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Completely Monotone Families of Solutions of n-th Order Linear Differential Equations and Infinitely Divisible Distributions (*).

PHILIP HARTMAN (**)

dedicated to Hans Lewy

1. – Introduction.

It was shown in [9] that if $x = I_\mu(t)$ is the unique solution of the modified Bessel differential equation

$$t^2x'' + tx' - (t^2 + \mu^2)x = 0 \tag{1.1}$$

satisfying $x \sim (t/2)^\mu/\Gamma(1+\mu)$ as $t \to 0$, then $I_\mu(t)$ is a completely monotone function of $\lambda = \mu^2 \geqq 0$ (for fixed $t > 0$); for the definitions of the solutions $I_\mu(t)$, $K_\mu(t)$ of (1.1), see [15], pp. 77-80. Thus, by the theorem of Hausdorff-Bernstein (cf. [16], p. 160), there exists a (unique) distribution function $W(r) = W(r, t, \infty)$ on $r \geqq 0$ satisfying

$$I_\mu(t) = I_0(t)\int_0^\infty \exp(-r\mu^2)\, W(dr, t, \infty) \qquad \text{for } \mu \geqq 0\,,$$

so that $W(0) = 0$, $W(\infty) = 1$, $W(r-0) = W(r)$, and $W(dr) \geqq 0$. The results below will imply the following generalization:

THEOREM 1.0. *(a)* $0 < t < \tau \leqq \infty$. *Then* $I_\mu(t)/I_\mu(\tau)$ *is a completely monotone function of* $\lambda = \mu^2 \geqq 0$, *so that there exists a (unique) distribution func-*

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tion $W(r) = W(r, t, \tau)$ *on* $r \geqq 0$ *satisfying*

$$I_\mu(t)/I_\mu(\tau) = [I_0(t)/I_0(\tau)] \int_0^\infty \exp(-r\mu^2)\, W(dr, t, \tau) \qquad \text{for } \mu \geqq 0\,;$$

the distribution function

$$W(r) = W(r, t, \tau) \quad \text{is infinitely divisible} \tag{1.2}$$

for $0 < t \leqq \tau \leqq \infty$;

$$W(\cdot, \tau_1, \tau_n) = W(\cdot, \tau_1, \tau_2) * W(\cdot, \tau_2, \tau_3) * \ldots * W(\cdot, \tau_{n-1}, \tau_n) \qquad \text{for } \tau_1 < \ldots < \tau_n \tag{1.3}$$

with $\tau_1 > 0$, $\tau_n \leqq \infty$;

(1.4) $W(r, t, \tau)$ is non-decreasing in r and t, and non-increasing in τ;

$$W(r, t, \tau) \to 0 \quad \text{as } t \to +0 \tag{1.5}$$

for $0 < \tau \leqq \infty$;

$$W(r, t, \tau) \to W(r, t, \infty) \quad \text{as } \tau \to \infty \text{ for } r \geqq 0\,, \tag{1.6}$$

and $t > 0$; *and*

$$W(r, t, \tau) \to \delta_0(r) \quad \text{as } t\uparrow\sigma,\ \tau\downarrow\sigma \tag{1.7}$$

with $0 < \sigma < \infty$,

$$W(r, t, \tau) \to \delta_0(r) \quad \text{as } \tau > t \to \infty\,, \tag{1.8}$$

where $\delta_0(r)$ *is the unit distribution function* (*i.e.*, $\delta(0)=0$ *and* $\delta(r)=1$ *for* $r>0$).

(*b*) *Also*, $1/K_\mu(t)$ *and* $K_\mu(\tau)/K_\mu(t)$ *are completely monotone functions of* $\lambda = \mu^2 \geqq 0$ *for* $0 < t < \tau < \infty$, *so that there exists a distribution function* $W(r) = W(r, t, \tau)$ *on* $r \geqslant 0$ *satisfying*

$$1/K_\mu(t) = [1/K_0(t)] \int_0^\infty \exp(-r\mu^2)\, W(dr, t, \infty) \qquad \text{for } \mu \geqslant 0\,,$$

$$K_\mu(\tau)/K_\mu(t) = [K_0(\tau)/K_0(t)] \int_0^\infty \exp(-r\mu^2)\, W(dr, t, \tau) \qquad \text{for } \mu \geqq 0\,.$$

The distribution function $W(r) = W(r, t, \tau)$ *satisfies* (1.2), (1.3) *with* $\tau_1 > 0$, $\tau_n \leqslant \infty$; (1.4), (1.5) *for* $0 < \tau \leqq \infty$; (1.6); (1.7) *with* $0 < \sigma < \infty$; *and* (1.8).

The arguments of [9] can be used to show that if the definition of $W(r, t, \tau)$ for $r \geqq 0$, $t > 0$ is extended by putting $W(r, t, \tau) = 0$ for $r < 0$ and/or $t = 0$, then $W(r, t, \tau)$ is continuous for $-\infty < r < \infty$ and $0 \leqq t < \tau (\leqq \infty)$, of class C^∞ for $-\infty < r < \infty$, $0 < t < \tau (\leqq \infty)$, and satisfies the parabolic equation

$$t^2 W_{tt} + t W_t - t^2 W = W_r \,,$$

for fixed τ, on $-\infty < r < \infty$ and $0 < t < \tau (\leqq \infty)$.

The result of [9] concerning the complete monotony of $I_\mu(t)$ as a function of $\lambda = \mu^2 \geqq 0$ (for fixed $t > 0$) suggests the questions as to when a family of solutions $x = X(t, \lambda)$ of a 1-parameter family of differential equations

$$(1.9) \qquad D^n x + q_{n-1}(t) D^{n-1} x + \ldots + q_1(t) D x - q(t, \lambda) x = 0 \,, \qquad D = d/dt \,,$$

$(t, \lambda) \in T \times \Lambda$, is a completely monotone function of $\lambda \in \Lambda$ (for fixed $t \in T$) or when such a completely monotone family of solutions $x = X(t, \lambda)$ exists, and when the analogue of Theorem 1.0 is valid.

Notation. – Unless otherwise specified, an « interval » can be bounded or unbounded, and closed or open or neither. T is a t-interval with endpoints α and β, $-\infty \leqq \alpha < \beta \leqq \infty$ and $T^0 = (\alpha, \beta)$ is its interior. Λ is a λ-interval and Λ^0 its interior.

We shall make some of the following assumptions from time to time.

(A1) $q_1(t), \ldots, q_{n-1}(t) \in C^0(T)$ *and* $q(t, \lambda) \in C^0(T \times \Lambda)$. *For fixed* $\lambda \in \Lambda$, (1.9) *is disconjugate on* T; *i.e., no solution* $x(t) \not\equiv 0$ *has more than* $n-1$ *zeros on* T.

(A2_k) $\partial^m q / \partial \lambda^m$ *exists, is continuous, and satisfies* $(-1)^{m+k} \partial^m q / \partial \lambda^m \geqq 0$ *on* $T \times \Lambda^0$ *for* $m = 1, 2, \ldots$ *and a fixed* k *(in particular,* $(-1)^{k+1} \partial q(t, \cdot)/\partial \lambda$ *is completely monotone on* Λ^0 *for fixed* $t \in T$*).*

It is immaterial in assumption (A1) whether or not zeros are counted with multiplicities. For a more general result, see [6]; and for a simple proof for the linear case at hand, see [13].

