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Existence of Strong Solutions for a Class of Nonlinear Partial Differential Equations Satisfying Nonlinear Boundary Conditions.

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dedicated to Hans Lewy

1. – Introduction.

We denote by Ω an open bounded set in the N-dimensional euclidean space $\boldsymbol{R}^N$ and by Γ the boundary of Ω. We assume that Γ is a C^2 manifold and that locally Ω is only on one side of Γ (see § 5).

Let β be a maximal monotone graph (see § 2) on $\boldsymbol{R}\times\boldsymbol{R}$ verifying $0\in\beta(0)$, let $f\in L^2(\Omega)$ and let $\mu>0$ be a fixed constant. Our aim is to study the following boundary value problem: We seek *strong solutions of*

$$(1.1)\qquad \begin{cases} -\Delta u(x)+g\big(x,u(x)\big)+\mu u(x)=f(x)\,, & \text{a.e. in } \Omega\,,\\ -\dfrac{\partial u}{\partial n}(\xi)\in\beta\big(u(\xi)\big)\,, & \text{a.e. on } \Gamma\,,\end{cases}$$

where $g(x,y)$ is a real function defined on a subset Λ of $\Omega\times\boldsymbol{R}$, monotone in y for any $x\in\Omega$ (¹). For the meaning of boundary conditions in the form used in (1.1) refer to [6], I.2.1; we recall that this formulation includes as particular cases the Dirichlet and Neumann conditions and the third boundary value problem.

Similar problems have been studied by several authors. For the existence of strong solutions when g depends only on y we refer the reader to [7]; see also [11]. If g depends also on x the existence of weak solutions for some related problems is known (see [9, 10]). Other interesting results related with our problem were obtained in [1, 2] and references.

In this paper we prove the existence of strong solutions in the last case. More precisely, under suitable conditions on $g(x,y)$ (²) we prove that there exists a unique strong solution $u(x)\in W^{2,2}(\Omega)$ of the problem (1.1). To prove

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(¹) See condition (4.4).

(²) See theorem 7.2; see also [4].

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this result we assume that $g(x, y)$ is differentiable on x_i, $1 \leqslant i \leqslant N$; however this condition is used in a very weak form (see 4.10) which is usefull in applications.

The results obtained in our paper, in particular the theorem 7.2 and the corollary 7.3, are usefull for the study of the bifurcation points μ for equation (1.1); see [5].

Finally we refer the reader to the remark 7.5 (added in proofs) at the end of the § 7.

We shall use the notation c to denote a constant which may change from case to case.

2. – Some definitions and known results on maximal monotone operators on Hilbert spaces.

For the reader's convenience some known results on maximal monotone operators are summarized in this section. Let H be a real Hilbert space and $A: H \to 2^H$ a multivalued operator on H; to any $u \in H$ the operator A associates the set $Au \subset H$. We put $D(A) = \{u \in H: Au \neq \emptyset\}$ and $R(A) = \bigcup_{u \in H} Au$. A is said to be monotone if for any $u_1, u_2 \in D(A)$ we have $(v_1 - v_2, u_1 - u_2) \geqslant 0$, $\forall v_1 \in Au_1$, $\forall v_2 \in Au_2$. A monotone operator is maximal monotone (m.m.) if it is maximal in the sense of graph's inclusion. The following result holds (see for instance [16]):

THEOREM 2.1. *Let A be monotone. Then A is m.m. if and only if $R(A + \lambda I) = H$ for all $\lambda > 0$ (or equivalently for one $\lambda > 0$).*

Let now $\Phi: H \to]-\infty, +\infty]$, $\Phi \not\equiv +\infty$, be a convex lower semicontinuous (l.s.c.) functional and put $D(\Phi) = \{u \in H: \Phi(u) < +\infty\}$. The subdifferential $\partial\Phi(u_0)$ of Φ at a point $u_0 \in D(\Phi)$ is by definition the set

$$\partial\Phi(u_0) = \{v \in H: \Phi(u) - \Phi(u_0) \geqslant (v, u - u_0), \forall u \in H\} .$$

We have the following result (see [17]):

THEOREM 2.2. *The operator $\partial\Phi$ is m.m. on H.*

If A is m.m. the resolvent of A is the operator $A^{(\lambda)} = (I + \lambda A)^{-1}$, $\lambda > 0$. The resolvent is a contraction defined on all of H. The Yosida approximation is by definition

$$A_\lambda u = \frac{u - A^{(\lambda)} u}{\lambda}, \qquad \lambda > 0 . \tag{2.1}$$

The Yosida approximation is a univalued operator defined on all of H; moreover A_λ is m.m. and Lipschitz continuous with constant $1/\lambda$.

The following theorem is a particular case of a result of H. Brezis, M. Crandall and A. Pazy (see [8]):

THEOREM 2.3. *Let A and B be univalued m.m. operators on H. Let $f \in H$ and $\mu > 0$. Then for any $\lambda > 0$ the equation*

$$Bu_\lambda + A_\lambda u_\lambda + \mu u_\lambda = f \tag{2.2}$$

has a unique solution u_λ. Moreover the equation

$$Bu + Au + \mu u = f \tag{2.3}$$

has a solution u if and only if $A_\lambda u_\lambda$ is bounded as $\lambda \to 0$. In this case $u_\lambda \to u$, $A_\lambda u_\lambda \to Au$ and $Bu_\lambda \to Bu$.

From the theorems 2.1 and 2.3 we obtain a necessary and sufficient condition for the maximal monotony of the sum $A + B$.

Finally, notice that if A is a univalued m.m. operator then

$$A_\lambda u = AA^{(\lambda)} u\,, \qquad \forall u \in H\,. \tag{2.4}$$

3. – The B operator.

We assume that the $L^p(\Omega)$ spaces ($p \in [1, +\infty]$) and the Sobolev spaces $W^{k,p}(\Omega)$ (k positive integer and $p \in [1, +\infty[$) are familiar to the reader (³); we denote by $\|\ \|_p$ and $\|\ \|_{k,p}$ the usual norms in these spaces and we put $H = L^2(\Omega)$ and $\|\ \| = \|\ \|_2$. We consider also the spaces $L^p(\Gamma)$ and the fractionary Sobolev spaces $W^{1-(1/p),p}(\Gamma)$ (³) with the usual norms. If $u(x)$ is a function defined on Ω we denote by $u(\xi)$ (or by u only) the trace of $u(x)$ on Γ.

Consider now a m.m. graph β on $\boldsymbol{R}\times\boldsymbol{R}$, so that $0 \in \beta(0)$, and define an operator B on H as follows:

$$B = -\Delta \tag{3.1}$$

with

$$D(B) = \left\{u \in W^{2,2}(\Omega)\colon -\frac{\partial u}{\partial n}(\xi) \in \beta\big(u(\xi)\big) \text{ a.e. on } \Gamma\right\}.$$

(³) See for instance [18].

The B operator is m.m. on H and furthermore, fixed $\mu > 0$, we have

$$\|u\|_{2,2} \leqslant c\|-\Delta u + \mu u\|\,, \qquad \forall u \in D(B)\,, \tag{3.2}$$

as follows from theorem I. 10 of [6]. Note that B is the subdifferential of the convex l.s.c. functional $\Phi: H \to]-\infty, +\infty]$ defined by

$$\Phi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Gamma j(u)\, d\Gamma\,, & \text{if } u \in W^{1,2}(\Omega) \text{ and } j(u) \in L^1(\Gamma)\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{3.3}$$

where $j: \boldsymbol{R} \to]-\infty, +\infty]$ is convex l.s.c., $\beta = \partial j$ and $j(0) = 0$ (see [7], theorem 12).

REMARK 3.1. For some particular boundary conditions the maximal monotony of B and the estimate (3.2) follows from classical results. This is the case when the boundary condition is linear and also when the boundary condition is the well known condition $u \geqslant 0$, $\partial u/\partial n \geqslant 0$, $u(\partial u/\partial n) = 0$ on Γ (this boundary condition corresponds in (1.1) to the m.m. graph $\beta(r) = 0$ if $r > 0$, $\beta(0) = \{t: t \leqslant 0\}$, $\beta(r) = \emptyset$ if $r < 0$). In the last case (see [4]) the maximal monotony of B follows from the $W^{1,2}$ existence results proved in [20], [15], from the $W^{2,2}$ regularity result proved in [14] and from theorem 2.1.

REMARK 3.2. The condition $\mu > 0$ in equation (1.1) can be weakened in some cases. Consider for instance the operator $B = -\Delta - \lambda_0$ with $D(B) = \{u \in W^{2,2}(\Omega): u(\xi) = 0 \text{ a.e. on } \Gamma\}$ (4). On the other hand let c_0 be the smallest constant verifying

$$\int_\Omega u^2\, dx \leqslant c_0 \int_\Omega |\nabla u|^2\, dx\,.$$

Then if we put $\lambda_0 = 1/c_0$ the operator B is m.m. in H and, fixed $\mu > 0$, the estimate $\|u\|_{2,2} \leqslant c\|-\Delta u + (\mu - \lambda_0)u\|$ holds on $D(B)$. Consequently for the Dirichlet problem we can put $-\Delta - \lambda_0$ instead of $-\Delta$ in theorem 7.2; analogous remarks hold for other boundary value problems.

