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Elliptic Quasi-Variational Inequalities and Application to a Non-Stationary Problem in Hydraulics (*).

AVNER FRIEDMAN - ROBERT JENSEN (**)

dedicated to Hans Lewy

Introduction.

In this paper we consider the elliptic quasi-variational inequality: Find a function $w(x,y)$ in a rectangle $0<x<a$, $0<y<H^*$ and a curve $y=\varphi(x)$ $(0<x<a)$ such that

$$(0.1)\qquad \begin{cases} -\Delta w+\alpha w=-1+\alpha G-\alpha\varphi & \text{if } w>0\,, \\ 0\geqslant -1+\alpha G-\alpha\varphi & \text{if } w=0\,, \\ w\geqslant 0\,, & \\ w(0,y)=\frac{1}{2}(H-y)^2 & \text{if } 0<y<H\,, \\ w(a,y)=\frac{1}{2}(h-y)^2 & \text{if } 0<y<h\,, \\ -w_{xx}+\alpha w=\alpha G-\alpha\varphi+l & \text{if } y=0,\ 0<x<a\,, \\ w=0 \quad \text{elsewhere on the boundary of the rectangle}\,, & \\ w(x,y)>0 \quad \text{if } y<\varphi(x),\ w(x,y)=0 & \text{if } \varphi(x)<y<H^*\,. \end{cases}$$

Here $\alpha>0$, $0<h<H<H^*$, and α, h, H, $l(x)$, $G(x,y)$ are given. The functions l, G are assumed to satisfy:

$$(0.2)\qquad 1+l(x)>0\,,\qquad G(x,0)>0\,,\qquad \alpha G(x,0)+l(x)>0\,,$$
$$G(x,y)\geqslant 0\,,\qquad G_y(x,y)\leqslant 0\,,$$

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and

$$G_y(x, y) < 0 \quad \text{if } 0 < y < G(x, y) . \tag{0.3}$$

The problem (0.1) arises in a natural way when one considers a non-stationary filtration problem of water in a dam with vertical walls. This problem leads to a parabolic quasi-variational inequality. Introducing a finite difference scheme with respect to the variable t, one is led to a problem of the form (0.1).

The main result of this paper (Theorem 1.1) asserts that *there exists a unique solution of* (0.1) *with* $\varphi(x)$ *which is continuous and positive valued for* $0 \leqslant x \leqslant a$.

In Section 1 we state this main result more carefully. In Sections 2-4 we prove the existence part of Theorem 1.1 under the additional restriction that $G_y(x, y) < 0$. The existence proof for general G (satisfying (0.2), (0.3)) is completed in Section 5. In Section 6 we prove the uniqueness of the solution.

The non-stationary filtration problem mentioned above is introduced in Section 7 where it is also reduced to a parabolic quasi-variational inequality. Finally, in Section 8 we apply Theorem 1.1 to the finite difference approximation of the parabolic problem.

A non-stationary filtration problem was recently studied by Torelli [6], [7] by other methods; for more details, see the remark at the end of Section 7.

1. - The main result.

Let α, a, h, H, H^* be positive numbers, $h < H < H^*$. Let

$$B = \{(x, y);\ 0 < x < a,\ 0 < y < H^*\}$$

and denote by $(\ ,\ )$ and $(\ ,\ )_0$ the scalar products in $L^2(B)$ and $L^2(0, a)$, respectively. Let $l(x)$, $G(x, y)$ be given functions defined for $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant H^*$. Consider the following problem:

Find functions $w(x, y)$, $\varphi(x)$ such that

$$(-\Delta w + \alpha w,\ v - w) \geqslant (-1 + \alpha G - \alpha\varphi,\ v - w) \tag{1.1}$$

$$\text{for any } v \in L^2(B),\ v \geqslant 0,$$

$$w(x, y) \geqslant 0 \quad \text{in } B, \tag{1.2}$$

$$\begin{aligned} w(0, y) &= \tfrac{1}{2}(H - y)^2 && \text{if } 0 \leqslant y \leqslant H, \\ &= 0 && \text{if } H < y < H^*, \end{aligned} \tag{1.3}$$

$$(1.4)\quad \begin{aligned} w(a,y) &= \tfrac{1}{2}(h-y)^2 && \text{if } 0\leqslant y\leqslant h\,, \\ &= 0 && \text{if } h<y<H^*\,, \end{aligned}$$

$$(1.5)\quad w(x,H^*)=0 \qquad \text{if } 0\leqslant x\leqslant a\,,$$

$$(1.6)\quad \begin{aligned} &\big(-w_{xx}(\cdot,0)+\alpha w(\cdot,0),\ v-w(\cdot,0)\big)_0\geqslant \\ &\quad\geqslant\big(\alpha G(\cdot,0)-\alpha\varphi(\cdot)+l(\cdot),\ v-w(\cdot,0)\big)_0 \\ &\qquad\qquad \text{for any } v=v(x)\in L^2(0,a),\ v\geqslant 0\,, \end{aligned}$$

$$(1.7)\quad w(x,y)>0 \quad \text{if } 0<y<\varphi(x),\ w(x,y)=0 \quad \text{if } \varphi(x)\leqslant y\leqslant H^*,$$

$$(1.8)\quad w\in W^{2,p}(B) \qquad \text{for any } 1<p<\infty\,.$$

Notice that, by Sobolev's inequality, w is necessarily continuously differentiable in $\bar{B}$. Notice also that if $w(x,0)>0$ for $0<x<a$ then the variational inequality (1.6) becomes

$$(1.9)\quad -w_{xx}(x,0)+\alpha w(x,0)=\alpha G(x,0)-\alpha\varphi(x)+l(x) \qquad (0<x<a)\,,$$

or, in view of (1.1),

$$(1.10)\qquad w_{yy}(x,0)=1+l(x) \qquad (0<x<a)\,.$$

For a given $\varphi(x)$, the condition (1.6) together with $w(0,0)=H^2/2$, $w(a,0)=h^2/2$ determine $w(x,0)$ for $0<x<a$. Thus system (1.1)-(1.6) (for a given $\varphi(x)$) is just an elliptic variational inequality for w and (1.8) is a regularity condition. The additional condition (1.7) on the free boundary $x=\varphi(x)$ changes the problem into (what we call) a quasi-variational inequality (Q.V.I.).

In the sequel we shall assume that

(1.11) $l(x)$ is continuous for $0\leqslant x\leqslant a$, $1+l(x)>0$;

(1.12) $G(x,y)$ is continuously differentiable in $\bar{B}$ and $G(x,0)>0$,
$\alpha G(x,0)+l(x)>0$, $\quad G(x,y)\geqslant 0$;

$$(1.13)\quad \begin{aligned} &G_y(x,y)<0 && \text{if } y<G(x,y)\,, \\ &G_y(x,y)\leqslant 0 && \text{if } y>G(x,y)\,. \end{aligned}$$

The main result of this paper is the following:

THEOREM 1.1. *Let* (1.11)-(1.13) *hold. Then there exists one and only one solution* w, φ *of the Q.V.I.* (1.1)-(1.8) *with* $\varphi(x)$ *continuous and positive valued for* $0 \leqslant x \leqslant a$.

Theorem 1.1 is proved in Sections 2-6.

The motivation for studying the Q.V.I. (1.1)-(1.8) comes from a non-stationary problem in hydraulics; this will be explained in detail in Section 7. In Section 8 we shall apply Theorem 1.1 to the hydraulic problem.

The method of proof of Theorem 1.1 has some general features that can be employed also to solve other Q.V.I. In particular, the boundary conditions (1.3), (1.4) can be replaced by more general boundary conditions without essentially affecting the proof.

In Sections 2-4 we shall prove the existence part of Theorem 1.1 when the assumption (1.1) is replaced by the stronger assumption

$$G_y(x, y) < 0 \quad \text{in } \bar{B}\,. \tag{1.14}$$

In Section 5 we shall complete the proof of existence (under the assumption (1.13)) by approximating G by functions $G - \varepsilon y$ ($\varepsilon \downarrow 0$) which, of course satisfy (1.14). The uniqueness part of Theorem 1.1 will be proved in Section 6.

2. – An approximating Q.V.I.

A general approach to solve Q.V.I. is by a fixed point theorem. In the case of (1.1)-(1.8) this approach proceeds as follows:

Given φ in some class A, we solve the variational inequality (1.1)-(1.6), (1.8) and denote its solution by w_φ. One shows that $w_\varphi(x, y)$ is monotone decreasing in y; hence there is a curve $y = \tilde{\varphi}(x)$ such that

$$\begin{aligned} w_\varphi(x, y) &> 0 \quad \text{if } y < \tilde{\varphi}(x)\,, \\ &= 0 \quad \text{if } y \geqslant \tilde{\varphi}(x)\,. \end{aligned}$$

Write $\tilde{\varphi} = T\varphi$. The problem of solving (1.1)-(1.8) is then equivalent to the problem of finding a fixed point for the mapping T.

When T is continuous, some standard fixed point theorems may be applicable. If T is not continuous, a fixed point theorem due to Tartar [5] may be applicable in case T is a monotone mapping. For the present problem, however, T is neither continuous nor monotone. We shall therefore proceed in a different manner. First we shall introduce an auxiliary Q.V.I. which approximates the original Q.V.I. In this auxiliary Q.V.I.

we have replaced the free boundary $y=\varphi(x)$ by a polygonal curve $y=\psi^n(x)$ (n positive integer):

$$\text{(2.1)}\qquad \begin{cases} y=\psi^n(x) \text{ is a polygonal curve with } n+1 \text{ vertices } \big(x_i,\ \psi^n(x_i)\big), \\ \text{where } x_i=\dfrac{a}{n}i,\ 0\leqslant i\leqslant n, \text{ and } \psi^n(0)=H,\ \psi^n(a)=h\,. \end{cases}$$

Thus, (1.1) is reduced to

$$\text{(2.2)}\quad (-\Delta w+\alpha w,\ v-w)\geqslant(-1+\alpha G-\alpha\psi^n,\ v-w)\quad \text{for any } v\in L^2(B),\ v\geqslant 0$$

and (1.6) is reduced to

$$\text{(2.3)}\quad \big(-w_{xx}(\cdot,0)+\alpha w(\cdot,0),\ v-w(\cdot,0)\big)_0\geqslant$$
$$\geqslant\big(\alpha G(\cdot,0)-\alpha\psi^n(\cdot)+l(\cdot),\ v-w(\cdot,0)\big)_0$$
$$\text{for any } v(x)\in L^2(0,a),\ v\geqslant 0\,.$$

We shall replace the condition (1.7) by the weaker condition

$$\text{(2.4)}\qquad \begin{aligned} w(x_i,y)&>0 \quad \text{if } 0<y<\psi^n(x_i)\,,\\ &=0 \quad \text{if } \psi^n(x_i)\leqslant y\leqslant H^* \quad (0\leqslant i\leqslant n)\,. \end{aligned}$$

DEFINITION. The system (2.2), (1.2)-(1.5), (2.3), (2.4), (1.8) will be called the *n-approximating Q.V.I.* and will be denoted by Π_n.

THEOREM 2.1. *Let* (1.11), (1.12), (1.14) *hold. Then there exists a solution* $w^n(x,y)$, $\psi^n(x)$ *of the problem* Π_n, *and* $\partial w^n/\partial y\leqslant 0$.

PROOF. We proceed by the approach outlined at the beginning of this section. Given an $(n+1)$-polygonal curve $y=\psi^n(x)$ with $\psi^n(0)=H$, $\psi^n(a)=h$, $0\leqslant\psi^n(x_i)\leqslant H^*$, we shall prove that there exists a unique solution w^n of the variational inequality (2.2), (1.2)-(1.5), (2.3), (1.8) and $w^n(x,y)$ is monotone decreasing in y. Let $y=\tilde\psi^n(x)$ be the $(n+1)$-polygonal curve with vertices $\big(x_i,\tilde\psi^n(x_i)\big)$, where

$$\tilde\psi^n(x_i)=\inf\{y;\ w^n(x_i,y)=0\}\,.$$

Thus,

$$\begin{aligned} w^n(x_i,y)&>0 \quad \text{if } 0<y<\psi^n(x_i)\,,\\ &=0 \quad \text{if } \psi^n(x_i)\leqslant y\leqslant H^*\,. \end{aligned}$$

Let $\tilde{\psi}^n = T\psi^n$. We shall show later on that T has a fixed point.

LEMMA 2.2. *There exists a unique solution* w^n *of* (2.2), (1.2)-(1.5), (2.3), (1.8) *and* $\partial w^n(x, y)/\partial y \leqslant 0$.

