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Maximal Submanifolds and Submanifolds with Constant Mean Extrinsic Curvature of a Lorentzian Manifold.

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dedicated to Jean Leray

1. – Introduction.

The existence of a maximal submanifold (with respect to area) is an important property for a space time, hyperbolic riemannian manifold satisfying Einstein equations.

On such an initial submanifold the system of constraints can be split into a linear system and a non linear equation, following conformal techniques initiated by Lichnerowicz [5] and developped in [8] and also [12]. The solution of the initial value problem, fundamental in General Relativity, then rests on the global solution of this non linear elliptic equation on the initial 3-manifold. Theorems of existence, uniqueness, or non existence in the presence of inadequate sources, can be proved for this equation (cf. [9]), using a method given in [3], essentially based on Leray-Schauder degree theory [1], [2].

The existence of a maximal submanifold is also essential in the proof of the positivity conjecture for the gravitational mass of an asymptotically flat space time (cf. [13], [14]).

We prove in this paper some theorems concerning the existence, uniqueness, or non existence, of maximal submanifolds, and more generally of submanifolds with given mean extrinsic curvature.

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2. – Definitions.

We consider a differentiable (C^∞) manifold V_l of dimension l endowed with a lorentzian metric g, pseudo-riemannian metric of signature $(+, --, \cdots)$, which we suppose time-orientable.

Let S be a (C^∞), $l-1$ submanifold of V_l. We suppose S space-like and we denote by n_S its unitary time oriented normal. We denote by g_S the (negative definite) riemannian metric induced by g on S and by K_S the second fundamental form (extrinsic curvature) of S as a submanifold of (V_l, g); K_S is a symmetric 2-tensor field on S given by:

$$\boldsymbol{K}_S = -\tfrac{1}{2}\pi\mathcal{L}_n g$$

where $\mathcal{L}$ is the Lie derivative operator, n a differentiable vector field in a neighborhood of S, identical with n_S on S, π the canonical projection (with respect to g) from the space of covariant 2-tensors on V_n onto the space of covariant 2-tensors on S, K_S does not depend on the choice of n.

The mean extrinsic curvature of S in (V_n, g) is a function on S:

$$P_S = \operatorname{tr}\boldsymbol{K}_S = -\operatorname{div} n$$

where the divergence is relative to the metric g.

Another equivalent expression for P_S is:

$$P_S = g(\tau, n_S)$$

where $\tau = \delta f^*$ is the vector field of tensions on S relative to its pseudo-riemannian immersion f in (V_l, g) (cf. Eells and Sampson [15], Lichnerowicz [6]).

The submanifold S is said to be maximal, the immersion f is an harmonic mapping from (S, g_S) into (V_n, g), if on S:

$$P_S = 0\,.$$

3. – Local coordinates.

If the equation of S in local coordinates is

$$s(x^\alpha) = 0\,,$$

we set

$$\mu_S = g^{\lambda\mu}\partial_\lambda s \partial_\mu s > 0\,, \qquad \partial_\lambda = \frac{\partial}{\partial x^\lambda}\,, \qquad n_\alpha = (\mu_S)^{-\frac{1}{2}}\partial_\alpha s$$

then

$$P_S = -\mu_S^{-\frac{1}{2}}\gamma_S^{\alpha\beta}\nabla_\alpha \partial_\beta s\,, \qquad \gamma_S^{\alpha\beta} = g^{\alpha\beta} - n_S^\alpha n_S^\beta\,.$$

4. – Mean extrinsic curvature as a function of g and S.

Let V_l be given, with a lorentzian metric $\hat{g}$ time orientable. Let S_0 be an $l-1$ submanifold of V_l, spatial for $\hat{g}$. We identify a neighborhood U of S_0 in V_l with an open set Ω of $S_0 \times \mathbf{R}$, through the trajectories of a vector field $\hat{n}$, identical with $\hat{n}_{S_0}$ on S_0. We define a family of $l-1$ submanifolds $S_\varphi \subset U$, called t-homotopic to S_0, as the submanifolds of V_l with the equation, in the above identification

$$t = \varphi(x)\,, \qquad x \in S_0\,,\ (x, t) \in \Omega\,.$$

Let now g be another lorentzian metric on V_l; the mean extrinsic curvature of S_φ as a submanifold of (V_l, g) is a function P_{S_φ}; of g and φ, with values in the space of functions on S_0.