When (A1) holds, (1.9) has first, second, ..., n-th principal solutions at $t = \beta$, say $x = \xi_1(t, \lambda), \ldots, \xi_n(t, \lambda)$; i.e., solutions satisfying $\xi_j(t, \lambda) \neq 0$ for t near β and $\xi_j(t, \lambda)/\xi_{j+1}(t, \lambda) \to 0$ as $t \to \beta$ for $j = 1, \ldots, n$; [7], pp. 328-347 or pp. 353-355 or [11]. (In [7], pp. 353-355, the assumption of « disconjugacy » is relaxed to « non-oscillatory at $t = \beta$ »). Furthermore, if $\tau \in T^0$, then there exist unique principal solutions $\eta_1(t, \tau, \lambda), \ldots, \eta_n(t, \tau, \lambda)$ at $t = \beta$ such that $x = \eta_j(t, \tau, \lambda)$ satisfies

$$(1.10) \qquad D^{i-1}x = 0 \quad \text{for } 1 \leqq i < j \quad \text{and} \quad D^{j-1}x = 1 \quad \text{at } t = \tau\,;$$

[7], Theorem 7.2_n(vii), p. 332. The solution $x = \eta_j(t, \tau, \lambda)$ is called the j-th *special principal solution* at $t=\beta$, determined by τ. A first principal solution $x = \xi_1(t, \lambda)$ is unique up to a non-zero constant factor and $\xi_1(t, \lambda) \neq 0$ for $t \in T^0$ ([7], Theorem 7.1_n(i), pp. 331-332), so that $\eta_1(t, \tau, \lambda) = \xi_1(t, \lambda)/\xi_1(\tau, \lambda)$. Sometimes it will be convenient to assume

($A3_\sigma$) *Let $\sigma = \alpha$ or $\sigma = \beta$. There exists a family of non-negative first principal solutions $x = \xi_1(t, \lambda)$ at $t = \beta$ satisfying*

$$(1.11_\sigma) \qquad \xi_1(t, \lambda)/\xi_1(t, \mu) \to 1 \quad \text{as } t \to \sigma \text{ for } \lambda, \mu \in \Lambda\,.$$

This assumption is equivalent to the requirement that

$$(1.12) \qquad h(\lambda, \mu) \equiv \lim_{t\to\sigma} x_1(t, \lambda)/x_1(t, \mu) \text{ exists and } \neq 0 \text{ for } \lambda, \mu \in \Lambda$$

holds for arbitrary first principal solutions $x_1(t, \lambda) \geqq 0$ of (1.9). For in this case, $\xi_1(t, \lambda) = x_1(t, \lambda)/h(\lambda, \nu)$, for a fixed $\nu \in \Lambda$, satisfies (1.11_σ). The asymptotic integration theory of (1.9) gives simple sufficient conditions for the validity of (1.12), hence of (A3). Also, if $T = (\alpha, \beta]$ is closed and bounded on the right and $x = \xi_1(t, \lambda)$ is the solution of (1.9) satisfying the initial conditions (1.10) with $j = n$ and $\tau = \beta$, then (1.11_β) holds; contrast part (b) of the following theorem with the result in the Appendix 2 below.

THEOREM 1.1. *Assume* (A1) *and* ($A2_k$) *for a fixed k, $0 < k < n$.* (a) *Let $\tau \in T^0$ and $x = \eta_{n-k}(t, \tau, \lambda)$ be the special* ($n-k$)-th *principal solution at $t = \beta$ determined by τ. Then $D^i\eta_{n-k}(t, \tau, \lambda) \in C^0(T \times T^0 \times \Lambda^0)$ for $i = 0, \ldots, n$, and $\eta_{n-k}(t, \tau, \cdot)$ is completely monotone on Λ^0, for fixed (t, τ) with $\alpha < \tau \leqq t < \beta$.* ($b$) *If, in addition, $n - k = 1$, $\Lambda^0 \supset [0, \infty)$, and $x = \xi_1(t, \lambda)$ is a first principal solution, then, for $\alpha < t \leqq \tau < \beta$, there exists a distribution function $W(r) = W(r, t, \tau)$ on $r \geqq 0$ such that*

$$(1.13) \qquad \xi_1(\tau, \lambda)/\xi_1(t, \lambda) = [\xi_1(\tau, 0)/\xi_1(t, 0)]\int_0^\infty \exp(-\lambda r)\, W(dr, t, \tau) \quad \text{for } \lambda \geqq 0\,.$$

The distribution function $W(r) = W(r, t, \tau)$ satisfies (1.2), (1.3) *with $\tau_1 > \alpha$ and $\tau_n < \beta$;* (1.4); *and* (1.7) *with $\alpha < \sigma < \beta$.*

REMARK 1. In Theorem 1.1(b), condition « $\Lambda^0 \supset [0, \infty)$ » can be replaced by « $\Lambda = (0, \infty)$ » provided that $\xi_1(t, \lambda)$, defined on $T \times \Lambda = T \times (0, \infty)$, has a continuous extension to $T \times [0, \infty)$; see [7], p. 335, or [5], p. 360, Corollary 6.6 and the comment following it. A similar remark applies to Theorem 1.2(b) and the last part of Lemma 2.2 below.

REMARK 2. If ($\mathrm{A2}_k$) holds, then ($\mathrm{A2}_{n-k}$) holds after the change of variables $t \to -t$. Thus, part (a) implies that if $x = \zeta_k(t, \tau, \lambda)$ is the k-th special principal solution of (1.9) at $t = \alpha$ determined by τ, i.e., $x(t) = \zeta_k(t, \tau, \lambda)$ satisfies the conditions $D^{i-1}x = 0$ for $1 \leqq i < k$ and $D^{k-1}x = (-1)^{k-1}$ at $t = \tau$, then $D^i\zeta_k(t, \tau, \lambda) \in C^0(T \times T \times \Lambda^0)$ for $i = 0, \ldots, n$, and $\zeta_k(t, \tau, \cdot)$ is completely monotone on Λ^0, for fixed (t, τ) with $\alpha < t \leqq \tau < \beta$.

THEOREM 1.2. *Assume* (A1), ($\mathrm{A2}_{n-1}$) *and* ($\mathrm{A3}_\sigma$) *with* $\sigma = \alpha$ [*or* $\sigma = \beta$]. (a) *Then* $D^i\xi_1(t, \lambda) \in C^0(T \times \Lambda^0)$ *for* $0 \leqq i \leqq n$, *and* $\xi_1(t, \cdot)$ [*or* $1/\xi_1(t, \cdot)$] *is completely monotone on* Λ^0 *for fixed* $t \in T^0$. (b) *If, in addition,* $\Lambda^0 \supset [0, \infty)$, *then there exists a distribution function* $W(r) = W(r, \alpha, t)$ [*or* $W(r) = W(r, \tau, \beta)$] *on* $r \geqq 0$ *such that*

$$\xi_1(\tau, \lambda) = \xi_1(\tau, 0)\int_0^\infty \exp(-\lambda r)\, W(dr, \alpha, \tau) \quad \text{for } \lambda \geqq 0 \tag{1.14}$$

and $\tau \in T^0$ [*or such that*

$$1/\xi_1(t, \lambda) = [1/\xi_1(t, 0)]\int_0^\infty \exp(-\lambda r)\, W(dr, t, \beta) \quad \text{for } \lambda \geqq 0 \tag{1.15}$$

and $t \in T_0$]; *also* $\tau_1 = \alpha$ [*or* $\tau_n = \beta$] *is admissible in* (1.3); $W(r, t, \tau) \to \delta_0(r)$ *as* $t < \tau \to \alpha$ [*or* $\tau > t \to \beta$]; *and* $W(\cdot, \alpha, \tau)$ [*or* $W(\cdot, t, \beta)$] *is infinitely divisible.*

REMARK 3. It will be clear from the proofs that Theorems 1.1 and 1.2 remain valid if (1.9) is replaced by an equation involving quasi-derivatives of the form

$$\left\{p_{M+1}^{-1} D p_M^{-1} D \ldots p_1^{-1} D^K + \sum_{j=1}^{K-1} q_j D^j - q\right\} x = 0\,, \tag{1.9$'$}$$

the operators D^i for $i = 0, \ldots, n$ are replaced by $L_i = D^i$ for $0 \leqq i < K$ and $L_{i+K} = p_{i+1} D \ldots D p_1^{-1} D^K$ for $0 \leqq i \leqq M$, and assumption (A1) is replaced by

(A1$'$) $K \geqq 1$, $M \geqq 0$, *and* $n = K + M$; $0 < p_j = p_j(t) \in C^0(T)$ *for* $j = 1, \ldots, M+1$; $q_j = q_j(t) \in C^0(T)$ *for* $j = 1, \ldots, K-1$; $q = q(t, \lambda) \in C^0(T \times \Lambda)$; *and* (1.9$'$) *is disconjugate on* T *for fixed* $\lambda \in \Lambda$.