(4) The Dirichlet condition $u = 0$ on Γ corresponds to the graph $\beta(0) = \boldsymbol{R}$, $\beta(r) = \emptyset$ if $r \neq 0$.

4. – The $\bar{g}$ operator.

Let $\psi: \Omega \times \boldsymbol{R} \to]-\infty, +\infty]$ be a convex, l.s.c. functional in the second variable y for almost all $x \in \Omega$ and measurable in the first variable x for all $y \in \boldsymbol{R}$. Moreover suppose for convenience that for almost all $x \in \Omega$ we have $\psi(x, 0) = 0$ and also $\psi(x, y) \geqslant 0$, $\forall y \in \boldsymbol{R}$.

Then for almost all $x \in \Omega$ the functional $y \to \psi(x, y)$ admits a subdifferential which will be denoted by $\partial\psi(x, y)$.

Now define a functional $\Psi: H \to [0, +\infty]$ in the following way:

$$\Psi(u) = \int_\Omega \psi\big(x, u(x)\big)\, dx \; ; \tag{4.1}$$

This functional is convex and l.s.c. (see appendix I, proposition 2) and his subdifferential at a point $u \in D(\Psi)$ is (see appendix I, corollary 2)

$$\partial\Psi(u) = \{f \in H: f(x) \in \partial\psi\big(x, u(x)\big) \text{ a.e. in } \Omega\} \, . \tag{4.2}$$

In the following we put $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $x = (x^i, x_i)$, $1 \leqslant i \leqslant N$. Furthermore the expression «for almost all x^i» means «for all $x^i \in \Omega_0^i$» where Ω_0^i is a subset of Ω^i such that $\Omega^i - \Omega_0^i$ is a set of $(N-1)$-dimensional measure zero and Ω^i is the orthogonal projection of Ω into the hyperplane $\{x: x_i = 0\}$.

We consider now two measurable functions $a(x)$ and $b(x)$ defined on Ω with range in $[-\infty, +\infty]$. We suppose that for almost all x^i the function $x_i \to b(x)$ is l.s.c. and the function $x_i \to a(x)$ is upper semicontinuous ($1 \leqslant i \leqslant N$). We suppose also that $a(x) < 0 < b(x)$ for all $x \in \Omega$.

Let now $g(x, y)$ be a real function defined on

$$\Lambda = \{(x, y) \in \Omega \times \boldsymbol{R}: a(x) < y < b(x)\}$$

and assume that $g(x, y)$ verifies the following conditions:

$$g(x, 0) = 0 \, , \qquad \forall x \in \Omega \; ; \tag{4.3}$$

(4.4) For any fixed $x \in \Omega$ the function $y \to g(x, y)$, defined on $]a(x), b(x)[$, is continuous and nondecreasing. If $-\infty < a(x)$ then $\lim\limits_{y \to a(x)} g(x, y) = -\infty$; if $b(x) < +\infty$ then $\lim\limits_{y \to b(x)} g(x, y) = +\infty$ [5].

[5] Putting $g(x, y) = \emptyset$ if $y \notin]a(x), b(x)[$ the hypothesis (4.4) becomes equivalent to the maximal monotonicity of the graph $y \to g(x, y)$ in $\boldsymbol{R} \times \boldsymbol{R}$ (for each $x \in \Omega$).

(4.5) For almost all x^i, $1 \leqslant i \leqslant N$, and for all $y \in \mathbf{R}$, $g(x, y)$ is a continuous function of the single coordinate x_i (6).

This last condition implies that for each $y \in \mathbf{R}$ the function $x \to g(x, y)$ is measurable in its domain $\{x \in \Omega : a(x) < y < b(x)\}$ (for, apply proposition 1 of appendix II with $B = \{x \in \Omega : a(x) < y < b(x)\}$ and $g(x) = g(x, y)$, $\forall x \in B$).

Consider now the function $\psi : \Omega \times \mathbf{R} \to]-\infty, +\infty]$ defined as follows:

$$\psi(x, y) = \begin{cases} \int_0^y g(x, \eta)\, d\eta & \text{if } y \in [a(x), b(x)] , \\ +\infty & \text{otherwise} . \end{cases} \tag{4.6}$$

The integral on the right hand side of (4.6) may be equal to $+\infty$; furthermore $\psi(x, y) \geqslant 0$. It is easy to see that $\psi(x, y)$ satisfies all the conditions listed at the beginning of this section (7). Furthermore for all $x \in \Omega$

$$\partial\psi(x, y) = g(x, y) \tag{4.7}$$

being implicit in (4.7) that $\partial\psi(x, y) = \emptyset$ if $y \notin]a(x), b(x)[$. Therefore the functional Ψ defined by (4.1) is convex and l.s.c. Moreover, from (4.2) and (4.7), it follows that

$$\partial\Psi = \bar{g} \tag{4.8}$$

where $\bar{g}$ is the multivalued operator

$$\bar{g}[u](x) = \begin{cases} g(x, u(x)) & \text{a.e. in } \Omega \text{, if } g(x, u(x)) \in H , \\ \emptyset & \text{otherwise (8)} . \end{cases} \tag{4.9}$$

Notice that $u(x) \in H$ and $g(x, u(x)) \in H$ imply that $\Psi(u) < +\infty$. Moreover on writing $g(x, u(x)) \in H$ it is implicit that $u(x) \in]a(x), b(x)[$ a.e. in Ω. Finally $\bar{g}$ is univalued on $D(\bar{g}) = \{u \in H : g(x, u(x)) \in H\}$ and is m.m. (by theorem 2.2).

(6) Notice that the domain of this function is $\{x_i : x \in \Omega,\ a(x) < y < b(x)\}$ with x^i and y fixed, and hence is an open subset of $\mathbf{R}$.

(7) In order to verify the measurability of $x \to \psi(x, y)$ in Ω notice that (suppose $y > 0$) for all $x \in \{x \in \Omega : y \leqslant b(x)\}$ we have

$$\psi(x, y) = \int_0^y g(x, \eta)\, d\eta = \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{y}{n}\, g\left(x, \frac{iy}{n}\right) .$$

(8) Obviously all definitions are coherent with the fact that elements of H are equivalence classes of functions.

In the sequel we shall also need the following hypothesis:

(4.10) For almost all x^i, $1 \leqslant i \leqslant N$, and for all $y \in \boldsymbol{R}$ the function $g(x, y)$ is differentiable as a function of the single coordinate x_i except eventually in points which belong to a (at most) denumerable set $E(x^i) \subset \{x_i = (x^i, x_i) \in \Omega\}$. We put by definition

$$g_i(x, y) = \frac{\partial}{\partial x_i} g(x, y) .$$

5. – Preliminary results.

In the sequel, for each fixed $x \in \Omega$, we denote by $g^{(\lambda)}(x, y)$ and $g_\lambda(x, y)$ the resolvent and the Yosida approximation of the m.m. graph $y \to g(x, y)$; it is implicit in this definition the convention referred in note (5).

By using (4.9) we easily verify that for any $u \in H$

$$\text{(5.1)} \qquad \begin{cases} \bar{g}^{(\lambda)}[u](x) = g^{(\lambda)}\big(x, u(x)\big) & \text{a.e. in } \Omega , \\ \bar{g}_\lambda\,[u](x) = g_\lambda\,\big(x, u(x)\big) & \text{a.e. in } \Omega . \end{cases}$$

LEMMA 5.1. *Let $y \in \boldsymbol{R}$ be fixed and put $v_0(x) = g^{(\lambda)}(x, y)$, $w_0(x) = g_\lambda(x, y)$. If x_0, $x \in \Omega$ and $v_0(x_0) \in \,]a(x), b(x)[$ then*

$$|w_0(x) - w_0(x_0)| \leqslant |g\big(x, v_0(x_0)\big) - g\big(x_0, v_0(x_0)\big)| .$$

PROOF. We have $y = v_0(x_0) + \lambda g\big(x_0, v_0(x_0)\big) = v_0(x) + \lambda g\big(x, v_0(x)\big)$ and consequently a straightforward computation yields

$$\big[v_0(x) + \lambda g\big(x, v_0(x)\big)\big] - \big[v_0(x_0) + \lambda g\big(x, v_0(x_0)\big)\big] = - \lambda\big[g\big(x, v_0(x_0)\big) - g\big(x_0, v_0(x_0)\big)\big] ;$$

therefore

$$|v_0(x) - v_0(x_0)| \leqslant \lambda |g\big(x, v_0(x_0)\big) - g\big(x_0, v_0(x_0)\big)|$$

since the resolvent is a contraction.

From this last estimate and from the identity $w_0(x) - w_0(x_0) = \big(v_0(x_0) - - v_0(x)\big)/\lambda$ the thesis follows.

