u = u_1 + u_2$, where

$$u_1(x, y, t) = \sum_{l=0}^{b-1} \{ (T_{\mu l} g_{\mu l})(x, y, t) + \int_0^t \iint_{R^{n+1}_+} K(z, v, s; x-z, y, v, t-s) (I-J)^{-1} (L T_{\mu l} g)_{\mu l}\, dz\, dv\, ds \} ,$$

$u_2(x, y, t) =$

$$= -\int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s)(I-J)^{-1}(Lu)(z, v, s)\, dz\, dv\, ds\,,$$

with

$$g_{\mu l}(x, y, t) = \sum_{j=0}^{b-1} \big(\mho_{lj} \Lambda^{b-1+\mu-j} D_y^j u(\cdot, 0, \cdot)\big((x, y, t)\,, \qquad \text{(cf. § 20)}\,.$$

Furthermore, for $0 \leqslant \beta < \gamma - 1/p < \mu$,

$$\sum_{k=}^{b-1} \|\Lambda^{b-1+\beta-k} D_y^k u(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha|\leqslant b-1} \|D_{xy}^\alpha \Lambda^\beta u(\cdot, y, \cdot)\|_{L(S_T)} +$$
$$+ \sum_{b\leqslant|\alpha|\leqslant 2b-1} \|d_p(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^\alpha \Lambda^\beta u(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant$$
$$\leqslant C \sum_{=0}^{b-1} \|\Lambda^{b-1+\mu-k} D_y^k u(\cdot, 0, \cdot)\|_{L^p(S_T)} + C(T) \cdot \|d_p^{b+1-\gamma}(Lu)\|_{L^p(R_+^{n+1}\times(0,T))}\,,$$

where $C(T) \to 0$ as $T \to 0$, d_p can be replaced by y or $t^{\gamma_p/2b}$, and C, $C(T)$ depend only on the parameter of parabolicity π and on the $\max\limits_{|\alpha|=2b} \sup|a_\alpha|$.

Proof. We assume first $u \in C_0^\infty(R_+^{n+1} \times (0, \infty))$, and let $u^\lambda = u_1^\lambda + u_2^\lambda$, u_1^λ and u_2^λ being the terms in the decomposition above, with

$$T_{\mu l}^\lambda,\ g_{\mu l}^\lambda,\ \mho_{lj}^\lambda = \big((S_{lj*}^\lambda) \cdot [I + (J_{lj}^{0\lambda}) + (J_{lj}^\lambda) \cdot (S_{lj*}^\lambda)]^{-1}\big)_{lj}$$

being the expressions corresponding to $F^\lambda(z, 0, s; x, y, t)$ (cf. §§ 13 and 20).

We observe that

$$g_{\mu l}^\lambda(x, y, t) = \sum_{j=0}^{b-1} \{\mho_{lj}^\lambda \Lambda^{b-1+\mu-j} D_y u(\cdot, 0, \cdot)\}(x, y, t)$$

vanishes near $t = 0$ and has all derivatives in $L^p(R_+^{n+1} \times (0, T))$.

By an argument similar to that in § 17, we see that

$$T_{\mu l}^\lambda g_{\mu l}^\lambda \in \mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))\,.$$

Hence $u_1^\lambda \in \mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$. u_2^λ belongs to $\mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$ by Theorem, § 16 ($Lu \in L^p$).

Therefore $u^\lambda \in \mathring{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$, and it is clear that $Lu^\lambda \equiv Lu$ for $y > 0$.

Also $\Lambda^{b-1+\mu-k} D_y^k u^\lambda$ tends to $\Lambda^{b-1+\mu-k} D_y^k u(\cdot, 0, \cdot)$ in $L^p(S^T)$ as $y \to 0^+$. On the other hand, $D_y^k u^\lambda$ and $D_y^k u$ converge in $L^p(S_T)$ as $y \to 0^+$, being functions in $\mathring{L}^p_{2b,1}(R^{n+1}_+\times(0,T))$. Hence we must have $D^k_y u^\lambda(\cdot,0,\cdot)=D^k_y u(\cdot,0,\cdot)$, and we see that $u-u^\lambda$ satisfies a homogeneous initial-boundary value problem with homogeneous data. By [3] it follows that

$$u\equiv u^\lambda\equiv u^\lambda_1+u^\lambda_2\,.$$

Now as $\lambda\to 0$, $g^\lambda_{\mu l}\to g_{\mu l}$ in $L^p(S_T)$ (it would be enough to show that $S^{\lambda *}\to S^*$, $J^\lambda\to J$ as operators on $L^p(S_T)\times\ldots\times L^p(S_T)$, $(b-1)$ times; we observe that

$$\|J^\lambda_{lh}(X_{(a,a+\varepsilon)}f)\|_{L^p(R^n\times(a,a+\varepsilon))}\leqslant\omega(\varepsilon)\|f\|_{L^p(R^n\times(a,a+\varepsilon))}\,,$$

ω in dependent of a and λ, cf. definition of J_{lh} in [3]).

