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The Regularity of the Trace for Minimal Surfaces.

JOHANNES C. C. NITSCHE (*)

dedicated to Hans Lewy

1. – This paper deals with the regularity properties of the free boundaries for minimal surfaces under assumptions which are weaker than those accessible heretofore.

Consider a configuration in Euclidean 3-space consisting of a surface T and of a rectifiable Jordan arc $\Gamma = \{\mathfrak{x} = \mathfrak{z}(t); 0 \leq t \leq \pi\}$ having its end points on T, but no other points in common with T. Denote by P the semi-disc in the (u, v)-plane $\{u, v; u^2 + v^2 < 1, v > 0\}$, by $\partial' P$ and $\partial'' P$ its boundary portions $\{u, v; u^2 + v^2 = 1, v > 0\}$ and $\{u, v; -1 < u < 1, v = 0\}$, respectively, and by P' the domain $P \cup \partial' P$. A surface $S = \{\mathfrak{x} = \mathfrak{x}(u, v); (u, v) \in P'\}$ is said to be bounded by the configuration, or chain, $\langle \Gamma, T \rangle$ if its position vector $\mathfrak{x}(u, v) = \{x(u, v), y(u, v), z(u, v)\}$ satisfies the following conditions:

i) $\mathfrak{x}(u, v) \in C^0(P')$.

ii) $\mathfrak{x}(u, v)$ maps the arc $\partial' P$ onto the open arc $(\Gamma) = \{\mathfrak{x} = \mathfrak{z}(t); 0 < t < \pi\}$ monotonically in such a way that

$$\lim_{\theta \to +0} \mathfrak{x}(\cos\theta, \sin\theta) = \mathfrak{z}(0)\,, \qquad \lim_{\theta \to \pi - 0} \mathfrak{x}(\cos\theta, \sin\theta) = \mathfrak{z}(\pi)\;;$$

that is, there exists a continuous increasing function $t = t(\theta)$ which maps the interval $0 < \theta < \pi$ onto the interval $0 < t < \pi$ such that $\mathfrak{x}(\cos\theta, \sin\theta) = \mathfrak{x}(t(\theta))$.

iii) The relation $\lim_{n\to\infty} d_T[\mathfrak{x}(u_n, v_n)] = 0$ holds for every sequence of points (u_n, v_n) in P' converging to a point on $\overline{\partial'' P}$.

Here $d_T[\mathfrak{x}] = \inf_{\mathfrak{t}\in T} |\mathfrak{x} - \mathfrak{t}|$ denotes the distance between the point $\mathfrak{x}$ and the surface T.

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Although the distance function $d_T[\mathfrak{x}(u, v)]$ is continuous in the closure $\overline{P}$, the same cannot generally be said about the vector $\mathfrak{x}(u, v)$. In fact, the trace of S on T, that is, the set of limit points on T for all sequences $\mathfrak{x}(u_n, v_n)$ as in iii) above, may well look quite bizarre.

We shall denote by $\mathfrak{A}$ the collection of all surfaces $S = \{\mathfrak{x} = \mathfrak{x}(u, v); (u, v) \in P'\}$ bounded by the chain $\langle \Gamma, T\rangle$ whose position vector belongs to $C^0(P') \cap H_2^1(P)$. If the end points of Γ can be connected on T by a rectifiable Jordan arc, then the solution of Plateau's problem for the resulting closed contour represents a surface of class $\mathfrak{A}$. In 1938 R. Courant proved that whenever the class $\mathfrak{A}$ is not empty there exists in $\mathfrak{A}$ a surface S minimizing the value of Dirichlet's integral

$$D_P[\mathfrak{x}] = \iint_P (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\, du\, dv\,;$$

see [1], [2], pp. 87-96 and [3], pp. 201-203. It was later shown in [13]—and this is crucial here—that the solution surface S has also the smallest (Lebesgue) area among all disc-type surfaces bounded by the chain $\langle \Gamma, T\rangle$. The position vector of S possesses the following additional properties:

ii′) The mapping of $\partial' P$ onto (Γ) is topological.

iv) $\mathfrak{x}(u, v)$ is harmonic in P and satisfies in P the conditions $\mathfrak{x}_u^2 = \mathfrak{x}_v^2$, $\mathfrak{x}_u \mathfrak{x}_v = 0$.

In the terminology of [10], pp. 231-232, S is a generalized minimal surface. Naturally, there may be more than one solution.

For the last thirty-five years it has been a problem of great challenge to study the regularity of the solution surface on its free boundary and the nature of its trace. Today a large body of results exists concerning these questions. A detailed description and a complete bibliography can be found in chapter VI.2 and on p. 707 of [14]. In the simplest case, if T is a plane, that part of the trace which corresponds to the open arc $\partial'' P$ is an analytic curve, and the solution vector $\mathfrak{x}(u, v)$ permits an analytic extension across $\partial'' P$. This has been proved by I. F. Ritter [15]. The case of an analytic surface T was subsequently studied by H. Lewy [8]. It is obvious that the smoothness of the trace will depend on the regularity properties of the supporting surface T. From [4] and [13] we know that the solution vector $\mathfrak{x}(u, v)$ which minimizes Dirichlet's integral in the class $\mathfrak{A}$ has a continuous extension to the closure of each domain $P_a = \{u, v;\ u^2 + v^2 < a^2,\ v > 0\}$, $0 < a < 1$, belonging to the Hölder class $C^{0,\gamma}(\overline{P}_a)$, if the supporting surface satisfies a chord-arc condition with constant $c > 0$. Here $\gamma = 2(2 + c)^{-2}$.