LEMMA 5.2. *Let $u \in H$ and put*

$$\text{(5.2)} \qquad \begin{cases} v(x) \;= g^{(\lambda)}\big(x, u(x)\big) , \\ w(x) = g_\lambda\,\big(x, u(x)\big) . \end{cases}$$

Suppose that x_0, $x \in \Omega$ *and that* $v(x_0) \in]a(x), b(x)[$. *Then we have*

$$(5.3)\qquad \begin{cases} |v(x) - v(x_0)| \leqslant |u(x) - u(x_0)| + \lambda |g(x, v(x_0)) - g(x_0, v(x_0))| \,, \\ |w(x) - w(x_0)| \leqslant \dfrac{1}{\lambda} |u(x) - u(x_0)| + |g(x, v(x_0)) - g(x_0, v(x_0))| \,. \end{cases}$$

PROOF. We have

$$u(x) = v(x) + \lambda g(x, v(x)) \,,$$
$$u(x_0) = v(x_0) + \lambda g(x_0, v(x_0))$$

and therefore

$$v(x) - v(x_0) = u(x) - u(x_0) - \lambda[g(x, v(x)) - g(x, v(x_0))] - \\ - \lambda[g(x, v(x_0)) - g(x_0, v(x_0))] \,.$$

Multiplying both sides of the last equation by $v(x) - v(x_0)$ and recalling that $y \to g(x, y)$ is nondecreasing we obtain

$$(5.4)\quad |v(x) - v(x_0)|^2 \leqslant \{u(x) - u(x_0) - \lambda[g(x, v(x_0)) - g(x_0, v(x_0))]\} (v(x) - v(x_0)) \,;$$

thus

$$|v(x) - v(x_0)| \leqslant |u(x) - u(x_0) - \lambda[g(x, v(x_0)) - g(x_0, v(x_0))]|$$

and in particular the first estimate (5.3) follows, as desired.

On the other hand recalling that the Yosida approximation is Lipschitz continuous with constant $1/\lambda$ we obtain easily

$$|w(x) - w(x_0)| \leqslant |u(x) - u(x_0)|/\lambda + |g_\lambda(x, u(x_0)) - g_\lambda(x_0, u(x_0))| \,.$$

From this estimate and from lemma 5.1 the second estimate (5.3) follows.

To prove the next theorem we need some results which we recall. Let $f(x)$ be a real function defined on $[\alpha, \beta]$. The upper bound and the lower bound of $(f(x+h) - f(x))/h$ when $h \to 0$ are denoted by $\overline{D}f(x)$ and $\underline{D}f(x)$ respectively and are called the upper derivate and the lower derivate of f at the point x. Obviously the derivative $f'(x)$ exists if and only if $\overline{D}f(x)$ and $\underline{D}f(x)$ are finite and equal. The following result holds (cf. [19], chapter I, n. 9):

THEOREM 5.1 *If* $\overline{D}f(x)$ *and* $\underline{D}f(x)$ *are finite a.e. on* $[\alpha, \beta]$ *then* $\overline{D}f(x) = \underline{D}f(x)$ *a.e. on* $[\alpha, \beta]$.

This theorem is a particular case of a result due to Saks. If we assume the continuity of f, the result is due to Denjoy and G. C. Young independently.

We need also the following result:

THEOREM 5.2. *Let $f(x)$ be a continuous function on $[\alpha, \beta]$ and suppose that the following conditions are fulfilled*:

(i) $\overline{D}f(x)$ *and* $\underline{D}f(x)$ *are finite on* $[\alpha, \beta]$ *except at most on a finite or denumerable subset* E;

(ii) *the derivative* $f'(x)$ (9) *is summable over* $[\alpha, \beta]$.

Then

$$\int_{\alpha}^{\beta} f'(x)\,dx = f(\beta) - f(\alpha)\,, \tag{5.5}$$

and consequently $f(x)$ is absolutely continuous (a.c.) on $[\alpha, \beta]$ (10).

This result is proved on [12], theorem 264, by assuming that $f'(x)$ exists if $x \notin E$. The proof given in [12] is easily adapted to the weaker condition (i).

Now let $u(x)$ be a function defined on Ω and let ω be a straight line such that $\omega \cap \Omega \neq \emptyset$. $u(x)$ is said to be a.c. on ω if $u(x)$ is a.c. on any closed interval contained in $\omega \cap \Omega$.

In the sequel we denote by $\partial u/\partial x_i$ and u_i' the derivative in the distribution's sense and the derivative in the classical sense respectively. The following result is well known (see for instance [18], chapter 2, theorem 2.2):

THEOREM 5.3. *Let $u(x) \in L^1(\Omega)$. If $\partial u/\partial x_i \in L^1(\Omega)$ then $u(x)$, eventually modified on a set of zero measure, is a.c. on almost all the parallels to the x_i axis; moreover $\partial u/\partial x_i = u_i'$ a.e. in Ω.*

Reciprocally, if $u(x) \in L^1(\Omega)$ is a.c. on almost all the parallels to the x_i axis and if $u_i'(x) \in L^1(\Omega)$ then $u_i' = \partial u/\partial x_i$ a.e. in Ω.

Our purpose is now to prove the following result:

THEOREM 5.4. *Let $u(x) \in H$ and let $v(x)$ and $w(x)$ be defined by* (5.2). *Assume that $u(x) \in W^{2,1}(\Omega)$* (11) *and that*

$$g_i(x, v(x)) \in L^1(\Omega)\,, \qquad i = 1, \ldots, N. \tag{5.6}$$

(9) Which exists a.e. by theorem 5.1.
(10) Since (5.5) holds on any subinterval.
(11) This condition can be weakened.

Then $v(x) \in W^{1,1}(\Omega)$, $w(x) \in W^{1,1}(\Omega)$ *and, for any index* i,

$$(5.7)\qquad \begin{cases} \left|\dfrac{\partial v}{\partial x_i}(x)\right| \leqslant \left|\dfrac{\partial u}{\partial x_i}(x)\right| + \lambda |g_i(x, v(x))| \,, \\ \left|\dfrac{\partial w}{\partial x_i}(x)\right| \leqslant \dfrac{1}{\lambda}\left|\dfrac{\partial u}{\partial x_i}(x)\right| + |g_i(x, v(x))| \,, \end{cases}$$

a.e. in Ω.

Notice that, independently from (5.6), the function $g_i(x, v(x))$ is defined a.e. in Ω and measurable in Ω. Notice also that the theorem 5.4 holds for any index i separately.

PROOF OF THEOREM 5.4. Let i be fixed and assume $u(x)$ modified on a set of measure zero so that the property described on the first part of theorem 5.3 holds. Then $u_i'(x) \in W^{1,1}(\Omega)$ and consequently $u_i'(x)$ modified on a subset of Ω of measure zero is a.c. (hence continuous) on almost all the parallels to x_i. Then on almost all this parallels $u(x)$ is a.c. and $u_i'(x)$, modified on a set of linear measure zero, is continuous. But then this last property holds without modify $u_i'(x)$.

On the other hand, by Fubini's theorem, the function $x_i \to g_i(x, v(x))$ is summable on almost all the parallels to the x_i axis.

Now let ω be a parallel in which the described properties and the conditions (4.5) and (4.10) hold. Furthermore let ω_0 be a closed interval contained in $\omega \cap \Omega$. We will prove that $v(x)$ is a.c. on ω_0 and that the derivative $v_i'(x)$ verifies the first estimate (5.7) a.e. in ω_0.

Let $x_0 \in \omega_0$ and put $x = x_0 + h_i$ where h_i is an increment in the x_i direction. By (5.3) we have

$$(5.8)\qquad |v(x_0 + h_i) - v(x_0)| \leqslant |u(x_0 + h_i) - u(x_0)| + \\ + \lambda |g(x_0 + h_i, v(x_0)) - g(x_0, v(x_0))| \,,$$

if $|h_i|$ is sufficiently small. This implies that the derivatives $\overline{D}v(x_0)$ and $\underline{D}v(x_0)$ are finite except at most on a finite or denumerable set $E \subset \omega_0$. Consequently the derivative $v_i'(x_0)$ exists a.e. in ω_0 (by theorem 5.1) and verify the first relation (5.7) a.e. in ω_0. In particular $v_i'(x_0)$ is summable on ω_0 and by theorem 5.2 it follows that $v(x)$ is a.c. in ω_0. Therefore, by the second part of theorem 5.3, $\partial v/\partial x_i$ exists in Ω and verifies (5.7) since it coincides a.e. in Ω with $v_i'(x)$.

Finally the existence of $\partial w/\partial x_i$ in $L^1(\Omega)$ follows directly from the existence of $\partial v/\partial x_i$ in $L^1(\Omega)$ and from the identity $w = (u - v)/\lambda$. Under these circumstances (5.3) implies trivialy the second relation (5.7). We can also prove the statement concerning $w(x)$ directly, as for $v(x)$.

PROPOSITION 5.1. *Suppose that the assumptions of theorem* 5.4 *hold. Then for all index i*

$$-\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_i} \leqslant |g_i(x, v(x))| \left|\frac{\partial u}{\partial x_i}\right| \tag{5.9}$$

a.e. in Ω.