PROOF. For any $0 < \varepsilon < 1$, let $\beta_\varepsilon(t)$ be a C^∞ function satisfying:

$$\begin{aligned}
&\beta_\varepsilon(t) \to 0 && \text{if } t > 0,\ \varepsilon \downarrow 0\,,\\
&\beta_\varepsilon(t) \to -\infty && \text{if } t < 0,\ \varepsilon \downarrow 0\,,\\
&\beta_\varepsilon'(t) \geqslant 0\,,\\
&\beta_\varepsilon(0) < -1 - \alpha H^*\,.
\end{aligned}$$

Consider the Dirichlet problem

$$-\Delta w + \alpha w + \beta_\varepsilon(w) = -1 + \alpha G - \alpha\psi^n \quad \text{in } B\,, \tag{2.5}$$

$$w \text{ satisfies (1.3)-(1.5)}\,, \tag{2.6}$$

$$\begin{aligned}
-w_{xx}(x, 0) + \alpha w(x, 0) + \beta_\varepsilon\big(w(x, 0)\big) = \\
= \alpha G(x, 0) - \alpha\psi^n(x) + l(x) \quad \text{in } 0 < x < a\,.
\end{aligned} \tag{2.7}$$

Using standard elliptic estimates and Schauder's fixed point theorem one can show that (2.5)-(2.7) has a unique solution $w = w_\varepsilon$ in $W^{2,p}(B)$, for any $1 < p < \infty$. Set $\zeta = \partial w_\varepsilon/\partial y$. Then

$$-\Delta\zeta + \alpha\zeta + \beta_\varepsilon'(w_\varepsilon)\zeta = \alpha G_y < 0\,.$$

Hence, by the maximum principle, ζ takes its maximum in $\bar{B}$ on the boundary. By (1.3), (1.4), $\zeta \leqslant 0$ on $x = 0$ and on $x = a$. By (2.2), (2.3),

$$\zeta_y = 1 + l > 0 \quad \text{on } y = 0\,,$$

so that ζ cannot take its maximum at $y = 0$. Finally, at $y = H^*$, (2.5) gives $\zeta_y < 0$, since

$$\beta_\varepsilon(0) + 1 - \alpha G(x, H^*) + \alpha\psi^n(x) < 0$$

(by the choice of $\beta_\varepsilon(0)$); consequently ζ cannot take its maximum at $y = H^*$. We conclude that $\zeta \leqslant 0$ on the boundary of B and, therefore, $\zeta \leqslant 0$ in B.

By standard arguments (see, for instance, [3]) one can show that, as $\varepsilon \downarrow 0$, w_ε converges to a function w^n which is the unique solution of (2.2), (1.2)-(1.5), (2.3), (1.8). Since $\partial w_\varepsilon/\partial y \leqslant 0$, we also have $\partial w^n/\partial y \leqslant 0$. This completes the proof.

Denote by X the space of all polygonal curves (2.1) and introduce the uniform norm $\| \ \|$ in X. Then X is a closed, bounded convex subset of the finite dimensional Banach space consisting of $(n+1)$-polygonal curves with vertices at $x = x_i$, $-\infty < x_i < \infty$, $0 \leqslant i \leqslant n$. The mapping T maps X into itself.

LEMMA 2.3. – *T is a continuous mapping.*

PROOF. Let ψ_j^n, $\bar{\psi}^n$ be in X,

$$\|\psi_j^n - \bar{\psi}^n\| \to 0 \quad \text{if } j \to \infty .$$

We have to show that

$$\|T\psi_j^n - T\bar{\psi}^n\| \to 0 \quad \text{if } j \to \infty . \tag{2.8}$$

Denote by w_j^n and $\overline{w}^n$ the solutions of the variational inequalities (2.2), (1.2)-(1.5), (2.3), (1.8) corresponding to ψ_j^n and $\bar{\psi}^n$ respectively, and let φ_j^n, $\bar{\varphi}^n$ be the polygonal curves with vertices $(x_i, \varphi_j^n(x_i))$, $(x_i, \bar{\varphi}^n(x_i))$ given by

$$\varphi_j^n(x_i) = \inf \{y; w_j^n(x_i, y) = 0\} ,$$
$$\bar{\varphi}^n(x_i) = \inf\{y; \overline{w}^n(x_i, y) = 0\} .$$

Since

$$|w_j^n|_{W^{2,p}(B)} \leqslant C < \infty \qquad (C \text{ independent of } j) ,$$

$w_j^n \to \overline{w}^n$ uniformly in $\bar{B}$. Recalling that

$$w_j^n(x_i, \varphi_j^n(x_i)) = 0 ,$$

we then conclude that

$$\overline{w}^n(x_i, \varphi_*^i) = 0 \qquad \text{where } \varphi_*^i = \varliminf_{j\to\infty} \varphi_j^n(x_i) .$$

Thus

$$\bar{\varphi}^n(x_i) \leqslant \varliminf_{j\to\infty} \varphi_j^n(x_i) ,$$

which implies that

$$\bar{\varphi}^n(x) \leqslant \varliminf \varphi_j^n(x) . \tag{2.9}$$

In order to complete the proof of (2.8), it remains to show that

$$\overline{\lim}\, \varphi^n(x) \leqslant \bar{\varphi}^n(x)\,. \tag{2.10}$$

Let

$$\gamma_j(x) = \inf\{y;\, w_j^n(x,y) = 0\}\,,$$
$$\gamma(x) = \inf\{y;\, \overline{w}^n(x,y) = 0\}\,.$$

Since

$$\varphi_j^n(x_i) = \gamma_j(x_i)\,, \qquad \bar{\varphi}^n(x_i) = \gamma(x_i)\,,$$

(2.10) will follow from

$$\overline{\lim}\, \gamma_j(x) = \gamma(x) \qquad \text{for all } x \in (0, a)\,. \tag{2.11}$$

The argument giving (2.9) also establishes that

$$\gamma(x) \leqslant \varliminf_{j\to\infty} \gamma_j(x) \qquad \text{for all } x\,. \tag{2.12}$$

We shall next prove that

$$\gamma_j(x) \to \gamma(x) \qquad \text{in measure}\,. \tag{2.13}$$

If this is false then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ such that

$$\text{measure } \{x;\, |\gamma_{j_i}(x) - \gamma(x)| > \delta_0\} > \varepsilon_0$$

for a subsequence $j_i \uparrow \infty$. In view of (2.12) we then have

$$\text{measure } \{x;\, \gamma_{j_i}(x) - \gamma(x) > \delta_0\} > \varepsilon_0\,. \tag{2.14}$$

Let

$$S = \{(x, y);\, \gamma(x) < y < H^*\}\,,$$
$$Q_i = \{(x, y);\, \gamma(x) < y < \gamma_{j_i}(x)\}\,.$$

The variational inequality for $w_{j_i}^n$ gives

$$\begin{aligned}\iint\limits_S (-\Delta w_{j_i}^n + \alpha w_{j_i}^n)\, dx\, dy &= \iint\limits_{Q_i} (-\Delta w_{j_i}^n + \alpha w_{j_i}^n)\, dx\, dy \\ &= \iint\limits_{Q_i} (-1 + \alpha G - \alpha \psi_{j_i}^n)\, dx\, dy \\ &= \iint\limits_{Q_i} (-1 + \alpha G - \alpha \bar{\psi}^n)\, dx\, dy + \iint\limits_{Q_i} (\alpha \bar{\psi}^n - \alpha \psi_{j_i}^n)\, dx\, dy\,.\end{aligned} \tag{2.15}$$

Since $\alpha\psi^n_{j_i} - \alpha\bar{\psi}^n \to 0$ uniformly and $-1 + \alpha G - \alpha\bar{\psi}^n \leqslant 0$ a.e. in S, the right-hand side is $\leqslant \sup(\alpha G_y)\delta_0\varepsilon_0$ if i is sufficiently large; here we used (2.14). Since $w^n_{j_i} \to w^n$ weakly in $W^{2,2}(B)$, we get from (2.15), upon taking $i \to \infty$,

$$\iint_S (-\Delta\bar{w}^n + \alpha\bar{w}^n)\,dx\,dy \leqslant \sup(\alpha G_y)\,\delta_0\varepsilon_0 < 0\ ;$$

this is impossible since $\bar{w}^n = 0$ in S and measure $S \geqslant \varepsilon_0\delta_0 > 0$ by (2.14) (since $\gamma_{j_i}(x) \leqslant H^*$).

Having proved (2.13), we shall now establish (2.12). If (2.12) is false then there exists a point $x_0 \in (0, a)$ such that

$$\lambda \equiv \overline{\lim}\, \gamma_j(x_0) - \gamma(x_0) > 0\ . \tag{2.16}$$

We shall need that fact that $\gamma(x)$ is continuous; the proof is similar to the proof of Lemma 5.3 (given in Section 5) and will be omitted.

From (2.16) we deduce, upon using (2.13) and the continuity of $\gamma(x)$, that for any $\delta > 0$ there exist points $x_1 < x_0 < x_2$ with $x_2 - x_1 < \delta$ such that

$$\min\{\gamma_{j_0}(x_0) - \gamma_{j_0}(x_1),\ \gamma_{j_0}(x_0) - \gamma_{j_0}(x_2)\} > \frac{\lambda}{2}\,. \tag{2.17}$$

Since $|w^n_j|_{W^{1,p}} \leqslant$ const independent of j,

$$|w^n_j(x, y)| \leqslant K\delta \qquad \text{if } x_1 < x < x_2,\ y > y_1$$

where K is a constant independent of j and $y_1 = \max\{\gamma_{j_0}(x_1), \gamma_{j_0}(x_2)\}$. Let

$$E = \{(x, y);\ x_1 < x < x_2,\ y > y_2\}\,, \qquad y_2 = y_1 + \delta^{\frac{1}{3}}\,.$$

Then $-1 + \alpha G - \alpha\psi^n_j \leqslant -c_1\delta^{\frac{1}{3}}$, $c_1 > 0$ in E, where c_1 is independent of δ. We compare, in E, $w^n_{j_0}$ with the function

$$v(y) = \begin{cases} C(\mu - (y - y_2))^2 & \text{if } y_2 < y < y_2 + \mu\,, \\ 0 & \text{if } y > y_2 + \mu\,, \end{cases}$$

where $C\mu^2 = K\delta$, $C/2 = c_1\delta^{\frac{1}{3}}$ (cf. the argument following (3.5)). We conclude that $w^n_{j_0} \leqslant v$, so that, in particular,

$$\gamma_j(x_0) \leqslant y_1 + \mu = y_0 + K_1\delta^{\frac{1}{3}},$$

where K_1 is a constant independent of δ. Since δ is arbitrary, we get a contradiction to (2.17).

Having proved Lemma 2.3, we recall that X is a closed, bounded convex set in a finite dimensional Banach space. Since T maps X into itself and is continuous, Brower's fixed point theorem yields the existence of a fixed point ψ^n for T. Denote the corresponding solution of the variational inequality by w^n. Then $w^n(x, y)$, $\psi^n(x)$ form a solution of the problem Π_n. This completes the proof of Theorem 2.1.

3. – Behavior of ψ^n as $n \to \infty$.

Let

$$\varphi^n(x) = \inf\{y; w^n(x, y)\} = 0$$

where (w^n, ψ^n) is a solution of problem Π_n. Thus,

$$\varphi^n(x_i) = \psi^n(x_i) \quad \text{if } 0 \leqslant i \leqslant n .$$

LEMMA 3.1. *At each point x_0 where $\varphi^n(x_0) < H^*$,*

$$-1 + \alpha G(x_0, \varphi^n(x_0)) - \alpha\psi^n(x_0) \leqslant 0 . \tag{3.1}$$

PROOF. Since $w^n(x, y) = 0$ if $y > \varphi^n(x)$, the variational inequality for w^n gives, for almost all x with $\varphi^n(x) < H^*$,

$$-1 + \alpha G(x, y) - \alpha\psi^n(x) \leqslant 0 \qquad \text{if } \varphi^n(x) < y < H^* .$$

Taking $y \downarrow \varphi^n(x)$ we conclude that (3.1) holds for almost all x_0 with $\varphi^n(x_0) < H^*$. To prove (3.1) for *all* x_0 we shall use the continuity of ψ^n and the inequality $G_y < 0$. These two properties imply that the curve $y = \zeta(x)$ defined by

$$-1 + \alpha G(x, \zeta(x)) - \alpha\psi^n(x) = 0$$

is uniquely defined and continuous for x in a neighborhood of x_0, provided

i) $$-1 + \alpha G(x_0, 0) \quad - \alpha\psi^n(x_0) > 0 ,$$

ii) $$-1 + \alpha G(x_0, H^*) - \alpha\psi^n(x_0) < 0 .$$

Let us first assume that (i) and (ii) hold. Then, for any $\varepsilon > 0$, the rectangle

$$R_\delta\colon \; |x - x_0| < \delta, \; \left|y - (\zeta(x_0) - \varepsilon)\right| < \delta$$

lies below the curve $y=\zeta(x)$ if δ is sufficiently small. But then, since $G_y<0$,

$$-1+\alpha G(x,y)-\alpha\psi^n(x)>\eta \quad \text{in } R_\delta$$

for some $\eta>0$. Hence, if $(x,y)\in R_\delta$ then

$$\begin{aligned} -\Delta w^n+\alpha w^n &> \eta \quad \text{if } w^n>0\,,\\ &=0 \quad \text{a.e. if } w^n=0\,; \end{aligned}$$

also, $w^n\neq 0$ a.e. in R_δ since $\varphi^n(x)\geqslant\zeta(x)$ a.e. if $|x-x_0|<\delta$.

Since w^n is in $C^1(\bar{R}_\delta)\cap W^{2,p}(R_\delta)$ $(1<p<\infty)$, we can represent it by means of Green's function G_δ of $-\Delta w+\alpha w=0$ in R_δ:

$$w^n=\iint\limits_{R_\delta} G_\delta(-\Delta w^n+\alpha w^n)\,dx\,dy-\int\limits_{\partial R_\delta}\frac{\partial G_\delta}{\partial\nu}\,w^n\,ds\,.$$

It follows (since $w^n\geqslant 0$ on ∂R_δ) that $w^n>0$ in R_δ. In particular, $w^n(x_0,\zeta(x_0)-\varepsilon)>0$, i.e. $\varphi^n(x_0)>\zeta(x_0)-\varepsilon$. Since ε is arbitrary, $\varphi^n(x_0)\geqslant\zeta(x_0)$.

We have assumed so far that (i) and (ii) hold. Now, if (i) does not hold then (3.1) is trivially true. If, on the other hand, (ii) is not satisfied then we can take $\zeta(x)\equiv H^*$ in the above proof and conclude that $\varphi^n(x_0)\geqslant\zeta(x_0)\equiv H^*$, a contradiction; thus (ii) must be satisfied.