Thus we have the representation $u=u_1+u_2$ for $u\in C^\infty_0(R^{n+1}\times(0,\infty))$. Any function in $\mathring{L}^p_{2b,1}(R^{n+1}_+\times(0,T))$ being a limit of $C^\infty_0(R^{n+1}\times(0,\infty))$-functions in the $L^p_{2b,1}$ sense, the general result follows by a density argument, recalling the obvious convergence in L^p of sequences like $(g_{\mu l})_\nu$ and Lu_ν, to their corresponding expressions for $u\in L^p_{2b,1}$, and the fact that

$$\|y^{b+1-\gamma}LT_{\mu l}((g_{\mu l})_\nu-g_{\mu l})\|_{L^p(R^n\times(0,T))}\to 0\,.$$

The estimate is an immediate consequence.

§ 22. – Using the representation above, we can prove the uniqueness of the solution to the problem (cf. Theorem, § 20): $u\in L^p_{2b,1}(R^n\times(\delta,\infty)\times\times(0,T))$ for every $\delta>0$,

$$Lu=f\in L^p(R^{n+1}_+\times(0,T))\qquad\text{for } y>0\,,$$

$$\|\Lambda^{b-1+\beta-k}(D^k_y u(\cdot,y,\cdot)-w^k(\cdot,\cdot))\|_{L^p(S_T)}\to 0\qquad\text{as } y\to 0^+$$

for every β, $0\leqslant\beta<\mu<1$, and $k=0,\ldots,b-1$.

Here $\Lambda^{b-1+\mu-k}w^k\in L^p(S_T)$.

THEOREM. If

(i) $u(x,y,t)\in\mathring{L}^p_{2b,1}(R^n\times(\delta,\infty)\times(0,T))$ for every $\delta>0$,

(ii) for $k=0,\ldots,b-1$, and some μ, $0<\mu<1$,

$$\|\Lambda^{b-1+\mu-k}D^k_y u(\cdot,y,\cdot)\|_{L^p(S_T)}\to 0\qquad\text{as } y\to 0^+,$$

and

(iii) $Lu = 0$ for $y > 0$,

Then $u \equiv 0$.

PROOF. We observe that for $\delta > 0$,

$$u(x, y + \delta, t) \in \mathring{L}^p_{2b,1}(R^{n+1}_+ \times (0, T))$$

is a solution of

$$L^\delta g \equiv \sum_{|\alpha| \leqslant 2b} a_\alpha(x, y + \delta, t) D^\alpha_{x,y} g + D_t g = 0 \qquad \text{for } y > 0 .$$

By the estimate of previous theorem, with $\beta = 0$, we have

$$\|u(\cdot, y + \delta, \cdot)\|_{L^p(S_T)} \leqslant C \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k} D^k_y u(\cdot, \delta, \cdot)\|_{L^p(S_T)} .$$

Hence,

$$\|u(\cdot, y, \cdot)\|_{L^p(S_T)} \leqslant \varliminf_{\delta \to 0^+} \|u(\cdot, y + \delta, \cdot)\|_{L^p(S_T)} = 0 \qquad \text{for } y > 0 ,$$

proving the theorem (we recall that the constants in the representation theorem depend only on π and $\max_{|\alpha| = 2b} \sup |a_\alpha|$).

VII. - An elliptic estimate.

§ 23. - Let now

$$\mathcal{E} \equiv \sum_{|\alpha| \leqslant 2b} a_\alpha(x, y) D^\alpha_{x,y}$$

be an operator in R^{n+1}_+, strongly elliptic in the sense that

$$\operatorname{Re}\Big(\sum_{|\alpha| = 2b} a_\alpha(x, y)(i\xi)^\alpha \Big) < -\pi \cdot |\xi^{2b} ,$$

$\pi > 0$ and independent of (x, y), $\xi \in R^{n+1}$, $\xi \neq 0$. We assume each $a_\alpha(x, y)$ to be bounded and measurable, and for $|\alpha| = 2b$, uniformly continuous in $\overline{R^{n+1}_+}$.

For $1 < p < \infty$ we define $\bar{d}_p(y) = \min(y, T^{\gamma_p/2b})$, T being a constant.

For B any real number, we will denote by $G_{-B}(x)$ the Bessel potential, defined by

$$\mathcal{F}(G_{-B}f)(x) = \left(1 + |x|^2\right)^{B/2} \cdot \mathcal{F}(f)(x) .$$

Now set $L \equiv \mathcal{E} - D_t$ and assume $u(x, y) \in L^p_{2b}(R^{n+1}_+)$.
Clearly $t \cdot u(x, y) \in \mathring{L}^p_{2b,1}\left(R^{n+1}_+ \times (0, T)\right)$ and we have (cf. § 21)