PROOF. From (2.1) it follows that $\lambda w = u - v$ and therefore

$$\lambda \frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^2 - \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,.$$

By using the first estimate (5.7) we obtain easily

$$\lambda \frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_i} \geqslant -\lambda\, |g_i(x, v(x))| \left|\frac{\partial u}{\partial x_i}\right|.$$

Multiplying by $-1/\lambda$ we obtain (5.9), as desired.

Suppose that the conditions of theorem 5.4 hold. Then $w(x) \in W^{1,1}(\Omega)$ and thus $w(x)$ has a trace on Γ; it is well known that this trace $w(\xi)$ belongs to $L^1(\Gamma)$.

THEOREM 5.5. *Suppose that the assumptions of theorem* 5.4 *hold. Then we have a.e. on* Γ

$$\begin{cases} w(\xi) \geqslant 0 & \text{if } u(\xi) > 0\,, \\ w(\xi) \leqslant 0 & \text{if } u(\xi) < 0\,, \\ w(\xi) = 0 & \text{if } u(\xi) = 0\,. \end{cases} \tag{5.10}$$

Obviously it is sufficient to prove that for any $x_0 \in \Gamma$ there exists a neighbourhood of x_0 on which (5.10) holds. Let $x_0 \in \Gamma$ and choose an index i so that the normal to Γ at x_0 is not orthogonal to the x_i direction. Then the boundary Γ has a representation $x_i = \gamma(x^i)$ in a neighbourhood of x_0. More precisely: There exists a neighbourhood (in $\boldsymbol{R}^{N-1}$) U_0 of x_0^i, a C^2 real function γ defined on U_0 and a real constant $\delta > 0$ such that for $x^i \in U_0$ we have $(x^i, x_i) \in \Gamma$ if and only if $x_i = \gamma(x^i)$. Furthermore, for any $x^i \in U_0$, the points (x^i, x_i) such that $x_i \in\,]\gamma(x^i), \gamma(x^i) + \delta]$ are in Ω and the points (x^i, x_i) such that $x_i \in [\gamma(x^i) - \delta, \gamma(x^i)[$ are in $\sim(\Omega \cup \Gamma)$, or reciprocally, the first points are in $\sim(\Omega \cup \Gamma)$ and the last points are in Ω; in the sequel we suppose, without loss of generality, that the first case holds. We put

$$\Gamma_0 = \left\{(x^i, \gamma(x^i)) : x^i \in U_0\right\}.$$

The following result holds (cf. by instance [18], chapter 2, theorem 4.3):

PROPOSITION 5.2. *Let $u(x) \in W^{1,1}(\Omega)$. Then $u(x)$, eventually modified on a set of measure zero contained in Ω, verifies the following property: The function $u(x)$ (extended to Γ_0) is a.c. on the closed intervals $x_i \in [\gamma(x^i), \gamma(x^i) + \delta]$ for almost all $x^i \in U_0$. Furthermore the trace of $u(x)$ on Γ_0 and the pointwise value of $u(x)$ on Γ_0 coincides a.e. on Γ_0.*

PROOF OF THEOREM 5.5. Assume $u(x)$ modified as indicated in proposition 5.2. Then $w(x) = g_\lambda(x, u(x))$ verifies the property described in proposition 5.2, without modification on Ω. For, if $u(x)$ is continuous in a segment $[\gamma(x^i), \gamma(x^i) + \delta]$, with $x^i \in \Omega_0^i \cap U_0$, it follows from (5.3) and (4.5) that $w(x)$ is continuous on the segment $]\gamma(x^i), \gamma(x^i) + \delta]$. From this result and from the proposition 5.2 (recall that $w(x) \in W^{1,1}(\Omega)$) the desired statement follows.

Fix now $x^i \in U_0$ in order to have $u(x)$ and $w(x)$ a.c. in the corresponding interval. If $u(x^i, \gamma(x^i)) > 0$ then $u(x^i, x_i) > 0$ for all x_i in a neighbourhood of $\gamma(x^i)$ and consequently $w(x^i, x_i) = g_\lambda(x, u(x^i, x_i)) \geq 0$ in this neighbourhood. Hence

$$w(x^i, \gamma(x^i)) = \lim_{x_i \to \gamma(x^i)} w(x^i, x_i) \geq 0 .$$

The proof of the second statement (5.10) is similar.

Suppose now that $u(x^i, \gamma(x^i)) = 0$. Let $x_i \in]\gamma(x^i), \gamma(x^i) + \delta[$. Putting $x = (x^i, x_i)$ it follows that

$$|w(x^i, x_i)| = \left| g_\lambda(x, u(x^i, x_i)) - g_\lambda\big(x, u(x^i, \gamma(x^i))\big) \right|$$

because $g_\lambda(x, 0) = 0$. Since the Yosida approximation is Lipschitz contitinuous we obtain

$$|w(x^i, x_i)| \leq \frac{1}{\lambda} |u(x^i, x_i) - u(x^i, \gamma(x^i))|$$

and consequently

$$w(x^i, \gamma(x^i)) = \lim_{x_i \to \gamma(x^i)} w(x^i, x_i) = 0 ,$$

as desired.

6. – The $\bar{g}$ operator (continued).

Let $\theta(x, y)$ be a real function defined on $\Omega \times \mathbf{R}$, measurable on x for all y and continuous on y for almost all x. If $u(x)$ is a real measurable function defined on Ω it is well known that $\theta(x, u(x))$ is measurable. Put $\bar{\theta}[u](x) =$

$=\theta(x, u(x))$. It is said that $\bar{\theta}$ acts from $L^p(\Omega)$ into $L^q(\Omega)$, $p, q \in [1, +\infty[$, if $\bar{\theta}$ transforms every function in $L^p(\Omega)$ into a function in $L^q(\Omega)$. The following result holds (cf. Krasnosel'skii [13], chapter I, § 2, theorems 2.1 and 2.2):

THEOREM 6.1. *If $\bar{\theta}$ acts from $L^p(\Omega)$ into $L^q(\Omega)$ then $\bar{\theta}$ is bounded and continuous.*

On the other hand the following necessary condition holds (cf. Krasnosel'skii [13], chapter I, § 2, theorem 2.3):

THEOREM 6.2. *If $\bar{\theta}$ acts from $L^p(\Omega)$ into $L^q(\Omega)$ then*

$$|\theta(x, y)| \leqslant d(x) + c|y|^{p/q}$$

where c is a positive constant and $d(x) \in L^q(\Omega)$.

It is obvious that this condition is also a sufficient condition. Another sufficient condition is the following (cf. [13], chapter I, § 2, n. 4):

PROPOSITION 6.1. *If*

$$|\theta(x, y)| \leqslant \sum_{j=1}^{m} d_j(x)|y|^{p/q_j} \tag{6.1}$$

with $q \leqslant q_j \leqslant +\infty$ and $d_j(x) \in L^{r_j}(\Omega)$, $r_j = qq_j/(q_j - q)$, then $\bar{\theta}$ acts from $L^p(\Omega)$ into $L^q(\Omega)$.

Let $s \in [1, +\infty]$ and define s' by $1/s' = 1 - (1/s)$. Furthermore if $1 \leqslant s < N$ define s^* by $1/s^* = (1/s) - (1/N)$. It is well known (Sobolev's embedding theorem) that $W^{1,s}(\Omega) \hookrightarrow L^{s^*}(\Omega)$.

We introduce now two exponents p and q related to the dimension N. We suppose p and q fixed as follows:

$$\begin{cases} p = 2N/(N-4) & \text{if } N > 4\,, \\ 1 < p < +\infty & \text{if } N \leqslant 4\,; \end{cases} \tag{6.2}$$

$$\begin{cases} q = 2N/(N+2) & \text{if } N > 2\,, \\ 1 < q \leqslant 2 & \text{if } N = 2\,, \\ q = 1 & \text{if } N = 1\,. \end{cases} \tag{6.3}$$

Notice that $p = (2^*)^*$ if $N > 4$ and $q = (2^*)'$ if $N > 2$. Furthermore $W^{2,2}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^{q'}(\Omega)$ for all N. If $N \leqslant 4$ in the first inclusion or $N \leqslant 2$ in the second one these embedding results are not the best possible and we can utilize the embedding on spaces of Hölder continuous functions to generalize our results.

We suppose finally that $g(x, y)$ verifies the following condition:

(6.4) For all index i, $1 \leqslant i \leqslant N$, and for almost all $x \in \Omega$, $|g_i(x, y)| \leqslant \theta(x, y) + c|g(x, y)|$, $\forall y \in]a(x), b(x)[$, where c is a positive constant and $\theta(x, y)$ acts from $L^p(\Omega)$ into $L^q(\Omega)$.

The following *Green's formulae* will be useful in the sequel: If $u(x) \in W^{2,2}(\Omega)$ and $w(x) \in W^{1,q}(\Omega)$ then

$$-\int_\Omega \Delta u \cdot w \, dx = \sum_i \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} dx - \int_\Gamma \frac{\partial u}{\partial n} w \, d\Gamma . \tag{6.5}$$

The integrands in (6.5) are summable functions over the corresponding domain of integration. For the reader's convenience we verify the validity of (6.5) in appendix III.