As an application, different from that concerning $I_\mu(t)$, we might mention that the results above imply that the Legendre differential equation

$$[(t^2-1)x']'-[\nu(\nu+1)+\mu^2/(t^2-1)]x=0$$

has a family of principal solutions $x=X(t,\mu,\nu)$ at $t=1$ on $T=(1,\infty)$ such that $X(t,\mu,\nu)$ is a completely monotone function of $\lambda=\mu^2\geqq 0$ for fixed $t>1$, $\nu\geqq 0$. In fact, from Theorem 1.2, we obtain $X(t,\mu,\nu)$ in the form $c(\mu,\nu)P_\nu^{-\mu}(t)$, where $c(\mu,\nu)$ can be given explicitly in terms of the Γ-function and $P_\nu^{-\mu}$ is a Legendre function of the first kind; cf. [4], p. 122, (3) and (5). Also, if $Q_\nu^\mu(t)$ is a Legendre function of the second kind, then $Q_\nu^\mu(t)$ is a principal solution at $t=\infty$ for fixed $\mu,\nu\geqq 0$ and, by Theorem 1.2, $\Gamma(\nu+\mu+1)\cdot$ $\cdot\exp(\pi i\mu)/Q_\nu^\mu(t)$ is a completely monotone function of $\lambda=\mu^2>0$ for fixed $t>1$, $\nu\geqq 0$; cf. [4], p. 122, (5). We could also formulate an analogue of Theorem 1.0.

In Section 2, we state the main Lemma 2.1 concerning the complete monotony of certain Green's functions with respect to λ (for fixed s,t). Section 3 contains the proofs of Theorems 1.1 and 1.2. Section 4 gives somewhat different criteria for a family of solutions $X(t,\lambda)$ of second order differential equations to be completely monotone with respect to λ. Theorem 5.1 deals with a generalization of (1.1) in a neighborhood of a regular singular point. Finally, Section 6 contains interesting related observations about the Γ-function.

Appendix 1 concerns the situation when m is restricted to a finite range in condition $(\mathrm{A2}_k)$. Appendix 2 deals with the Cauchy functions of a 1-parameter family of n-th order linear differential equations. Appendix 3 extends from $j=1$ to $1\leqq j<n$ a known characterization of a j-th special principal solution of a disconjugate equation; cf. Theorem 7.1_n, [7], p. 330 for the case $j=1$.

2. – On Green's functions.

Assume (A1) and fix $[a,b]\subset T$ and k, $0<k<n$. Write the left side of (1.9) as $L(\lambda)x$, where $L(\lambda)=D^n+q_{n-1}D^{n-1}+\ldots+q_1D-q$ is a differential operator. Consider the boundary value problem on $[a,b]$ consisting of the differential equation $L(\lambda)x=0$ and boundary conditions

$$(2.1)\qquad (D^{i-1}x)(a)=0 \quad \text{for } i=1,\ldots,n-k,$$

$$(2.2)\qquad (D^{i-1}x)(b)=0 \quad \text{for } i=1,\ldots,k.$$

The fact that (1.9) is disconjugate on T, hence on $[a, b]$, implies the existence of a Green's function $G_{ab}(t, s, \lambda)$ defined on $[a, b]\times[a, b]\times\Lambda$ such that if $h(t)\in C^0[a, b]$ or $h(t)\in L^2(a, b)$, then

$$L(\lambda)x = h \tag{2.3}$$

has a unique solution satisfying (2.1), (2.2), and x is given by

$$x(t) = \int_a^b G_{ab}(t, s, \lambda)\, h(s)\, ds\,. \tag{2.4}$$

Recall that G_{ab} is uniquely determined by the conditions: (i) as a function of t, $x(t) = G_{ab}(t, s, \lambda)$ is a solution of (1.4) on $[a, s)$, $(s, b]$ and satisfies (2.1), (2.2); (ii) $D^{i-1}x(t) = D^{i-1}G_{ab}(t, s, \lambda)\in C^0\big([a, b]\times[a, b]\big)$ for fixed $\lambda\in\Lambda$ and $i = 1, \ldots, n-1$, and $D^{n-1}G_{ab}(s+0, s, \lambda) - D^{n-1}G_{ab}(s-0, s, \lambda) = 1$.

The main lemma is the following

LEMMA 2.1. *Assume* (A1), (A2_k) *with fixed* k, $0 < k < n$, *and* $[a, b]\subset T$. *Then* $(-1)^k G_{ab}(t, s, \cdot)$ *is completely monotone on* Λ *for fixed* s, $t\subset[a, b]$.

McKean [12] contains a similar result for self-adjoint (possibly singular) boundary value problems with $n = 2$, $q(t, \lambda) = \lambda + Q(t) \geqq \lambda \geqq 0$. We simplify his operator-theoretic proof.

REMARK. Lemma 2.1 implies that if $h(t) = h(t, \lambda)\in C^0\big([a, b]\times\Lambda\big)$ has the property that $h(t, \cdot)$ is completely monotone on Λ for fixed $t\in[a, b]$, then the unique solution $x(t) = x(t, \lambda)$ of (2.1)-(2.3) given by (2.4) is completely monotone on Λ for fixed $t\in[a, b]$.

The proofs of Lemma 2.1 and Theorem 1.1-1.2 can be modified to obtain the following:

LEMMA 2.2. *Let* $0 < p(t)\in C^0(T)$, $q(t, \lambda)\in C^0(T\times\Lambda)$ *satisfy* (A2_1), *and*

$$\big(p(t)x'\big)' - q(t, \lambda)x = 0 \tag{2.5}$$

disconjugate on T *for fixed* λ. *Let* $x = Y(t, \lambda)$, $Z(t, \lambda)$ *be non-negative (first) principal solutions of* (2.5) *at* $t = \alpha, \beta$, *and suppose that* Y, Z *are linearly independent and normalized by* $p(t)[ZY' - Z'Y] = 1$. *Then* $G(t, s, \lambda)$, *defined as* $Y(t, \lambda)Z(s, \lambda)$ *or* $Y(s, \lambda)Z(t, \lambda)$ *according as* $t\leqq s$ *or* $t\geqq s$, *is a completely monotone function of* $\lambda\in\Lambda^0$ *for fixed* $s, t\in T$. *If either* $s\leqq\tau\leqq t$ *or* $t\leqq\tau\leqq s$, *then* $G(t, s, \cdot)/G(\tau, \tau, \cdot)$ *is completely monotone on* Λ^0. *If also* $\Lambda^0 \supset [0, \infty)$, *then there exists a distribution function* $P(r) = P(r, t, s, \tau)$ *on* $r \geqq 0$ *satisfying*

$$G(t, s, \lambda)/G(\tau, \tau, \lambda) = [G(t, s, 0)/G(\tau, \tau, 0)]\int_0^\infty \exp(-r\lambda)P(dr, t, s, \tau)$$

for $\lambda \geqq 0$, *and* $P(\cdot, t, s, \tau)$ *is infinitely divisible.* Note that

$$G(t, s, \lambda)/G(\tau, \tau, \lambda) = [Y(s, \lambda)/Y(\tau, \lambda)]\cdot[Z(t, \lambda)/Z(\tau, \lambda)] \quad \text{if } s \leqq \tau \leqq t.$$

PROOF OF LEMMA 2.1. It will be clear from the explicit construction of $G_{ab}(t, s, \lambda)$ in the proof of Theorem 1.1 that $\partial^{i+j+m}G_{ab}(t, s, \lambda)/\partial t^i \partial s^j \partial\lambda^m$ exists and is continuous on $[a, b]\times[a, b]\times\Lambda$[or Λ^0] if $m = 0$ [or $m > 0$] and either $0 \leqq i, j \leqq n-2$ or $0 \leqq i, j \leqq n$ and $t \neq s$. It is known ([1], [10], [14]; cf. [2], p. 105) that $(-1)^k G_{ab}(t, s, \lambda) \geqq 0$ on $[a, b]\times[a, b]\times\Lambda$.

If $t \neq s$, then $L(\lambda)G(\cdot, s, \lambda) = 0$. Differentiation of this equation with respect to $\lambda \in \Lambda^0$ gives (2.3), where $x = \partial G_{ab}(\cdot, s, \lambda)/\partial\lambda$ and $h = G_{ab}(\cdot, s, \lambda)\cdot \cdot\partial q(\cdot, \lambda)/\partial\lambda$. As functions of t, h is continuous and $x \in C^n[a, b]$ satisfies (2.1)-(2.2). Hence (2.4) holds, i.e.,

$$(2.6) \qquad (-1)^k \partial G_{ab}(t, s, \lambda)/\partial\lambda = \\ = \int_a^b [(-1)^k G_{ab}(t, r, \lambda)]\cdot[(-1)^k q_\lambda(r, \lambda)]\cdot[(-1)^k G_{ab}(r, s, \lambda)]\,dr \leqq 0\,.$$

Successive differentiations with respect to $\lambda \in \Lambda_0$ and an obvious induction give Lemma 2.1.