For a regular surface T of class C^1 it follows that $\mathfrak{x}(u, v)$ belongs to $C^{0,\beta}(\bar{P}_a)$, where the exponent β can be arbitrarily chosen in the open interval $(0, \frac{1}{2})$ and is independent of a. In [6] W. Jäger proved a Kellogg-type theorem: If T is an « admissible » surface of class $C^{m,\alpha}(m \geqslant 3, 0 \leqslant \alpha \leqslant 1)$, then $\mathfrak{x}(u, v) \in C^{m,\alpha}(\bar{P}_a)$ if $\alpha > 0$, and $\mathfrak{x}(u, v) \in C^{m-1,\beta}(\bar{P}_a)$ for every $\beta \in (0, 1)$ if $\alpha = 0$. Jäger's proof, as well as similar methods employed in connection with related problems, require, among other things, that the normal vector of the supporting surface T have continuous second derivatives, and thus lose their applicability if T does not possess a certain « starting regularity »-C^3 at least. If one wants to go further, new difficulties have to be overcome. Nevertheless, it would be of great interest to settle the cases $m = 2$ and $m = 1$. After all, in the related situation of Plateau's problem a Kellogg-type theorem is known to be valid for all $m \geqslant 1$; see [11], [12].

It is the purpose of the present paper to extend the validity of Jäger's theorem to the case $m = 2$. Our result provides a partial answer to the question formulated in § 921 of [14]. The proof consists of two parts. In the first part an initial regularity property is ascertained which then permits an application of the transversality condition in its strong form. The second part concerns itself with the higher regularity. Here the ideas of [11], [12] are employed in a suitably modified form with the effect that any reference to the differentiability theory for solutions of elliptic systems —a formidable subject not amenable to transparent demonstrations—can be entirely avoided. Our result is as follows (the concept of an admissible surface will be explained in section 2):

THEOREM. *If T is an admissible surface of class $C^{m,\alpha}$ $(m \geqslant 2, 0 \leqslant \alpha \leqslant 1)$ then the solution vector $\mathfrak{x}(u, v)$ has a continuous extension to the closure of each domain P_a, $0 < a < 1$, belonging to the Hölder class $C^{m,\alpha}$ if $0 < \alpha < 1$, to the Hölder class $C^{m,\beta}$ for every $\beta \in (0, 1)$ if $\alpha = 1$ and to the Hölder class $C^{m-1,\beta}$ for every $\beta \in (0, 1)$ if $\alpha = 0$.*

REMARK 1. For $m > 2$ the cases $C^{m,0}$ and $C^{m-1,1}$ are essentially equivalent. In fact, the conclusion $\mathfrak{x} \in C^{m,\beta}$ is an immediate consequence of the assumption $T \in C^{m,1}$. Moreover, a perusal of our proofs will show that the $C^{0,1}$-character—and then, as a consequence, the $C^{1,\beta}$-character—of the trace is already assured by the assumption that T belong to class $C^{1,1}$. It goes without saying that the improvement from $C^{2,0}$ to $C^{1,1}$ is less significant than the improvement from $C^{2,\alpha}$ to $C^{1,\alpha}$ would be. Since our demonstration of the higher regularity rests in part on the function theoretic methods developed in [11], one could also resort to further facts known from the theory of complex functions and replace the various Hölder conditions by suitable Dini conditions. For Plateau's problem this has been done by F. D. Lesley [7].

REMARK 2. It can be proved that the vector $\mathfrak{x}(u, v)$ is Hölder continuous in the closure of all of P provided the chain $\langle \Gamma, T \rangle$ itself satisfies a chord-arc condition (see [4], [13]) so that, in particular, the arc Γ meets the surface T subject to a « lift-off » condition (see [13], p. 134). It is not possible to achieve a higher regularity for $\mathfrak{x}(u, v)$ near the points $(u = \pm 1, v = 0)$ even if Γ and T themselves should possess such a regularity, unless Γ meets T orthogonally. This can already be observed in the special case where the supporting surface is a plane. Here the vector $\mathfrak{x}(u, v)$ permits an analytic extension, and the extended vector appears as a solution of Plateau's problem for a contour consisting of Γ and its image under reflection across T. Since this contour will have two corners, unless Γ meets T orthogonally, the vector $\mathfrak{x}(u, v)$ can at best be Hölder continuous in $\overline{P}$. The trace $\{\mathfrak{x} = \mathfrak{x}(u, 0); -1 \leq u \leq 1\}$, however, is rectifiable; by the theorem of Fejér-Riesz (see [14], § 318) its length is seen to be majorized by that of Γ itself. A similar estimate holds doubtlessly also for more general supporting surfaces T. A detailed discussion of this question will be the subject of an independent investigation.

2. – A surface T imbedded in 3-dimensional Euclidean space is said to be admissible of class $C^{m,\alpha}$ $(m \geq 2, 0 \leq \alpha \leq 1)$ if it satisfies the following conditions:

i) For every point $\mathfrak{x}_0 = \{x_0, y_0, z_0\}$ of T there exist an open sphere S containing $\mathfrak{x}_0$ and a function $f(\mathfrak{x}) \equiv f(x, y, z) \in C^{m,\alpha}$ with non-vanishing gradient such that the statements $\mathfrak{x} \in S$, $f(\mathfrak{x}) = 0$ and $\mathfrak{x} \in T \cap S$ are equivalent.

ii) There is a positive number $p = p_T$ such that every point $\mathfrak{x}$ in the parallel set $T_p = \{\mathfrak{y}; d_T[\mathfrak{y}] < p\}$ can be uniquely expressed in the form

$$\mathfrak{x} = \mathfrak{a}^*(\mathfrak{x}) + \lambda^*(\mathfrak{x}) \mathfrak{N}^*(\mathfrak{x}) . \tag{1}$$

The vectors $\mathfrak{a}^*(\mathfrak{x})$, $\mathfrak{N}^*(\mathfrak{x})$, as well as the functions $\lambda^*(\mathfrak{x})$ are of class $C^{m-1,\alpha}$ in T_p. $\mathfrak{a}^*(\mathfrak{x})$ defines a point of T (the foot of $\mathfrak{x}$ on T), $\mathfrak{N}^*(\mathfrak{x})$ is the unit normal vector of T in this point and $|\lambda^*(\mathfrak{x})| = d_T[\mathfrak{x}]$.

iii) There is a positive constant $\mathfrak{C}_0 < \infty$ such that

$$\sup_{\mathfrak{x} \in T_p} |\operatorname{grad} \mathfrak{N}^*(\mathfrak{x})|^2 = \sup_{\mathfrak{x} \in T_p} \{(\mathfrak{N}_x^*)^2 + (\mathfrak{N}_y^*)^2 + (\mathfrak{N}_z^*)^2\} \leq \mathfrak{C}_0^2 . \tag{2}$$

Note that an admissible surface which is not compact cannot have finite boundary points.