7. – The existence theorem.

From theorem 2.3 and from the results of sections 3 and 4 it follows that for every $f \in H$ the problem

$$Bu_\lambda + \vec{g}_\lambda[u_\lambda] + \mu u_\lambda = f \tag{7.1}$$

has a unique solution u_λ and the problem

$$Bu + \vec{g}[u] + \mu u = f \tag{7.2}$$

has a solution u if and only if $\vec{g}_\lambda[u_\lambda]$ is bounded in H as $\lambda \to 0$. In this section we utilize the preliminary results of section 5 to prove this last property.

Notice that (7.1) is equivalent to

$$\begin{cases} -\Delta u_\lambda(x) + g_\lambda\big(x, u_\lambda(x)\big) + \mu u_\lambda(x) = f(x) & \text{a.e. in } \Omega , \\ -\dfrac{\partial u_\lambda}{\partial n}(\xi) \in \beta\big(u_\lambda(\xi)\big) & \text{a.e. in } \Gamma , \end{cases} \tag{7.1$'$}$$

with $u_\lambda \in W^{2,2}(\Omega)$, and (7.2) is equivalent to (1.1) plus $u \in W^{2,2}(\Omega)$.

In the sequel we put $v_\lambda = \bar{g}^{(\lambda)}[u_\lambda]$ and $w_\lambda = \bar{g}_\lambda[u_\lambda]$, i.e.,

$$\begin{cases} v_\lambda(x) = g^{(\lambda)}(x, u_\lambda(x)) , \\ w_\lambda(x) = g_\lambda\,(x, u_\lambda(x)) . \end{cases}$$

The following result holds:

THEOREM 7.1. *Assume that*

$$g_i(x, v_\lambda(x)) \in L^q(\Omega) , \qquad i = 1, \dots, N. \tag{7.3}$$

Then

$$\int_\Omega g_\lambda(x, u_\lambda(x))^2 \, dx \leqslant \int_\Omega f(x) g_\lambda(x, u_\lambda(x)) \, dx + \sum_i \int_\Omega \left| g_i(x, v_\lambda(x)) \frac{\partial u_\lambda}{\partial x_i} \right| dx . \tag{7.4}$$

PROOF. From theorem 5.4 and from (7.3) it follows that $w_\lambda \in W^{1,q}(\Omega)$. Multiplying both sides of the first equation (7.1′) by $w_\lambda(x)$ and integrating in Ω it follows that

$$-\int_\Omega \Delta u_\lambda \cdot w_\lambda \, dx + \int_\Omega w_\lambda^2 \, dx + \mu \int_\Omega u_\lambda w_\lambda \, dx = \int_\Omega f w_\lambda \, dx .$$

Applying now the Green's formulae (6.5) to the first integral we obtain

$$\int_\Omega w_\lambda^2 \, dx \leqslant \int_\Omega f w_\lambda \, dx - \sum_i \int_\Omega \frac{\partial u_\lambda}{\partial x_i} \frac{\partial w_\lambda}{\partial x_i} \, dx - \int_\Gamma \left(- \frac{\partial u_\lambda}{\partial n} \right) w_\lambda \, d\Gamma \tag{7.5}$$

since $u_\lambda w_\lambda \geqslant 0$ a.e. in Ω. On the other hand

$$\int_\Gamma \left(- \frac{\partial u_\lambda}{\partial n} \right) w_\lambda \, d\Gamma \geqslant 0 . \tag{7.6}$$

For, from the second relation (7.1′) it follows that $-(\partial u_\lambda/\partial n)(\xi) \geqslant 0$ if $u_\lambda(\xi) > 0$ and $-(\partial u_\lambda/\partial n)(\xi) \leqslant 0$ if $u_\lambda(\xi) < 0$ (a.e. in Γ) and consequently the integrand in (7.6) is almost everywhere nonnegative in Γ, by theorem 5.5.

Finally from (7.5), (7.6) and proposition 5.1 we obtain (7.4), as desired.

LEMMA 7.1. *Let u and u_1 be solutions of* (7.2) *with data f and f_1 respectively. Then*

$$\|u - u_1\|_{1,2} \leqslant c \|f - f_1\| .$$

The same result holds for equation (7.1) *and in particular*

$$\|u_\lambda\|_{1,2} \leq c\|f\| . \tag{7.7}$$

The proof is an easy exercise.

PROPOSITION 7.1. *Let u and u_1 be solutions of* (7.2) *with data f and f_1 respectively. Then the following estimate holds*:

$$\|u - u_1\|_p \leq c\|f - f_1\| . \tag{7.8}$$

REMARK 7.1. We shall see later (theorem 7.2) that the problem (1.1) has a unique solution $u \in W^{2,2}(\Omega)$ for each data $f \in L^2(\Omega)$. Therefore the proposition 7.1 says that the operator $f \to u$ is Lipschitz continuous from $L^2(\Omega)$ into $L^p(\Omega)$ (It is also Lipschitz continuous from $L^2(\Omega)$ into $W^{1,2}(\Omega)$, by lemma 7.1).

PROOF OF PROPOSITION 7.1. (See also [6]). If $N \leq 2$ the estimate (7.8) follows directly from lemma 7.1. Hence we suppose that $N > 2$. Put $z = u - u_1$ and $h = f - f_1$. The real function $\zeta(y) = |y|^s\, y$, $s > 0$, is continuously differentiable on $\boldsymbol{R}$ and $\zeta'(y) = (s+1)|y|^s$. Put $s = 2p/N$ and consider the function $\zeta[z](x) = \zeta(z(x))$. We have

$$\frac{\partial \zeta[z]}{\partial x_i} = \left(\frac{2p}{N} + 1\right) |z|^{2p/N} \frac{\partial z}{\partial x_i}$$

and consequently $\zeta[z] \in W^{1,q}(\Omega)$. For, $|z|^{2p/N}$ and $\partial z/\partial x_i$ are integrable with exponents $N/2$ and 2^* respectively and consequently its product belongs exactly to $L^q(\Omega)$. Hence applying (6.5) with w replaced by $\zeta[z]$ and u replaced by z we obtain

$$-\int_\Omega \Delta z \cdot \zeta[z]\, dx \geq \left(\frac{2p}{N} + 1\right) \int_\Omega |z|^{2p/N} |\nabla z|^2\, dx \tag{7.9}$$

since $-(\partial z/\partial n)\zeta[z] \geq 0$ a.e. in Γ. In fact $\zeta[z](\xi) = |z(\xi)|^{2p/N} z(\xi)$ a.e. in Γ and furthermore

$$-\frac{\partial z}{\partial n} z \equiv \left[\left(-\frac{\partial u}{\partial n}\right) - \left(-\frac{\partial u_1}{\partial n}\right)\right] (u - u_1) \geq 0 \qquad \text{a.e. in } \Gamma .$$

On the other hand from the first equation (1.1) written for u, f and u_1, f_1 it follows that $-\Delta z(x) + g(x, u(x)) - g(x, u_1(x)) + \mu z(x) = h(x)$ a.e. in Ω.

Multiplying both terms by $\zeta[z](x)$, integrating on Ω and applying (7.9) we obtain

$$(7.10)\qquad \left(\frac{2p}{N}+1\right)\int_\Omega |z|^{2p/N}|\nabla z|^2\,dx+\mu\int_\Omega |z|^{(2p/N)+2}\,dx\leqslant\int_\Omega |z|^{(2p/N)+1}\,|h|\,dx\,.$$

Consider now the function $\eta[z]=|z|^{p/N}z$. We have

$$\frac{\partial\eta[z]}{\partial x_i}=\left(\frac{p}{N}+1\right)|z|^{p/N}\frac{\partial z}{\partial x_i}\,,$$

and from (7.10) it follows that

$$(7.11)\qquad \|\eta[z]\|^2_{1,2}\leqslant c\|h\|\,\|z\|^{(2p/N)+1}_{2(2p/N+1)}\,.$$

Next we will prove the estimate

$$(7.12)\qquad \|z\|_{2(2p/N+1)}\leqslant c\|h\|\,.$$

It is easy to see that (equality holds if $N>4$)

$$(7.13)\qquad 2^*\left(\frac{p}{N}+1\right)\geqslant 2\left(\frac{2p}{N}+1\right).$$

Since $\|\eta[z]\|_{2^*}\leqslant c\|\eta[z]\|_{1,2}$ it follows that

$$\|z\|^{2(N+p)/N}_{2^*(p/N+1)}\leqslant c\|\eta[z]\|^2_{1,2}\,.$$

From this estimate and from (7.13) we obtain

$$(7.14)\qquad \|z\|^{2(N+p)/N}_{2(2p/N+1)}\leqslant c\|\eta[z]\|^2_{1,2}\,.$$

Finally (7.12) follows from (7.11) and (7.14).

Now (7.8) coincides with (7.12) if $N>4$, since $2(2p/N+1)=p$. If $N\leqslant 4$ then $p<2(2p/N+1)$ and (7.8) follows from (7.12).