LEMMA 3.2. *There exists a constant M such that, for any $0\leqslant x'<\bar{x}<x''\leqslant a$,*

$$\psi^n(\bar{x})\leqslant\max\{\psi^n(x'),\psi^n(x'')\}+M|x''-x'|^{\frac{1}{2}}\,. \tag{3.2}$$

PROOF. We shall first prove (3.2) in case $x'=x_i$, $x''=x_j$. Let $\delta=x''-x'$. For definiteness we take $\psi^n(x')\leqslant\psi^n(x'')$. We may also assume that $\psi^n(\bar{x})>\psi^n(x'')$ and $(\bar{x},\psi^n(\bar{x}))$ is a vertex. Set $y_0=\psi^n(x'')$. Let

$$I_y=\{(x,y);\,\bar{x}-\delta_{1y}<x<\bar{x}+\delta_{2y}\}$$

where

$$w^n(x,y)>0 \text{ if } \bar{x}-\delta_{1y}<x<\bar{x}+\delta_{2y},\quad w^n(\bar{x}-\delta_{1y},y)=0,\quad w^n(\bar{x}+\delta_{2y},y)=0\,.$$

Thus $\delta_{1y}\leqslant\bar{x}-x'$, $\delta_{2y}\leqslant x''-\bar{x}$ if $y\geqslant y_0$; indeed, this follows from the fact that $w^n(x,y)=0$ if $x=x_i$, $y>\psi^n(x_i)$ and if $x=x_j$, $y>\psi^n(x_j)$.

Consider all the vertices $(x_l,\psi^n(x_l))$ with $\bar{x}-\delta_{1y_0}<x_l<\bar{x}+\delta_{2y_0}$, say $l=l_0,l_0+1,\dots,l_1$. They must all lie above I_{y_0}, i.e., $\psi^n(x_l)>y_0$. Indeed, if $\psi^n(x_l) < y_0$ then $w^n(x_l, y_0) = 0$, which is impossible (since $w^n > 0$ on I_{ν_0}). We conclude that

$$\psi^n(x) - y_0 \geqslant 0 \quad \text{if } x_{l_0} \leqslant x \leqslant x_{l_1}.$$

Consequently, by (3.1),

$$\begin{aligned}(3.3)\qquad & -1 + \alpha G(x, y_0) - \alpha\psi^n(x)\\ &= [-1 + \alpha G(x_j, y_0) - \alpha\psi^n(x_j)] + \alpha[G(x, y_0) - G(x_j, y_0)]\\ &\quad - [\alpha\psi^n(x) - \alpha\psi^n(x_j)]\\ &\leqslant O(\delta)\end{aligned}$$

if $x_{l_0} \leqslant x \leqslant x_{l_1}$. The same inequality holds for $x_{l_1} < x < \bar{x} + \delta_{2\nu_0}$ if $\psi^n(x) > \psi^n(x_j)$, i.e., if $\psi^n(x_{l_1+1}) \geqslant y_0$. Suppose $\psi^n(x_{l_1+1}) < y_0$. Then, $\psi^n(x) - \psi^n(x_{l_1+1}) \geqslant 0$ if $x_{l_1} < x < \bar{x}$ and, by (3.1),

$$-1 + G\big(x_{l_1+1}, \psi^n(x_{l_1+1})\big) - \alpha\psi^n(x_{l_1+1}) \leqslant 0 .$$

Clearly $x_{l_1+1} < x_j$ so that $x_{l_1+1} - x < \delta$ if $(x, y_0) \in I_\nu$. It follows that

$$\begin{aligned}& -1 + \alpha G(x, y_0) - \alpha\psi^n(x)\\ &\quad = \big[-1 + G\big(x_{l_1+1}, \psi^n(x_{l_1+1})\big) - \alpha\psi^n(x_{l_1+1})\big]\\ &\qquad + [G(x, y_0) - G(x_{l_1+1}, y_0)] - [\alpha\psi^n(x) - \psi^n(x_{l_1+1})]\\ &\quad \leqslant O(\delta)\end{aligned}$$

provided $x_{l_1+1} \leqslant x \leqslant \bar{x} + \delta_{2\nu_0}$. Similarly one handles the case where $\bar{x} - \delta_{1\nu_0} < < x < x_{l_0}$. We have thus proved that

$$(3.4)\qquad -1 + G(x, y_0) - \alpha\psi^n(x) \leqslant O(\delta) \quad \text{if } (x, y_0) \in I_{\nu 0} .$$

Since $G_y < 0$ in $\bar{B}$, we now deduce that there is a positive constant c such that

$$(3.5)\quad -1 + G(x, y) - \alpha\psi^n(x) \leqslant -c\delta^{\frac{1}{2}} \quad \text{if } \bar{x} - \delta_{1\nu_0} < x < \bar{x} + \delta_{2\nu_0},\ y - y_0 > \delta^{\frac{1}{2}} .$$

Let $y_1 = y_0 + \delta^{\frac{1}{2}}$. Consider the function

$$v(y) = \begin{cases} C[\gamma - (y - y_1)]^2 & \text{if } 0 < y - y_1 < \gamma ,\\ 0 & \text{if } y - y_1 > \gamma .\end{cases}$$

We claim that, for a suitable C, v satisfies the variational inequality

$$-v_{yy}+\beta(v)\ni -c\delta^{\alpha} \quad \text{if } y>y_1,$$

where $\beta(v)$ is the monotone operator $\beta(v)=\{0\}$ if $v>0$, $\beta(0)=(-\infty, 0]$. In fact, it suffices to choose C so that

$$\tag{3.6} \tfrac{1}{2}C = c\delta^{\frac{1}{3}}.$$

Since w^n is Lipschitz continuous (with coefficient independent of n) and since it vanishes at the endpoints of the interval I_{y_1} (whose length is $\leqslant \delta$), we have

$$w^n(x, y_1)\leqslant A\delta \quad \text{if } (x, y_1)\in I_{y_1}$$

where A is a constant independent of n. Also,

$$w^n(x, y)=0 \quad \text{if } x=\bar{x}-\delta_{1y_1} \text{ or } x=\bar{x}+\delta_{2y_1}, \quad y>y_1.$$

Hence, if $v(y_1)=A\delta$, i.e., if

$$\tag{3.7} Ck^2 = A\delta$$

then a standard comparison theorem for variational inequalities gives

$$w^n(x, y)\leqslant v(y).$$

Since $(\bar{x}, \psi^n(\bar{x}))$ is a vertex, $w^n(\bar{x}, \psi^n(\bar{x}))=0$ so that

$$\tag{3.8} \psi^n(\bar{x})\leqslant \gamma + y_1.$$

Solving for γ^2 from (3.7), (3.6) we get

$$\gamma^2=\frac{A\delta}{C}=\frac{A\delta}{2c\delta^{\frac{1}{3}}}=\frac{A}{2c}\delta^{\frac{2}{3}}.$$

Hence, by (3.8),

$$\psi^n(\bar{x})\leqslant y_0+\delta^{\frac{1}{3}}+\gamma\leqslant y_0+M\delta^{\frac{1}{3}}.$$

This completes the proof of the lemma in case $x'=x_i$, $x''=x_j$.

Consider next the case where x', x'' are not necessarily vertices, but $x''-x'>1/n$. The point $(x', \psi^n(x'))$ lies on a segment of the polygonal curve

$y=\psi^n(x)$; one of the endpoints $(x_i, \psi^n(x_i))$ is such that $\psi^n(x_i)\leqslant\psi^n(x')$. Since we may assume that $\psi^n(x')<\psi^n(\bar{x})$ and that $(\bar{x}, \psi^n(\bar{x}))$ is a vertex, we have: $x_i<\bar{x}$. Also, clearly, $|x'-x_i|<1/n$.

Similarly let x_j be such that $\psi^n(x_j)<\psi^n(x'')$, $\bar{x}<x_j$, $|x_j-x''|<1/n$. Applying the special case proved above for x_i, $\bar{x}$, x_j, and noting that $|x_j-x_i|<3|x''-x'|$, the assertion (3.2) follows.

It remains to consider the case where $|x'-x''|<1/n$. Without loss of generality we may take $x'<x_k<x''$, $\bar{x}=x_k$. We also assume for definiteness that $\psi^n(x')\leqslant\psi^n(x'')$. By what we have already proved,

$$\psi^n(x_k)\leqslant\psi^n(x_{k+1})+M\left(\frac{1}{n}\right)^{\frac{1}{3}}\qquad (M \text{ constant})\,.$$

Since

$$\frac{\psi^n(\bar{x})-\psi^n(x'')}{\psi^n(\bar{x})-\psi^n(x_{k+1})}=\frac{x''-\bar{x}}{x_{k+1}-x_k}=\frac{x''-\bar{x}}{1/n}\,,$$

we get

$$\psi^n(\bar{x})-\psi^n(x'')\leqslant M\left(\frac{1}{n}\right)^{\frac{1}{3}}\frac{x''-\bar{x}}{1/n}\leqslant M(x''-x')^{\frac{1}{3}}\,.$$

LEMMA 3.3. *Let $f_n(x)$ be a sequence of functions in $L^\infty(a,b)$ (where $-\infty<$ $<a<b<\infty$) satisfying:*

(i) $\|f_n\|_{L^\infty}\leqslant N$ *(N constant);*

(ii) $f_n\to g$ *weakly in $L^p(a,b)$, for some $1<p<\infty$;*

(iii) *there exists a continuous non-negative function $\omega(t)$, for $0\leqslant t<\infty$, such that $\omega(0)=0$ and*

$$f_n(x)\leqslant\max\{f_n(x'), f_n(x'')\}+\omega(x''-x')$$

for any $a\leqslant x'<x<x''\leqslant b$. Then $f_n\to g$ in measure in (a,b).

PROOF. Suppose.

$$f_n\not\to g \quad\text{in measure}\,. \tag{3.9}$$

Let

$$B=\{x\in(a,b);\,\overline{\lim_{n\to\infty}}f_n(x)>g(x)\}\,,$$

$$C_\varepsilon^n=\{x\in(a,b);\,f_n(x)-g(x)<-\varepsilon\}\,,$$

and denote by μ the Lebesgue measure on the real line.

If

$$\mu(B) = 0 \tag{3.10}$$

then, by (3.9), there exist $\varepsilon > 0$, $\delta > 0$ and a subsequence $n_j \uparrow \infty$ such that $\mu(C_\varepsilon^{n_j}) > \delta$. Using this and (3.10), we find that

$$\overline{\lim_{j\to\infty}} \int_{(a,b)} (f_{n_j} - g)\, dx \leqslant -\varepsilon\delta\ ,$$

thus contradicting (ii). Consequently, if (3.9) holds then $\mu(B) > 0$. Hence also

$$\mu(B_\varepsilon) > 0 \tag{3.11}$$

for some $\varepsilon > 0$, where

$$B_\varepsilon = \left\{x \in (a, b)\,;\ \overline{\lim_{n\to\infty}} f_n(x) - g(x) > \varepsilon\right\}\,.$$

Let x_0 be a point of B_ε such that

$$\lim_{r\to 0} \frac{1}{2r} \int_{x_0 - r}^{x_0 + r} |g(y) - g(x_0)|\, dy = 0\ , \tag{3.12}$$

i.e., x_0 is a Lebesgue point of g. Let $\eta > 0$ be so small that

$$\omega(\eta') < \frac{\varepsilon}{3} \qquad \text{if } 0 < \eta' < \eta \tag{3.13}$$

and choose r so small that

$$2r < \eta\ , \tag{3.14}$$

$$\frac{1}{2r} \int_{x_0 - r}^{x_0 + r} |g(y) - g(x_0)|dy < \frac{\varepsilon}{20}\,. \tag{3.15}$$

Let

$$D = \left\{x \in (x_0 - r,\, x_0 + r)\,;\ |g(x) - g(x_0)| \geqslant \frac{\varepsilon}{3}\right\}\,.$$

Then, from (3.15) we deduce that

$$\mu(D) < \left[\int_{x_0 - r}^{x_0 + r} |g(y) - g(x_0)|\, dy\right] \Big/ \left(\frac{\varepsilon}{3}\right) \leqslant \frac{3}{10}\, r < \frac{1}{2}\, r\,. \tag{3.16}$$

Hence, for the set

$$G = (x_0 - r, x_0 + r) - D$$

we have

$$\mu[G \cap (x_0 - r, x_0)] > 0 , \qquad \mu[G \cap (x_0, x_0 + r)] > 0 . \tag{3.17}$$

Since $x_0 \in B_\varepsilon$, there exists a sequence $n_j \uparrow \infty$ such that

$$f_{n_j}(x_0) > g(x_0) + \varepsilon . \tag{3.18}$$

Let χ_G, χ^-, χ^+ be the characteristic functions for the sets G, $(x_0 - r, x_0)$ and $(x_0, x_0 + r)$ respectively. By the weak convergence of f_n to g,

$$\int \chi_G \chi^-(f_{n_j} - g)\, dx \to 0 ,$$

$$\int \chi_G \chi^+(f_{n_j} - g)\, dx \to 0$$

as $j \to \infty$. These relations together with (3.17) imply that there exist points x' in $G \cap (x_0 - r, x_0)$ and x'' in $G \cap (x_0, x_0 + r)$ and j_0 such that

$$f_{n_{j_0}}(x') - g(x') < \frac{\varepsilon}{3} ,$$

$$f_{n_{j_0}}(x'') - g(x'') < \frac{\varepsilon}{3} .$$

Recalling (3.18) we derive

$$\begin{aligned} f_{n_{j_0}}(x_0) - f_{n_{j_0}}(x') = [f_{n_{j_0}}(x_0) - g(x_0)] + [g(x_0) - g(x')] + \\ + [g(x') - f_{n_{j_0}}(x')] > \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3} , \end{aligned}$$

and similarly

$$f_{n_{j_0}}(x_0) - f_{n_{j_0}}(x'') \geqslant \frac{\varepsilon}{3} .$$

Thus

$$f_{n_{j_0}}(x_0) - \max \left\{ f_{n_{j_0}}(x'), f_{n_{j_0}}(x'') \right\} \geqslant \frac{\varepsilon}{3} .$$

Since $x'' - x' < 2r$, $\omega(x'' - x') < \varepsilon/3$ (by (3.13), (3.14)). We have thus derived a contradiction to the assumption (iii). This proves that (3.9) cannot occur.