The reader may find it convenient to represent T locally with the aid of isothermal Gaussian parameters ξ and η in the form $\mathfrak{x} = \mathfrak{t}(\xi, \eta)$ where $\mathfrak{t}_\xi^2 = \mathfrak{t}_\eta^2 \equiv E > 0$, $\mathfrak{t}_\xi \mathfrak{t}_\eta \equiv F = 0$. If $\mathfrak{X}(\xi_0, \eta_0)$ denotes the unit normal vector of the surface T at one of its points $\mathfrak{t}(\xi_0, \eta_0)$, every point $\mathfrak{x} = \{x, y, z\}$ in a neighborhood has a representation

$$\mathfrak{x} = \mathfrak{t}(\xi, \eta) + \zeta \mathfrak{X}(\xi, \eta) . \tag{3}$$

This representation establishes a one-to-one correspondence between the coordinates x, y, z of the point $\mathfrak{x}$ and the triples ξ, η, ζ: $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$, $\zeta = \zeta(x, y, z)$. Obviously,

$$\left\{\begin{aligned} &\mathfrak{t}\big(\xi(x, y, z), \eta(x, y, z)\big) = \mathfrak{a}^*(\mathfrak{x}) \\ &\mathfrak{X}\big(\xi(x, y, z), \eta(x, y, z)\big) = \mathfrak{X}^*(\mathfrak{x}) \\ &\zeta(x, y, z) = \lambda^*(\mathfrak{x}) . \end{aligned}\right. \tag{4}$$

A simple computation employing the differential geometric quantities L, M, N (coefficients of the second fundamental form of T) and H, K (mean and Gaussian curvatures of T) shows that

$$\left\{\begin{aligned} &\operatorname{grad} \xi(x, y, z) = \frac{1}{\Delta}\left(\mathfrak{t}_\xi + \frac{\zeta}{E}[-N\mathfrak{t}_\xi + M\mathfrak{t}_\eta]\right) , \\ &\operatorname{grad} \eta(x, y, z) = \frac{1}{\Delta}\left(\mathfrak{t}_\eta + \frac{\zeta}{E}[M\mathfrak{t}_\xi - L\mathfrak{t}_\eta]\right) , \\ &\operatorname{grad} \zeta(x, y, z) = \mathfrak{X} . \end{aligned}\right. \tag{5}$$

Here

$$\Delta = E(1 - 2\zeta H + \zeta^2 K) . \tag{6}$$

Consider a vector $\mathfrak{x}(u, v)$ which has continuous first derivatives in a domain of the (u, v)-plane and maps this domain into the parallel set T_p. In the representation

$$\mathfrak{x}(u, v) = \mathfrak{a}^*\big(\mathfrak{x}(u, v)\big) + \lambda^*\big(\mathfrak{x}(u, v)\big(\mathfrak{N}^*\big(\mathfrak{x}(u, v)\big)$$

then the abbreviations $\mathfrak{a}(u, v) = \mathfrak{a}^*\big(\mathfrak{x}(u, v)\big)$, $\mathfrak{N}(u, v) = \mathfrak{N}^*\big(\mathfrak{x}(u, v)\big)$ and $\lambda(u, v) = \lambda^*\big(\mathfrak{x}(u, v)\big)$ will be used.

3. – We now assume that the supporting surface T is admissible of class C^2 (abbreviation for $C^{2,0}$). The set $\mathfrak{A}$ of comparison surfaces is obviously not empty in this case. Let $S = \{\mathfrak{x} = \mathfrak{x}(u, v);\ (u, v) \in P'\}$ be a solution surface whose position vector minimizes Dirichlet's integral subject to the conditions i), ii′), iii), iv) of section **1**. Setting $u + iv = w = \varrho e^{i\theta}$ we shall interchangeably use the notations $\mathfrak{x}(u, v)$, or $\mathfrak{x}(w)$, or $\mathfrak{x}(\varrho, \theta)$ (and later also $\mathfrak{a}(w)$ etc.)—whichever is most convenient. Denoting by $C(w_0; \varrho)$ the domain $\{w;\ w \in P,\ |w - w_0| < \varrho\}$ we introduce the abbreviation

$$D[\mathfrak{x};\ w_0,\ \varrho] = \tfrac{1}{2}\iint\limits_{C(w_0;\varrho)} (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\, du\, dv\ .$$

From the remarks in the introduction it is clear that $\mathfrak{x}(u, v)$ has a Hlöder continuous extension to every domain $\overline{P}_a$, $0 < a < 1$, so that $|\mathfrak{x}(w_2) - \mathfrak{x}(w_1)| \leqslant \mathfrak{C}_1 |w_2 - w_1|^\beta$ for $w_1, w_2 \in \overline{P}_a$ and arbitrary $\beta \in (0, \frac{1}{2})$. (The constant $\mathfrak{C}_1$ depends on a and β: $\mathfrak{C}_1 = \mathfrak{C}_1(a, \beta)$.) We shall show here that $\mathfrak{x}(u, v)$ satisfies in fact a Lipschitz condition:

Lemma. *If T is an admissible surface of class C^2—or of class $C^{1,1}$; see remark* 1 *of section* **1**—*then the solution vector $\mathfrak{x}(u, v)$ has a continuous extension to each domain $\overline{P}_a$, $0 < a < 1$, belonging to class $C^{0,1}(\overline{P}_a)$.*

In view of a well-known lemma of C. B. Morrey ([9], pp. 134-135) this lemma will be a consequence of the following assertion:

For every number a in $0 < a < 1$ there exist positive constants d and M such that

$$D[\mathfrak{x};\ w_0,\ \varrho] \leqslant M\varrho^2$$

for all $w_0 \in P_a$, $0 < \varrho < d$.