$$\begin{aligned}
&\sum_{k=0}^{b-1} \|D^k_y \Lambda^{b-1+\beta-k}(s \cdot u)(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha| \leqslant b-1} \|\Lambda^\beta(s \cdot D^\alpha_{x,y} u)(\cdot, y, \cdot)\|_{L^p(S_T)} + \\
&\qquad + \sum_{b \leqslant |\alpha| \leqslant 2b-1} \left\| \left(d_p(y, \cdot)\right)^{|\alpha|-(b-1)} \Lambda^\beta(s \cdot D^\alpha_{x,y} u)(\cdot, y, \cdot)\right\|_{L^p(S_T)} \\
&\qquad \leqslant C \sum_{k=0}^{b-1} \|D^k_y \Lambda^{b-1+\mu-k}(s \cdot u)(\cdot, 0, \cdot)\|_{L^p(S_T)} + \\
&\qquad + C(T) \left\| \left(d_p(y, t)\right)^{b+1-\gamma} u(x, y)\right\|_{L^p(R^{u+1}_+ \times (0,T))} + \\
&\qquad + C(T) \left\| \left(d_p(y, t)\right)^{b+1-\gamma} tLu(x, y)\right\|_{L^p(R^{np1}_+ \times (0,T))} \quad (*) ,
\end{aligned}$$

where

$$0 \leqslant \beta < \gamma < 1/p < \mu , \qquad d_p(y, t) = \min\left(y, t^{\gamma_p/2b}\right) \ (§ 5) ,$$

and $C(T) \to 0$ as $T \to 0^+$.

THEOREM. If $\mathcal{E}$ is a strongly elliptic operator in R^{n+1}_+, and $(x, y) \in L^p_{2b}(R^{n+1}_+)$, then, with $0 \leqslant \beta < \gamma - 1/p < \mu < 1$,

$$\begin{aligned}
&\sum_{k=0}^{b-1} \|G_{-(b-1+\beta-k)}(D^k_y u)(\cdot, y)\|_{L^p(R^n)} + \sum_{|\alpha| \leqslant b-1} \|G_{-\beta}(D^\alpha_{x,y} u)(\cdot, y\|_{L^p(R^n)} + \\
&\qquad + \sum_{b \leqslant |\alpha| \leqslant 2b-1} \left(\bar{d}_p(y)\right)^{|\alpha|-(b-1)} \|G_{-\beta}(D^{x,y}_\alpha u)(\cdot, y)\|_{L^p(R^n)} \leqslant \\
&\qquad \leqslant C \sum_{k=0}^{b-1} \|G_{-(b-1+\beta-k)}(D^k_y u)(\cdot, 0)\|_{L^p(R^n)} + C \left\| \left(\bar{d}_p(y)\right)^{b+1-\gamma} \cdot \right. \\
&\qquad \left. \cdot \mathcal{E}u(x, y) \right\|_{L^p(R^{n+1}_+ \times (0,T))} + C \|u\|_{L^p(R^{u+1}_+ \times (0,T))} .
\end{aligned}$$

PROOF. Consider first the following inequalities (see [5] for the first two)

$$\text{(i)} \ \frac{1}{C}\left(\bar{d}(y)\right)^{|\alpha|} T^{1/p} \leqslant \left(\int_\tau^T \left(d(y, t)\right)^{|\alpha|p} dt\right)^{1/p} \leqslant C\left(\bar{d}_p(y)\right)^{|\alpha|} T^{1/p}$$

where $\tau = 0, T/2$, and $|\alpha|$ can be replaced with $b+1-\gamma$;

(ii) $\|\Lambda^{-k} f\|_{L^p(S_T)} \leqslant C \cdot \|G_k f\|_{L^p(R^n)}$;

(iii) $\|(d_p(y,t))^{b+1-\gamma} \cdot t \cdot \mathcal{E}u(x,y)\|_{L^p(R_+^{n+1}\times(0,T))} \leqslant CT^{1+1/p} \|(\bar{d}_p(y))^{b+1-\gamma} \cdot$
$\cdot \mathcal{E}u(x,y)\|_{L^p(R_+^{n+1})}$;

(iv) $\|(d_p(y,t))^{b+1-\gamma} u(x,y)\|_{L^p(R_+^{n+1}\times(0,T))} \leqslant CT^{(\gamma_p/2b)(b+1-\gamma)+1/p} \cdot$
$\cdot \|u(x,y)\|_{L^p(R_+^{n+1})}$;

(v) for $0 < B < 2b$, $\|\Lambda^{\beta}(sf(z))\|_{L^p(S_T)} \leqslant C\|G_{-B} f\|_{L^p(R^n)}$.

(i)-(v) show that the right hand side of the parabolic estimate (∗) is majorized by the right hand side of the elliptic one. It is clear that the proof will be completed by the following result, whose proof follows the lines of [5]. Appendix

LEMMA. Let $f(x) \in C_0^\infty(R^n)$. Then for $1 < p < \infty$, $0 \leqslant B \leqslant 2b$,

$$\|G_{-B} f\|_{L^p(R^n)} \leqslant C_T \|\Lambda^B(s\cdot f(z))\|_{L^p(R^n\times(T/2,T))}.$$

REMARK. If u is assumed to have support contained in $\{(x,y)\colon |x|^2 + y^2 \leqslant r^2, y \geqslant 0\}$, the term $\|u\|_{L^p(R_+^{n+1})}$ in the estimate may be replaced by $\|u\|_{L^1(R_+^{n+1})}$, with a change in the constants.