If (x^i) are local coordinates on S_0, and $(x^\alpha) = (x^0, x^i)$, $x^0 = t$ the corresponding local coordinates in U, we have, still denoting by φ the expression of $x \mapsto \varphi(x)$ in local coordinates:

$$P_{S_\varphi} \equiv P(g, \varphi) = \{\mu_\varphi^{-\frac{1}{2}}(\gamma_\varphi^{ij}\partial_{ij}^2\varphi - \gamma_\varphi^{\alpha\beta}\Gamma_{\alpha\beta}^i\,\partial_i\varphi + \gamma_\varphi^{\alpha\beta}\Gamma_{\alpha\beta}^0)\} \tag{1}$$

with $\Gamma_{\alpha\beta}^\lambda$ Christoffel symbols of g,

$$\mu_\varphi \equiv g^{00} - 2g^{0i}\partial_i\varphi + g^{ij}\partial_i\varphi\,\partial_j\varphi > 0\,,$$
$$\gamma_\varphi^{\alpha\beta} = g^{\alpha\beta} - n_{S_\varphi}^\alpha n_{S_\varphi}^\beta$$

in particular γ_φ^{ij} are the contravariant components of the metric induced on S_φ by g, γ_φ^{00} and γ_φ^{0i} are respectively a scalar and a vector field on S_φ. If we set:

$$V = (g^{00})^{-\frac{1}{2}}\,, \qquad \gamma^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}$$

then

$$\gamma_\varphi^{00} = \mu_\varphi^{-1} V^{-2} \gamma^{ij} \partial_i \varphi \, \partial_j \varphi ,$$
$$\gamma_\varphi^{0i} = \mu_\varphi^{-1} (V^{-2} \gamma^{ij} \partial_j \varphi + g^{0i} \gamma^{hj} \partial_h \varphi \, \partial_j \varphi - g^{0j} \gamma^{ih} \partial_j \varphi \, \partial_h \varphi)$$
$$\gamma_\varphi^{ij} = g^{ij} - \mu_\varphi^{-1} (g^{0i} - g^{ih} \partial_h \varphi)(g^{0j} - g^{jk} \partial_k \varphi) .$$

If φ is the constant map $x \mapsto \varphi(x) \equiv \tau$, a constant, then $\gamma_\tau^{00} = 0$, $\gamma_\tau^{0i} = 0$; $\gamma_\tau^{ij} = \gamma^{ij}$ and:

$$P(g, \tau) = \{V \gamma^{ij} \Gamma_{ij}^0\}_{x_0 = \tau} .$$

5. – Existence theorems.

DEFINITION. Let F and G be Banach spaces of scalar functions on S_0 and E be a Banach space of 2-tensor fields on Ω. The triple (E, F, G) is said to be $\hat{g}$ regular if there exists a neighborhood $X \times Y \subset E \times F$ of $(0, 0)$ such that $(g - \hat{g}, \varphi) \mapsto P(g, \varphi)$ is a C^1 mapping from $X \times Y$ into G, whose partial derivative $P'_\varphi(\hat{g}, 0)$ at $g - \hat{g} = 0$, $\varphi = 0$ is the operator of formal linearization.

The *formal linearization* (with respect to φ) of $P(g, \varphi)$ is a linear partial differential operator which, for $g = \hat{g}$, $\varphi = 0$, has the following expression in the chosen coordinates (where $\hat{g}^{0i} = 0$, $\hat{g}^{00} = V^{-2}$):

$$P'_\varphi(\hat{g}, 0) \cdot \psi = \{V(\hat{g}^{ij} \partial^2_{ij} \psi - \hat{g}^{hk} \hat{\Gamma}^i_{hk} \partial_i \psi)\}_{x^0 = \varphi(x) = 0}$$
$$+ \left\{ \frac{\partial}{\partial x^0} (\hat{\mu}_\varphi^{-1/2} \hat{\gamma}_\varphi^{\alpha\beta} \Gamma^0_{\alpha} \ )_{\partial_i \varphi = 0} \right\}_{x^0 = \varphi(x) = 0} \psi$$
$$+ \left\{ \frac{\partial}{\partial(\partial_i \varphi)} (\hat{\mu}_\varphi^{-1/2} \hat{\gamma}_\varphi^{\alpha\beta} \Gamma^0_{\alpha\beta})_{x^0 = \varphi(x) = 0} \right\}_{\partial_i \varphi = 0} \partial_i \psi .$$

But

$$(\hat{\mu}_\varphi^{-\frac{1}{2}} \hat{\gamma}_\varphi^{\alpha\beta} \hat{\Gamma}^0_{\alpha\beta})_{\partial_i \varphi = 0} = (\hat{V}^{-1} \hat{\gamma}^{ij} \hat{\Gamma}^0_{ij})_{x^0 = \tau} = P(\hat{g}, \tau) .$$

If $\widehat{\text{Ricc}}$ is the Ricci tensor of $\hat{g}$, we have the identity (cf. [1], [6]),

$$\left\{ \frac{\partial}{\partial \tau} P(\hat{g}, \tau) \right\}_{\tau = 0} = \hat{\Delta}_0 V + V \hat{K}_{S_0} \cdot \hat{K}_{S_0} + V \widehat{\text{Ricc}} (n_{S_0}, n_{S_0})$$

where $\hat{\Delta}_0$ is the Laplace operator of the metric $\hat{g}_0$ induced by $\hat{g}$ on S_0.