LEMMA 3.4. *There exists a set $N \subset (0, a)$ of measure zero and a subsequence ψ^n such that, as $n_j \to \infty$,*

(3.19) $$\psi^{n_j}(x) \to \psi(x) \quad \textit{for all } x \in S = [0, a] - N\,;$$

the set S contains all the points ia/n, $0 \leqslant i \leqslant n$, $1 \leqslant n < \infty$. The function ψ, restricted to S, has a right limit at each point $x \in S$.

PROOF. The assertion (3.19) follows from Lemma 3.2, 3.3 by taking a suitable subsequence ψ^{n_j} of ψ^n. The function $\psi(x)$ must clearly satisfy the inequality

(3.20) $$\psi(\bar{x}) \leqslant \max\{\psi(x'), \psi(x'')\} + M(x'' - x')^{\frac{1}{2}}$$

for any $0 \leqslant x' < \bar{x} < x'' \leqslant a$, $x' \in S$, $\bar{x} \in S$, $x'' \in S$. Let $x_0 \in S$. We claim that $\lim \psi(x)$ exists as $x \in S$, $x \downarrow x_0$. Indeed, otherwise there exist sequences ξ_n, η_n such that

$$\xi_n \in S\,, \quad \eta_n \in S\,, \quad \xi_n \downarrow x_0\,, \quad \eta_n \downarrow x_0\,, \quad \psi(\xi_n) \to \alpha\,, \quad \psi(\eta_n) \to \beta$$

where $\alpha > \beta$. Taking in (3.20) $x' = \eta_i$, $x'' = \eta_j$, $\bar{x} = \xi_k$ with suitable i, j, k increasing to infinity, we get $\beta \leqslant \alpha$; a contradiction.

4. – Existence of a solution for the Q.V.I. in case $G_y < 0$.

There exists a sequence $n_j \uparrow \infty$ such that, as $n = n_j \to \infty$,

(4.1) $$\psi^n(x) \to \psi(x) \qquad \text{if } x \in S\,,$$

(4.2) $$w^n \to w \qquad \text{uniformly in } \bar{B}\,,$$

(4.3) $$w^n \to w \qquad \text{weakly in } W^{2,p}(B)\,,$$

(4.4) $$w^n(\cdot, 0) \to w(\cdot, 0) \qquad \text{weakly in } W^{2,p}(0, a)$$

for any $1 < p < \infty$; here S is as in Lemma 3.4. Indeed, (4.1) follows from Lemma 3.4 and (4.3) $\big((4.4)\big)$ follows from the fact that $w^n(x, y)$ $\big(w^n(x, 0)\big)$ is a solution of the variational inequality (1.1) $\big((1.6)\big)$ with $\varphi = \psi^n$ and, consequently, its $W^{2,p}$ norm is bounded uniformly with respect to n.

For simplicity we shall take $n_j = n$. Clearly $w_y \leqslant 0$. Let

$$\varphi(x) = \inf\{y\,; \, w(x, y) = 0\}\,.$$

Notice that w is a solution of the variational inequality (1.1)-(1.6), (1.8) with φ replaced by ψ in both (1.1) and (1.6); in particular,

$$(4.5) \qquad (-\Delta w + \alpha w, v - w) \geqslant (-1 + \alpha G - \alpha\psi, v - w) \quad \text{for any } v \in L^2(B),\ v \geqslant 0 .$$

If we show that

$$(4.6) \qquad \varphi = \psi \quad \text{a.e.},$$

then w, φ constitute a solution of the Q.V.I. (1.1)-(1.8). In the remaining part of this section we shall prove the relation (4.6).

Let $x_0 \leqslant x \leqslant x_{i+1}$ where $x_j = (a/n)j$. Then $w^n(x, y) = 0$ if $x = x_i$, $y > \psi^n(x_i)$ and if $x = x_{i+1}$, $y > \psi^n(x_{i+1})$. Hence, if $\psi^n(x_i) \leqslant \psi(x_{i+1})$,

$$\begin{aligned} w^n(x, \psi^n(x)) &= w^n(x, \psi^n(x)) - \psi^n(x_i, \psi^n(x)) \\ &\leqslant A(x - x_i) \leqslant A/n \end{aligned}$$

where $A = \sup |w_x|$. Similarly one shows that $w^n(x, \psi^n(x)) \leqslant A/n$ if $\psi^n(x_{i+1}) < < \psi^n(x_i)$. Taking $n \to \infty$ and using (4.1), (4.2) we find that a.e. $w(x, \psi(x)) = 0$; consequently $\varphi(x) \leqslant \psi(x)$. It remains to show that

$$(4.7) \qquad \psi(x) \leqslant \varphi(x) \quad \text{a.e.}.$$

Denote by S^* the set of all points $x \in [0, a]$ for which either $\varphi(x) = H^*$ or

$$-1 + \alpha G(x, \varphi(x)) - \alpha\psi(x) \leqslant 0 .$$

The variational inequality (4.5) for w implies that almost all x in $[0, a]$ belong to S^*. Let

$$S_0 = S \cap S^* .$$

We shall need the following lemma.

LEMMA 4.1. *The function φ restricted to S_0 has a right limit at each point of S_0.*

PROOF. If the assertion is not true then there is point $x_0 \in S_0$ and a $\delta_0 > 0$ such that for any $\varepsilon > 0$ we can find points $x' < \bar{x} < x''$ in S_0 with $x_0 < x'$, $x'' - x_0 < \varepsilon$, satisfying

$$(4.8) \qquad 2\delta_0 + \max\{\varphi(x'), \varphi(x'')\} < \varphi(\bar{x}) .$$

Assume for definiteness that

$$y_0 \equiv \varphi(x'') \geqslant \varphi(x') .$$

Since $x'' \in S_0 \subset S^*$,

$$-1 + \alpha G\big(x'', \varphi(x'')\big) - \alpha\psi(x'') \leqslant 0 .$$

Since ψ, restricted to S_0, has a right limit at x_0, we obtain

$$-1 + \alpha G(x, y_0) - \alpha\psi(x) \leqslant O(\varepsilon) + o(1) \qquad (x' < x < x'',\ x \in S_0)$$

where $o(1) \to 0$ if $\varepsilon \to 0$. Recalling that $G_y < 0$, we deduce that there exists a $\delta > 0$ such that

$$-1 + \alpha G(x, y_0 + \delta_0) - \alpha\psi(x) \leqslant -\delta \qquad (x' < x < x'',\ x \in S_0)$$

provided ε is sufficiently small.

Since $w(x'', y_0) = 0$, also $w(x'', y_0 + \delta_0) = 0$ and, consequently,

$$w(x, y_0 + \delta_0) \leqslant A\varepsilon \quad \text{ if } x' < x < x''$$

where $A = \sup |w_x|$.

We shall compare w with

$$v = v(y) = \begin{cases} C(y - (y_0 + \delta_0) - \gamma)^2 & \text{if } y_0 + \delta_0 < y < y_0 + \delta_0 + \gamma , \\ 0 & \text{if } y > y_0 + \delta_0 + \gamma . \end{cases}$$

Choosing C, γ such that

$$\tag{4.9} \tfrac{1}{2} C = \delta , \qquad C\gamma^2 = A\varepsilon ,$$

we have

$$-v_{yy} + \beta(v) \ni -\delta ,$$

$$v(y_0 + \delta_0) = A\varepsilon .$$

Hence, by a comparison theorem for variational inequalities,

$$w(x, y) \leqslant v(y) \quad \text{ if } x' < x < x'',\ y_0 + \delta_0 < y < H^* .$$

In particular,

$$\varphi(\bar{x}) \leqslant y_0 + \delta_0 + \gamma .$$

Solving (4.9) for γ^2 we find that

$$\gamma^2 = \frac{A\varepsilon}{C} = \frac{A\varepsilon}{2\delta} < (\delta_0)^2$$

if ε is sufficiently small, so that $\varphi(\bar{x}) < y_0 + 2\delta_0$. This contradicts (4.8) Thus the lemma is proved.

Suppose now that (4.7) is false. Then there is a subset K of S_0 of positive measure and a positive number δ such that

$$\varphi(x) < \psi(x) - 5\delta \quad \text{if } x \in K. \tag{4.10}$$

Let x_0 be a point of density of K. Then (4.10) holds at a sequence of points $x = \xi_n \in K$, $\xi_n \downarrow x_0$: Now, by Lemmas 3.4, 4.1, φ and ψ, when restricted to S_0, have a right limit at x_0. Consequently, for any $\varepsilon > 0$ there exists an interval $(x_0, x_0 + \varepsilon')$ such that

$$|\varphi(x) - \varphi(x_0 + 0)| < \varepsilon, \quad |\psi(x) - \psi(x_0 + 0)| < \varepsilon, \tag{4.11}$$

$$\varphi(x) < \psi(x) - 4\delta \tag{4.12}$$

for all $x \in S_0 \cap (x_0, x_0 + \varepsilon')$, where $\varphi(x_0 + 0)$, $\psi(x_0 + 0)$ are the right limits of φ, ψ (when restricted to S_0) at x_0.

Suppose there exist vertices $(x_i, \psi^n(x_i))$, $(x_j, \psi^n(x_j))$ with x_i, x_j in $(x_0, x_0 + \varepsilon')$ and with

$$\psi^n(x_i) < \varphi(x_i) + 2\delta, \quad \psi^n(x_j) < \varphi(x_j) + 2\delta.$$

Then, by Lemma 3.2, (4.11) and the fact that x_i, x_j belong to S_0,

$$\psi^n(x) < \varphi(x) + 3\delta \quad \text{if } x_i < x < x_j, \ x \in S_0,$$

provided ε' is sufficiently small. Taking $n \to \infty$ we get $\psi(x) \leqslant \varphi(x) + 3\delta$, which contradicts (4.12) unless $x_j - x_i \to 0$.

The above remark implies that there exists an interval (β, γ) in $(x_0, x_0 + \varepsilon')$ such that

$$\psi^n(x_i) \geqslant \varphi(x_i) + 2\delta \quad \text{for all } x_i \in (\beta, \gamma). \tag{4.13}$$

We shall take n sufficiently large so that there are indeed vertices x_i in (β, γ). By decreasing (β, γ) if necessary we conclude from (4.13), (4.11) that

$$\psi^n(x) \geqslant \varphi(x) + \delta \quad \text{if } \beta < x < \gamma, \ x \in S_0 \tag{4.14}$$

provided ε' is sufficiently small, and $n \geqslant n_0(\varepsilon')$.

Let (β', β'') be any subinterval of (β, γ). Suppose

$$\varphi^n(x) \geqslant \varphi(x) + \frac{\delta}{4} \qquad \text{if } \beta' < x < \beta'' , \ x \in S_0 \tag{4.15}$$

for a sequence $n = n_j \uparrow \infty$. The variational inequality for w^n then gives

$$\iint_A (-\Delta w^n + \alpha w^n) v \, dx \, dy = \iint_A (-1 + \alpha G - \alpha \psi^n) v \, dx \, dy$$

for any $v \in L^2(A)$, where A is the region defined by $\beta' < x < \beta''$, $\varphi(x) < y < < \varphi(x) + \delta/4$. Taking $n = n_j \to \infty$ and using (4.1), (4.3), and recalling that $w(x, y) = 0$ in A, we get

$$0 = \iint_A (-1 + \alpha G - \alpha \psi) v \, dx \, dy \, .$$

Since v is arbitrary, $-1 + \alpha G - \alpha \psi = 0$ a.e. in A. Hence $G_y = 0$ in A, which contradicts the assumption (1.14). We have thus proved that (4.15) cannot hold in any subinterval (β', β'') and for any sequence $n = n_j \uparrow \infty$.

We conclude that for any n sufficiently large there exist points $\tilde{x}_{1n}$, $\tilde{x}_{2n}$ in the interval $(\beta, (\beta + \gamma)/2)$ and in S_0 with $\tilde{x}_{1n} < \tilde{x}_{2n}$ such that

$$\varphi^n(\tilde{x}_{in}) \geqslant \varphi(\tilde{x}_{in}) + \frac{\delta}{4} \qquad (i = 1, 2) \, , \ \tilde{x}_{2n} - \tilde{x}_{1n} > 2n \, . \tag{4.16}$$

For definiteness we shall take $\varphi^n(\tilde{x}_{1n}) \leqslant \varphi^n(\tilde{x}_{2n})$.

Let $(\bar{x}_{3n}, \psi_n(\bar{x}_{3n}))$ be the vertex with $\tilde{x}_{1n} \leqslant \bar{x}_{3n} \leqslant \bar{x}_{2n}$ and with the smallest y-coordinate. Then $\psi^n(\bar{x}_{3\varphi}) = \varphi^n(\bar{x}_{3n})$. Now

$$\psi^n(\bar{x}_{3n}) > \varphi(\bar{x}_{3n}) + \delta \, ,$$

by (4.14). If ε' is sufficiently small then

$$\varphi(\bar{x}_{3n}) + \delta > \varphi(\tilde{x}_{in}) + \frac{\delta}{4} \quad i = 1, 2)$$

so that, by (4.16), $\psi^n(\bar{x}_{3n}) > \varphi^n(\tilde{x}_{in})$. It follows that $\bar{x}_{3n} \neq \tilde{x}_{in}$ $(i = 1, 2)$, i.e., $\tilde{x}_{1n} < \bar{x}_{3n} < \tilde{x}_{2n}$. Let $x = \bar{x}_{1n}$ be the smallest number of the form ja/n (j non-negative integer) such that $\tilde{x}_{1n} < x \leqslant \bar{x}_{3n}$ and let $x = \bar{x}_{2n}$ be the largest number of the form ja/n such that $\bar{x}_{3n} \leqslant x < \tilde{x}_{2n}$. Since $\tilde{x}_{2n} - \tilde{x}_{1n} > 2/n$, $\bar{x}_{3n} > \bar{x}_{1n}$.