3. – Proof of Theorem 1.1(*a*).

Let $x(t) = X(t, \tau, \lambda)$ be the solution of (1.4) satisfying the initial conditions (1.10) with $j = n$, so that $X(t, \tau, \lambda)$ is the Cauchy function for (1.9) for fixed $\lambda \in \Lambda$. It is clear that (A1) implies that $\partial^i X/\partial t^j \partial\tau^k \in C^0(T\times T\times\Lambda)$ for $j + k = i = 0, \ldots, n$. Also if (A2$_k$) holds, then $\partial^{i+j+m}X/\partial t^i \partial\tau^j \partial\lambda^m \in C^0(T\times T\times\Lambda^0)$ for $i, j = 0, \ldots, n$ and arbitrary m.

Let $x(t) = Y(t, a, s, b, \lambda)$, $Z(t, a, s, b, \lambda)$ be the solutions of (1.9) such that

$$(3.1) \qquad G_{ab}(t, s, \lambda) = Y(t, a, s, b, \lambda) \quad \text{or } Z(t, a, s, b, \lambda)$$

according as $a \leqq t \leqq s$ or $s \leqq t \leqq b$. Thus, condition (ii), following (2.4) and defining G_{ab}, implies that $Z - Y = X(t, s, \lambda)$, while (i) implies that $x(t) = Y$ satisfies (2.1) and $x(t) = Z$ satisfies (2.2). Since $Y = Z - X$, $x = Z$ is determined by the boundary conditions

$$\text{(3.2)} \qquad \begin{aligned} [D^{i-1}Z]_{t=a} &= [D^{i-1}X]_{t=a} \qquad 1 \leqq i \leqq n-k\,, \\ [D^{i-1}Z]_{t=b} &= 0 \qquad \text{if } 1 \leqq i \leqq k\,. \end{aligned}$$

Similarly, $x = Y$ is determined by

$$\text{(3.3)} \qquad \begin{aligned} [D^{i-1}Y]_{t=a} &= 0 \qquad &&\text{if } 1 \leqq i \leqq n-k\,, \\ [D^{i-1}Y]_{t=b} &= -[D^{i-1}X]_{t=b} \qquad &&\text{if } 1 \leqq i \leqq k\,. \end{aligned}$$

Let $x(t) = X^j(t, s, \tau, \lambda)$ be the unique solution of (1.9) satisfying the boundary conditions

$$\text{(3.4)} \qquad \begin{aligned} [D^{i-1}X^j]_{t=s} &= 0 \qquad &&\text{if } 1 \leqq i \leqq n-j\,, \\ [D^{i-1}X^j]_{t=\tau} &= 0 \qquad &&\text{or } 1 \text{ if } 1 \leqq i < j \text{ or } i = j\,. \end{aligned}$$

Thus, by (3.2) and (3.3), we have

$$\text{(3.5)} \qquad Z(t, a, s, b, \lambda) = \sum_{j=1}^{n-k} X^j(t, b, a, \lambda) \cdot D^{j-1}X(a, s, \lambda)\,,$$

$$\text{(3.6)} \qquad Y(t, a, s, b, \lambda) = -\sum_{j=1}^{k} X^j(t, a, b, \lambda) \cdot D^{j-1}X(b, s, \lambda)\,.$$

This makes it clear that Y, Z have smoothness properties similar to those of X.

PROPOSITION 3.1. *Assume* (A1) *and* ($A2_k$) *with k fixed,* $0 < k < n$. *Then* $(-1)^{k+1}X^k(t, a, s, \cdot)$, *for* $a < t < s$, *and* $X^{n-k}(t, b, s, \cdot)$, *for* $s < t < b$, *are completely monotone on* Λ.

PROOF. The definition of the Cauchy function $X(t, \tau, \lambda)$ implies that $D^{j-1}X(t, \tau, \lambda) \sim (t-\tau)^{n-j}/(n-j)!$ as $t \to \tau$ uniformly on compacts of Λ. Hence, by (3.5) and (3.6),

$$\begin{aligned} X^k(t, a, s, \lambda) &= -\lim_{b \to s} Y(t, a, s, b, \lambda)(n-k)!/(b-s)^{n-k}\,, \\ X^{n-k}(t, b, s, \lambda) &= \lim_{a \to s} (-1)^k Z(t, a, s, b, \lambda)\, k!/(s-a)^k\,. \end{aligned}$$

Thus, Proposition 3.1 follows from Lemma 2.1 and (3.1); cf. [16], pp. 147 ff., or Appendix 1 below.

PROPOSITION 3.2. *Assume* (A1) *and let* $\lambda \in \Lambda$ *be fixed. Then*

$$\eta_j(t, s, \lambda) = \lim_{b \to \beta} X^j(t, b, s, \lambda) \qquad \text{for } j = 1, \ldots, n-1 . \tag{3.7}$$

This limit is valid in C^n *on arbitrary* (t, s)*-compacts of* $T \times T^0$.

REMARK. This proposition and (3.5) imply, for example, that

$$(-1)^k \sum_{j=1}^{n-k} \eta_j(t, s, \cdot) D^{j-1} X(a, s, \cdot)$$

is completely monotone on Λ^0 for fixed (a, s, t), $a < s < t$ and $a, t \in T$. Proposition 3.2 is contained in Appendix 3 below.

It is now easy to complete the proof of Theorem 1.1(a). Propositions 3.1, 3.2 imply that $\eta_{n-k}(t, s, \cdot)$ is completely monotone on Λ^0 for fixed (s, t), $\alpha < s \leqq t < \beta$. In particular, it is continuous and non-increasing with respect to λ for fixed (s, t). It therefore follows from Dini's theorem that $\eta_{n-k}(t, s, \lambda) \to \eta_{n-k}(t, s, \lambda_0)$, as $\lambda \to \lambda_0 \in \Lambda^0$, uniformly on (t, s)-compacts of $\alpha < s \leqq t < \beta$, hence of $T \times T^0$. Consequently, $\eta_{n-k}(t, s, \lambda) \in C^0(T \times T^0 \times \Lambda^0)$.

Hence, $D^i \eta_{n-k}(t, s, \lambda) \in C^0(T \times T^0 \times \Lambda^0)$ when $i = 0, \ldots, n$, for a solution of (1.9) is uniquely determined by its values at n distinct t-values $\big($i.e., if $x_1(t, \lambda), \ldots, x_n(t, \lambda)$ are linearly independent solutions of (1.9) and $t_1 < \ldots < t_n$, then $\det\big(x_i(t_j, \lambda)\big) \neq 0\big)$. $\big($For an analogous non-linear result, see the proof of Lemma I 1.1, [8], pp. 207-210. In the linear case at hand, a proof need not use the disconjugacy of (1.4), cf. Lemma 7.1, [5], p. 479.$\big)$

PROOF OF THEOREM 1.1(b). Since $\eta_1(t, t, \cdot) = \xi_1(\tau, \cdot)/\xi_1(t, \cdot)$ is completely monotone on $\Lambda = [0, \infty)$ for fixed (t, τ) with $\alpha < t \leqq \tau < \beta$ by part (a), the existence of a unique distribution function $W(r) = W(r, t, \tau)$ satisfying (1.13) follows from the Hausdorff-Bernstein theorem. The relation (1.3) for $\tau_1 < \ldots < \tau_n$ with $\tau_1, \tau_n \in T$ is a consequence of (1.13) and the standard theorem on the products of Laplace-Steltjes transforms. The statement (1.4) follows from (1.3); in fact, if $\tau_1 < \sigma < \tau_2$, then (1.3) implies that

$$W(r, \tau_1, \tau_2) = \int_0^r W(r - \varrho, \tau_1, \sigma)\, dW(\varrho, \sigma, \tau_2) = \int_0^r W(r - \varrho, \sigma, \tau_2)\, dW(\varrho, \tau_1, \sigma)$$

so that

$$W(r, \tau_1, \tau_2) \leqq W(r, \tau_1, \sigma) \qquad \text{and} \qquad W(r, \tau_1, \tau_2) \leqq W(r, \sigma, \tau_2) ,$$

which implies (1.4). The statement (1.7) is clear from (1.13) since $\eta_1(\tau, t, \lambda) = \xi_1(\tau, \lambda)/\xi_1(t, \lambda) \to 1$ as $t, \tau \to \sigma$ uniformly on (σ, λ)-compacts of $T^0 \times [0, \infty)$. Furthermore, the monotony condition (1.4) and (1.7) imply, by virtue of Dini's theorem, that (1.7) holds uniformly on compact (r, σ)-subsets of $\{r > 0\} \times T^0$. The infinite divisibility of $W(\cdot, t, \tau)$, $\alpha < t \leqq \tau < \beta$, follows from (1.3) and the uniformity of (1.7); cf. [3], pp. 128-135.