COROLLARY 7.1. *If* u_λ *is the solution of* (7.1) *we have*

$$(7.15)\qquad \|v_\lambda\|_p\leqslant\|u_\lambda\|_p\leqslant c\|f\|\,.$$

PROOF. Since $|v_\lambda(x)|\leqslant|u_\lambda(x)|$ the first estimate is obvious. Applying the proposition 7.1 with g replaced by g_λ and $u_1\equiv f_1\equiv 0$ the second estimate follows.

COROLLARY 7.2 (see also [6] proposition I.4). *Let* $Bu + \mu u = f$ *and* $Bu_1 + \mu u_1 = f_1$. *Then*

$$\|u - u_1\|_p \leq c\|f - f_1\| . \tag{7.16}$$

PROOF. Apply proposition 7.1 with $g \equiv 0$.

LEMMA 7.2. *Let* u_λ *be the solution of* (7.1). *Then the following estimate holds*:

$$\|w_\lambda\|^2 \leq c\|f\|\;\|w_\lambda\| + \sum_i \int_\Omega \theta(x,\; v_\lambda(x)) \left|\frac{\partial u_\lambda}{\partial x_i}\right| dx . \tag{7.17}$$

PROOF. From (2.4) it follows that $w_\lambda(x) = g(x, v_\lambda(x))$ and consequently we deduce from (6.4) that

$$|g_i(x, v_\lambda(x))| \leq \theta(x, v_\lambda(x)) + c|w_\lambda(x)| \quad \text{a.e. in } \Omega, \tag{7.18}$$

for any index i. Consequently condition (7.3) holds and by theorem 7.1

$$\|w_\lambda\|^2 \leq \|f\|\;\|w_\lambda\| + c\|w_\lambda\|\;\|\nabla u_\lambda\| + \sum_i \int_\Omega \theta(x, v_\lambda(x)) \left|\frac{\partial u_\lambda}{\partial x_i}\right| dx .$$

From this estimate and from lemma 7.1 we obtain (7.17).

THEOREM 7.2. *Assume that* $g(x, y)$ *verifies the conditions* (4.3), (4.4), (4.5), (4.10) *and* (6.4). *Then for every* $f \in H$ *there is a unique solution* $u \in W^{2,2}(\Omega)$ *of* (1.1). *Moreover the following estimate hold*:

$$\|u\|_{2,2} \leq c\Big(\|f\| + \|\theta(x, u(x))\|_q\Big) , \tag{7.19}$$

and

$$\|u\|_p \leq c\|f\| . \tag{7.19'}$$

PROOF. – From (7.15) it follows that $\|v_\lambda\|_p$ is bounded as $\lambda \to 0$. Therefore applying theorem 6.1 we obtain

$$\|\theta(x, v_\lambda(x))\|_q \leq c \quad \text{as } \lambda \to 0 . \tag{7.20}$$

On the other hand

$$\left\|\frac{\partial u_\lambda}{\partial x_i}\right\|_{q'} \leq c\|u_\lambda\|_{2,2} , \qquad \forall i ,$$

since $W^{1,2}(\Omega) \hookrightarrow L^{q'}(\Omega)$. This last estimate, (3.2) and (7.1) yield

$$\left\| \frac{\partial u_\lambda}{\partial x_i} \right\|_{q'} \leqslant c(\|f\| + \|w_\lambda\|) . \tag{7.21}$$

Finally from (7.17), (7.20) and (7.21) it follows that w_λ is bounded in H as $\lambda \to 0$ and therefore, by theorem 2.3, the problem (7.2) has a unique solution u. Moreover

$$\begin{cases} u_\lambda \to u & \text{in } H, \\ w_\lambda \to \bar{g}[u] & \text{in } H. \end{cases} \tag{7.22}$$

In order to prove (7.19) we shall see that

$$\lim_{\lambda \to 0} v_\lambda = u \qquad \text{in } L^p(\Omega) . \tag{7.23}$$

From (7.1), (7.2) and corollary 7.2 it follows that $\|u - u_\lambda\|_p \leqslant c\|\bar{g}[u] - \bar{g}_\lambda[u_\lambda]\|$. Hence by (7.22) $u_\lambda \to u$ in $L^p(\Omega)$ as $\lambda \to 0$. Therefore

$$\limsup_{\lambda \to 0} \|v_\lambda\|_p \leqslant \|u\|_p . \tag{7.24}$$

On the other hand $v_\lambda \to u$ in H as $\lambda \to 0$ since $v_\lambda = u_\lambda + \lambda w_\lambda$. Therefore by the reflexivity of $L^p(\Omega)$,

$$v_\lambda \to u \quad \text{weakly in } L^p(\Omega) . \tag{7.25}$$

Since $\| \ \|_p$ is lower semicontinuous for the weak convergence, (7.25) and (7.24) yield

$$\|v_\lambda\|_p \to \|u\|_p \qquad \text{as } \lambda \to 0 . \tag{7.26}$$

Finally from (7.25) and (7.26) we obtain (7.23) (see by instance [19], p. 78) and consequently, by theorem 6.1, $\theta(x, v_\lambda(x)) \to \theta(x, u(x))$ in $L^q(\Omega)$ as $\lambda \to 0$. On the other hand it follows from (7.17) and (7.21) that

$$\|w_\lambda\|^2 \leqslant c\left(\|\theta(x, v_\lambda(x))\|_q^2 + \|f\|^2\right) .$$

Passing to the limit in this inequality we obtain

$$\|g(x, u(x))\|^2 \leqslant c\left(\|\theta(x, u(x))\|_q^2 + \|f\|^2\right) . \tag{7.27}$$

Consequently (7.19) holds since $\|u\|_{2,2}\leqslant c\|f-g(x,u(x))\|$ as follows from (7.2) and (3.2).

COROLLARY 7.3. *Assume that the hypothesis of theorem 7.2 hold and that $\theta(x,y)$ verifies (6.1). Then the solution u of (1.1) verifies the estimate*

$$\|u\|_{2,2}\leqslant c\sum_{j=1}^{m}\|f\|^{p/q_j}\,. \tag{7.28}$$

PROOF. – Using Hölder's inequality we obtain easily the estimate

$$\|\theta(x,u(x))\|_q\leqslant\sum_{j=1}^{m}\|u\|_p^{p/q_j}\|d_j\|_{r_j}\,.$$

From this estimate and from (7.19′) the estimate (7.28) follows.

THEOREM 7.3. *Suppose that the assumptions of theorem 7.2 hold. Let Φ be defined by (3.3) and Ψ be defined by (4.1), (4.6). Then the functional $\Phi+\Psi$ is convex and l.s.c. in H and furthermore*

$$\partial(\Phi+\Psi)=B+\bar{g}\,.$$

PROOF. Since $B+\bar{g}\subset\partial(\Phi+\Psi)$ and $\partial(\Phi+\Psi)$ is monotone (by theorem 2.2) it suffices to prove that $B+\bar{g}$ is m.m. But this statement follows from theorem 2.1 since $R(B+\bar{g}+\mu I)=H$ as proved in theorem 7.2.

REMARK 7.2. The proposition 7.1 and the related corollaries are not necessary if $N\leqslant 2$. In this case the lemma 7.1 suffices since $W^{1,2}(\Omega)\hookrightarrow L^p(\Omega)$. The same remark would hold also for $N>2$ if we had assumed the stronger condition $p=2^*$.

On the other hand, for small values of N, the proposition 7.1 is not the best possible one and the results can be generalized as remarked in section 6.

REMARK 7.3. Let $E\subset\Omega$ be a set such that the projections E^i, $1\leqslant i\leqslant N$, are sets of $(N-1)$-dimensional measure zero. Then in theorem 7.2 the conditions $a(x)<0<b(x)$, (4.3) and (4.4) may fail on E $\big(g(x,y)$ can be arbitrarily given for $x\in E\big)$. The proof is obvious.

REMARK 7.4 (*the evolution case*). The study of evolution equations related to the stationary case (1.1) can be done using well known abstract results based on the maximal monotony of $B+\bar{g}$, which was proved in this paper. To write results in this direction is then an easy exercise, and we leave it to the reader.

REMARK 7.5 (added in proofs). In his interesting paper *On the solvability of semilinear elliptic equations with nonlinear boundary conditions*, to appear in the « Mathematische Annales », H. BRILL obtains (independently and by a different method) a general existence result for related problems. In his paper the function g depends also on the gradient ∇u, moreover, differentiability (and monotonicity) of g, as a function of the variable u, is not required.

Under these general conditions the existence (unicity and bounds) of a solution u is not always true and depends on the proof of the existence of a suitable subsolution and a suitable supersolution (see theorem 2.3 in the Brill's paper). A simple condition under which existence holds is described in the remark 2.4 of that paper. However if one assumes that all the conditions of theorem 7.2 hold (plus $a(x) \equiv -\infty$, $b(x) \equiv +\infty$ in our notation) the result following from that remark does not, in general, imply the existence of a solution u for the problem (1.1) since it requires the existence of two constants $R_1 \leqslant 0$ and $R_2 \geqslant 0$ such that $g(x, R_1)$ and $g(x, R_2)$ belong to $L^2(\Omega)$ and such that $g(x, R_1) \leqslant f(x) \leqslant g(x, R_2)$ a.e. on Ω.