Next, by (4.14), (4.11), (4.16),

$$\psi^n(\bar{x}_{3n}) > \delta + \varphi(\bar{x}_{3n}) > \delta + \varphi(\tilde{x}_{3n}) - \frac{\delta}{4} \geqslant \gamma\delta + \varphi^n(\tilde{x}_{2n}) - \frac{\delta}{4} - \frac{\delta}{4} \, ,$$

i.e.,

$$\psi^n(\bar{x}_{3n}) \geqslant y_0 + \frac{\delta}{2}, \qquad y_0 = \varphi^n(\tilde{x}_{2n}) . \tag{4.17}$$

Let $\bar{y} = y_0 + \delta/4$. We claim that

$$-1 + \alpha G - \alpha\psi^n \geqslant 0 \quad \text{at } (\bar{x}_{3n}, \bar{y}) \tag{4.18}$$

(if n is sufficiently large). Indeed, otherwise we shall have (since $G_y < 0$ and $\psi^n \geqslant \psi^n(\bar{x}_{3n})$)

$$\begin{aligned} -1 + \alpha G - \alpha\psi^n &< -c(y - \bar{y}) + O(\varepsilon') \\ &< -c_1\delta \qquad \text{if } y > \bar{y} + \frac{\delta}{8},\ \bar{x}_{1n} < x < \bar{x}_{2n} \end{aligned}$$

where c, c_1 are positive constants, provided ε' is sufficiently small (depending on δ). We can now use a comparison argument (as in the proof of Lemma 3.2) in the region $\bar{x}_{1n} < x < \bar{x}_{2n}$, $y > \bar{y} + \delta/8$ and deduce that

$$\varphi^n(\bar{x}_{3n}) - \bar{y} - \frac{\delta}{8} \leqslant C\varepsilon' .$$

Hence, if ε' is sufficiently small,

$$\psi^n(\bar{x}_{3n}) = \varphi^n(\bar{x}_{3n}) < \bar{y} + \frac{\delta}{4} = y_0 + \frac{\delta}{2},$$

which contradicts (4.17).

Having proved (4.18), we recall that $G_y < 0$ and thus deduce the inequality

$$-1 + \alpha G - \alpha\psi^n \geqslant \mu\delta \quad \text{at } (\bar{x}_{3n}, y_0)$$

where μ is a positive constant depending only on $\sup|G_y|$. Since $\tilde{x}_{2n} \in S_0 \subset S^*$,

$$-1 + \alpha G - \alpha\psi^n \leqslant 0 \quad \text{at } (\tilde{x}_{2n}, y_0) .$$

We must therefore have

$$\psi^n(\tilde{x}_{2n}) > \psi^n(\bar{x}_{3n}) + \frac{\mu}{2}\delta \tag{4.19}$$

if ε' is sufficiently small. Denoting by x_{1n}^* the point for which

$$\begin{aligned} w^n(x, y_0) &> 0 \quad \text{if } x_{1n}^* < x < \bar{x}_{3n} \\ &= 0 \quad \text{if } x = x_{1n}^*, \end{aligned}$$

we also have

$$\psi^n(x_{1n}^*) > \psi^n(\bar{x}_{3n}) + \frac{\mu}{2}\delta . \tag{4.20}$$

Inequalities similar to (4.19), (4.20) can be obtained for three ponits in the interval $((\beta+\gamma)/2, \gamma)$, say, for definiteness, $\tilde{y}_{1n} < \bar{y}_{3n} < y_{2n}^*$. Taking $x_1 = \bar{x}_{3n}$, $x_2 = \tilde{x}_{2n}$, $x_3 = \tilde{y}_{1n}$, $x_4 = \bar{y}_{3n}$, we then have

$$\psi^n(x_1) + \frac{\mu}{2}\delta < \psi^n(x_2) ,$$

$$\psi^n(x_4) + \frac{\mu}{2}\delta < \psi^n(x_3)$$

and $x_1 < x_2 < x_3 < x_4$, $x_4 - x_1 < \varepsilon'$. But this contradicts the inequality (3.2) with $x' = x_1$, $x'' = x_4$ and either $\bar{x} = x_2$ (if $\psi^n(x_2) \geqslant \psi^n(x_3)$) or $\bar{x} = x_3$ (if $\psi^n(x_3) \geqslant \psi^n(x_2)$), provided ε' is sufficiently small, depending on δ. This contradiction shows that (4.10) is false; this completes the proof of (4.7).

We sum up:

LEMMA 4.2. *Let* (1.11), (1.12), (1.14) *hold. Then there exists a solution* w, φ *of the Q.V.I.* (1.1)-(1.8).

In the next section we shall prove the same assertion when (1.14) is replaced by the weaker condition (1.13). We shall also prove in that section that $\varphi(x)$ is continuous and positive valued.

5. – Existence of a solution of the Q.V.I.

In this section we shall complete the existence part of Theorem 1.1. Let

$$G_\varepsilon = G - \varepsilon y \qquad (0 < \varepsilon < 1) .$$

By Lemma 4.2 there exists a solution w^ε, φ^ε of the Q.V.I. (1.1)-(1.8) with G replaced by G_ε.

Denote by S_ε the set of points $x \in [0, a]$ such that either $\varphi^\varepsilon(x) = H^*$ or $\varphi^\varepsilon(x) < H^*$ and

$$-1 + \alpha G(x, \varphi^\varepsilon(x)) - \alpha\varphi^\varepsilon(x) \leqslant 0 .$$

From the variational inequality for w^ε it follows that $[0, a] - S_\varepsilon$ has measure zero.

LEMMA 5.1. *There exists a constant M (independent of ε) such that for any three points x', $\bar{x}$, x'' in S_ε with $0 \leqslant x' < \bar{x} < x'' \leqslant a$, $0 < \varepsilon < 1$,*

$$\varphi^\varepsilon(\bar{x}) \leqslant \max\{\varphi^\varepsilon(x'), \varphi^\varepsilon(x'')\} + M(x'' - x')^{\frac{1}{3}} . \tag{5.1}$$

PROOF. We shall assume that $\varphi^\varepsilon(x') \leqslant \varphi^\varepsilon(x'')$; the case $\varphi^\varepsilon(x'') < \varphi^\varepsilon(x')$ can be handled similarly. We may assume that $\varphi^\varepsilon(x'') < \varphi^\varepsilon(\bar{x})$.

Set $y_0 = \varphi^\varepsilon(x'')$, $\bar{y} = \varphi^\varepsilon(\bar{x})$. Since $-1 + \alpha G_\varepsilon - \alpha y \leqslant 0$ at (x'', y_0),

$$-1 + \alpha G_\varepsilon(x, y) - \alpha y \leqslant O(\delta) \qquad \text{if } x' < x < x'' ,\ y \geqslant y_0 \tag{5.2}$$

where $\delta = x'' - x'$. If $\bar{x} < y_0 + \delta^{\frac{1}{3}}$ then (5.1) follows with $M = 1$. Suppose then that

$$\bar{x} > y_0 + \delta^{\frac{1}{3}} \equiv y_1 . \tag{5.3}$$

Let $I = \{(x, y_1); \bar{x} - \delta_1 < x < \bar{x} + \delta_2\}$ be an interval on which $w^\varepsilon > 0$, such that $w^\varepsilon = 0$ at the endpoints. Then $x' \leqslant \bar{x} - \delta_1 < \bar{x} < \bar{x} + \delta_2 \leqslant x''$, and

$$w^\varepsilon(x, \bar{y}) \leqslant A\delta \qquad \text{on } I .$$

Also, by (5.2), (5.3),

$$-1 + \alpha G_\varepsilon(x, y) - \alpha\varphi^\varepsilon(x) \leqslant -c\delta^{\frac{1}{3}}$$

if $\bar{x} - \delta_1 < x < \bar{x} + \delta_2$, $y > y_1$, $w^\varepsilon(x, y) > 0$, where c is a positive constant; here we take δ to be sufficiently small.

We can now compare w^ε with a function

$$v(y) = \begin{cases} C(y - y_1 - \gamma)^2 & \text{if } y_1 < y < y_1 + \gamma , \\ 0 & \text{if } y > y_1 + \gamma , \end{cases}$$

where $C/2 = c\delta^{\frac{1}{3}}$, $C\gamma^2 = A\delta$. We conclude that

$$\bar{y} \leqslant y_1 + \gamma = y_0 + \delta^{\frac{1}{3}} + \gamma , \qquad \gamma = A_1\delta^{\frac{1}{3}} \qquad (A_1 \text{ constant}) .$$

This completes the proof of the lemma.

Using Lemma 4.1 we conclude (by a variant of Lemma 3.3 with $[a, b]$ replaced by $\cap S_\varepsilon$ for a sequence of ε's) that there is a sequence $\varepsilon_n \downarrow 0$ such that if $\varepsilon=\varepsilon_n\downarrow 0$ then

$$\varphi^\varepsilon \to \varphi_0 \text{ for almost all } x\in(0,a) \, ;$$

also

$$\begin{aligned} w^\varepsilon &\to w && \text{uniformly in } \bar{B} \, , \\ w^\varepsilon &\to w && \text{weakly in } W^{2,p}(B) \, , \\ w^\varepsilon(\cdot,0) &\to w(\cdot,0) && \text{weakly in } W^{2,p}\big((0,a)\big) \end{aligned}$$

for any $1<p<\infty$. Clearly, $w_y\leqslant 0$. Let

$$\varphi(x)=\inf\{y;\, w(x,y)=0\} \, .$$

Since $w^\varepsilon\big(x,\varphi^\varepsilon(x)\big)=0$ we deduce that $w\big(x,\varphi(x)\big)=0$ a.e., i.e.,

$$\varphi(x)\leqslant\varphi^0(x) \text{ a.e.} \, . \tag{5.4}$$

We shall now prove that

$$\varphi^0(x)\leqslant\varphi(x) \text{ a.e.} \, . \tag{5.5}$$

If (5.5) is false then the set where $\varphi<\varphi^0$ has positive measure. Consequently there is a $\delta>0$ such that

$$\varphi(x)<\varphi^0(x)-\delta \qquad \text{if } x\in K\subset[0,a] \, , \; \mu(K)>0 \, . \tag{5.6}$$

Since, by Egoroff's theorem, $\varphi^\varepsilon\to\varphi^0$ almost uniformly, as $\varepsilon=\varepsilon_n\downarrow 0$, there is a subset K_0 of K of positive measure such that

$$\varphi^0(x)<\varphi^\varepsilon(x)+\delta \qquad \text{if } x\in K_0 \, , \tag{5.7}$$

provided $\varepsilon=\varepsilon_n$ is sufficiently small.

We now apply the variational inequality for w^ε ($\varepsilon=\varepsilon_n$ small) in the region $x\in K_0$, $\varphi(x)<y<\varphi^0(x)-\delta$ (which, by (5.7), lies below $y=\varphi^\varepsilon(x)$) and take $\varepsilon=\varepsilon_n\downarrow 0$; cf. the proof of (4.15). We find that

$$-1+\alpha G(x,y)-\alpha\varphi^0(x)=0 \tag{5.8}$$

for almost all (x,y), $x\in K_0$, $\varphi(x)<y<\varphi^0(x)-\delta$. It follows that

$$G_y(x,y)=0 \qquad \text{if } y=\varphi(x) \, , \; x\in K_0 \, .$$

In view of (1.13) we must therefore have $\varphi(x) > G(x, \varphi(x))$, if $x \in K_0$. Hence

$$-1 + \alpha G(x, \varphi(x)) - \alpha\varphi^0(x) = \\ = -1 + \alpha[G(x, \varphi(x)) - \varphi(x)] + \alpha[\varphi(x) - \varphi^0(x)] < 0 ,$$

thus contradicting (5.8).

We have proved that $\varphi^0 = \varphi$ a.e. Hence w, φ form a solution of the Q.V.I. (1.1)-(1.8). Next we prove:

LEMMA 5.2. *If* $x_0 \in [0, a]$, $\varphi(x_0) < H^*$ *then*

$$-1 + \alpha G(x_0, \varphi(x_0)) - \alpha\varphi(x_0) \leqslant 0 . \tag{5.9}$$

PROOF. We shall take $0 < x_0 < a$; the cases $x_0 = 0$, $x_0 = a$ can be handled in the same way with trivial changes. Suppose first that

(i) $-1 + \alpha G(x_0, 0) > 0$,

(ii) $-1 + \alpha G(x_0, H^*) - \alpha H^* < 0$.

Then there exists a point ζ_0 such that

$$-1 + \alpha G(x_0, \zeta_0) - \alpha\zeta_0 = 0 , \qquad 0 < \zeta_0 < H^* .$$

Since $\zeta_0 < G(x_0, \zeta_0)$, $G_y(x_0, \zeta_0) < 0$ by (1.13). It follows that in a neighborhood of $x = x_0$ there is a unique continuous curve $y = \zeta(x)$ such that

$$-1 + \alpha G(x, \zeta(x)) - \alpha\zeta(x) = 0 .$$

For any $\varepsilon > 0$ there is a $\delta > 0$ sufficiently small so that the rectangle

$$R_\delta \colon |x - x_0| < \delta , \qquad |y - (\zeta(x_0) - \varepsilon)| < \delta$$

lies under $y = \zeta(x)$.