On the other hand, because γ_φ^{00} and γ_φ^{ij} depend only on $\partial_i \varphi$ through quadratic terms if $g^{0i} = 0$, we have:

$$\left\{ \frac{\partial}{\partial(\partial_i \varphi)} (\hat{\mu}_\varphi^{-1/2} \hat{\gamma}_\varphi^{\alpha\beta} \hat{\Gamma}^0_{\alpha\beta}) \right\}_{\partial_i \varphi = 0} = 2 \left(\hat{V} \frac{\partial}{\partial(\partial_i \varphi)} \hat{\gamma}_\varphi^{0j} \hat{\Gamma}^0_{0j} \right)_{\partial_i \varphi = 0} = 2 \hat{g}^{ij} \partial_i V$$

finally the formal linearization is (1):

$$P'_\varphi(\hat{g}, 0)\cdot\psi = \hat{\Delta}_0(V\psi) + V(K_{S_0}\cdot K_{S_0} + \widehat{\operatorname{Ricc}}(n_{S_0}, n_{S_0}))\psi\ .$$

6. – Case o S_0 compact space like section for $\hat{g}$.

If S_0 is a compact submanifold of V_l, everywhere spatial for $\hat{g}$, with or without boundary, the following triple of Banach spaces is $\hat{g}$ regular, for any $\hat{g}$ which is itself in E:

I) $E = 2$ tensors fields in the Hölder classes $C^{2,\alpha}(\Omega)$;

F scalar functions in the Hölder classes $C^{2,\alpha}(S_0)$ (vanishing on the boundary ∂S_0 if ∂S_0 is not empty);

G scalar functions in the Hölder classes $C^{0,\alpha}(S_0)$.

Another interesting $\hat{g}$ regular triple, for $\hat{g}\in E$ is in the physical case $n = 4$.

II) E: 2 tensor fields in the Sobolev space $H_3(\Omega)$;

F: scalar functions in the Sobolev spaces $\mathring{H}_3(S_0)$ (which is identical with $H_3(S_0)$ if $\partial S_0 = 0$);

G: scalar functions in $H_1(S_0)$.

Note. The hypothesis on g and $\hat{g}$ are made after the identification of a neighborhood U of S_0 in V_l with $\Omega \subset \mathbf{R}\times S_0$ through the trajectories of a vector field $\hat{n}$ normal to S_0 with respect to $\hat{g}$.

In the cases I and II the following lemma is valid (recall that $\hat{g}_0$ is negative definite).

Isomorphism Lemma. The linear mapping $\psi \mapsto P'_\varphi(\hat{g}, 0).\ \psi$ is an isomorphism between F and G under one of the following hypothesis:

1) ∂S_0 is not empty and, on S_0:

$$\hat{K}_{S_0}\cdot\hat{K}_{S_0} + \widehat{\operatorname{Ricc}}(n_{S_0}, n_{S_0}) \geqslant 0$$

2) S_0 has empty boundary and on S_0:

$$\hat{K}_{S_0}\cdot\hat{K}_{S_0} + \widehat{\operatorname{Ricc}}(n_{S_0}, n_{S_0}) \geqslant 0 \quad \text{and} \quad \neq 0\ .$$

(1) Such an expression with $V = 1$ (identification through geodesics) has been obtained, by a different method, in the properly riemannian case by Duschek [16].

Except when $\hat{K}_{S_0} \equiv 0$ (i.e. S_0 is a totally geodesic submanifold of V_l) we have:

$$\hat{K}_{S_0} \cdot \hat{K}_{S_0} \geqslant 0 , \qquad \hat{K}_{S_0} \cdot \hat{K}_{S_0} \not\equiv 0 .$$

On the other hand if $\hat{g}$ satisfies on V_l the Einstein equations (physical case: $n = 4$)

$$\widehat{\mathrm{Ricc}} = T - \tfrac{1}{2} \hat{g} \operatorname{tr} T$$

where T is a given 2-tensor (stress energy tensor) we have in all realistic physical situations

(2) $$\widehat{\mathrm{Ricc}}(u, u) \geqslant 0 \quad \text{for every time like vector } u$$

with

$$\widehat{\mathrm{Ricc}}(u, u) = 0$$

only in regions devoid of energy sources.