VIII. – The main results in a general domain.

§ 24. – We now consider the Initial-Dirichlet boundary-value problem with initial data zero, for a parabolic equation where the cylinder is defined as the product of a domain in R^{n+1} with the time interval $(0, T)$. P shall denote a point inside that domain, $D^\alpha = D_P^\alpha$ a spatial derivative of order $|\alpha|$, Q a point on the boundary. The differential operator, L, is assumed to satisfy Petrovski's condition and the conditions on the coefficients stated in § 5 with (x, y, t) replaced by (P, t).

From here on, Ω will denote a bounded, smooth domain in R^{n+1}. By this we mean

(i) There exists a finite number of functions f_i having continuous and bounded derivatives up to order $2b+2$, and each mapping the disc $\{(x,0)\colon |x|^2 < r_i^2, r_i > 0\} \subset R^{n+1}$ into $\partial\Omega$ in a $1-1$ manner, such that every point $Q \in \partial\Omega$ can be written $Q = f_i(x,0)$, $|x|^2 < r_i^2$, for some i;

(ii) If N_Q denotes the unit inner normal to $\partial\Omega$ at Q, and $\Omega_\delta = \{P \in \Omega\colon \operatorname{dist}(P, \partial\Omega) > \delta, \delta \geqslant 0\}$, then there is a number $\delta_0 > 0$ such that for each i

the function

$$(x, y) \to f_i(x, y) = Q + y \cdot N_Q, \qquad Q = f_i(x, 0) \in \partial\Omega$$

maps the set $\{(x, y): |x|^2 < r_i^2,\ 0 \leqslant y \leqslant 4\delta_0\}$ into $\bar{\Omega} - \Omega_{4\delta_0}$ in a $1-1$ manner. We assume further that every point $P \in \bar{\Omega} - \Omega_{4\delta_0}$ can be uniquely written as $P = Q + y \cdot N_Q$, $Q \in \partial\Omega$.

We consider the finite covering of the set $\bar{\Omega} - \Omega_{\delta_0}$ by the sets U_i, image of

$$\{(x, y): |x|^2 < r_i^2,\ 2\delta_0 > y > \varkappa,\ \varkappa \text{ a suitable number } < 0\}$$

under $f_i(x, y) = f_i(x, 0) + y \cdot N_{f_i(x,0)}$. We let $\{\varphi_i\}$ denote a fixed C_0^∞ partition of unity subordinate to $\{U_i\}$, and we denote by $\{\zeta_i\}$ a family of functions such that $\zeta_i \in C_0^\infty(U_i)$ and $\zeta_i \equiv 1$ in a neighborhood of the support of φ_i. We point out here that when $0 < \delta < \delta_0$, for the domain Ω_δ we can associate the sets $U_{\delta i}$, and the families $\{\varphi_{\delta i}\}$, $\{\zeta_{\delta i}\}$ obtained from the ones above by the transformation $P \to P + \delta N_Q$, where $P = Q + yN_Q \in U_i$. It follows that the derivatives of $\varphi_{\delta i}$, $\zeta_{\delta i}$ can be bounded uniformly on δ, $0 < \delta < \delta_0$.

If $u(P)$ is a function defined in $\bar{\Omega}$ we will set, for simplicity, $u_i^\sim(x, y) = u \circ f_i(x, y)$. For the functions φ_i, ζ_i, we will set, e.g., $(\varphi_i) = \varphi_i^\sim$ when there is no confusion. We will also write $u_i^\sim(x, y, t)$ for $u(f_i(x, y), t)$. We observe that $(D_{N_Q} u_i^\sim)(x, y) = D_y(u_i^\sim)(x, y)$.

The L^p-norm of a function $u(Q, t)$ defined on $\partial\Omega \times (0, T)$ is equivalent to $\sum_i \|\varphi_i u\|$, or $\sum_i \|\zeta_i u\|$, which are computed as integrals over $R^n \times (0, T) = S_T$.

We introduce now the operators Λ on $\partial\Omega$.

Definition. If $u(Q, t) \in L^p(\partial\Omega \times (0, T))$, $1 < p < \infty$, and $0 \leqslant \beta \leqslant 2b$,

$$(\Lambda^{-\beta} u)(Q, t) = \sum_i \zeta_i(Q) \Lambda^{-\beta}[\varphi_i^\sim \cdot u_i^\sim](fi^{-1}(Q), t),$$

where $\Lambda^{-\beta}[\cdot]$ is the operator already defined on S_T.

Definition. For functions $u(Q, t) \in C^\infty(\partial\Omega \times (0, T))$ which are identically zero for t near zero,

$$(\Lambda^{\beta} u)(Q, t) = \sum_i \zeta_i(Q) \Lambda^{\beta}[\varphi_i^\sim \cdot u_i^\sim](fi^{-1}(Q), t).$$

It can be shown that $\Lambda^\beta \Lambda^{-\beta}$ is not in general the identity on C^∞ functions that vanish near $t = 0$, but it is extendible to an invertible operator on $L^p(\partial\Omega \times (0, T))$, $1 < p < \infty$.

The Bessel potentials are defined in a similar way.