We proceed to prove this assertion.

Choose a number d in the interval $0 < d < (1 - a)/2$ which is so small that the domain $\{u, v;\ u^2 + v^2 < (a + 2d)^2,\ 0 < v < 2d\}$ is mapped into the parallel set T_p and that $\mathfrak{C}_1(a + 2d, \beta)(2d)^\beta < \frac{1}{2}$. Now consider a point $w_0 = u_0 + iv_0$ in P_a. For $v_0 \geqslant d$ and $0 < \varrho < d$ we have (see [13], p. 140)

$$D[\mathfrak{x};\ w_0,\ \varrho] \leqslant \left(\frac{\varrho}{d}\right)^2 D[\mathfrak{x};\ w_0,\ d] \leqslant M_0 \varrho^2\ , \qquad M_0 = d^{-2} D_p[\mathfrak{x}]\ .$$

Assuming now that $0 < v_0 < d$ and $v_0 < \varrho < d$, we find

$$D[\mathfrak{x};\ w_0,\ \varrho] \leqslant D[\mathfrak{x};\ u_0,\ v_0 + \varrho]\ .$$

For a fixed point u_0 on $\partial'' P$ and $0 < r < 1 - |u_0|$ we shall introduce the abbreviation $\Phi(r) = D[\mathfrak{x}; u_0, r]$. By virtue of iv),

$$\Phi(r) = \int_0^r d\varrho \int_0^\pi \mathfrak{x}_\varrho^2(u_0 + \varrho e^{i\theta})\, \varrho \, d\theta \, .$$

Since $D_p[\mathfrak{x}] < \infty$, the following is true for almost all values of r in $0 < r < 1 - |u_0|$:

1) There are no branch points of S on the arc $\Lambda_r = \{\mathfrak{x} = \mathfrak{x}(u_0 + re^{i\theta}); 0 < \theta < \pi\}$, that is, $|\mathfrak{x}_r(u_0 + re^{i\theta})| > 0$ for $0 < \theta < \pi$.

2) The derivative $\Phi'(r)$ exists and is equal to

$$\Phi'(r) = r \int_0^\pi \mathfrak{x}_r^2(u_0 + re^{i\theta})\, d\theta \, .$$

For such values of r let $l(r)$ be the length of the curve Λ_r:

$$l(r) = \int_0^\pi |\mathfrak{x}_\theta(u_0 + re^{i\theta})|\, d\theta \, .$$

Using Schwarz's inequality, we find

$$l^2(r) \leqslant \pi \int_0^\pi \mathfrak{x}_\theta^2(u_0 + re^{i\theta})\, d\theta = \pi r^2 \int_0^\pi \mathfrak{x}_r^2(u_0 + re^{i\theta})\, d\theta = \pi r \Phi'(r) \, . \tag{7}$$

We shall next modify our surface S. The part of S which corresponds to the semi-disc $\overline{C(u_0; r)} = \{u, v; (u - u_0)^2 + v^2 \leqslant r^2, v \geqslant 0\}$ will be replaced by a suitably chosen surface $\Sigma_r = \{\mathfrak{x} = \mathfrak{y}(\varrho, \theta); 0 \leqslant \varrho \leqslant r, 0 \leqslant \theta \leqslant \pi\}$ bounded by the chain $\langle \Lambda_r, T \rangle$. (It is immaterial here whether Λ_r is a Jordan curve or not, or has points in common with T.) Σ_r is a ruled surface generated by the normals to T through the points of Λ_r:

$$\mathfrak{y}(\varrho, \theta) = \mathfrak{a}(u_0 + re^{i\theta}) + \frac{\varrho}{r} \lambda(u_0 + re^{i\theta}) \mathfrak{N}(u_0 + re^{i\theta}) \, .$$

(Remember the abbreviations $\mathfrak{a}^*(\mathfrak{x}(w)) = \mathfrak{a}(w)$ etc.). For $0 < \theta < \pi$ we have

$$\mathfrak{y}_\varrho = \frac{1}{r} \lambda \mathfrak{N} \, , \qquad \mathfrak{y}_\theta = \mathfrak{a}_\theta + \frac{\varrho}{r} \lambda \mathfrak{N}_\theta + \frac{\varrho}{r} \lambda_\theta \mathfrak{N}$$

and thus

$$|\mathfrak{y}_\varrho \times \mathfrak{y}_\theta| = \frac{1}{r}|\lambda|\left|\mathfrak{a}_\theta + \frac{\varrho}{r}\lambda\mathfrak{N}_\theta\right|.$$

Now

$$\begin{aligned}\left(\mathfrak{a}_\theta + \frac{\varrho}{r}\lambda\mathfrak{N}_\theta\right)^2 &= \mathfrak{a}_\theta^2 + 2\frac{\varrho}{r}\lambda\mathfrak{a}_\theta\mathfrak{N}_\theta + \frac{\varrho^2}{r^2}\lambda^2\mathfrak{N}_\theta^2 \leqslant \\ &\leqslant (1+\sigma)\mathfrak{a}_\theta^2 + \left(1+\frac{1}{\sigma}\right)\frac{\varrho^2}{r^2}\lambda^2\mathfrak{N}_\theta^2 \leqslant \\ &\leqslant (1+\sigma)\mathfrak{a}_\theta^2 + \lambda^2\left(1+\frac{1}{\sigma}\right)\mathfrak{C}_\theta^2\mathfrak{x}_\theta^2 .\end{aligned}$$