Definition. A lorentzian manifold $(V_l, \hat{g})$, time orientable and with a Ricci tensor satisfying (2) is called a space-time. A space-time $(V_l, \hat{g})$ is source free if $\widehat{\mathrm{Ricc}} \equiv 0$.

When the mapping $P'_\varphi(\hat{g}, 0)$ is an isomorphism between F and G we can apply to the C^1 mapping $(g, \varphi) \mapsto P(g, \varphi)$ the implicit function theorem. Thus we have proved.

Theorem 1. Let $(V_l, \hat{g})$ be a space-time, and S_0 be a compact space like submanifold of V_l with mean extrinsic curvature C in the metric $\hat{g}$. Let (E, F, G) be a $\hat{g}$ regular triple of Banach spaces, $C \in G$. If S_0 is not a totally geodesic submanifold without boundary of a source free space time $(V_l, \hat{g})$, then there exists a neighborhood X of 0 in E such that for every g with $g - \hat{g} \in X$ the lorentzian manifold (V_l, g) admits a space like submanifold S_φ, $\varphi \in F$, with mean curvature C.

The case $C = 0$ gives the following corollary, valid in the same function spaces:

Corollary. If a space-time $(V_l, \hat{g})$ admits a compact space like maximal submanifold, which is not a totally geodesic submanifold without boundary for a source-free space-time, all neighbouring lorentzian manifolds have the same property.

When a source-free space time admits a closed spatial submanifold which is totally geodesic it is not always true that all neighbouring space-times admit a maximal submanifold. We shall prove that they admit a submanifold of constant mean curvature, but the constant cannot be given a priori.

THEOREM. Let $(V_l, \hat{g})$ be a source-free space-time, admitting a totally geodesic space like submanifold, compact without boundary S_0, let E, F, G be a $\hat{g}$ regular triple of Banach spaces. There exists a neighborhood X of zero in E such that if $g-\hat{g}\in X$, then (V_l, g) admits a space like submanifolds, S_φ, $\varphi \in F$, with constant mean curvature.

PROOF. Denote by $\tilde{F}$ [resp. $\tilde{G}$] the subspace of F [resp. G] of functions with vanishing integral on S_0. Consider the C^1 mapping $\tilde{P}$ from $X\times\tilde{Y}$ to $\tilde{Z}$ (with $\tilde{Y}$, $\tilde{Z}$ open sets of $\tilde{F}$, $\tilde{G}$) defined by:

$$\tilde{P}\colon (g, \varphi) \mapsto P(g, \varphi) - (\widehat{\mathrm{Vol}}\, S_0)^{-1}\int_{S_0} P(g, \varphi)\hat{\eta}_0$$

($\hat{\eta}_0$ and $\widehat{\mathrm{Vol}}\, S_0$, volume form, and total volume of S_0 in the metric induced by $\hat{g}$)

$\tilde{P}$ is a C^1 mapping, with derivative P'_φ at the point $g-\hat{g}=0$, $\varphi=0$ the Laplace operator $\hat{\Delta}_0$, therefore an isomorphism from $\tilde{F}$ onto $\tilde{G}$. The implicit function theorem applied to the equation

$$\tilde{P}(g, \varphi) = 0$$

gives the result.

7. – Case of a non compact S_0.

The general case of a non compact S_0 in a general lorentzian manifold (V_l, g) is difficult to handdle, due to the lack of sufficient knowledge of the invertibility of the corresponding Laplace operator. However a case of particular physical interest is the case where S_0 is diffeomorphic to $\mathbf{R}^3$ and the metrics g_S' induced on S_0 by g are « asymptotically euclidean ». We then have (cf. [20]).

LEMMA. The following triple of Banach spaces is η-regular (η Minkowski metric on $\mathbf{R}^3\times\mathbf{R}$).

1) $h = g - \eta$ is in E if $h \in C^{2,\alpha}(\mathbf{R}^3\times\mathbf{R})$ and

$$\sup_{x^0\in\mathbf{R}}\ \sup_{x\in\mathbf{R}^3} |x|^{2+a}|D^\lambda h_{\alpha\beta}| < \infty\,, \quad 0 \leqslant |\lambda| \leqslant 2\,,\ 0 < a < 1;$$

2) φ is in F if $\varphi \in C^{2,\alpha}(\mathbf{R}^3)$ and

$$\operatorname*{Sup}_{x\in\mathbf{R}^3} \left(|x|^a\varphi| + |x|^{1+a}|D^i\varphi| + |x|^{2+a}|\Delta_0\varphi|\right) < \infty ;$$