DEFINITION. If $u(Q) \in C^\infty(\partial\Omega)$,

$$(G_{-\beta}u)(Q) = \sum_i \zeta_i(Q)\, G_{-\beta}[\varphi_i^{\sim}\cdot u_i^{\sim}](f_i^{-1}(Q), t)\,,$$

where $\mathcal{F}_x(G_{-\beta}[\varphi_i^{\sim}\cdot u_i^{\sim}]) = (|x|^2+1)^{b/2}\cdot \mathcal{F}_x[\varphi_i^{\sim}\cdot u_i^{\sim}]$, $x \in R^n$.

In the next paragraphs we follow the method used in [5], article (4.1), for the corresponding problem with initial data zero.

§ 25. – THEOREM. If $u \in \mathring{L}^p_{2b,1}(\Omega\times(0,T))$, $1<p<\infty$, $0\leqslant\beta<\mu<1$, and $Lu=0$ in $\Omega\times(0,T)$, then

$$\sup_{y>\delta_0}\Big\{\sum_{|\alpha|\leqslant 2b-1} y^{[|\alpha|-(b-1)]}\|D^\alpha(\Lambda^\beta u)(Q+yN_Q,t)\|_{L^p(\partial\Omega\times(0,T))} + \sum_{k=0}^{b-1}\|\Lambda^{b-1+\beta-k}[D^k_{N_Q}u](Q+yN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}\Big\} \leqslant C\cdot\sum_{k=0}^{b-1}\|\Lambda^{b-1+\mu-k}[D^k_{N_Q}u](Q,t)\|_{L^p(\partial\Omega\times(0,T))}\,.$$

(Here and in the following, the expression $y^{[|\alpha|-(b-1)]}$ is meant to be replaced by 1 for $|\alpha|\leqslant b-1$).

PROOF. We shall sketch the proof of this result, which proceeds first for small T (this §), and then in the general case (next §).

By applying the definition of Λ^β (§ 24), dropping continuous functions of compact support, and introducing a new constant C_{δ_0}, we can see that the left hand side $(LHS^\beta_T(u;\Omega))$ of the estimate above is less than or equal to

$$\|u\|^{\sim} \equiv C_{\delta_0}\sum_i \sup_{y<2\delta_0}\Big\{\sum_{|\alpha|\leqslant 2b-1} y^{[|\alpha|-(b-1)]}\|D^\alpha_{xy}\Lambda^\beta(\varphi_i^{\sim}\cdot u_i^{\sim})(\cdot,y,\cdot)\|_{L^p(S_T)} + \sum_{k=0}^{b-1}\|\Lambda^{b-1+\beta-k}D^k_y(\varphi_i^{\sim}\cdot u_i^{\sim})(\cdot,y,\cdot)\|_{L^p(S_T)} + \sum_{k=1}^{b-1}\sum_{m=1}^{k}\|\Lambda^{b-1+\beta-k}(D^m_y\varphi_i^{\sim}\cdot D^{k-m}_y u_i^{\sim})(\cdot,y,\cdot)\|_{L^p(S_T)}\Big\}\,.$$

We now

(i) Define a parabolic operator $L_i^{\sim}$ on $R^{n+1}_+\times(0,T)$, with coefficients bounded and measurable, and those of the leading terms, uniformly continuous in $\overline{R^{n+1}_+}\times[0,T]$, that satisfies

$$L_i^{\sim}(u_i^{\sim})(x,y,t) = L(u)(f_i(x,y),t) \qquad \text{for } f_i(x,y)\in U_i\,;$$

(ii) Apply the estimates for $L_i^{\sim}$ (§ 20) to the first two terms above;

(iii) Omit somewhat lengthy considerations to obtain, with $0<\delta<\delta_0$ and $C(T)\to 0$ as $T\to 0^+$

$$\begin{aligned} LHS_T^\beta(u;\Omega) &\leqslant \|u\|^{\sim} \leqslant C\sum_{k=0}^{b-1}\|\Lambda^{b-1+\mu-k}(D^k_{N_Q}u)(Q,t)\|_{L^p(\partial\Omega\times(0,T))}+\\ &+C_{\delta,\delta_0}C(T)\sup_{\delta<y<2\delta_0}\sum_{|\alpha|\leqslant 2b-1}\|(D^\alpha u)(Q+yN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}+\\ &+C_{\delta_0}C(T)\delta^{1-\gamma+1/p}\sup_{y<\delta}\sum_{|\alpha|=\leqslant 2b-1}y^{[|\alpha|-(b-1)]}\|(D^\alpha u)(Q+yN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}+\\ &+C_{\delta_0}\sum_i\sup_{y<2\delta_0}\sum_{k\geqslant m\geqslant 0}\|\Lambda^{-1+\mu-\beta}\Lambda^{b-1+\beta-(k-m)}(D_y^m\varphi_i^{\sim}D_y^{k-m}u_i^{\sim})(\cdot,y,\cdot)\|_{L^p(S_T)}; \end{aligned}$$

(iv) Take δ small enough so we can move the third term in (iii) over to the left hand sides. We modify the constants (introducing C_δ in front of $\|u\|^{\sim}$) and fix this δ from now on. And finally

(v) Claim. There exists a $T_0>0$ such that the estimate in the Theorem holds for $T\leqslant T_0$.