Here σ is an arbitrary positive quantity. Differentiating the identity $\mathfrak{x}(r, \theta) = \mathfrak{a}(r, \theta) + \lambda(r, \theta)\mathfrak{N}(r, \theta)$ we obtain

$$\mathfrak{x}_\theta = \mathfrak{a}_\theta + \lambda\mathfrak{N}_\theta + \lambda_\theta\mathfrak{N}$$

and, since $\mathfrak{a}_\theta$ and $\mathfrak{N}_\theta$ are both vectors orthogonal to $\mathfrak{N}$,

$$\mathfrak{x}_\theta^2 = \mathfrak{a}_\theta^2 + 2\lambda a_\theta\mathfrak{N}_\theta + \lambda^2\mathfrak{N}_\theta^2 + \lambda_\theta^2 \tag{8}$$

so that

$$\begin{aligned}\mathfrak{x}_\theta^2 &\leqslant (1+\sigma)\mathfrak{a}_\theta^2 + \left(1+\frac{1}{\sigma}\right)\lambda^2\mathfrak{N}_\theta^2 + \lambda_\theta^2, \qquad \sigma > 0, \\ \mathfrak{x}_\theta^2 &\geqslant (1-\sigma)\mathfrak{a}_\theta^2 - \left(\frac{1}{\sigma}-1\right)\lambda^2\mathfrak{N}_\theta^2 + \lambda_\theta^2, \qquad 0 < \sigma < 1,\end{aligned}$$

and finally

$$\begin{aligned}\left[1-\left(1+\frac{1}{\sigma}\right)\mathfrak{C}_0^2\lambda^2\right]\mathfrak{x}_\theta^2 &\leqslant (1+\sigma)\mathfrak{a}_\theta^2 + \lambda_\theta^2, \qquad \sigma > 0, \\ \left[1+\left(\frac{1}{\sigma}-1\right)\mathfrak{C}_0^2\lambda^2\right]\mathfrak{x}_\theta^2 &\geqslant (1-\sigma)\mathfrak{a}_\theta^2 + \lambda_\theta^2, \qquad 0 < \sigma < 1.\end{aligned}$$

A combination of the above inequalities leads to

$$\left(a_\theta + \frac{\varrho}{r}\lambda\mathfrak{N}_\theta\right)^2 \leqslant (1+\sigma)\left[\frac{1}{1-\sigma} + \frac{2}{\sigma}\mathfrak{C}_0^2\lambda^2\right]\mathfrak{x}_\theta^2 - \frac{1}{1-\sigma}\lambda_\theta^2,$$

where the number σ is again arbitrary but restricted to the interval $0 < \sigma < 1$.

Since the point $w = u_0$ has its image on the supporting surface T, we have $\lambda(u_0) = \lambda^*(\mathfrak{x}(u_0)) = 0$ and

$$\lambda(u_0 + re^{i\theta}) = \lambda^*(\mathfrak{x}(u_0 + re^{i\theta})) - \lambda^*(\mathfrak{x}(u_0))$$

so that, by (4) and (5),

$$|\lambda(u_0 + re^{i\theta})| \leq |\mathfrak{x}(u_0 + re^{i\theta}) - \mathfrak{x}(u_0)| \leq \mathrm{C}_1(a + 2d, \beta) r^\beta .$$

By our assumption $\mathrm{C}_1(a + 2d, \beta) r^\beta < \frac{1}{2}$ for $0 < r < 2d$. We now choose $\sigma = |\lambda|$ whenever $\lambda \neq 0$. Then, whenever $\lambda(r, \theta) \neq 0$,

$$\left(a_\theta + \frac{\varrho}{r}\lambda\mathfrak{N}_\theta\right)^2 \leq (1 + 2\mathrm{C}_1 r^\beta)\{[1 + (1 + 2\mathrm{C}_0^2)\mathrm{C}_1 r^\beta]\mathfrak{x}_\theta^2 - \lambda_\theta^2\} \leq$$
$$\leq (1 + 2\mathrm{C}_1 r^\beta)[1 + (1 + 2\mathrm{C}_0^2)\mathrm{C}_1 r^\beta](\mathfrak{x}_\theta^2 - \mu_\theta^2) .$$

The function $\mu(r, \theta)$ in the last inequality is defined by the relation

(9) $$\lambda(r, \theta) = [1 + (1 + 2\mathrm{C}_0)^2 \mathrm{C}_1 r^\beta]^{\frac{1}{2}} \mu(r, \theta) .$$

For the area

$$A(\Sigma_r) = \int_0^r \int_0^\pi |\mathfrak{y}_\varrho \times \mathfrak{y}_\theta| \, d\varrho \, d\theta$$

of Σ_r we now obtain the estimates

$$A(\Sigma_r) \leq [1 + (1 + 2\mathrm{C}_0^2)\mathrm{C}_1 r^\beta][1 + 2\mathrm{C}_1 r^\beta]^{\frac{1}{2}} \int_0^\pi |\mu(r, \theta)| \sqrt{\mathfrak{x}_\theta^2(r, \theta) - \mu_\theta^2(r, \theta)} \, d\theta \leq$$
$$\leq (1 + \mathrm{C}_2 r^\beta) \int_0^\pi |\mu(r, \theta)| \sqrt{\mathfrak{x}_\theta^2(r, \theta) - \mu_\theta^2(r, \theta)} \, d\theta .$$

The new compound constant C_2 depends in a simple way on C_0, C_1 and d. Since $|\mathfrak{x}_\theta(r, \theta)| = r|\mathfrak{x}_r(r, \theta)| > 0$ for $0 < \theta < \pi$ and since, by (7), (8), (9), $\mu_\theta(r, \theta)$ is a square integrable function of θ, we can introduce the arc length on Λ_r

$$s(\theta) = \int_0^\theta |\mathfrak{x}_\theta(u_0 + re^{i\theta})| \, d\theta , \qquad s(0) = 0 , \qquad s(\pi) = l(r)$$

as parameter in the integral. Then

$$A(\Sigma_r) \leq (1 + \mathrm{C}_2 r^\beta) \int_0^{l(r)} |\mu| \sqrt{1 - \mu_s^2} \, ds .$$