3) G is the Banach space of functions $f \in C^{0,\alpha}(\mathbf{R}^3)$ such that

$$\operatorname*{Sup}_{x\in\mathbf{R}^3} |x|^{2+a}|f| < \infty .$$

The norm in G is

$$\|f\|_G = \|f\|_{C^{0,\alpha}(\mathbf{R}^3)} + \operatorname*{Sup}_{x\in\mathbf{R}^3} |x|^{2+a}|f|$$

with

$$\|f\|_{C^{0,\alpha}(\mathbf{R}^3)} = \operatorname*{Sup}_{x\in\mathbf{R}^3} |f| + \operatorname*{Sup}_{\substack{x\neq y\\ |x-y|\leqslant 1}} \frac{|f(x)-f(y)|}{|x-y|}$$

the norms in E and F are defined analogously.

PROOF. $(\eta - g, \varphi) \mapsto P(g, \varphi)$ is a C^1 mapping from a neighborhood of zero in $(E\times F)$ into G. Its partial derivative with respect to φ, at $g - \eta = 0$, $\varphi = 0$, spatial hyperplane of the Minkovski space time, is the flat space Laplace operator, isomorphism from F onto G. Therefore:

THEOREM. Every lorentzian manifold $(\mathbf{R}^4, g)$ in an E-neighborhood of the Minkovski space time $(\mathbf{R}^4, \eta)$ admits a maximal space like submanifold.

Indications for an alternate proof of this theorem, in weighted Sobolev spaces, has been given (cf. [21]).

8. – Non existence theorem.

DEFINITION. A lorentzian manifold (V_l, g) is said to admit a slicing if there exists a diffeomorphism Λ between V_l and a product $S_0\times\mathbf{R}$

$$\Lambda\colon V_l \mapsto S_0\times\mathbf{R} \quad \text{by } y \mapsto (x, t)$$

such that the submanifolds $S_\tau = \Lambda^{-1}(\{t=\tau\})$ are space like for g. The submanifolds S_τ, are the slices defining the slicing, the space like submanifolds S_φ (equation $t = \varphi(x)$ in the image by $\Lambda($ with φ non constant are also called slices.

THEOREM. If a lorentzian manifold (V_l, g) admits a slicing by slices compact without boundary of mean extrinsic curvature (m.e.c.) uniformly bounded below [resp. above] by k, it admits no slice with m.e.c. less than k [resp. greater than k].

PROOF. The function φ defining a slice S_φ, by $t = \varphi(x)$ attains on S_0 its maximum and its minimum. At an extremum point $\bar{x}$ of $\varphi, \bar{t} = \varphi(\bar{x})$, the m.e.c. of S_φ is:

$$P(g, \varphi)(\bar{x}) = \{(g^{00})^{-\frac{1}{2}}(\gamma^{ij}\partial^2_{ij}\varphi + \gamma^{ij}\Gamma^0_{ij})\}_{\substack{x=\bar{x}\\ t=\bar{t}}}$$

but

$$(\gamma^{ij}\Gamma^0_{ij})_{\substack{x=\bar{x}\\ t=\bar{t}}} = P(g, \bar{t})(\bar{x}) \geqslant k \qquad [\text{resp. } \leqslant k] .$$

Thus if $P(g, \varphi)(x) < k$ [resp. $> k$] we have, at an extremum point of φ:

$$\gamma^{ij}\partial^2_{ij}\varphi < 0 \qquad [\text{resp. } > 0]$$

which is impossible in a maximum [resp. minimum], since γ^{ij} is negative definite.

EXAMPLE. The metric product $T_{l-1}\times \mathbf{R}$, with T_{l-1} flat torus, admits no slice with every where positive, or everywhere negative, mean extrinsic curvature.

9. – Uniqueness theorems.

DEFINITION. A space-time (V_l, g) is said to be nowhere source-free if $\operatorname{Ricc}(g)(u, u) > 0$ for every time like vector u.

Given a space like submanifold S_0 of a lorentzian manifold (V_l, g) there exists always a neighborhood U of S_0 in which the geodesics normal to S_0 define a diffeomorphism between U and $\Omega \subset S_0 \times \mathbf{R}$. Such a neighborhood U is called a *gaussian neighborhood.* The submanifolds S_φ, $t = \varphi(x)$ are defined through this identification.

THEOREM. If a nowhere source free space time of class C^2 admits a space like submanifold S_0 compact without boundary, with mean extrinsic curvature a constant k, it admits no other space like submanifold S with m.e.c. k in a gaussian neighborhood of S_0.