Clearly, the second term in (iii) (modified as in (iv)) can be bounded with

$$C_{\delta,\delta_0}C(T)LHS_T^\beta(u;\Omega)\leqslant C_\delta C_{\delta,\delta_0}C(T)\cdot\|u\|^{\sim}.$$

Also, the fourth term in (iii) can be shown to be bounded by

$$C_{\beta,\mu}(T)C_{\delta,\delta_0}C_\delta\|u\|^{\sim},\qquad \text{with } C_{\beta,\mu}(T)\to 0 \text{ as } T\to 0^+.$$

Therefore, if T is selected so that

$$C_{\delta_0}C_{\beta,\mu}(T)+C_{\delta,\delta_0}C(T)<1,$$

then

$$(\#)\qquad LHS_T^\beta(u;\Omega)\leqslant C_{\delta,\delta_0,\beta,\mu}\sum_{k=1}^{b-1}\|\Lambda^{b-1+\mu-k}(D^k_{N_Q}u)(Q,t)\|_{L^p(\partial\Omega\times(0,T))},$$

i.e. there exists a $T_0>0$, depending on δ, δ_0, β, μ such that the theorem is true for $\partial\Omega\times(0,T)$, $T\leqslant T_0$.

Remark. For $T\leqslant T_0$, (#) is also true for all the domains $\Omega_{\delta_0^1}$, $0\leqslant\delta_0^1<\delta_0$, with the same constant $C_{\delta,\delta_0,\beta,\mu}$ (which, we recall, depends on the families $\{U_i\}$, $\{\varphi_i\}$, $\{\zeta_i\}$, cf. definitions in § 24).

§ 26. – To prove the theorem for any T, we rewrite the estimates in (iii) with third term deleted, see (iv), as

$$LHS_T^\beta(u;\Omega)\leqslant C_{\delta,\delta_0}\left\{\sum_{k=0}^{b-1}\|\Lambda^{b-1+\mu-k}(D_{N_Q}^k u)\|_{L^p(\partial\Omega\times(0,T))}+\|u\|^{\approx}\right\},$$

where we have set

$$\|u\|^{\approx}\equiv\sup_{\delta<y<2\delta_0}\sum_{|\alpha|\leqslant 2b-1}\|(D^\alpha u)(Q+yN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}+$$
$$+\sum_i\sup_{y<2\delta_0}\sum_{k\geqslant m\geqslant 0}^{b-1}\|\Lambda^{-1+\mu-\beta}\Lambda^{b-1+\beta-(k-m)}(D_y^m\varphi_{\tilde{i}}D_y^{k-m}u_{\tilde{i}})(\cdot,y,\cdot)\|_{L^p(S_T)},$$

and observe that the desired result is a consequence of the following lemma, whose proof is also omitted

LEMMA – Given $\varepsilon>0$, there exists a constant C_ε such that for all $u\in\mathring{L}^p_{2b,1}(\Omega\times(0,T))$ that satisfy $Lu=0$ we have

$$\|u\|^{\approx}\leqslant\varepsilon LHS_T^\beta(u;\Omega)+C_\varepsilon\sum_{k=0}^{b-1}\|\Lambda^{b-1+\mu-k}(D_{N_Q}^k u)\|_{L^p(\partial\Omega\times(0,T))}.$$

REMARK. The a priori estimate in Theorem, § 25, holds for all domains $\Omega_{\delta_0^1}$, with $\delta_0^1\ll\delta_0$. The constant in the theorem is independent of δ_0^1 (cf. also Remark to § 25).

§ 27. – We are now in a position to prove the main results.

THEOREM 1. Let $w^k\in L^p(\partial\Omega\times(0,T))$, $k=0,\dots,b-1$, be such that $\Lambda^{b-1+\mu-k}w^k\in L^p(\partial\Omega\times(0,T))$ for some μ, $0<\mu<1$. Then there exists a solution to the problem

(i) $u(P,t)\in\mathring{L}^p_{2b,1}(\Omega^*\times(0,T))$ for every subdomain Ω^* such that $\bar{\Omega}^*\subset\Omega$,

(ii) $Lu=0$ in $\Omega\times(0,T)$, and

(iii) for every β, $0\leqslant\beta<\mu$,

$$\lim_{y\to 0^+}\|\Lambda^{b-1+\beta-k}[D_{N_Q}^k u(Q+yN_Q,t)-w^k(Q,t)]\|_{L^p(\partial\Omega\times(0,T))}=0.$$

PROOF. Let $w_j^k\in C_0^\infty(\partial\Omega\times(0,\infty))$, such that $w_j^k\to w^k$ and $\Lambda^{b-1+\mu-k}w_j^k\to$ $\to\Lambda^{b-1+\mu-k}w^k$, in $L^p(\partial\Omega\times(0,T))$ as $j\to\infty$, $0\leqslant 1\leqslant b-1$. It is known [10] that we can find $u_j(P,t)\in\mathring{L}^p_{2b,1}(\Omega\times(0,T))$ satisfying (ii) and (iii) with w^k replaced by w^k.