The function $y - G(x, y)$ increases in y and is negative on $y = \zeta(x)$. Hence $y < G(x, y)$ in R_δ. By (1.13) it follows that

$$G_y < 0 \qquad \text{in } R_\delta . \tag{5.10}$$

From the variational inequality for w we have

$$-1 + \alpha G(x, \varphi(x)) - \alpha\varphi(x) \leqslant 0 \text{ a.e.} ,$$

so that $\zeta(x) \leqslant \varphi(x)$ a.e. Therefore, $w > 0$ a.e. in R_δ. But then also $w_y < 0$ a.e. in R_δ.

Next, taking Δw_y as a distribution derivative and using (5.10), we find that

$$- \Delta w_y + \alpha w_y = \alpha G_y(x, y) < 0 \qquad \text{in } R_\delta.$$

Hence (cf. the argument in the proof of Lemma 3.1) $w_y < 0$ in R_δ. It follows that $w > 0$ in R_δ and, consequently, $\varphi(x_0) \geqslant \zeta(x_0) - \varepsilon$. Since ε is arbitrary, (5.9) follows.

So far we made the assumptions (i), (ii). Now, if (i) does not hold then (5.9) is trivially true. On the other hand, (ii) is always satisfied; indeed, if (ii) does not hold then the previous proof remains valid with $\zeta(x) \equiv H^*$, thus leading to $\varphi(x_0) = H^*$, which is impossible.

LEMMA 5.3. *$\varphi(x)$ is continuous for $0 \leqslant x \leqslant a$.*

PROOF. We shall first prove that $\varphi(x_0 + 0)$ exists if $0 \leqslant x_0 < a$. Suppose this is not true. Then there exists a $\delta > 0$ such that for any $\varepsilon' > 0$ there are points $x_0 < x' < \bar{x} < x'' - x_0 < \varepsilon'$ for which

$$\max\{\varphi(x'), \varphi(x'')\} < \varphi(\bar{x}) - 2\delta_0 . \tag{5.11}$$

For definiteness we take

$$\varphi(x') \leqslant \varphi(x'') \equiv y_0 . \tag{5.12}$$

From Lemma 5.2

$$-1 + \alpha G(x'', y_0) - \alpha y_0 \leqslant 0 .$$

Hence, for any $\delta > 0$,

$$-1 + \alpha G(x, y) - \alpha y \leqslant -\alpha\delta_0 \qquad \text{if } y > y_0 + \delta_0 . \tag{5.13}$$

Set $y_1 = y_0 + \delta_0$ and let $I = \{x; \bar{x} - \delta_1 < x < \bar{x} + \delta_2\}$ be the interval such that $w(x, y_1)$ is positive in I and vanishes at the endpoints. Clearly $\delta_1 + \tilde{x}_2 < \varepsilon'$. Since w is Lipschitz continuous,

$$w(x, y_1) < A\varepsilon' \qquad \text{if } x \in I .$$

From (5.13) we also have

$$- \Delta w + \alpha w \leqslant -\alpha\delta_0 \qquad \text{if } x \in I\,,\ y > y_1\,,\ w > 0 .$$

We now compare w with

$$v(y) = \begin{cases} C(y - y_1 - \gamma)^2 & \text{if } y_1 < y < y_1 + \gamma\,, \\ 0 & \text{if } y > y_1 + \gamma \end{cases}$$

where

$$\tfrac{1}{2} C = \alpha\delta_0\,, \qquad Ck^2 = A\varepsilon'\,.$$

By a comparison theorem for variational inequalities we conclude that $w \leqslant v$ if $x \in I$, $y_1 < y < H^*$. Consequently,

$$\varphi(\bar{x}) \leqslant y_1 + \gamma = y_0 + \delta_0 + \left(\frac{A\varepsilon'}{2\alpha\delta_0}\right)^{\frac{1}{2}} < y_0 + 2\delta_0$$

if ε' is sufficiently small; this contradicts (5.11), (5.12).

We have proved that $\varphi(x_0 + 0)$ exists. We shall now show that $\varphi(x_0 + 0) = \varphi(x_0)$. We suppose that

$$\varphi(x_0 + 0) \neq \varphi(x_0) \tag{5.14}$$

and derive a contradiction. Since $\varphi(x_0) \leqslant \varphi(x_0 + 0)$, (5.14) implies that $\varphi(x_0) < \varphi(x_0 + 0)$.

Let $R = \{(x, y);\ x_0 < x < x_0 + \delta,\ \varphi(x_0) + \delta < y < \varphi(x_0 + 0) - \delta\}$ be a rectangle lying in the region where $w > 0$, and denote by Γ the segment $x = x_0$, $\varphi(x_0) + \delta < y < \varphi(x_0 + 0) - \delta$. Then

$$\begin{aligned} -\Delta w_y + \alpha w_y &= \alpha G_y < 0 && \text{in } R\,, \\ w_y &\leqslant 0 && \text{in } R\,, \\ w_y &= 0 && \text{on } \Gamma\,. \end{aligned}$$

The function w_y is smooth in $R \cup \Gamma$. By the strong maximum principle it follows that $w_{yx} > 0$ on Γ, i.e., $(w_x)_y > 0$ along Γ. But this is impossible since w_x vanishes at both endpoints of Γ. We have thus proved that $\varphi(x_0 + 0) = \varphi(x_0)$.

Similarly one can prove that $\varphi(x_0 - 0)$ exists and equals $\varphi(x_0)$, if $0 < x_0 \leqslant a$.

We have proved that $\varphi(x)$ is continuous in $[0, a]$. In order to complete the existence part of Theorem 1.1 it remains to show that $\varphi(x) > 0$ for $0 \leqslant x \leqslant a$. Suppose this is not true. Since $\varphi(0) > 0$, $\varphi(a) > 0$, and φ is continuous, there exists a point x_0, $0 < x_0 < a$, such that $\varphi(x) > 0$ if $0 < x < x_0$, $\varphi(x_0) = 0$. But then

$$-w_{yy} = -\alpha w + \alpha G - \alpha\varphi + l > 0 \qquad \text{if } y = 0\,,\ x_0 - \delta < x < x_0$$

for some small δ $\big($since $\alpha G(x, 0) + l(x) > 0\big)$. Since also $w = w_x = 0$ at $(x_0, 0)$, we conclude that $w(x, 0) \leqslant 0$ if $x < x_0$, $x_0 - x$ sufficiently small; this is of course impossible.

REMARK. Let y_0 be a positive number such that

$$-1 + \alpha G(x, y) - \alpha y \leqslant -1 \qquad \text{if } y \geqslant y_0 \tag{5.15}$$

and let $N \geqslant \sup_x w(x, y_0)$. Since

$$-w_{xx}(x, 0) + \alpha w(x, 0) \leqslant \alpha G(x, 0) + l(x)\,, \qquad w(0, 0) = \tfrac{1}{2}H^2\,,\ w(a, 0) = \tfrac{1}{2}h^2\,,$$

$w(x, 0)$ is bounded above by a constant depending only on H, h, α, and $\sup[\alpha G(x, 0) + l(x)]$. Since $w(x, y_0) \leqslant w(x, 0)$, N can be chosen to depend only on H, h, α, $\sup[\alpha G(x, 0) + l(x)]$. We claim that

$$\varphi(x) \leqslant y_0 + \sqrt{2N}\,. \tag{5.16}$$

Indeed, in the region $y \geqslant y_0$,

$$-\Delta w + \alpha w + \beta(w) \ni f\,, \qquad f \leqslant -1 \text{ by (5.15)}\,.$$

Comparing w with

$$v(y) = \begin{cases} \tfrac{1}{2}(y - y_0 - \sqrt{2N})^2 & \text{if } y_0 < y < y_0 + \sqrt{2N}\,, \\ 0 & \text{if } y > y_0 + \sqrt{2N}\,, \end{cases}$$

we find that $w \leqslant v$, from which (5.16) follows.

6. – Uniqueness.

In this section we shall prove the general comparison theorem:

THEOREM 6.1. *Let* (w, φ) *be a solution of the Q.V.I.* (1.1)-(1.8) *with* $\varphi \in C[0, a]$, $\varphi > 0$, *and let* $(\hat{w}, \hat{\varphi})$ *be a solution of the Q.V.I.* (1.1)-(1.8) *with*

G, l, H, h replaced by $\hat{G}$, $\hat{l}$, $\hat{H}$, $\hat{h}$, and with $\hat{\varphi} \in C[0, a]$, $\hat{\varphi} > 0$. If G, $\hat{G}$, l, $\hat{l}$ are continuous for $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant H^*$ and if

$$G(x, 0) \geqslant \hat{G}(x, 0) , \qquad G(x, 0) - G(x, y) \geqslant \hat{G}(x, 0) - \hat{G}(x, y) , \tag{6.1}$$

$$l(x) \geqslant \hat{l}(x) , \quad H \geqslant \hat{H} , \quad h \geqslant \hat{h} , \tag{6.2}$$

then

$$w(x, 0) - w(x, y) \geqslant \hat{w}(x, 0) - \hat{w}(x, y) . \tag{6.3}$$

The uniqueness part of Theorem 1.1 is a consequence of Theorem 6.1. Indeed, if $G = \hat{G}$, $l = \hat{l}$, $H = \hat{H}$, $h = \hat{h}$, then (6.3) becomes an equality. Taking $y = H^*$ we conclude that $w(x, y) = \hat{w}(x, y)$. Hence also $\varphi(x) = \hat{\varphi}(x)$.

To prove Theorem 6.1, let

$$\zeta(x, y) = w(x, 0) - w(x, y) ,$$
$$\hat{\zeta}(x, y) = \hat{w}(x, 0) - \hat{w}(x, y) ,$$
$$\eta = \zeta - \hat{\zeta} .$$

We have to show that $\eta \geqslant 0$. Since $w_y \leqslant 0$,

$$\eta(x, y) \geqslant \eta\big(x, \hat{\varphi}(x)\big) \qquad \text{if } \hat{\varphi}(x) < y < H^* . \tag{6.4}$$

Suppose η takes a negative minimum in $\bar{B}$. In view of (6.4), the negative minimum is attained at a point (x_0, y_0) satisfying

$$y_0 \leqslant \hat{\varphi}(x_0) .$$

Since $H \geqslant \hat{H}$, $h \geqslant \hat{h}$, we must have $0 < x_0 < a$. Since $\eta(x, 0) = 0$, we must also have $y_0 > 0$. Next,

$$-\Delta\eta + \alpha\eta = \alpha\big(G(x, 0) - G(x, y)\big) - \alpha\big(\hat{G}(x, 0) - \hat{G}(x, y)\big) \geqslant 0$$

if $0 < y < \min\big(\varphi(x), \hat{\varphi}(x)\big)$, so that, by the maximum principle,

$$y_0 \geqslant \min\big(\varphi(x_0), \hat{\varphi}(x_0)\big) .$$

Thus, we must have one of the possibilities:

$$y_0 = \hat{\varphi}(x_0) < \varphi(x_0) , \qquad 0 < x_0 < a , \qquad y_0 > 0 , \tag{6.5}$$

$$y_0 = \varphi(x_0) \leqslant \hat{\varphi}(x_0) , \qquad 0 < x_0 < a , \qquad y_0 > 0 , \tag{6.6}$$

$$\varphi(x_0) < y_0 \leqslant \hat{\varphi}(x_0) , \qquad 0 < x_0 < a . \tag{6.7}$$

We shall derive a contradiction in all cases.

Suppose first that (6.5) holds. Then $\eta_y(x_0, y_0) \leqslant 0$, i.e.,

$$\hat{w}_y(x_0, y_0) - w_y(x_0, y_0) \leqslant 0 .$$

But this is impossible since

$$\hat{w}_y(x_0, y_0) = \hat{w}_y(x_0, \hat{\varphi}(x_0)) = 0 , \qquad w_y(x_0, y_0) < 0 \qquad (\text{for } y_0 < \varphi(x_0)) .$$

Consider next a subcase of (6.6) where

$$y_0 = \varphi(x_0) < \hat{\varphi}(x_0) . \tag{6.8}$$

Since $\eta(x_0, y_0) < 0$, $w(x_0, 0) - \hat{w}(x_0, 0) < -\hat{w}(x_0, y_0) < 0$. Let $I = (b, c)$ be the largest interval containing x_0 such that $\varphi(x) - \hat{\varphi}(x) < 0$. Then

$$\begin{aligned} &-\big(w(x, 0) - \hat{w}(x, 0)\big)_{xx} + \alpha\big(w(x, 0) - \hat{w}(x, 0)\big) = \\ &\qquad = \alpha\big(G(x, 0) - \hat{G}(x, 0)\big) - \alpha\big(\varphi(x) - \hat{\varphi}(x)\big) + \big(l(x) - \hat{l}(x)\big) > 0 \qquad \text{in } I . \end{aligned}$$

Hence, the strict minimum of $w(x, 0) - \hat{w}(x, 0)$ in $\bar{I}$ occurs at the boundary, say at $x = b$. Since $\varphi(b) = \hat{\varphi}(b)$,

$$\begin{aligned} \eta\big(b, \varphi(b)\big) &= w(b, 0) - \hat{w}(b, 0) < w(x_0, 0) - \hat{w}(x_0, 0) \leqslant \\ &\leqslant w(x_0, 0) - \big(\hat{w}(x_0, 0) - \hat{w}(x_0, y_0)\big) = \eta(x_0, y_0) , \end{aligned} \tag{6.9}$$

which is impossible since η takes its minimum at (x_0, y_0). Similarly one derives a contradiction in case $w(x, 0) - \hat{w}(x, 0)$ takes its strict minimum at the other endpoint $x = c$ of I.