Proof of Theorem 1.2(*a*). By Theorem 1.1, $\eta_1(\tau, t, \cdot) = \xi_1(\tau, \cdot)/\xi_1(t, \cdot)$ is completely monotone on Λ^0 if $\alpha < t \leqq \tau < \beta$. Thus if $(\mathrm{A3}_\alpha)$ holds and $\mu \in \Lambda$ is fixed, $\xi_1(\tau, \cdot) = \lim \eta_1(\tau, t, \cdot)\xi_1(t, \mu)$, as $t \to \alpha$, is completely monotone on Λ^0 for fixed $t \in T^0$. Similarly, if $(\mathrm{A3}_\beta)$ holds, then $1/\xi_1(t, \cdot)$ is completely monotone on Λ^0 for fixed $t \in T^0$. The asserted smoothness properties of $\xi_1(t, \lambda)$ follow as in the proof of Theorem 1.1(*a*).

Proof of Theorem 1.2(*b*). This is similar to the proof of Theorem 1.1(*b*) and will be omitted.

4. – Use of asymptotic estimates.

It was mentioned in [9], although no details were given, that a proof that $I_\mu(t)$ is a completely monotone function of $\lambda = \mu^2 \geqq 0$ could be based on a knowledge of the asymptotic behavior of $I_\mu(t)$ near $t = 0$ and $t = \infty$, instead of using Green's functions. A generalization of such a proof can be obtained from the following simple proposition.

Theorem 4.1. *In the differential equation*

$$(4.0) \qquad \big(p(t)x'\big)' - q(t, \lambda)x = 0\,,$$

let $0 < p(t) \in C^0(T)$, $q(t, \lambda)$ *satisfies assumption* $(\mathrm{A2}_1)$ *and* $q(t, \lambda) \geqq 0$. *Let* $x = X(t, \lambda)$ *be a solution of* (4.0) *such that* $X, X' \in C^0(T \times \Lambda)$, *and* $\partial^n X/\partial\lambda^n$ *exists and is continuous on* $T \times \Lambda^0$ *for* $n = 0, 1, \ldots$. *Then* $X(t, \cdot)$ *is completely monotone on* Λ (*for fixed* $t \in T$) *if and only if*

$$(4.1_\sigma) \qquad 0 \leqq \limsup_{t \to \sigma}\, (-1)^n \partial^n X(t, \lambda)/\partial\lambda^n (\leqq \infty) \qquad \text{for } n = 0, 1, \ldots$$

holds for $\sigma = \alpha, \beta$ *and all* $\lambda \in \Lambda^0$.

In contrast to Theorems 1.1 and 1.2, it is not supposed that $x = X(t, \lambda)$ is a principal solution. A variant of Theorem 4.1 is the following:

THEOREM 4.2. *Let $p(t)$, $q(t, \lambda)$ be as in Theorem 4.1 and $x = X(t, \lambda)$ a principal solution (4.0) at $t = \beta$ such that $\partial^n X/\partial\lambda^n$ exists and is continuous on $T \times \Lambda^0$ for $n = 0, 1, ..$ and that either*

$$(4.2) \qquad \int^{\beta} dt/p(t) = \infty \qquad \text{or} \qquad \lim_{t\to\beta} X(t, \lambda) = 0 \qquad \text{for } \lambda \in \Lambda .$$

Then $X(t, \cdot)$ is completely monotone on Λ if and only if

$$(4.3) \qquad 0 \leqq \limsup_{t\to\alpha}\, (-1)^n \partial^n X(t, \lambda)/\partial\lambda^n (\leqq \infty) \qquad \text{for } \lambda \in \Lambda^0,\ n = 0, 1, \ldots .$$

PROOF OF THEOREM 4.1. Since the necessity of (4.1_α) and (4.1_β) for $\lambda \in \Lambda^0$ is clear, we only prove the sufficiency. Since $q(t, \lambda) \geqq 0$, a standard maximum principle shows that a solution of (4.0), for fixed λ, cannot have a negative minimum interior to T. Thus (4.1_σ) for $n = 0$ and $\sigma = \alpha, \beta$ imply the case $n = 0$ of

$$(4.4_n) \qquad (-1)^n X^{(n)} \geqq 0 \quad \text{on} \quad T \times \Lambda^0, \quad \text{where } X^{(n)} = \partial^n X/\partial\lambda^n .$$

Let $n - 1 \geqq 0$ and assume (4.4_k) for $k = 0, \ldots, n-1$. If $x = X(t, \lambda)$ in (4.0), n differentiations with respect to λ on Λ^0 give

$$(4.5) \qquad (p(t)y')' - q(t, \lambda)y = -F_n(t, \lambda),$$

where

$$(4.6) \qquad y = (-1)^n X^{(n)}, \qquad F_n = \sum_{k=0}^{n-1} C_{nk}(-1)^{n-k+1} q^{(n-k)}(-1)^k X^{(k)} \geqq 0,$$

$q^{(m)} = \partial^m q/\partial\lambda^m$, and $C_{nk} = n!/k!(n-k)!$. Another appeal to the maximum principle shows that, for a fixed λ, y cannot have a negative minimum interior to T. Hence (4.1_σ) for $\sigma = \alpha, \beta$ implies (4.4_n). This completes the proof.

PROOF OF THEOREM 4.2. Since one-half of Theorem 1.2 is trivial, we prove only the sufficiency of (4.3) for the complete monotony of $X(t, \cdot)$. We first consider the case

$$(4.7) \qquad X(t, \lambda) \to 0 \quad \text{as } t \to \beta \text{ for } \lambda \in \Lambda$$

of (4.2). By induction, we shall prove (4.4_n) and

$$(4.8_n) \qquad X^{(n)}(t, \cdot) \to 0 \quad \text{as } t \to \beta \text{ in } L^1_{\mathrm{loc}}(\Lambda^0).$$

Of course, (4.8_n) is trivial if $T = (\alpha, \beta]$ is closed on the right, by virtue of (4.7) and the continuity of $X^{(n)}$ on $T \times \Lambda$.

The case $n = 0$ of (4.4_n) follows as in Theorem 4.1. Also the case $n = 0$ of (4.8_n) follows from (4.7) and Lebesgue's monotone convergence theorem. Assume (4.4_k) and (4.8_k) for $k = 0, \dots, n-1$, and differentiate (4.0) with respect to $\lambda \in \Lambda^0$ to obtain (4.5), (4.6). Suppose, if possible, that

$$(4.9) \qquad y(\tau, \lambda_0) < 0 \quad \text{for some } (\tau, \lambda_0) \in T \times \Lambda^0 .$$

Note that if $T = (\alpha, \beta]$ is closed on the right, then $\tau \neq \beta$ by (4.7). Since y cannot have a negative minimum at an interior point of T, it follows from (4.3) that $y'(t, \lambda_0) \leqq 0$ and $y(t, \lambda_0) \leqq y(\tau, \lambda_0) < 0$ for $t \geqq \tau$. By continuity, $y(\tau, \lambda) < 0$ and similarly $y(t, \lambda) \leqq y(\tau, \lambda) \leqq y(\tau, \lambda_0)/2 < 0$ for $t \geqq \tau$ and λ near λ_0.