Now if $\theta(P) \in C_0^\infty(\Omega)$, $\theta \equiv 1$ in Ω_δ, then

$$\|\theta u_j\|_{L^p_{2b,1}(\Omega\times(0,T))} \leqslant C_\delta \sum_{\substack{|\beta|>0\\|\beta|+|\alpha|\leqslant 2b}} \|(D^\beta\theta)(D^\alpha u_j)\|_{L^p(\Omega\times(0,T))}\,.$$

This, together with the theorem, § 25, imply that if $\bar{\Omega}^* \subset \Omega$,

$$\|u_j - u_l\|_{L^p_{2b,1}(\Omega^*\times(0,T))} \leqslant C_{\Omega^*} \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k}(w_j^k - w_l^k)\|_{L^p(\partial\Omega\times(0,T))}\,.$$

Hence $\{u_i\}$ is a Cauchy sequence in $L^p_{2b,1}(\Omega^*\times(0,T))$, for every subdomain Ω^* with $\bar{\Omega}^* \subset \Omega$. Let $u(P,t)$ denote the limit of $\{u_j\}$. Clearly $u(P,t)$ satisfies (i) and (ii).

To prove (iii) we observe that

$$\sup_{y<\delta_0} \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k} D^k_{N_Q}(u_j - u_l)(Q + yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} \leqslant$$
$$\leqslant C \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k}[w_j^k - w_l^k]\|_{L^p(\partial\Omega\times(0,T))}\,,$$

which implies

$$\sup_{y<\delta_0} \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k} D^k_{N_Q}(u_j - u)(Q + yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} \leqslant$$
$$\leqslant C\cdot \lim_{l-\infty} \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k}[w_j^k - w_l^k]\|_{L^p(\partial\Omega\times(0,T))} \to 0 \quad \text{as } j\to\infty\,.$$

Now we observe that

$$\Lambda^{b-1+\beta-k} D^k_{N_Q} u = \Lambda^{b-1+\beta-k} D^k_{N_Q}(u - u_j) + \Lambda^{b-1+\beta-k}[D^k_{N_Q} u_j]\,,$$

and that

$$\lim_{y\to 0^+} \|\Lambda^{b-1+\beta-k}[D^k_{N_Q}(u - u_j)]\|_{L^p(\partial\Omega\times(\ ,T))} \leqslant \sup \|\Lambda^{b-1+\beta-k} D^k_{N_Q}(u - u_j)\|_{L^p(\partial\Omega\times(0,T))}\,,$$

and we see that

$$\lim_{y\to 0^+} \Lambda^{b-1+\beta-k}[D^k_{N_Q} u] = \lim_{j\to\infty} \Lambda^{b-1+\beta-k} w_j^k = \Lambda^{b-1+\beta-k} w^k \quad \text{in } L^p(\partial\Omega\times(0,T))\,.$$

THEOREM 2. Let $u(P,t)$ be any function such that

(i) $u(P,t)$ belongs in $\mathring{L}^p_{2b,1}(\Omega^*\times(0,T))$ for every subdomain Ω^* such that $\bar{\Omega}^* \subset \Omega$,

(ii) $Lu \equiv 0$ in $\Omega \times (0, T)$,

(iii) for $k = 0, \dots, b-1$, and for some μ, $0 < \mu < 1$,

$$\varliminf_{y\to 0^+} \|\Lambda^{b-1+\mu-k}[D^k_{N_Q} u](Q + yN_Q, t)\|_{L^p(\partial\Omega\times(0,\chi T))} < \infty .$$

Then, for $0 \leqslant \beta < \mu$,

$$\sup_{y<\delta_0}\Big\{\sum_{|\alpha|\leqslant 2b-1} y^{[|\alpha|-(b-1)]}\|D^\alpha(\Lambda^\beta u)(Q + yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} + \\ + \sum_{k=0}^{b-1}\|\Lambda^{b-1+\beta-k}[D^k_{N_Q}u](Q+yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))}\Big\} \leqslant \\ \leqslant C\cdot\sum_{k=0}^{b-1}\varliminf_{y\to 0^+}\|\Lambda^{b-1+\mu-k}[D^k_{N_Q}u](Q+yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} .$$

COROLLARY. The solution to problem (i)-(iii) in Theorem 1 is unique.

PROOF OF THEOREM 2. We recall that $(\cdot)^{[|\alpha|-(b-1)]} \equiv 1$ for $|\alpha| \leqslant b-1$ (§ 25). We consider any fixed y, $0 < y < \delta_0$, we take $0 < \delta_0^1 \ll y$, and study $\Omega_{\delta_0^1}\times(0, T)$; clearly $u \in \mathring{L}^p_{2b,1}(\Omega_{\delta_0^1}\times(0, T))$ and $Lu = 0$. By the a priori estimate in Theorem, § 25 (cf. also the Remark to § 26), and with $0 \leqslant \beta < \mu$,

$$\sum_{|\alpha|\leqslant 2b-1}(y-\delta_0^1)^{[|\alpha|-(b-1)]}\|D^\alpha(\Lambda^\beta u)(Q+yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} + \\ + \sum_{k=0}^{b-1}\|\Lambda^{b-1+\beta-k}[D^k_{N_Q}u](Q+yN_Q, t)\|_{L^p(\partial\Omega\times(0,T))} \leqslant \\ \leqslant C_{\beta,\mu,\delta_0}\cdot\sum_{k=0}^{b-1}\|\Lambda^{b-1+\mu-k}[D^k_{N_Q}u](Q+\delta_0^1 N_Q, t)\|_{L^p(\partial\Omega\times(0,T))} .$$

The theorem follows by taking $\varliminf$ on both sides and recalling that C_{β,μ,δ_0} does not depend on δ_0^1 (cf. loc. cit.).