Note. If S_0 is compact with boundary, the same theorem is valid, if $\varphi = 0$ on ∂S_0.

PROOF. The mean curvature $P(g, \varphi)$ of S_φ is given by (1). If S_0 is compact φ takes on S_0 its maximum and its minimum. If $\varphi \not\equiv 0$ either its maximum is positive, either its minimum is negative.

Let $x = \bar{x}$ be an extremum for φ. Set $\bar{t} = \varphi(\bar{x})$. For $x = \bar{x}$, $\partial_i \varphi = 0$, thus, if $P(g, \varphi) = k$:

$$P(g, \varphi)(\bar{x}) = (g^{ij}\partial^2_{ij}\varphi + g^{ij}\Gamma^0_{ij})_{\substack{x=\bar{x}\\ x^0\equiv t=\bar{t}}} = k$$

with, in gaussian coordinates ($g_{00} = 1$, $g_{0i} = 0$)

$$(g^{ij}\Gamma^0_{ij})_{\substack{x=\bar{x}\\ x^0=\bar{t}}} = P(g, \bar{t})(\bar{x})$$

if g is C^2 whe have:

$$P(g, \tau)(x) = P(g, 0)(x) + \tau\left\{\frac{\partial}{\partial\tau}P(g, \tau)(x)\right\}_{\tau=h}, \qquad 0 \leqslant h \leqslant \tau$$

with, if S_0 has curvature k,

$$P(g, 0)(x) = k$$

but, as we have recalled before

$$\frac{\partial P}{\partial t}(g, h) = K_{S_h}\cdot K_{S_h} + \operatorname{Ricc}(g)(n_{S_h}, n_{S_h})$$

thus, under the hypothesis we have made

$$\frac{\partial P}{\partial t}(g, h) > 0 \qquad \forall x \in S_0 .$$

and, at a positive maximum ($\bar{t} > 0$) [resp. negative minimum $\bar{t} < 0$]

$$(g^{ij}\partial^2_{ij}\varphi)_{\substack{x=\bar{x}\\ x^0=\bar{t}}} = -\bar{t}\frac{\partial P}{\partial t}(g, h)\,\bar{x} < 0 \qquad [\text{resp.} > 0]$$

which is incompatible with $\bar{x}$ a maximum, [resp. a minimum] since g^{ij} is negative definite.

The same theorem is true (same proof) when S_0 is compact with boundary and $\varphi(x) = 0$ for $x \in \partial S_0$.

If S_0 is not compact we say that $\varphi\colon S_0 \to R$ tens to zero at infinity if, given any $\varepsilon > 0$ there exists a compact $K \subset S_0$ such that

$$\operatorname*{Sup}_{x \in \complement K} |\varphi(x)| < \varepsilon .$$

If φ is not identically zero it takes on S_0 a positive value a [or a negative value]. We choose K such that $\operatorname*{Sup}_{x\in\complement K} |\varphi(x)| < a$, and we conclude that φ attains a maximum at an interior point of K — the same reasoning than in the above theorem proves the contradiction when $\varphi \not\equiv 0$ defines like $\varphi = 0$ a submanifold of m.e. curvature k.

We remark on the other hand that the requirement for the validity of the proof is

$$K_{S_h} \cdot K_{S_h} + \operatorname{Ricc}(g)(n_{S_h}, n_{S_h}) > 0$$

This inequality will be certainly satisfied if $\operatorname{Ricc}(g)(u, u) \geqslant 0$ for every time like vector u, and $K_{S_h} \cdot K_{S_h} > 0$ on the family of submanifolds S_h. A case of physical interest is the following uniqueness theorem.

THEOREM. The mass hyperboloïd $(x^0)^2 - \Sigma(x^i)^2 = m^2$, $x^0 > 0$, is the only slice in the quadrant $(x^0)^2 - \Sigma(x^i)^2 > 0$, $x^0 > 0$ of Minkovski space time with constant mean curvature $-1/3m$, such that $(x^0)^2 - \Sigma(x^i)^2$ tends to m^2 at infinity on this slice.

PROOF. Take polar coordinates in the open set $(x^0)^2 - \Sigma(x^i)^2 > 0$, $x^0 > 0$ of Minkovski space, by setting first

$$\varrho^2 = (x^0)^2 - \Sigma(x^i)^2$$

and denoting by u three angular coordinates.

The mean curvature of S_ϱ ($\varrho = \text{cte}$) is easily computed to be

$$P(\varrho) = -\frac{3}{\varrho}.$$

We then define a Cauchy surface in $(x^0)^2 - \Sigma(x^i)^2 > 0$, $x^0 > 0$ by $\varrho = \varphi(u)$. We apply the preceding reasoning, but here we can compute directly

$$\frac{\partial P}{\partial \varrho} = \frac{3}{\varrho^2} > 0 \qquad \forall \varrho .$$

10. – Robertson Walker manifolds.