We have proved that (6.8) cannot occur. We shall now prove that, if $y_0 = \varphi(x_0)$, the following possibility cannot occur:

$$\varphi(x) \leqslant \hat{\varphi}(x) \qquad \text{in an interval containing } x_0 . \tag{6.10}$$

The proof proceeds as in the case of (6.8). We now denote by $I = (b, c)$ the largest interval containing x_0 such that $\varphi(x) - \hat{\varphi}(x) \leqslant 0$ if $x \in \bar{I}$. If $\varphi \not\equiv \hat{\varphi}$ in I then the strong maximum principle can be applied, as before, to the function $w(x, 0) - \hat{w}(x, 0)$, and we get a contradiction (using (6.9)).

If $\varphi \equiv \hat{\varphi}$ in an interval containing w_0, then $\eta\big(x, \varphi(x)\big) = w(x, 0) - \hat{w}(x, 0)$ in this interval. Consequently, $w(x, 0) - \hat{w}(x, 0)$ takes a negative minimum at $x = x_0$. But this is impossible since

$$-\big(w(x,0)-\hat{w}(x,0)\big)_{xx} = -\alpha\big(w(x,0)-\hat{w}(x,0)\big) + \alpha\big(G(x,0)-G(x,0)\big) + \\ + \big(l(x)-\hat{l}(x)\big) > 0 \qquad \text{at } x = x_0 .$$

We shall now prove that, when $y_0 = \varphi(x_0)$, the following situation cannot occur:

$$(6.11) \qquad \varphi(x) \leqslant \hat{\varphi}(x) \qquad \text{in an interval } (x^*, x_0) .$$

Indeed, suppose (6.11) holds. Since (6.10) cannot occur, we must then have

$$(6.12) \qquad \hat{\varphi}(x_n) < \varphi(x_n) \qquad \text{for a sequence } x_n,\ x_n \downarrow x_0 .$$

If $\varphi \not\equiv \hat{\varphi}$ in (x^*, x_0) then the minimum of $w(x,0)-\hat{w}(x,0)$ in $[x^*, x_0]$ can occur only at the endpoints. Since

$$w(x^*,0)-\hat{w}(x^*,0) \geqslant \eta\big(x^*, \hat{\varphi}(x^*)\big) \geqslant \eta(x_0, y_0) = w(x_0,0)-\hat{w}(x_0,0) ,$$

the minimum is attained at x_0, and the strong maximum principle yields

$$\big(w(x,0)-\hat{w}(x,0)\big)_x < 0 \qquad \text{at } x = x_0 .$$

Consequently,

$$w(x,0)-\hat{w}(x,0) \leqslant w(x_0,0)-\hat{w}(x_0,0)-\gamma(x-x_0) = \eta(x_0,y_0)-\gamma(x-x_0)$$

if $x > x_0$, $x - x_0$ small. Using (6.12) we get

$$\eta\big(x_n, \hat{\varphi}(x_n)\big) = w(x_n,0)-\hat{w}(x_n,0)-w\big(x, \hat{\varphi}(x_n)\big) \leqslant \\ \leqslant w(x_n,0)-\hat{w}(x_n,0) \leqslant \eta(x_0,y_0)-\gamma(x_n-x_0) < \eta(x_0,y_0)$$

if n is sufficiently large, which is impossible.

If (6.11) holds and $\varphi \equiv \hat{\varphi}$ in (x^*, x_0), then ζ takes its negative minimum also at (x_1, y_1) where $x^* < x_1 < x_0$, $y_1 = \varphi(x_1)$, and $\hat{\varphi} = \varphi$ in an interval containing x_1. Thus we are back in the case (6.10), which was already ruled out.

We have proved that (6.11) cannot occur. Similarly, one can show that when $y_0 = \varphi(x_0)$ the following possibility cannot hold:

$$(6.13) \qquad \varphi(x) \leqslant \hat{\varphi}(x) \qquad \text{in an interval } (x_0, x^{**}) .$$

We return to the case (6.6). We have already ruled out the subcase (6.8). Thus it remains to show that the subcase

$$y_0 = \varphi(x_0) = \hat{\varphi}(x_0) \tag{6.14}$$

cannot happen. Since we have already ruled out (6.11), (6.13), we must have (in the event that (6.14) holds)

$$\begin{cases} \hat{\varphi}(x_n) < \varphi(x_0) & \text{for a sequence } x_n,\, x_{2n} > x_0\,, \\ x_{2n+1} < x_0\,, \quad x_n \to x_0 & \text{if } n \to \infty\,. \end{cases} \tag{6.15}$$

Now,

$$\begin{aligned} w(x_n, 0) - \hat{w}(x_n, 0) \geqslant w(x_n, 0) - w\big(x_n, \hat{\varphi}(x_n)\big) = \\ = \eta\big(x_n, \hat{\varphi}(x_n)\big) \geqslant \eta(x_0, y_0) = w(x_0, 0) - \hat{w}(x_0, 0)\,. \end{aligned} \tag{6.16}$$

Since $x_n \to x_0$ from both sides of x_0, we conclude that

$$w_x(x_0, 0) - \hat{w}_x(x_0, 0) = 0\,.$$

Also,

$$\begin{aligned} -\big(w(x, 0) - \hat{w}(x, 0)\big)_{xx} = -\alpha\big(w(x, 0) - \hat{w}(x, 0)\big) + \alpha\big(G(x, 0) - G(x, 0)\big) + \\ + \big(l(x) - \hat{l}(x)\big) > 0 \qquad \text{at } x = x_0\,. \end{aligned}$$

Hence $w(x, 0) - \hat{w}(x, 0)$ takes a local strict maximum at x_0. But this is impossible since, by (6.16),

$$w(x_n, 0) - \hat{w}(x_n, 0) \geqslant w(x_0, 0) - \hat{w}(x_0, 0)\,.$$

We have ruled out the case (6.6).

We shall now rule out the case (6.7). Suppose (6.7) holds. Since $\zeta(x_0, y)$, for $\varphi(x_0) < y < \hat{\varphi}(x_0)$, is strictly decreasing in y, we must have $y_0 = \hat{\varphi}(x_0)$. Let $I = (b, c)$ be the largest interval containing x_0 in which $\varphi(x) < \hat{\varphi}(x)$. The argument following (6.8) shows that $w(x, 0) - \hat{w}(x, 0)$ can take its minimum in $\bar{I}$ only at the endpoints; suppose it takes it at $x = b$. Then, since $\varphi(b) = \hat{\varphi}(b)$,

$$\zeta\big(b, \hat{\varphi}(b)\big) = w(b, 0) - \hat{w}(b, 0) < w(x_0, 0) - \hat{w}(x_0, 0) = \zeta(x_0, y_0)$$

which is impossible. Similarly $w(x, 0) - \hat{w}(x, 0)$ cannot take its minimum in $\bar{I}$ at $x = c$.

We have proved that if η is not $\geqslant 0$ in $\bar{B}$ then one of the conditions (6.5)-(6.7) must hold. Since we have now established that each of these conditions leads to a contradiction, the proof of Theorem 6.1 is complete.

7. – A non-stationary problem in hydraulics.

Let

$$B = \{(x, y);\ 0 < x < a,\ 0 < y < H^*\}\,,$$
$$Q = \{(x, y, t);\ (x, y) \in B,\ 0 < t < T\}$$

where a, H^*, T are given positive numbers. Consider the following problem: Find a function $u(x, y, t)$ and a surface $y = \varphi(x, t)$, $0 \leqslant \varphi \leqslant H^*$ such that

$$u_t - \Delta u = 0 \quad \text{in } \Omega \equiv \{(x, y, t) \in Q,\ 0 < y < \varphi(x, t)\}\,, \tag{7.1}$$

$$u(x, y, 0) = g(x, y) \quad \text{in } B\,, \tag{7.2}$$

$$u(0, y, t) = H(t) \quad \text{if } 0 < y < H(t)\,, \tag{7.3}$$

$$\begin{cases} u(a, y, t) = h(t) & \text{if } 0 < y < h(t)\,, \\ u(0, y, t) = y & \text{if } h(t) < y < \varphi(a, t)\,, \end{cases} \tag{7.4}$$

$$u_y(x, 0, t) = -l(x, t) \quad \text{if } 0 < x < a\,,\ 0 < t < T\,, \tag{7.5}$$

$$\varphi(0, t) = H(t)\,, \quad h(t) \leqslant \varphi(a, t) \leqslant H^* \quad \text{if } 0 < t < T\,, \tag{7.6}$$

$$\varphi(x, 0) = \gamma(x) \quad \text{if } 0 < x < a\,, \tag{7.7}$$

$$u(x, y, t) = y \quad \text{if } y = \varphi(x, t)\,,\ 0 < x < a\,,\ 0 < t < T\,, \tag{7.8}$$

$$u_t = u_x^2 + u_y^2 - u_y \quad \text{if } y = \varphi(x, t)\,,\ 0 < x < a\,,\ 0 < t < T\,, \tag{7.9}$$

$$\begin{cases} u \in C(\bar{\Omega})\,; \quad u_x, u_y, u_t \text{ belong to } C(\bar{\Omega} \cap \setminus \partial Q)\,, \\ u_{xx}, u_{yy} \text{ belong to } C(\Omega)\,, \end{cases} \tag{7.10}$$

$$\varphi \in C[0, a) \times [0, T]\,. \tag{7.11}$$

Here $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the functions g, H, h, l, γ are given, and

$$\begin{cases} h, H \in C[0, T[\,, \\ 0 < h(t) < H(t) < H^* \qquad \text{if } 0 < t < T\,, \end{cases} \tag{7.12}$$

$$g \in C(\bar{B}),\ g(y, x) > y \qquad \text{if } 0 < y < \gamma(x), \tag{7.13}$$

$$\gamma \in C[0, a],\ 0 < \gamma(x) < H^* \qquad \text{if } 0 \leqslant x \leqslant a,\ h(0) \leqslant \gamma(a); \tag{7.14}$$

$$l \in C([0, a] \times [0, T]),\ l(x, t) > -1 \qquad \text{if } 0 \leqslant x \leqslant a,\ 0 \leqslant t \leqslant T. \tag{7.15}$$

This problem represents a physical model which arises when compressible fluid is moving across an underground dam which separates two resesvoirs of the fluid. The levels of the reservoirs are $H(t)$ and $h(t)$ and the walls of the dam are vertical. The bottom of the dam is generally not impervious; the fluid is moving across the bottom at rate $|l(x, t)|$ $(l(x, t) > 0$ when the fluid is moving upward and $l(x, t) < 0$ when the fluid is moving downward). The function u represents the piezometric head and $-(u_x, u_y)$ is the velocity of the fluid. The problem is formulated, for instance, in [1].

The system (7.1) (7.11) is a *free boundary problem.* The surface $y = \varphi(x, t)$ is called the *free boundary.*

The condition $g(x, y) > y$ reflects the physical condition that the internal pressure in the fluid (at time $t = 0$) is positive. The condition $l > -1$ is needed to assure that the free boundary does not arise to infinity.

LEMMA 7.1. *Let* (u, φ) *satisfy* (7.1)-(7.8), (7.10)-(7.11). *Then*

$$u(x, y, t) > y \qquad \textit{in } \Omega. \tag{7.16}$$

If $\varphi \in C^2$ *then also*

$$1 - u_y > 0 \qquad \textit{if } x = \varphi(x, t). \tag{7.17}$$

If φ *is only assumed to belong to* C^1, *then* (7.17) *holds on a dense set of the free boundary.*

PROOF. Consider the function $v = u - y$ in Ω. It satisfies

$$v_t - \Delta v = 0 \qquad \text{in } \Omega.$$

We claim than $v \geqslant 0$ in Ω. Indeed, otherwise (by the maximum principle) v attains a negative minimum on the parabolic boundary of Ω, say at $\bar{P} = (\bar{x}, \bar{y}, \bar{t})$. By the conditions (7.3), (7.4) and (7.2), (7.13), the point $\bar{P}$ must lie on $y = 0$. But then $v_y > 0$ at $\bar{P}$, i.e., $-l(\bar{x}, \bar{t}) - 1 > 0$, which contradicts (7.15).

Having proved that $v \geqslant 0$ in Ω, the strong maximum principle now implies that $v > 0$ in Ω, i.e., (7.16) holds. The strict minimum of v in $\bar{\Omega}$ is thus obtained on the free boundary. If $\varphi \in C^2$ then the inside strong sphere property holds and therefore (see [2])

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(u - y) < 0 \qquad \text{on } y = \varphi(x) ,$$

i.e., (7.17) holds. If φ is only assumed to be in C^1, then the inside strong sphere property holds for a dense set of the free boundary. Hence (7.17) holds in this set.

LEMMA 7.2. *Let* (u, φ) *satisfy* (7.1)-(7.8), (7.10)-(7.11), *and let* $\varphi \in C^1$. *Then* (7.9) *holds if and only if*

$$-\varphi_t = u_y - u_x \varphi_x \qquad \textit{on } y = \varphi(x, t) . \tag{7.18}$$

PROOF. Differentiating the relation

$$u(x, \varphi(x, t), t) = \varphi(x, t)$$

we get

$$u_x + u_y \varphi_x = \varphi_x ,$$
$$u_t + u_y \varphi_t = \varphi_t .$$

Using (7.17) we can solve for φ_x, φ_t (on a dense subset of $y = \varphi(x, t)$):

$$\varphi_x = \frac{u_x}{1 - u_y} , \qquad \varphi_t = \frac{u_t}{1 - u_y} \qquad \text{at } y = \varphi(x, t) . \tag{7.19}$$

If we substitute this into (7.9), we get

$$u_y - u_x \varphi_x = -\varphi_t$$

for a dense set on the free boundary; by continuity this relation then holds everywhere on the free boundary.