Obviously, $\mu_s^2 \leqslant 1$. We set $\sigma = \pi s/l(r)$ and define a new function $\nu(\sigma)$:

$$\nu(\sigma) = \frac{\pi}{l(r)} \mu\big(r, \theta(\sigma l(r)/\pi)\big)$$

$(\theta(s)$ denotes the inverse function of $s(\theta))$. For the integral on the right hand side we now obtain

$$\int_0^{l(r)} |\mu| \sqrt{1-\mu_s^2}\, ds = \frac{l^2(r)}{\pi^2} \int_0^{\pi} |\nu| \sqrt{1-\nu'^2}\, d\sigma \leqslant$$

$$\leqslant \frac{l^2(r)}{2\pi^2} \int_0^{\pi} (1+\nu^2-\nu'^2)\, d\sigma \leqslant$$

$$\leqslant \frac{l^2(r)}{2\pi} + \frac{l^2(r)}{2\pi^2} \int_0^{\pi} (\nu^2-\nu'^2)\, d\sigma .$$

The function $\nu(\sigma)$ is analytic for $0<\sigma<\pi$ and continuous for $0 \leqslant \sigma \leqslant \pi$. The derivative $\nu'(\sigma)$ is square integrable. Since the end points of Λ_r lie on T, also the boundary conditions $\nu(0) = \nu(\pi) = 0$ are satisfied. Under these assumptions the inequality

$$\int_0^{\pi} \nu^2(\sigma)\, d\sigma \leqslant \int_0^{\pi} \nu'^2(\sigma)\, d\sigma$$

holds; see [5], p. 185. It now follows that

$$A(\Sigma_r) \leqslant \frac{1}{2\pi} (1 + \mathfrak{C}_2 r^{\beta})\, l^2(r) \leqslant \frac{r}{2} (1 + \mathfrak{C}_2 r^{\beta})\, \Phi'(r) .$$

The comparison surface $\bar{S} = \{\mathfrak{x} = \bar{\mathfrak{x}}(w);\, w \in P'\}$ with the position vector

$$\bar{\mathfrak{x}}(w) = \begin{cases} \mathfrak{x}(w) & \text{for } w \in P' \setminus C(u_0; r) \\ \mathfrak{y}(w) & \text{for } w \in C(u_0; r) \end{cases}$$

is bounded by the chain $\langle \Gamma, T \rangle$. The vector $\bar{\mathfrak{x}}(w)$ is linearly absolute continuous in P' with square integrable first derivatives so that the area of $\bar{S}$ curves to

$$A(\bar{S}) = D_{P \setminus C(u_0; r)}[\mathfrak{x}] + A(\Sigma_r) .$$

Recalling the minimizing property of the surface S ([13], theorem A) for which

$$A(S) = D_{P\setminus C(u_0;r)}[\mathfrak{x}] + \Phi(r)$$

we are led to the inequality

$$\Phi(r) \leqslant \frac{r}{2}(1 + C_2 r^{\beta})\,\Phi'(r)\,.$$

This inequality holds for almost all r in $v_0 < r < v_0 + d$. Integrating between the limits $r = v_0 + \varrho$ and $r = v_0 + d$, and recalling our current assumption $0 < v_0 < d$, we obtain

$$\Phi(v_0+\varrho) \leqslant \left(\frac{v_0+\varrho}{v_0+d}\right)^2 \Phi(v_0+d)\left(\frac{1+C_2(v_0+d)^{\beta}}{1+C_2(v_0+\varrho)^{\beta}}\right)^{2/\beta} \leqslant$$
$$\leqslant \left(\frac{2\varrho}{d}\right)^2 D_P[\mathfrak{x}](1+C_2(2d)^{\beta})^{2/\beta}$$

and therefore

$$D[\mathfrak{x};\, w_0,\, \varrho] \leqslant D[\mathfrak{x};\, u_0, v_0+\varrho] \leqslant M\varrho^2$$

where

$$M = \frac{4}{d^2}(1 + C_2(2d)^{\beta})^{2/\beta} D_p[\mathfrak{x}]\,.$$

We have now dealt with the cases $v \geqslant d$ and $0 < v_0 < \varrho < d$. For the last case $0 < \varrho \leqslant v_0 < d$ a combination of the preceding inequalities gives

$$D[\mathfrak{x};\, w_0,\, \varrho] \leqslant \left(\frac{\varrho}{v_0}\right)^2 D[\mathfrak{x};\, w_0,\, v_0] \leqslant \left(\frac{\varrho}{v_0}\right)^2 D[\mathfrak{x};\, u_0,\, 2v_0] \leqslant$$
$$\leqslant \left(\frac{\varrho}{v_0}\right)^2\left(\frac{2v_0}{d}\right)^2 (1 + C_2(2d)^{\beta})^{2/\beta} D_p[\mathfrak{x}] \leqslant M\varrho^2\,.$$

Since $M_0 < M$, we have in all cases

$$D[\mathfrak{x};\, w_0,\, \varrho] \leqslant M\varrho^2 \quad \text{for } w_0 \in P_a\,, \quad 0 < \varrho < d\,.$$

Our assertion, and with it our lemma are proved.

4. – At the present stage we know that the vector $\mathfrak{x}(u, v)$, which is harmonic in P, has uniformly bounded derivatives in each domain P_a, $0 < a < 1$. From this fact it can be concluded that the limits $\lim_{v\to+0} \mathfrak{x}_u(u, v)$,

$\lim_{v\to+0}\mathfrak{x}_v(u, v)$ as well as the derivatives $\mathfrak{x}_u(u, 0)$, $\mathfrak{x}_v(u, 0)$ exist and are equal, respectively, for almost all u in $-1 < u < 1$.