If $\lambda > \mu$, then we have

$$(4.10) \qquad (-1)^{n+1}\Big\{X(t, \lambda) - \sum_{k=0}^{n-1} X^{(k)}(t, \mu)(\lambda - \mu)^k/k!\Big\} = \\ = -\int_{\mu}^{\lambda} (\lambda - \nu)^{n-1} y(t, \nu)\, d\nu/(n-1)! .$$

The right side is not less than $-(\lambda - \mu)^n y(\tau, \lambda_0)/2n! > 0$ for λ, μ near λ_0 and $t \geqq \tau$. Integrate (4.10) with respect to μ over an interval $[\varrho, \lambda]$ to obtain

$$(4.11) \qquad (-1)^{n+1} X(t, \lambda)(\lambda - \varrho) - \sum_{k=0}^{n-1} \int_{\varrho}^{\lambda} X^{(k)}(t, \mu)(\lambda - \mu)^k\, d\mu/k! = \\ = -\int_{\varrho}^{\lambda} (\lambda - \nu)^{n-1}(\nu - \varrho)\, y(t, \nu)\, d\nu/(n-1)! ,$$

which is not less than $-(\lambda - \varrho)^{n+1} y(\tau, \lambda_0)/2(n+1)! > 0$. This contradicts (4.7) and (4.8_k) for $k = 0, \dots, n-1$, and so (4.9) cannot hold. Thus $y = (-1)^n X^{(n)} \geqq 0$ on $T \times \Lambda^0$, i.e., (4.4_n) holds. Furthermore, (4.7), (4.8_k) for $k = 0, \dots, n-1$, and (4.11) imply (4.8_n). This completes the induction and the proof of Theorem 1.2 in the case (4.7) of (4.2).

We now consider the case $\int^{\beta} dt/p(t) = \infty$ of (4.2). By induction, we prove (4.4_n) and

$$(4.12_n) \qquad X^{(n)}(t, \cdot) \text{ is bounded } t \to \beta \text{ in } L^1_{\mathrm{loc}}(\Lambda^0) .$$

This proof is similar to that above if it is noted that (4.9) implies not only $y'(t, \lambda_0) \leqq 0$ for $t \leqq \tau$, but also that $y(t, \lambda_0) \to -\infty$ as $t \to \beta$. This follows from a convexity argument since (4.5) shows that $d^2y(t, \lambda_0)/ds^2 \leqq 0$ for $0 \leqq s < \infty$, where $ds/dt = 1/p(t)$, $s(\tau) = 0$, and $t = t(s)$ is the inverse of $s = s(t)$.

5. – A regular singular point.

The next result conerns the case of the family of differential equations

$$t^2x'' + r_0tx' - \left[q_0(\lambda) + \sum_{n=1}^{\infty} q_n t^n\right] x = 0 . \tag{5.1}$$

THEOREM 5.1. *Let r_0 be arbitrary and $q_1, q_2, \ldots$ non-negative constants such that the power series in* (5.1) *is convergent for $|t| < \beta(\leqq \infty)$. Let $q_0(\lambda) \in C^\infty(\Lambda)$ satisfy*

$$q_0(\lambda) > 0 \quad \text{ and } \partial q_0/\partial\lambda \text{ is completely monotone on } \Lambda . \tag{5.2}$$

The indicial polynomial $P(\nu) = P(\nu, \lambda)$,

$$P(\nu, \lambda) = \nu(\nu - 1) + r_0\nu - q_0(\lambda) \tag{5.3}$$

has the (unique) positive zero $\nu = \nu(\lambda) > 0$,

$$\nu(\lambda) = (1 - r_0)/2 + [(1 - r_0)^2/4 + q_0(\lambda)]^{\frac{1}{2}} , \tag{5.4}$$

and $\partial\nu/\partial\lambda$ is completely monotone on Λ. Let $c(\lambda) \neq 0$ be arbitrary. Then, for fixed λ, (5.1) *has a solution*

$$X(t, \lambda) = c(\lambda)t^{\nu(\lambda)} \sum_{n=0}^{\infty} x_n(\lambda)t^n \tag{5.5}$$

on $0 < t < \beta$ such that $x_0(\lambda) = 1$ and $x_n(\lambda)$ is completely monotone on Λ for $n = 1, 2, \ldots$. Hence, if $c(\lambda)t^{\nu(\lambda)}$ is completely monotone on Λ for $0 < t < \beta_0(\leqq \beta)$, then the same is true of (5.5). *(This is the case if $c(\lambda) \equiv 1$ and $\beta_0 \leqq 1$.)*

PROOF. We shall make use of the following simple fact in this proof and in the next section.

PROPOSITIOH 5.1. *Let $g(\mu)$ be completely monotone for $\mu \geqq 0$. Let $\mu = \varphi(\lambda)$ be continuous for $\lambda \geqq 0$, $\varphi(0) = 0$, $\varphi \in C^\infty(0, \infty)$, and $d\varphi/d\lambda$ is completely monotone for $\lambda > 0$. Then $G(\lambda) = g(\varphi(\lambda))$ is completely monotone for $\lambda \geqq 0$.*

Substituting (5.5) into (5.1) gives the recursion formula

$$(5.6) \qquad x_n(\lambda) = \sum_{j=1}^{n} x_{n-j}(\lambda)\, q_j / P(n+\nu(\lambda), \lambda) \quad \text{and } x_0(\lambda) = 1 .$$

Here $P(\nu, \lambda)$ is the indicial polynomial (5.3) for fixed λ, so that, by (5.4),

$$(5.7) \qquad P(n+\nu(\lambda), \lambda) = 2n[(1-r_0)^2/4 + q_0(\lambda)]^{\frac{1}{2}} + n^2 .$$

An analogue of Proposition 5.1 with $g(\mu) = (2n\mu + n^2)^{-1}$ and $\varphi(\lambda) = [(1-r_0)^2/4 + q_0(\lambda)]^{\frac{1}{2}}$ shows that $G(\lambda) = g(\varphi(\lambda))$ is completely monotone. Another application gives that $q_j / P(n+\nu(\lambda), \lambda)$ is completely monotone on Λ. An induction and (5.6) imply that $x_0(\lambda) = 1$, $x_1(\lambda)$, ... are completely monotone.

6. – A property of the Γ-function.

The modified Bessel differential equation (1.1) is of the type (5.1) with $q_0(\lambda) = \lambda = \mu^2 > 0$. The solution $I_\mu(t)$ has an expansion,

$$(6.1) \qquad I_\mu(t) = [(t/2)^\mu / \Gamma(1+\mu)] \cdot \Big[1 + \sum_{n=1}^{\infty} (t/2)^n / n!(\mu+1) \ldots (\mu+n)\Big] ,$$

analogous to (5.6); cf. [15], p. 77. From Theorem 5.1, we can only deduce that $I_\mu(t)$ is a completely monotone function of $\lambda = \mu^2 > 0$ for $0 < t \leqq 2e^{-\gamma}$ (rather than for all $t > 0$). In fact, we have the following:

PROPOSITION 6.1. *Let $t > 0$ be fixed and $f(\mu) = t^\mu / \Gamma(1+\mu)$. Then $F(\lambda) = f(\lambda^{\frac{1}{2}})$ is completely monotone for $\lambda \geqq 0$ if and only if*

$$(6.2) \qquad 0 < t \leqq e^{-\gamma} = 0.56 \ldots ,$$

where γ is the Euler-Mascheroni constant. In this case, $-\partial \log F(\lambda)/\partial\lambda$ is also completely monotone for $\lambda > 0$.

REMARK 1. By the Hausdorff-Bernstein theorem (cf. [16], p. 60), it follows that if (6.2) holds, then there exists a distribution function $V(r) = V(r, t)$ for $r \geqq 0$ satisfying

$$t^\mu / \Gamma(1+\mu) = \int_0^\infty \exp(-r\mu^2)\, V(dr, t) \qquad \text{for } \mu > 0 ;$$

$V(r, t) = 0$ or $V(r, t) = V(r + \log(te^\gamma), e^{-\gamma})$ according as $r + \log(te^\gamma) \leqq 0$ or > 0.

REMARK 2. The exponent $\frac{1}{2}$ in $F(\lambda)=f(\lambda^{\frac{1}{2}})$ is the « best » possible in the sense that $f(\lambda^{\delta})$ is not completely monotone on $\lambda>0$, for any $\delta>\frac{1}{2}$ and any $t>0$. In fact, it is readily verified, by using the formulas in the proof to follow, that $d^2f(\lambda^{\delta})/d\lambda^2\to-\infty$ as $\lambda\to 0$ if $\frac{1}{2}<\delta<1$, $t>0$.