Appendix I.

Consider the measure space $(B, \mathcal{M})$ where B is a non empty set and $\mathcal{M}$ is a σ-algebra in B. The following result holds (see [3], appendix II):

PROPOSITION 1. *Let $\psi: B \times \boldsymbol{R} \to]-\infty, +\infty]$ be a convex l.s.c. function in the second variable y for all $x \in B$ and measurable on the first variable x for all $y \in \boldsymbol{R}$. Suppose that there exists at least a measurable function $\xi: B \to \boldsymbol{R}$ such that $\psi(x, \xi(x)) < +\infty$ $\forall x \in B$ and $\psi(x, \xi(x))$ is measurable. Then $\psi(x, u(x))$ is measurable for any measurable function $u: B \to \boldsymbol{R}$.*

LEMMA 1. – *Let $A \in \mathcal{M}$ and let $u: B \to \boldsymbol{R}$ be measurable. There exist measurable simple functions $u_n(x)$ (n positive integer) vanishing on $B - A$ and such that the following properties hold:*

a) $\lim_{n \to +\infty} u_n(x) = u(x)\,, \qquad \forall x \in A,$

b) for any $x \in A$ there exists a positive integer $N(x)$ such that

$$u_n(x) \leqslant u_{n+1}(x) \leqslant u(x) \tag{1}$$

if $n > N(x)$ ((1) can be replaced by $u(x) \leqslant u_{n+1}(x) \leqslant u_n(x)$).

PROOF. Put for each n

$$E_{n,i} = A \cap u^{-1}([(i-1)2^{-n}, i2^{-n}[) , \qquad 1 \leqslant i \leqslant n2^n ,$$
$$\tilde{E}_{n,i} = A \cap u^{-1}([-i2^{-n}, -(i-1)2^{-n}[) , \qquad 1 \leqslant i \leqslant n2^n .$$

Let χ_E be the characteristic function of a measurable set E and put

$$u_n = \sum_{j=1}^{n2^n} [(i-1)2^{-n}\chi_{E_{n,i}} - i2^{-n}\chi_{\tilde{E}_{n,i}}] .$$

The u_n are measurable simple functions vanishing on $B - A$. Fix $x_0 \in A$ and put $N(x_0) > |u(x_0)|$; for each $n > N(x_0)$ the point x_0 belongs to one and only one set $E_{n,i}$ or $\tilde{E}_{n,i}$ $(1 \leqslant i \leqslant n2^n)$. Since $u(x_0) - 2^{-n} < u_n(x_0) \leqslant u(x_0)$ $a)$ holds. Moreover $b)$ holds trivialy.

PROOF OF PROPOSITION 1. For each $x \in B$ put

$$\text{(2)} \qquad \begin{cases} a(x) = \inf\{y : \psi(x, y) < +\infty\} , \\ b(x) = \sup\{y : \psi(x, y) < +\infty\} . \end{cases}$$

Obviously $a(x) \leqslant \xi(x) \leqslant b(x)$ $\forall x \in B$. Moreover the function $y \to \psi(x, y)$ is continuous on $[a(x), b(x)]$.

Consider now the following measurable sets:

$$\text{(3)} \qquad \begin{aligned} B_1 &= \{x \in B : u(x) < \xi(x)\} , \\ B_2 &= \{x \in B : \xi(x) < u(x)\} , \\ B_3 &= \{x \in B : \xi(x) = u(x)\} . \end{aligned}$$

From lemma 1 it follows the existence of two sequences of measurable simple functions in B, $u_n^{(1)}(x)$ and $u_n^{(2)}(x)$, such that:

$a)$ $u_n^{(j)}(x) \to u(x)$ $\forall x \in B_j$, $j = 1, 2$.

$b)$ for each $x \in B_1$ [resp. B_2] there exists $N(x)$ such that

$$u_n^{(1)}(x) \geqslant u(x) \qquad [\text{resp. } u_n^{(2)}(x) \leqslant u(x)] \qquad \text{if } n > N(x) .$$

Moreover $u_n^{(j)}(x) = 0$ on $B - B_j$, $j = 1, 2$.

Let

$$\psi_n(x) = \begin{cases} \psi(x, u_n^{(j)}(x)) & \text{if } x \in B_j,\ j = 1, 2 , \\ \psi(x, \xi(x)) & \text{if } x \in B_3 . \end{cases}$$

We shall prove that these functions are measurable on B. On B_3 this is obvious. To prove the statement on B_1 (on B_2 the proof is similar) we shall see that the sets

$$\{x \in B_1 \colon \psi(x, u_n^{(1)}(x)) \in I\} \tag{4}$$

are measurable for any open set $I \subset]-\infty, +\infty]$. Since $u_n^{(1)}(x) = \sum_j \beta_j \chi_{E_j}$ on B_1, with $\beta_j \in \boldsymbol{R}$, $E_j \in \mathcal{M}$, $\cup E_j = B_1$ and $E_i \cap E_j = \emptyset$ if $i \neq j$, it follows that the set (4) coincides with the measurable set $\bigcup_j \{x \in E_j \colon \psi(x, \beta_j) \in I\}$.

Finally we shall see that

$$\lim_{n \to +\infty} \psi_n(x) = \psi(x, u(x)), \qquad \forall x \in B. \tag{5}$$

If $x \in B_3$ this is obvious. Suppose that $x \in B_2$; then

$$\xi(x) < u(x), \quad \lim u_n^{(2)}(x) = u(x) \quad \text{and} \quad u^{(2)}(x) \leqslant u(x) \quad \text{if } n > N(x).$$

In particular

$$a(x) \leqslant \xi(x) < u_n^{(2)}(x) \leqslant u(x) \tag{6}$$

for n sufficiently large. If $b(x) < u(x)$ we have $b(x) < u_n^{(2)}(x)$ for n sufficiently large and therefore $\psi(x, u_n^{(2)}(x)) = +\infty = \psi(x, u(x))$. If $u(x) \leqslant b(x)$ the relation (6) yields $a(x) < u_n^{(2)}(x) \leqslant u(x) \leqslant b(x)$; therefore $\lim \psi(x, u_n^{(2)}(x)) = \psi(x, u(x))$ since $y \to \psi(x, y)$ is continuous on $[a(x), b(x)]$.

COROLLARY 1. *Let μ be a complete measure on the σ-algebra $\mathcal{M}$. Then the proposition* 1 *holds if the conditions «$y \to \psi(x, y)$ is convex and l.s.c.» and «$\psi(x, \xi(x)) < +\infty$» hold only for almost all $x \in B$.*

In the following we suppose that

(7) the σ-additive measure μ is complete and positive ($\mu \colon \mathcal{M} \to [0, +\infty]$).

Furthermore we assume that the hypothesis of corollary 1 hold and that for a fixed p, $1 \leqslant p \leqslant +\infty$,

(8) $\xi(x) \in L^p(B)$, $\psi(x, \xi(x)) \in L^1(B)$ and for almost all $x \in B$ one has $\psi(x, y) \geqslant \psi(x, \xi(x))$, $\forall y \in \boldsymbol{R}$.

Put for any $u(x) \in L^p(B)$

$$\Psi(u) = \int_B \psi(x, u(x))\, d\mu(x). \tag{9}$$

Since $\psi(x, u(x)) \geqslant \psi(x, \xi(x))$ a.e. in B with $\psi(x, \xi(x)) \in L^1(B)$, the integral in (9) has a clear meaning.

PROPOSITION 2. *Suppose that the conditions of corollary* 1 *and the conditions* (7) *and* (8) *hold. Then the functional* $\Psi: L^p(B) \to]-\infty, +\infty]$ *is convex, l.s.c. and* $\Psi \not\equiv +\infty$.

PROOF. – The first and the last statements are trivially verified. Suppose now that $u_n \to u$ in $L^p(B)$ and that $i \equiv \liminf_{n\to+\infty} \Psi(u_n) < +\infty$. There exists a subsequence u_{n_k} such that $\lim \Psi(u_{n_k}) = i$ and $u_{n_k}(x) \to u(x)$ a.e. in B. Since $y \to \psi(x, y)$ is l.s.c. for almost all $x \in B$ it follows that $\psi(x, u(x)) \leqslant \liminf \psi(x, u_{n_k}(x))$ a.e. in B. Hence by Fatou's lemma $\Psi(u) \leqslant i$ (recall that $\psi(x, u_{n_k}(x)) \geqslant \psi(x, \xi(x)) \in L^1(B)$).

In the sequel we assume that

(10) the measure μ is σ-finite, i.e.,
$B = \bigcup_{i=1}^{+\infty} B_i$ with $B_i \in \mathcal{M}$ and $\mu(B_i) < +\infty$, $\forall i$.

Let $p' = p/(p-1)$. The following proposition holds:

PROPOSITION 3. *Assume that the hypothesis of corollary* 1 *and the conditions* (7), (8) *and* (10) *hold. Let* $f \in L^{p'}(B)$ *and* $u \in D(\Psi)$. *Then the following conditions are equivalent:*

(i) $\Psi(v) - \Psi(u) \geqslant \int_B f(v-u)\, d\mu, \ \forall v \in L^p(B)$.