IX. – The elliptic estimate.

§ 28. – As in § 23, we let $\mathcal{E} \equiv \sum_{|\alpha|\leqslant 2b} a_\alpha(P)D^\alpha$ be a strongly elliptic operator on Ω, that is, $\mathcal{E} - D_t$ is parabolic in the sense of Petrovski. Again $\pi > 0$ will denote the parameter of parabolicity of $\mathcal{E} - D_t$ and the a-'s are assumed to be bounded and measurable, and for $|\alpha| = 2b$, uniformly continuous in $\bar{\Omega}$. $G_{-\beta}$ shall denote the Bessel potentials defined in § 24.

As done in §§ 25-27, the expression $(\cdot)^{[|\alpha|-(b-1)]}$ is to be replaced by 1 for $|\alpha| \leqslant b-1$.

THEOREM. If $u \in L^p_{2b}(\Omega)$ and $\mathcal{E}u = 0$, then for $0 < \beta < \mu$,

$$\sup_{y<\delta_0}\Big\{\sum_{|\alpha|\leqslant 2b-1} y^{[|\alpha|-(b-1)]}\|D_\alpha[G_{-\beta}u](Q+yN_Q)\|_{L^p(\partial\Omega)} + \\ + \sum_{k=0}^{b-1}\|G_{-(b-1+\beta-k)}[D^k_{N_Q}u](Q+yN_Q)\|_{L^p(\partial\Omega)}\Big\} \\ \leqslant C\Big\{\sum_{k=0}^{b-1}\|G_{-(b-1+\mu-k)}[D^k_{N_Q}u](Q)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(\Omega)}\Big\}.$$

The proof of this result follows lines analogous to those in the parabolic estimate of § 25. We shall only sketch them.

By application of the definitions of the Bessel potentials to the left hand side $(LHS^\beta(u))$ of the inequality, a bound is obtained to whose terms the estimates of § 23 and Remark, § 23 apply with elliptic operators $\mathcal{E}_i^\sim$ defined by

$$\mathcal{E}_i^\sim(u_i^\sim)(x,y) = \mathcal{E}u(f_i(x,y)) \qquad \text{for } f_i(x,y) \in U_i.$$

Support considerations on the C_0^∞ functions in the definition of $G_{-\beta}$ (§ 24) lead to the estimate

$$LHS^\beta(u) \leqslant C\Big(\sum_{k=0}^{b-1}\|G_{-(b-1+\mu-k)}(D^k_{N_Q}u)(Q)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(R^{n+1}_+)}\Big) + \\ + \|u\|^{\approx} + C_{\delta_0}\delta^{1-\gamma+1/p}\sup_{y<\delta}\sum_{|\alpha|\leqslant 2b-1} y^{[|\alpha|-(b-1)]}\|D^\alpha u(Q+yN_Q)\|_{L^p(\partial\Omega)} \\ + C_{\delta,\delta_0}\sup_{\delta<\gamma<2\delta_0}\sum_{|\alpha|\leqslant 2b-1}\|D^\alpha u(Q+yN_Q)\|_{L^p(\partial\Omega)},$$

where we have set

$$\|u\|^{\approx} \equiv C_{\delta_0}\sum_i \sup_{y<2\delta_0}\sum_{k\geqslant m\geqslant 0}^{b-1}\|G_{1-\mu+\beta}G_{-(b-1+\beta-(k-m))}(D^m_y\varphi_i^\sim D^{k-m}_y u_i^\sim)(y)\|_{L^p(R^n)}.$$

Fixing δ small enough, the third term in the estimate above can be moved over to the left hand side. The last term can be shown to be $\leqslant \varepsilon\cdot C\|\mathcal{E}(\varphi u)\|_{L^p(\Omega)} + C_\varepsilon\|u\|_{L^1(\Omega)}$, where $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ on $\Omega' \supset\supset \Omega_\delta$, by using a trace theorem [11], and an estimate in [1], already needed for Remark, § 23.

But $\|\mathcal{E}(\varphi u)\|_{L^p(\Omega)} \leqslant C_{\delta\varphi}LHS^\beta(u)$, so choosing ε small enough we eventually get

$$LHS^{\beta}(u) \leqslant C\left\{\sum_{k=0}^{b-1} \|G_{-(b-1+\mu-k)}(D^k_{N_Q} u)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(\Omega)} + \|u\|^{\approx}\right\}.$$

The proof can be completed by proving the following

LEMMA. To every $\varepsilon' > 0$ there is a constant $C_{\varepsilon'}$ such that every $u \in L^p_{2b}(\Omega)$ with $\mathcal{E}u = 0$ satisfies

$$\|u\|^{\approx} \leqslant \varepsilon' \, LHS^{\beta}(u) + C_{\varepsilon'} \|u\|_{L^1(\Omega)}.$$

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