PROOF. Let $\psi(\mu)$ be the logarithmic derivative of $\Gamma(\mu)$, $\psi(\mu)=\Gamma'(\mu)/\Gamma(\mu)$. By a standard formula,

$$\psi(1+\mu)+\gamma=\sum_{n=0}^{\infty}\mu/(n+1)(n+1+\mu)\geqq 0 \quad \text{for } \mu\geqq 0\,; \tag{6.3}$$

cf. [4], p. 15. Put

$$\varrho(\mu)=\int_0^{\mu}[\psi(1+s)+\gamma]\,ds \quad \text{and } \varphi(\lambda)=\varrho(\lambda^{\frac{1}{2}})\,. \tag{6.4}$$

Then Proposition 5.1 is applicable to $\varphi(\lambda)$ for, in this case,

$$d\varphi/d\lambda=[\psi(1+\lambda^{\frac{1}{2}})+\gamma]/2\lambda^{\frac{1}{2}}=\sum_{n=0}^{\infty}1/2(n+1)(n+1+\lambda^{\frac{1}{2}}) \tag{6.5}$$

is completely monotone for $\lambda\geqq 0$. By (6.4) and $f(\mu)=t^{\mu}/\Gamma(1+\mu)$,

$$F(\lambda)=f(\mu)=\exp\big(-\varrho(\mu)\big((te^{\gamma})^{\mu} \quad \text{with } \mu=\lambda^{\frac{1}{2}}\geqq 0\,. \tag{6.6}$$

When (6.2) holds, $(te^{\gamma})^{\mu}$ is a completely monotone function of $\mu\geqq 0$, hence of $\lambda\geqq 0$ (by Proposition 5.1 with $\varphi(\lambda)=\lambda^{\frac{1}{2}}$). Also, $e^{-\varrho}$ is a completely monotone functjon of $\varrho\geqq 0$, so that $\exp\big(-\varrho(\mu)\big)$ is a completely monotone function of λ (by the case (6.4) of Proposition 5.1). Consequently, (6.6) is completely monotone for $\lambda\geqq 0$ when (6.2) holds. Also, in this case, $-F_{\lambda}(\lambda)/F(\lambda)=d\varphi/d\lambda+[\log\big(1/(te^{\gamma})\big]/2\lambda^{\frac{1}{2}}$ is completely monotone for $\lambda>0$, where $d\varphi/d\lambda$ is given by (6.5).

Let $te^{\gamma}>1$. Then, by (6.6), $df(\mu)/d\mu=[-d\varrho(\mu)/d\mu+\log(te^{\gamma})]f(\mu)$. Since $d\varrho/d\mu=0$ at $\mu=0$, by (6.3)-(6.4), and $\log(te^{\gamma})>0$, we have $df/du>0$ at $\mu=0$. Hence $F(\lambda)=f(\lambda^{\frac{1}{2}})$ is increasing for small $\lambda>0$ and cannot be completely monotone.

Appendix 1: Monotone families of solutions.

We use the notation of Sections 1-3.

DEFINITION. *The class $M_{\mu}(\Lambda)$.* A function $h(\lambda)$, $\lambda\in\Lambda$, is said to be of class $M_1=M_1(\Lambda)$ if h is nonnegative and nonincreasing. A function $h\in M_1(\Lambda)$

is said to be of class $M_2 = M_2(\Lambda)$ if it is continuous and convex on Λ. If $\mu = 2, 3, \ldots, h$ is said to be of class $M_\mu = M_\mu(\Lambda)$ if $h \in C^0(\Lambda) \cap C^{\mu-2}(\Lambda^0)$ and $(-1)^m h^{(m)} \in M_2(\Lambda^0)$ for $m = 0, \ldots, \mu - 2$. $M_\infty(\Lambda)$ is the class of completely monotone functions on Λ.

Note that $h \in M_\mu$ is continuous if $\mu > 1$. The class M_μ is closed under point-wise convergence if $\Lambda = \Lambda^0$ is open. This is false if if $\Lambda = [0, \infty)$ as is seen from the example $h_n(\lambda) = \exp(-n\lambda)$.

Lemma. *Let $\Lambda = \Lambda^0$ be an open interval. Let $h_n \in M_\mu$ for $n = 1, 2, \ldots$ such that $h(\lambda) = \lim h_n(\lambda)$ exists as $n \to \infty$ for $\lambda \in \Lambda$. Then $h \in M_\mu$. Furthermore, if $\mu \geqq 2$, then*

$$h_n^{(j)}(\lambda) \to h^{(j)}(\lambda) \quad \text{as } n \to \infty \text{ for } j = 0, \ldots, \mu - 2 \tag{1}$$

uniformly on compacts of Λ.

Proof. It is clear that $h \in M_\mu$ if $\mu = 1$ or $\mu = 2$. Let $\mu = 2$, $a < c < \beta$, and $[a, \beta) \subset \Lambda$. Then $c \leqq \sigma < \tau < \beta$ implies that

$$0 \leqq [h_n(\sigma) - h_n(\tau)]/(\tau - \sigma) \leqq [h_n(a) - h_n(c)]/(c - a)\,,$$

so that h_n is uniformly Lipschitz continuous on $[c, \beta)$ with a Lipschitz constant independent of n. Thus (1) with $j = 0$ holds uniformly on λ-compacts.

Let $\mu = 3$. The argument just completed shows that the sequence of first order derivatives $h_1', h_2', \ldots$ are uniformly bounded on λ-compacts. Thus, since they are convex, they are uniformly Lipschitz continuous with a Lipschitz constant independent of n on λ-compacts. Hence the Arzela selection theorem implies that there exist subsequences of $h_1', h_2', \ldots$ uniformly convergent on λ-compacts. But the limit of such a subsequence is necessarily h', independent of the subsequence. Consequently, (2) with $j = 1$ holds uniformly on λ-compacts. This proves the case $\mu = 3$. The proof of the Lemma can be completed by a simple induction.

($\mathrm{A2}_{k\mu}$) *Let $\mu \geqq 1$ and $k > 0$ be fixed integers. Let $\partial^m q/\partial\lambda^m$ exist, be continuous, and satisfy $(-1)^{m+k}\partial^m q/\partial\lambda^m \geqq 0$ on $T \times \Lambda^0$ for $m = 1, 2, \ldots, \mu$ (in particular, $(-1)^{k+1}\partial q(t, \cdot)/\partial\lambda \in M_{\mu-1}(\Lambda^0)$).*

The Lemma just proved and the arguments in Section 2 and 3 have the following consequences which are analogues of statements in Sections 1-3.

Theorem $1.1_{k\mu}$. *Assume* (A1) *and* ($\mathrm{A2}_{k\mu}$) *for fixed integers (k, μ),* $0 < k < n$ and $\mu > 1$. *Let $\tau \in T^0$. Then $D^i \eta_{n-k}(t, \tau, \lambda) \in C^0(T \times T^0 \times \Lambda^0)$ for $0 \leqq i \leqq n$ and $\eta_{n-k}(t, \tau, \cdot) \in M_\mu(\Lambda^0)$ for fixed (t, τ), $\alpha < \tau \leqq t < \beta$.*

THEOREM 1.2_μ. *Assume* (A1), $(A2_{k\mu})$ *with* $k = n-1$ *and* $\mu > 1$, *and* $(A3_\sigma)$ *with* $\sigma = \alpha$ [*or* $\sigma = \beta$]. *Then* $D^i\xi_1(t, \lambda) \in C^0(T \times \Lambda^0)$ *for* $0 \leqq i \leqq n$, *and* $\xi_1(t, \cdot)$ [*or* $1/\xi_1(t, \cdot)$] *is of class* $M_\mu(\Lambda^0)$ *for fixed* $t \in T^0$.

LEMMA $2.1_{k\mu}$. *Assume* (A1), $(A2_{k\mu})$ *with fixed* (k, μ), $0 < k < n$ *and* $\mu > 0$. *Let* $[a, b] \subset T$. *Then* $(-1)^k G_{ab}(t, s, \cdot) \in M_\mu(\Lambda)$.

PROPOSITION $3.1_{k\mu}$. *Assume* (A1), $(A2_{k\mu})$ *with fixed* (k, μ), $0 < k < n$ *and* $\mu > 0$. *Then* $(-1)^{k+1}X^k(t, a, s, \cdot) \in M_\mu(\Lambda)$ *for fixed* (t, a, s), $\alpha < a < t < s < \beta$, *and* $X^{n-k}(t, b, s, \cdot) \in M_\mu(\Lambda)$ *for fixed* (t, b, s), $\alpha < s < t < b$.

Note that Λ (rather than Λ^0) and $\mu > 0$ (rathe. than $\mu > 1$) occur in the last two assertions, since no limit process is involved in the proof of Lemma $2.1_{k\mu}$ and the limit process in the proof of Proposition $3.1_{k\mu}$ is uniform on compacts of Λ.

Appendix 2: Cauchy functions of n-th order equations.