Conversely, substituting φ_x, φ_y from (7.19) into (7.18), the relation (7.9) readily follows, at a dense subset of the free boundary; by continuity, (7.9) holds everywhere.

We shall now transform the problem (7.1)-(7.11) into a quasi-variational inequality. Let

$$\tilde{u}(x, y, t) = \begin{cases} u(x, y, t) & \text{in } \Omega\,, \\ y & \text{in } \bar{Q}\setminus\Omega\,, \end{cases} \tag{7.20}$$

$$w(x, y, t) = \int_y^{H^*} [\tilde{u}(x, \eta, t) - \eta]\, d\eta\,. \tag{7.21}$$

The following formulas are easily derived:

$$w_y(x, y, t) = \begin{cases} y - u(x, y, t) & \text{if } y < \varphi(x, t)\,, \\ 0 & \text{if } y > \varphi(x, t)\,; \end{cases} \tag{7.22}$$

$$w_t(x, y, t) = \begin{cases} \displaystyle\int_y^{\varphi(x,t)} u_t(x, \eta, t)\, d\eta + [u(x, \varphi(x, t), t) - \varphi(x, t)]\, \varphi_x(x, t) \\ \qquad\qquad\qquad\qquad \text{if } y < \varphi(x, t)\,, \\ 0 \qquad\qquad\qquad\qquad \text{if } y > \varphi(x, t)\,, \end{cases}$$

so that, by (7.8),

$$w_t(x, y, t) = \begin{cases} \displaystyle\int_y^{\varphi(x,t)} u_t(x, \eta, t)\, d\eta & \text{if } y < \varphi(x, t)\,, \\ 0 & \text{if } y > \varphi(x, t)\,; \end{cases} \tag{7.23}$$

$$w_x(x, y, t) = \begin{cases} \displaystyle\int_y^{\varphi(x,t)} u_x(x, \eta, t)\, d\eta & \text{if } y < \varphi(x, t)\,, \\ 0 & \text{if } y > \varphi(x, t)\,; \end{cases} \tag{7.24}$$

$$w_{yy}(x, y, t) = \begin{cases} 1 - u_y(x, y, t) & \text{if } y < \varphi(x, t)\,, \\ 0 & \text{if } y > \varphi(x, t)\,; \end{cases} \tag{7.25}$$

$$w_{tt}(x, y, t) = \begin{cases} \displaystyle\int_y^{\varphi(x,t)} u_{xx}(x, \eta, t)\, d\eta + u_x(x, \varphi(x, t), t)\, \varphi_x(x, t) \\ \qquad\qquad\qquad\qquad \text{if } y < \varphi(x, t)\,, \\ 0 \qquad\qquad\qquad\qquad \text{if } y > \varphi(x, t)\,. \end{cases} \tag{7.26}$$

It follows that, if $y < \varphi(x, t)$,

$$(-\Delta w + w_t)(x, y, t) = \int_y^{\varphi(x,t)} (-u_{xx} + u_t)(x, \eta, t)\, d\tilde{x} - 1 + u_y(x, y, t) + \\ + u_x(x, \varphi(x, t), t)\, \varphi_x(x, t)\,.$$

Using (7.1) and (7.18), we obtain

$$(7.27)\qquad (-\Delta w + w_t)(x, y, t) = \begin{cases} -1-\varphi_t(x,t) & \text{if } y<\varphi(x,t)\,, \\ 0 & \text{if } y>\varphi(x,t)\,. \end{cases}$$

We claim that

$$(7.28)\qquad 1+\varphi_t(x,t)\geqslant 0 \qquad \text{if } y<\varphi(x,t)\,.$$

Indeed, by (7.18),

$$1+\varphi_t = (1-u_y)+u_x\varphi_x \qquad \text{at } y=\varphi(x,t)\,,$$

and $1-u_y>0$ by (7.17), whereas

$$u_t\varphi_t = \frac{u_y^2}{1-u_y}\geqslant 0$$

by (7.19); thus (7.28) follows.

The relations (7.27), (7.28) can be recast in the form:

$$(7.29)\qquad (w_t-\Delta w, v-w)\geqslant(-1-\varphi_t, v-w) \qquad \text{for any } v\in L^2(Q)\,,\ v\geqslant 0\,;$$

we use here the fact that

$$(7.30)\qquad \begin{cases} w(x,y,t)>0 & \text{if } 0<y<\varphi(x,t)\,, \\ w(x,y,t)=0 & \text{if } \varphi(x,t)<y<H^*\,. \end{cases}$$

The notation (,) in (7.29) indicates the scalar product in $L^2(Q)$.

The initial and boundary conditions for w are:

$$(7.31)\qquad w(x,y,0) = k(x,y)\,,$$

$$k(x,y) = \begin{cases} \displaystyle\int_y^{\gamma(x)}[g(x,\eta)-\eta]\,d\eta & \text{if } y<\gamma(x)\,, \\ 0 & \text{if } y>\gamma(x)\,; \end{cases}$$

$$(7.32)\qquad w(0,y,t) = \begin{cases} \frac{1}{2}(H(t)-y)^2 & \text{if } y<H(t)\,, \\ 0 & \text{if } y>H(t)\,, \end{cases}$$

$$(7.33)\qquad w(a,y,t) = \begin{cases} \frac{1}{2}(h(t)-y)^2 & \text{if } y<h(t)\,, \\ 0 & \text{if } y>h(t)\,, \end{cases}$$

$$(7.34)\qquad w(x,H^*,t)=0\,.$$

Finally, to calculate $w(x, 0, t)$, we use (7.25), (7.5):

$$w_{yy} = 1 + l .$$

Making use also of (7.27), we get

$$(7.35) \qquad \begin{cases} w_t(x, 0, t) - w_{xx}(x, 0, t) = -\varphi_t(x, t) + l(x, t) & \text{in } B , \\ w(0, 0, t) = \dfrac{(H(t))^2}{2}, \quad w(a, 0, t) = \dfrac{(h(t))^2}{2}, & w(x, 0, 0) = k(0) . \end{cases}$$

We have proved:

THEOREM 7.3. *Let* (u, φ) *be a solution of* (7.1)-(7.11) *with* $\varphi \in C^1$. *Then* (w, φ) *is a solution of* (7.29)(-(7.35).

A solution (w, φ) of (7.29)-(7.35) is called *classical* if w_x, w_y, w_t are continuous in $\bar{Q}$; w_{xx}, w_{yy} are uniformly continuous in Ω, and φ is continuously differentiable.

THEOREM 7.4. *Let* (w, φ) *be a classical solution of* (7.29)-(7.35) *and let* $u = y - w_y$. *Then* (u, φ) *is a solution of* (7.1)-(7.9).

The proof is direct. The only condition that requires a proof is (7.18), and this is proved in the same way as in the case of one space dimension [4].

From now on we concentrate on the problem (7.29)-(7.35).

REMARK. A. Torelli [6] has considered a variant of (7.1)-(7.9) whereby $l \equiv 0$, (7.1) is replaced by $\Delta u = 0$, and the initial data $u(x, y, 0)$ is the stationary solution of the problem when $H \equiv H(0)$, $h \equiv h(0)$, $l \equiv 0$. He reduced the problem in (u, φ) into a problem of the form $D_t U + A(t) U = f$, $U(0) = U_0$ where $A(t)$ is a nonlinear pseudo-differential operator and announced a proof of existence and uniqueness for the latter problem. Torelli [7] also derived results similar to Lemmas 7.1, 7.2.

The system (7.29)-(7.35) is a Q.V.I. The analogous Q.V.I. in the case of one space dimension was studied in [4] by using a finite difference scheme with respect to the t variable. In the following section we shall introduce such a scheme for the problem (7.29)-(7.35), and then apply Theorem 1.1 in order to solve this scheme in a unique way. The problem of proving and uniqueness for (7.29)-(7.35) is still open.

8. – The finite difference approximations.

For any positive integer n, divide $(0, T)$ into n intervals of equal length T/n and let

$$\alpha = n/T , \qquad t_i = i\alpha \qquad (i = 0, 1, \ldots, n) .$$

The free boundary $y = \varphi(x, t)$ will be replaced by portions, piecewise linear in t, connecting $(\varphi^i(x), t_i)$ to $(\varphi^{i+1}(x), t_{i+1})$. Writing

$$w^i(x, y) = w(x, y, t_i)\,,$$

the quasi-variational problem (7.29)-(7.35) leads to the following finite difference scheme of elliptic quasi-variational inequalities:

$$(8.1)\qquad (-\Delta w^i + \alpha w^i, v - w^i) \geqslant (-1 + \alpha(w^{i-1} + \varphi^{i-1}) - \alpha\varphi^i, v - w^i)\quad \text{for any } v \in L^2(B)\,,\ v \geqslant 0\,,$$

where (,) is the scalar product in $L^2(B)$,

$$(8.2)\qquad w^i(x, y) > 0 \text{ if } 0 < y < \varphi^i(x)\,,\qquad w^i(x, y) = 0 \text{ if } \varphi^i(x) < y < H^*\,,$$

$$(8.3)\qquad w^i(0, y) = \tfrac{1}{2}(H_i - y)^2 \text{ if } 0 < y < H_i\,,\qquad w^i(0, y) = 0 \text{ if } H_i < y < H^*,$$

$$(8.4)\qquad w^i(a, y) = \tfrac{1}{2}(h_i - y)^2 \text{ if } 0 < y < h_i\,,\qquad w^i(a, y) = 0 \text{ if } h_i < y < H^*$$

where $H_i = H(t_i)$, $h_i = h(t_i)$,

$$(8.5)\qquad w^i(x, H^*) = 0\,,$$

$$(8.6)\qquad \begin{cases} -w^i_{xx}(x, 0) + \alpha w^i(x, 0) = \alpha[w^{i-1}(x, 0) + \varphi^{i-1}(x)] - \alpha\varphi^i(x) + l^i(x)\,, \\ w^i(0, 0) = \tfrac{1}{2}H_i^2\,,\qquad w^i(a, 0) = \tfrac{1}{2}h \end{cases}$$

where $l^i(x) = l(x, t_i)$.

The system (8.1)-(8.6) is to hold for $1 \leqslant i \leqslant n$; in addition,

$$(8.7)\qquad w^0(x, y) = k(x, y)\,,\qquad \varphi^0(x) = \gamma(x)\,.$$

We add the regularity condition:

$$(8.8)\qquad w^i \in W^{2,p}(B)\qquad \text{for any } 1 < p < \infty\,,$$

and also require that

$$(8.9)\qquad 0 < \varphi^i(x) \leqslant H^*\,.$$

The Q.V.I. (8.1)-(8.9) for a fixed i coincides with the Q.V.I. (1.1)-(1.8) provided $H = H_i$, $h = h_i$, $l = l^i$ and

$$G(x, y) = w^{i-1}(x, y) + \varphi^{i-1}(x) .$$

If $\varphi^{i-1}(x) > 0$ then $G(x, 0) > 0$. If $l(x) \geqslant 0$ then $\alpha G(x, 0) + l(x) > 0$. If $\partial w^{i-1}/\partial y \leqslant 0$ then

if $y < \varphi^{i-1}(x)$ then $G(x, y) > \varphi^{i-1}(x) > y$, and

if $y > \varphi^{i-1}(x)$ then $G(x, y) = \varphi^{i-1}(x) < y$.

Thus (1.13) follows; in fact,

$$G_y(x, y) < 0 \qquad \text{if } y < G(x, y) ,$$

$$G_y(x, y) = 0 \qquad \text{if } y > G(x, y) .$$

Notice that $G(x, y)$ is not known to be differentiable in x. However the proof of Theorem 1.1 remains valid (with minor changes) when G_x is not assumed to exist, but G, G_y are continuous; instead of using the relation $G(x, y) - G(\bar{x}, y) = 0(|\bar{x} - x|)$ (in Sections 3-5) one uses the relation $G(x, y) - G(\bar{x}, y) = \sigma(|\bar{x} - x|)$, where $\sigma(t) \to 0$ if $t \downarrow 0$. The condition (1.11) follows from (7.15). The condition $\partial w^0/\partial y \leqslant 0$ follows from (7.31), (7.13). We can therefore proceed to apply Theorem 1.1 (with $G(x, y)$, $G_y(x, y)$ continuous) inductively for $i = 1, 2, \ldots$. We obtain:

THEOREM 8.1 *Let* (7.12)-(7.14) *hold, and let* $l \in C[0, a] \times [0, T]$, $l \geqslant 0$. *Then there exists a unique solution* $\{w^i, \varphi^i; 0 \leqslant i \leqslant n\}$ *of the system of elliptic Q.V.I.* (8.1)-(8.9) *with* $\varphi^i \in C[0, a]$.

It is easy to show that

$$w^i(x, y) \leqslant M \tag{8.10}$$

where M is a constant independent of i, n. Indeed, from the Q.V.I. for w^i we get $\alpha(\varphi^{i-1}(x) - \varphi^i(x)) \leqslant C$, C a constant independent of i, n, so that

$$-w^i_{xx}(x, 0) + \alpha w^i(x, 0) \leqslant C + \alpha M_{i-1}$$

with a different C, where $M_j = \max w^j(x, 0)$, $0 \leqslant x \leqslant a$. By comparison we then get

$$w^i(x) \leqslant M_{i-1} + \frac{C}{\alpha};$$

consequently $M_i \leqslant M_{i-1} + C/\alpha$, and (8.10) follows.

Using a comparison argument one can also show that $w^i_y(x, y) \geqslant -N$ where N is a constant independent of i, N.

If the condition $l \geqslant 0$ (in Theorem 8.1) is replaced by the condition $l > -1$, then the assertions of Theorem 8.1 remain valid expect that (8.9) is replaced by $0 \leqslant \varphi_i(x) \leqslant H^*$.

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