A Robertson Walker manifold is a product $V_{l-1} \times \mathbb{R}$, with a lorentzian metric of the form

$$ds^2 = dt^2 - f(t)\, d\bar{s}^2$$

where f is a positive C^1 function on $\mathbf{R}$ and $d\bar{s}^2$ a (positive) C^1 riemannian metric $\bar{g}$ on $V_{l-1} \equiv S_0$.

A simple computation shows that the mean curvature of S_φ: $t = \varphi(x)$ is:

$$P_\varphi \equiv -f^{-1}(\bar{\delta}(\mu^{-\frac{1}{2}} d\varphi)) + \tfrac{1}{2}\mu^{-3/2} f'(3 - 4f^{-1}\bar{g}(d\varphi, d\varphi))$$

For the submanifold $t = c$ [constant], the equation $P_c = 0$ reduces to

$$f'(c) = 0 .$$

But, if $f'(c) = 0$, the submanifold $t = c$ is totally geodesic [$K_c = 0$]. More generally we shall prove:

THEOREM. If $V_l = S_0 \times \mathbf{R}$ is a Robertson Walker manifold, with S_0 compact without boundary, then V_l admits a maximal space like section S if and only if it admits a totally geodesic submanifold.

PROOF. If S_0 is compact without boundary φ attains on S_0 its maximum $\bar{t}_2$ at a point $\bar{x}_2$, and its minimum $\bar{t}_1$ at a point $\bar{x}_1$: But, at an extremum of φ and if S is a maximal submanifold:

$$(\bar{g}^{ij}\partial^2_{ij}\varphi)_{\substack{x=\bar{x} \\ t=\bar{t}}} = -\tfrac{3}{2} f'(\bar{t}) .$$

Therefore we must have:

$$(i) \qquad f'(\bar{t}_1) \leqslant 0 , \quad f'(\bar{t}_2) \geqslant 0 , \quad \bar{t}_1 \leqslant \bar{t}_2$$

and f' must vanish at least at one point $t = c$. The manifold $t = c$ is totally geodesic.

REMARK. This theorem is valid without any assumption on $\mathrm{Ricc}(g)$. The theorem does not say that a maximal submanifold is necessarily the manifold $t = c$, with $f'(c) = 0$. However by the uniqueness theorem proven before this conclusion follows if V_l is a nowhere source free space time. We remark here that the inequality $\mathrm{Ricc}(g)(u, u) \geqslant 0$, for all time like u, implies

$$f'' \leqslant 0 .$$

But (i) implies

$$f'(\bar{t}_1) = f'(\bar{t}_2) = 0$$

and $\bar{t}_1 = \bar{t}_2$ if f'' does not vanish on an interval of $\mathbf{R}$, thus

$$\varphi = \text{constant}, \qquad \text{uniquely determined}.$$

If $f'' \leqslant 0$ and f'' vanishes on an interval of $\mathbf{R}$ we may have $\bar{t}_1 < \bar{t}_2$, but then $f'(\bar{t}_1) = f'(\bar{t}_2) = 0$ implies $f'(t) = 0$ for $t \in (\bar{t}_1, \bar{t}_2)$, thus $f(t)$ is a constant on this interval, and the metric is static. But it is known (Lichnerowicz [4]) that there does not exist non flat static space times with closed space like sections. We have proved:

THEOREM. A non flat Robertson Walker space time $V_{l-1} \times R$ with V_{l-1} closed admits at most one maximal slice, which is totally geodesic.

The flat case will be included it the study of the next paragraph.

11. – Lorentzian manifolds with constant curvature.

The second fundamental form $\boldsymbol{K}$ of a maximal submanifold S of a lorentzian manifold (V_l, g) with constant curvature c is a solution of the equation (cf. for the properly riemannian case J. Simons [17], S. S. Chern [18])

$$\overline{\nabla}^k \overline{\nabla} K_{ij} + (\boldsymbol{K}\cdot\boldsymbol{K} - c) K_{ij} = 0$$

($\overline{\nabla}$ covariant derivation in the metric $\bar{g}$ induced on S by g, $\bar{g}$ is negative definite if S is space like)

Thus, if we set $H^2 = \boldsymbol{K}\cdot\boldsymbol{K} \geqslant 0$:

$$\overline{\nabla}^k \overline{\nabla}_k H^2 = \overline{\nabla}^k K^{ij} \overline{\nabla} K_i - (H^2 - c) H^2 .$$

The right hand side is $\leqslant 0$ if $H^2 \geqslant c$. We deduce immediately by integration on S, the following theorem:

THEOREM. Let S be a closed, space-like, maximal submanifold of a lorentzian manifold (V_l, g) with constant curvature c then

1) if $c \geqslant 0$ and $H^2 = K\cdot K \geqslant c$ then $H^2 \equiv c$ and $\overline{\nabla}_k K_{ij} = 0$.