In the N-th order linear differential equation

$$(1) \qquad Lx \equiv \left\{p_{m+1}^{-1} D p_m^{-1} D \ldots p_1^{-1} D^k - \sum_{j=0}^{k-1} q_j D^j\right\} x = 0\,,$$

let $D = d/dt$; $k \geqq 1$; $m \geqq 0$; $n = k + m$; $p_j = p_j(t, \lambda) > 0$, $q_j(t, \lambda) \geqq 0$ continuous on $T \times \Lambda$ such that $p_j(t, \cdot)$, $q_j(t, \cdot)$ are completely monotone on Λ (for fixed $t \in T$). For suitable functions $x(t)$ on T, define the vector $y(t; x) = (y_1, \ldots, y_n)$ by $y_j = D^{j-1}x$ for $1 \leqq j \leqq k$ and $y_{k+j} = p_1^{-1} D \ldots D p_j^{-1} D^k x$ for $1 \leqq j \leqq m$, so that (1) is equivalent to the first order system

$$(2) \qquad \begin{cases} y_j' = y_{j+1} & \text{for } 1 \leqq j < k\,; \\ y_{k+j}' = p_{j+1} y_{k+j+1} & \text{for } 0 \leqq j < m\,; \\ y_n' = p_{m+1} \sum_{j=0}^{k-1} q_j y_{j+1}\,; \end{cases}$$

cf., e.g., [5], pp. 309-310.

For fixed λ, let $X(t, s, \lambda)$ be the Cauchy function for (1), i.e., if $y(t; x)$ is the vector belonging to the solution $x(t) = X(t, s, \lambda)$ of (1), then y satisfies the initial conditions

$$(3) \qquad y_j = 0 \quad \text{for } 1 \leqq j < n \qquad \text{and} \qquad y_n = 1 \quad \text{at } t = s\,.$$

We denote this vector $y(t; x)$, with $x = X(t, s, \lambda)$, by $Y(t, s, \lambda) = (Y_1(t, s, \lambda), \ldots, Y_n(t, s, \lambda))$. For example, $Y_j(t, s, \lambda) = D^{j-1}X(t, s, \lambda)$ for $1 \leqq j \leqq k$.

THEOREM. *Under the conditions above, $Y_j(t, s, \cdot)$ is completely monotone on Λ for fixed $t, s \in T$, $t > s$, and $j = 1, \ldots, n$.*

Note that, in this theorem, $q_j(t, \cdot)$ is completely monotone, so that assumption $(A2_0)$ holds, but the disconjugacy of (1) is not assumed. It is clear that the theorem is contained in the following

LEMMA. *Let $A(t, \lambda)$ be a continuous $n \times n$ matrix function on $T \times \Lambda$ such that every entry is completely monotone on Λ, for fixed $t \in T$. Let $U(t) = U(t, s, \lambda)$ be the fundamental matrix of*

$$U' = A(t, \lambda)\, U \tag{4}$$

reducing to the identity at $t = s$. Then each entry of $U(t, s, \cdot)$ is completely monotone on Λ for fixed $s, t \in T$, $t > s$.

REMARK. – It follows that if $h(t, \lambda)$ is a continuous vector on $T \times \Lambda$ such that each component of $h(t, \cdot)$ is completely monotone on Λ and $c(\lambda)$ is a vector with completely monotone components, then

$$u' = A(t, \lambda) u + h(t, \lambda)\,, \qquad u(s) = c(\lambda)\,,$$

has the solution

$$u(t, s, \lambda) = U(t, s, \lambda) c(\lambda) + \int_s^t U(t, r, \lambda) h(r, \lambda)\, dr\,,$$

with completely monotone components on Λ, for fixed $t, s \in T$, $t \geqq s$.

PROOF OF LEMMA. Since each entry of $A(t, \lambda)$ is non-negative, it is clear that each entry of $U(t, s, \lambda)$ is non-negative for $t \geqq s$. Let $U = U(t, s, \lambda)$ in (4) and differentiate with respect to $\lambda \in \Lambda^0$ to get

$$U'_\lambda(\cdot, s, \lambda) = A(\cdot, \lambda)\, U_\lambda(\cdot, s, \lambda) + A_\lambda(\cdot, \lambda)\, U(\cdot, s, \lambda)\,.$$

Since $U_\lambda(t, s, \lambda) = 0$ when $t = s$, the variations of constants formula gives

$$U_\lambda(t, s, \lambda) = \int_s^t U(t, r, \lambda) A_\lambda(r, \lambda)\, U(r, s, \lambda)\, dr\,.$$

This shows that each entry of $U_\lambda(t, s, \lambda)$ is non-positive for $t \geqq s$. Repeated differentiations with respect to $\lambda \in \Lambda^0$ and an induction give the lemma.

Appendix 3: Special principal solutions.

Let the differential equation

$$D^n x + q_{n-1}(t) D^{n-1}x + \dots + q_1(t)Dx + q_0(t)x = 0 \tag{1}$$

have continuous coefficients on a t-interval T with endpoints α, β, where $-\infty \leqq \alpha < \beta \leqq \infty$. Let $T^0 = \operatorname{int} T$.

THEOREM. *Let* (1) *be disconjugate on T and $[s, b] \subset T_0$. Let $x = X^j(t, b, s)$ be the solution satisfying the boundary conditions*

$$\begin{aligned} &[D^{i-1}X^j]_{t=b} = 0 \quad \text{if } 1 \leqq i \leqq n-j\,, \\ &[D^{i-1}X^j]_{t=s} = 0 \quad \text{or } 1 \text{ if } 1 \leqq i < j \text{ or } i = j\,. \end{aligned} \tag{2}$$

Then

$$\eta_j(t, s) = \lim_{b\to\beta} X^j(t, b, s) \qquad \text{for } j = 1, \dots, n-1 \tag{3}$$

exists in C^n on arbitrary (t, s)-compacts of $T \times T^0$ and is the j-th special principal solution of (1) *at $t = \beta$, determined by s; cf.* (1.10) *above.*

PROOF. The case $j = 1$ follows from Theorem 7.1_n(ii), [7], p. 330. In particular, the theorem is correct if $n = 2$. Assume its validity for disconjugate differential equations of order $n-1(\geqq 1)$. Let $1 < j < n$.

Let $\eta_1 = \eta_1(t, s)$. There exists a disconjugate differential equation of order $n-1$, say $L_{n-1}v = 0$, such that x is a solution of (1) if and only if $v = W(\eta_1, x) \equiv \eta_1 x' - \eta_1' x$ is a solution of $L_{n-1}v = 0$; furthermore, for $1 < j < n$, x is a j-th principal solution of (1) at $t = \beta$ if and only if $v = W(\eta_1, x)$ is a $(j-1)$-st principal solution of $L_{n-1}v = 0$ at $t = \beta$; Theorem 7.2_n(iv), [7], p. 332. Put $V^{j-1}(t, b, s) = W(\eta_1, X^j(t, b, s))$ for $j = 2, \dots, n-1$. Then $v = V^{j-1}$ is a solution of $L_{n-1}v = 0$ and satisfies the same boundary conditions at $t = b, s$ as does $X^{j-1}(t, b, s)$, i.e., (2) with j replaced by $j-1$. By the induction hypothesis, $\psi_{j-1}(t, s) = \lim W(\eta_1, X^j(t, b, s))$, as $b \to \beta$, exists in C^{n-1} on arbitrary (t, s)-compacts of $T \times T^0$, and is the $(j-1)$-st special principal solution of $L_{n-1}v = 0$ at $t = \beta$, determined by s. This implies that $(X^j/\eta_1)' \to \psi_{j-1}/\eta_1^2$ as $b \to \beta$. Hence, also $\varrho(t) = \lim X^j(t, b, s)$ exists, as $b \to \beta$, in C^n on arbitrary (t, s)-compacts of $T \times T^0$, and

$$\varrho(t) = \eta_1(t, s) \int_s^t [\psi_{j-1}(r, s)/\eta_1^2(r, s)]\, dr + c\eta_1(t, s)\,,$$

where c is a constant. Since $\varrho(s) = 0$ and $\eta_1 > 0$, we have $c = 0$, so that ϱ satisfies the condition (1.10) with $\tau = s$. Also, from $\psi_{j-1} = W(\eta_1, \varrho)$, it follows that $x = \varrho(t)$ is a j-th principal solution of (1) at $t = \beta$. Hence $\varrho(t) = \eta_j(t, s)$, and the proof is complete.

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