(ii) $\psi(x, v(x)) - \psi(x, u(x)) \geqslant f(x)\, (v(x) - u(x))$ a.e. *in* B *for any* $v \in L^p(B)$.

(iii) $\psi(x, y) - \psi(x, u(x)) \geqslant f(x)(y - u(x))\ \forall y \in R$, *for almost all* $x \in B$.

PROOF ([12]). (i) $\Rightarrow$ (ii): Let $E \in \mathcal{M}$ and put

$$w(x) = \begin{cases} v(x) & \text{if } x \in E\,, \\ u(x) & \text{if } x \in B - E\,. \end{cases}$$

Obviously $w \in L^p(E)$. From (i) it follows that

$$\int_E \psi(x, v(x))\, d\mu - \int_E \psi(x, u(x))\, d\mu \geqslant \int_E f(v-u)\, d\mu$$

([12]) Cf. also the proof of proposition 3, appendix I [6].

and consequently

$$(11)\qquad \int_E [\psi(x, v(x)) - \psi(x, u(x)) - f(x)(v(x) - u(x))]\, d\mu(x) \geq 0\,, \qquad \forall E \in \mathcal{M}.$$

Therefore the integrand in (11) is non negative a.e. in B. (ii) $\Rightarrow$ (iii): Fix a B_i and consider the function

$$w_n(x) = \begin{cases} u(x)\,, & \text{if } x \in B - B_i\,, \\ r_n + \xi(x)\,, & \text{if } x \in B_i\,, \end{cases}$$

where $\{r_1, r_2, \ldots\}$ is the set of all the rational numbers. Since $\mu(B_i) < +\infty$ the functions $w_n \in L^p(B)$. Applying now (ii) it follows that for each index n

$$(12)\qquad \psi(x, r_n + \xi(x)) - \psi(x, u(x)) \geq f(x)(\xi(x) + r_n - u(x)) \qquad \text{a.e. in } B_i\,.$$

Obviously there exists a set of measure zero $A \subset B_i$ such that for all $x \in B_i - A$ the following properties hold: (j) the estimate (12) holds for any index n; (jj) $\psi(x, u(x)) < +\infty$; (jjj) $y \to \psi(x, y)$ is convex, l.s.c. and $\psi(x, \xi(x)) < +\infty$. Now define for any $x \in B_i - A$ the functions $a(x)$ and $b(x)$ as in (2) and fix $x \in B_i - A$. If $y \notin [a(x), b(x)]$ the property (iii) holds since $\psi(x, y) = +\infty$. Suppose now that $y \in [a(x), b(x)]$. Since $\xi(x) \in [a(x), b(x)]$ there exists a subsequence $\xi(x) + r_{n_k} \in [a(x), b(x)]$ such that $\xi(x) + r_{n_k} \to y$ $(a(x) = b(x)$ is not excluded$)$. Passing to the limit in (12) with this subsequence we obtain (iii).

Finally (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is trivially verified.

Let X be a real Banach space and let X' be the dual space of X The subdifferential of a functional $\Psi\colon X \to]-\infty, +\infty]$ at a point $u \in D(\Psi)$ is the set $\partial\Psi(u) = \{v' \in X'\colon \Psi(v) - \Psi(u) \geq \langle v', v - u\rangle,\ \forall v \in X\}$.

Corollary 2. *Assume that the conditions of proposition* 3 *are fullfiled and that* $1 \leq p < +\infty$. *Then for each* $u \in D(\Psi)$

$$(13)\qquad \partial\Psi(u) = \{f \in L^{p'}(B)\colon f(x) \in \partial\psi(x, u(x)) \text{ a.e. in } B\}\,.$$

Appendix II.

We shall denote the Lebesgue measure in $\boldsymbol{R}^N$ by m_N. If $B \subset \boldsymbol{R}^N$ we denote by B^i the orthogonal projection of B into the hyperplane $\{x\colon x_i = 0\}$. Moreover we put $\omega(x^i) = \{x \in B\colon x^i = \text{constant}\}$.

Proposition 1. *Let* $B \subset \boldsymbol{R}^N$ *be a* N*-measurable set and let* $g\colon B \to \boldsymbol{R}$. *Let* B_0^i *be a subset of* B^i *such that* $m_{N-1}(B^i - B_0^i) = 0$, $i = 1, \ldots, N$. *Finally as-*

sume that for all $x^i \in B_0^i$ the sets $\omega(x^i)$ are open (in $\boldsymbol{R}$) and the restrictions of $g(x)$ to $\omega(x^i)$ are continuous ($i = 1, \ldots, N$). Then g is N-measurable in B.

PROOF. For $N = 1$ the result is obvious. We suppose that the result holds for the value N and we shall prove it for the value $N+1$. Denote by $B(x_{N+1})$, $B^i(x_{N+1})$ and $B_0^i(x_{N+1})$ the intersections on B, B^i and B_0^i with the hyperplane $\{x \in \boldsymbol{R}^{N+1} : x_{N+1} = \text{constant}\}$, $1 \leqslant i \leqslant N$. For almost all x_{N+1} the sets $B(x_{N+1})$ are N-measurable. Furthermore $m_{N-1}(B^i(x_{N+1}) - B_0^i(x_{N+1})) = 0$ for almost all x_{N+1} since $m_N(B^i - B_0^i) = 0$ $(1 \leqslant i \leqslant N)$. Consequently the conditions of proposition 1 hold for almost all the sections $B(x_{N+1})$. Let us denote by A the set of the exceptional values x_{N+1} for which the referred conditions don't hold. Put

$$\bar{g}(x) = \begin{cases} g(x) & \text{if } x \in B\,, \\ 0 & \text{if } x \notin B\,. \end{cases}$$

If $x_{N+1} \notin A$ the function $g(x)$ is N-measurable in $B(x_{N+1})$ (by the induction hypothesis) and consequently $\bar{g}(x)$ is N-measurable on the corresponding hyperplane $\{x : x_{N+1} = \text{constant}\}$.

For each positive integer m fix a sequence a_j^m, j integer, such that $a_j^m \notin A$ and $0 < a_{j+1}^m - a_j^m < 1/m$. For each x_{N+1} define $a_{j(x_{N+1})}^m$ by $a_{j(x_{N+1})}^m \leqslant x_{N+1} < a_{j(x_{N+1})+1}^m$ and put

$$\bar{g}_m(x^{N+1}, x_{N+1}) = \bar{g}(x^{N+1}, a_{j(x_{N+1})}^m)\,.$$

Obviously $\bar{g}_m$ is $(N+1)$-measurable in $\boldsymbol{R}^{N+1}$. Fix now $x \in B$ such that $x^{N+1} \in B_0^{N+1}$. For m sufficiently large $\bar{g}_m(x^{N+1}, x_{N+1}) = g(x^{N+1}, a_{j(x_{N+1})}^m)$ since $\omega(x^{N+1})$ is open. On the other hand since the restriction of $g(x)$ to $\omega(x^{N+1})$ is continuous and $\lim_{m \to +\infty} a_{j(x_{N+1})}^m = x_{N+1}$ we have

$$\lim_{m \to +\infty} \bar{g}_m(x) = g(x)\,. \tag{1}$$

Therefore (1) holds for almost all $x \in B$.

Appendix III.

We shall verify that (6.5) holds if $u \in W^{2,2}(\Omega)$ and $w \in W^{1,q}(\Omega)$. Consider two sequences of regular functions, say $C^2(\bar{\Omega})$ functions, $u_m(x)$ and $w_m(x)$ such that $u_m(x) \to u(x)$ in $W^{2,2}(\Omega)$ and $w_m(x) \to w(x)$ in $W^{1,q}(\Omega)$. Obviously (6.5) holds with u and w replaced by u_m and w_m respectively; by passing to the limit when $m \to +\infty$ we obtain (6.5) as follows from the following remarks: In the first integral $\Delta u_m \to \Delta u$ in $L^2(\Omega)$ and $w_m \to w$ in $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$.

In the second integral $\partial w_m/\partial x_i \to \partial w/\partial x_i$ in $L^q(\Omega)$ and $\partial u_m/\partial x_i \to \partial u/\partial x_i$ in $W^{1,2}(\Omega) \hookrightarrow L^{q'},(\Omega)$. In the third integral $\partial u_m/\partial n \to \partial u/\partial n$ in $W^{\frac{1}{2},2}(\Gamma)$ and $w_m \to w$ in $W^{1-1/q,q}(\Omega)$. If $N > 2$ the embedding theorems for fractionary Sobolev spaces give $W^{\frac{1}{2},2}(\Gamma) \hookrightarrow L^{r'}(\Gamma)$ and $W^{1-1/q,q}(\Gamma) \hookrightarrow L^r(\Gamma)$ with $r = 2(N-1)/N$. If $N = 2$ the same inclusions hold for $r = q$. Finally if $N = 1$ (6.5) is nothing but the usual integration by parts formulae since w and du/dx are then a.c. on Ω.

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