2) if $c \leqslant 0$ then $H^2 = 0$, i.e. S is a totally geodesic submanifold.

In the case of $V_l = R^4$ and $c = 0$ (Minkovski space the same theorem (Bernstein theorem) has been proved, differently, by Calabi [19]).

12. – Maximization of area.

Let as before $(V_{l-1} \times \mathbf{R}, g)$ be a smooth lorentzian manifold. Denote by $\mathcal{A}(\varphi)$ the area (volume) of a smooth slice S_φ, in the positive metric $-\gamma_\varphi$

induced on S_φ by g:

$$\mathcal{A}(\varphi) = \int_{S_\varphi} i_{n_\varphi}\eta = \int_{S_\varphi} \eta_\varphi$$

(η volume element of g, η_φ volume element of $-\gamma_\varphi$) then (cf. Lichnerowicz [5], Avez [7]) the derivative of the mapping $\mathcal{A}$ is the linear mapping

$$\psi \mapsto \mathcal{A}'(\varphi)\cdot\psi = -\int_{S_\varphi} \psi P(g, \varphi)\eta_\varphi .$$

The critical points of $\mathcal{A}$ (extrema of $\varphi \mapsto \mathcal{A}(\varphi)$) are the maximal slices (slices of zero mean extrinsic curvature).

The second derivative of $\mathcal{A}$ is given by the quadratic form:

$$\mathcal{A}''(\varphi)\cdot(\psi, \psi) = -\int_{S_\varphi} \psi P'_\varphi(g, \varphi)\cdot\psi\eta_\varphi$$

thus (2), at $\varphi = 0$, if S_0 is compact without boundary, or if φ satisfies appropriate boundary [resp. asymptotic] conditions on ∂S_0 [resp. at infinity]

$$\{\mathcal{A}''(\varphi)(\psi, \psi)\}_{\varphi=0} = \int_{S_0} (\gamma_0(d\psi, d\psi) - \psi^2(K_0\cdot K_0 + \mathrm{Ricc}(g)(n_0, n_0))\eta_0 .$$

Where γ_0 is the negative definite metric of S_0. Therefore:

THEOREM. A maximal slice of a space time (V_l, g), $V_l = V_{l-1}\times R$ is a strict local (3) maximum for the area if it is not a totally geodesic submanifold of a source free space-time.

On the other hand if (V_l, g), $V_l = V_{l-1} + \mathbf{R}$, is a lorentzian manifold and S_0 a totally geodesic slice such that

$$\mathrm{Ricc}(g)(n_{S_0}, n_{S_0}) < 0$$

then S_0 can be deformed to a slice of greater area.

(2) The following expression is valid for $\{x\}\times\mathbf{R}$ g-orthogonal to S_0.

(3) Among all slices if it is closed, among slices with given boundary [resp. asymptotic] conditions otherwise.

EXAMPLE. Consider the lorentzian manifold $(S_3 \times \mathbf{R}, g)$ with g given by the De Sitter metric:

$$ds^2 = dt^2 - \alpha^2 \cos h^2(\alpha^{-1} t)\{dX^2 + \sin^2 X(d\theta^2 + \sin^2\theta\, d\varphi^2)\}\,.$$

The metric g has constant curvature and $\mathrm{Ricc}(g) = -3\alpha^{-2} g$. The instant of time symmetry $t = 0$ is a totally geodesic slice S_0 such that $\mathrm{Ricc}(g) \cdot \cdot (n_{S_0}, n_{S_0}) = -3\alpha^{-2}$. The area of S_0 is:

$$\sigma = 2\pi^2 \alpha^3$$

whereas the area of a slice $t = \varepsilon$, $\varepsilon > 0$ is:

$$\sigma_\varepsilon = 2\pi^2\alpha^3 \cos h^2(\alpha^{-1}\varepsilon) > \sigma\,.$$

Aknowledgement. Some of the theorems of this paper concerning the uniqueness of maximal submanifolds, and maximization of area, have also been obtained simultaneously and independently, by different methods by D. Brill and O'Flaherty. For their proofs and related results, and for a discussion of the physical implications see [22], [23], and references therein.

The theorems of this paper, with a sketch of the proofs have been announced in [11].

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