According to W. Jäger [6], the solution surface $S = \{\mathfrak{x} = \mathfrak{x}(u, v); (u, v) \in P'\}$ satisfies a certain weak transversality condition: For $-1 < u < 1$ and sufficiently small positive v the vector $\mathfrak{x}(u, v)$ can be expressed in the form

$$\mathfrak{x}(u, v) = \mathfrak{a}(u, v) + \lambda(u, v)\mathfrak{N}(u, v) .$$

Let now $\mathfrak{v}(u, v) \in C^1(\bar{P})$ be an arbitrary test vector with compact support in $P \cup \partial'' P$. Then

$$\lim_{v\to+0}\int_{-1}^{+1}\big(\mathfrak{v}(u, v)\mathfrak{a}_v(u, v)\big)\, du = 0,\ \mathfrak{v}(u, v) \in C_0^1(P \cup \partial'' P) . \tag{10}$$

Consider a value u, $-1 < u < 1$, for which the limit $\lim_{v\to+0}\mathfrak{x}_v(u, v)$ as well as the derivative $\mathfrak{x}_v(u, 0)$ exist and are equal. (Almost all u have this property.) In the neighborhood of the point $\mathfrak{x}(u, 0)$ on T we employ the representation (3). Obviously,

$$\mathfrak{a}_v(u, v) = (\mathfrak{x}_v \operatorname{grad}\xi)\,\mathfrak{t}_\xi + (\mathfrak{x}_v \operatorname{grad}\eta)\,\mathfrak{t}_\eta = \frac{1}{E}\,[(\mathfrak{x}_v\mathfrak{t}_\xi)\,\mathfrak{t}_\xi + (\mathfrak{x}_v\mathfrak{t}_\eta)\,\mathfrak{t}_\eta] + O(v) .$$

Here the arguments in $\mathfrak{x}_v$ are u and v and the arguments in E, $\mathfrak{t}_\xi$, $\mathfrak{t}_\eta$ are $\xi\big(x(u, v), y(u, v), z(u, v)\big)$ and $\eta\big(x(u, v), y(u, v), z(u, v)\big)$.

From the transversality condition (10) it can be concluded that the vector $\mathfrak{x}_v(u, 0)$ must be parallel to the normal vector $\mathfrak{N}$ of T at the point $\mathfrak{x}(u, 0)$. In other words:

The solution surface S is orthogonal to T in almost all points of the trace $\{\mathfrak{x} = \mathfrak{x}(u, 0); -1 < u < 1\}$.

5. – We can now turn to the proof of our theorem for which we shall scrutinize the properties of the position vector $\mathfrak{x}(u, v)$ in a fixed domain $\bar{P}_a$, $0 < a < 1$. Let $T^{(a)}$ be a compact portion of the supporting surface T which contains the subarc $\{\mathfrak{x} = \mathfrak{x}(u, 0); |u| \leqslant a\}$ of the trace of S on T. Consider an arbitrary point in $T^{(a)}$ and choose a coordinate system for which this point is the origin while the tangent plane to T becomes the (x, y)-plane. There exist positive constants r and $\mathfrak{C}$ such that the connected part of T containing the (new) origin and lying in the cylinder $x^2 + y^2 \leqslant r^2$ of (x, y, z)-space has a representation $z = \psi(x, y)$. If T is an admissible surface of class $C^{m,\alpha}$ $(m \geqslant 1, 0 \leqslant \alpha \leqslant 1)$, the function $\psi(x, y)$ possesses continuous derivatives up to those of order m and satisfies the relations $\psi(0, 0) = = \psi_x(0, 0) = \psi_y(0, 0) = 0$ as well as

$$|\partial^k \psi(x, y)| \leqslant \mathrm{C} \quad \text{for } k = 0, 1, \dots, m$$
$$|\partial^m \psi(\bar{x}, \bar{y}) - \partial^m \psi(x, y)| \leqslant \mathrm{C}[(\bar{x} - x)^2 + (\bar{y} - y)^2]^{\alpha/2} \quad \text{if } 0 < \alpha \leqslant 1$$

for $x^2 + y^2 \leqslant r^2$, $\bar{x}^2 + \bar{y}^2 \leqslant r^2$. Here ∂^k stands for any partial derivative of order k. The numbers r and C can be chosen to be *the same* for all points in $T^{(a)}$ and depend on the selection of $T^{(a)}$ only.

Let $(u_0, 0)$, $-a \leqslant u_0 \leqslant a$, be a point on $\partial'' P$ in whose neighborhood we wish to investigate the regularity properties of the trace of S on T. In the vicinity of the point $\mathfrak{x}(u_0, 0)$ on T we represent the surface T in the form $z = \psi(x, y)$ as explained above. The transition to the new coordinates, achieved by a translation and a rotation, does not affect the harmonicity or the regularity properties of $\mathfrak{x}(u, v)$. A number $\varepsilon = \varepsilon(a)$ can be chosen so small that the following conditions are satisfied:

i) $0 < \varepsilon < (1 - a)/2$.

ii) $x^2(u, v) + y^2(u, v) \leqslant r^2/4(1 + \mathrm{C}^2)$ for $(u, v) \in \overline{C(u_0; \varepsilon) \cap P}$.

iii) For $(u, v) \in \overline{C(u_0; \varepsilon) \cap P}$ the points $\mathfrak{x}(u, v)$ are contained in the parallel set T_p (the neighborhood of T in which the representation (1) is valid).

Since we can now restrict the investigation of $\mathfrak{x}(u, v)$ to the closure of the domain $C(u_0; \varepsilon) \cap P$ it is convenient to map this domain conformally onto the unit disc Q in a new $w' = u' + iv'$-plane in such a way that the points $w = u_0 - \varepsilon$, $w = u_0$, $w = u_0 + \varepsilon$ correspond to the points $w' = -i$, $w' = 1$, $w' = i$, respectively. The points $w = u_0 - \varepsilon/2$ and $w = u_0 + \varepsilon/2$ will then be mapped onto certain points $w' = e^{-i\bar{\theta}}$ and $w' = e^{i\bar{\theta}}$. The number $\bar{\theta}$, $0 < \bar{\theta} < \pi/2$, depends on a, but *does not depend* on the location of the point $(u_0, 0)$ on $\partial'' P$. The vector $\mathfrak{x}(u, v)$ becomes a harmonic vector of the new variables u', v' and retains all regularity properties, except in the points $w' = \pm i$. (We shall, however, restrict our study of the boundary behavior to the subarc $\{w' = e^{i\theta};\ |\theta| \leqslant \bar{\theta}\}$ of ∂Q.) For the sake of simplicity we shall again denote the new variables by u and v and the new vector by $\mathfrak{x}(u, v)$. Introducing polar coordinates according to $w = u + iv = \varrho e^{i\theta}$ we shall, as before, interchangeably use the notation $\mathfrak{x}(u, v)$, or $\mathfrak{x}(w)$ or $\mathfrak{x}(\varrho, \theta)$.

The harmonic components of the vector $\mathfrak{x}(u, v)$ are in Q real parts of analytic functions,

$$x(u, v) = \operatorname{Re} g_1(w)\,, \qquad y(u, v) = \operatorname{Re} g_2(w)\,, \qquad z(u, v) = \operatorname{Re} g_3(u, v)$$

which satisfy the relation

$$g_1'^2(w) + g_2'^2(w) + g_3'^2(w) = 0\,. \tag{11}$$

The fact that the vector $\mathfrak{x}(u, v)$ satisfies a Lipschitz condition implies the inequalities $|g_j'(w)| \leqslant N < \infty$ for $0 \leqslant \varrho < 1$, $|\theta| \leqslant \bar{\theta}$. The constant N is independent of the location of the point $(u_0, 0)$ on the segment $|u| \leqslant a$ of $\partial'' P$ and depends only on a (and on the choice of $T^{(a)}$); see « argument $\mathcal{A}$ » in [11], pp. 313-314. It then follows from the theory of complex functions that the derivatives $(\partial/\partial\theta) g_j(e^{i\theta})$ as well as the radial limits $\lim\limits_{\varrho\to 1} g_j'(\varrho e^{i\theta})$ exist for almost all values of θ in $-\pi/2 < \theta < \pi/2$ and that

$$\lim_{\varrho\to 1} \frac{\partial}{\partial\varrho} g_j(\varrho e^{i\theta}) = -i \frac{\partial}{\partial\theta} g_j(e^{i\theta})\,, \qquad \lim_{\varrho\to 1} \frac{\partial}{\partial\theta} g_j(\varrho e^{i\theta}) = \frac{\partial}{\partial\theta} g_j(e^{i\theta})$$

for these values.

Let θ_0, $|\theta_0| \leqslant \bar{\theta}$, be such a « good » value of θ. In the vicinity of the point $\mathfrak{x}(e^{i\theta_0})$ on T the supporting surface T will now be represented in the special form $z = \psi(x, y)$. Again we retain the notation $\mathfrak{x}(w)$ for the transformed (by a motion) position vector. We then have, in particular, $x(e^{i\theta_0}) = y(e^{i\theta_0}) = z(e^{i\theta_0}) = 0$. Note that in view of condition ii) above, $x^2(u, v) + y^2(u, v) \leqslant r^2$ for $(u, v) \in \bar{Q}$.

The transversality condition implies the relation $\mathfrak{x}_\varrho(e^{i\theta}) \times \mathfrak{N} = 0$ for almost all θ in $-\pi/2 < \theta < \pi/2$. Since the normal vector $\mathfrak{N}$ is proportional to the vector $\{-\psi_x, -\psi_y, 1\}$ we see that

$$\begin{aligned} x_\varrho(e^{i\theta}) + z_\varrho(e^{i\theta})\, \psi_x\big(x(e^{i\theta}), y(e^{i\theta})\big) &= 0 \\ y_\varrho(e^{i\theta}) + z_\varrho(e^{i\theta})\, \psi_y\big(x(e^{i\theta}), y(e^{i\theta})\big) &= 0 \end{aligned} \tag{12}$$

for almost all θ in $-\pi/2 < \theta < \pi/2$. For the same values of θ a differentiation of the identity $z(e^{i\theta}) = \psi\big(x(e^{i\theta}), y(e^{i\theta})\big)$ leads to

$$z_\theta(e^{i\theta}) = x_\theta(e^{i\theta})\, \psi_x\big(x(e^{i\theta}), y(e^{i\theta})\big) + y_\theta(e^{i\theta})\, \psi_y\big(x(e^{i\theta}), y(e^{i\theta})\big)\,. \tag{13}$$

Since $\psi(0, 0) = \psi_x(0, 0) = \psi_y(0, 0) = 0$, we find that $x_\varrho(e^{i\theta_0}) = y_\varrho(e^{i\theta_0}) = z_\theta(e^{i\theta_0}) = 0$. Consequently, for almost all θ in $|\theta| \leqslant \bar{\theta}$,

$$\begin{aligned} |z_\theta(e^{i\theta}) - z_\theta(e^{i\theta_0})| &\leqslant 2N\mathrm{C}\{|x(e^{i\theta}) - x(e^{i\theta_0})| + |y(e^{i\theta}) - y(e^{i\theta_0})|\} \leqslant \\ &\leqslant 4N^2\mathrm{C}|\theta - \theta_0| \end{aligned}$$

and similarly

$$|x_\varrho(e^{i\theta}) - x_\varrho(e^{i\theta_0})| \leq 2N^2 \mathrm{C} |\theta - \theta_0|$$
$$|y_\varrho(e^{i\theta}) - y_\varrho(e^{i\theta_0})| \leq 2N^2 \mathrm{C} |\theta - \theta_0| .$$

Since

$$\varrho x_\varrho(w) = \mathrm{Re}\{w g_1'(w)\} , \qquad \varrho y_\varrho(w) = \mathrm{Re}\{w g_2'(w)\} , \qquad z_\theta(w) = \mathrm{Re}\{i w g_3'(w)\}$$

it follows as in [11], esp. p. 320, that

$$|g_j''(\varrho e^{i\theta_0})| \leq \mathrm{C}_3 \log \frac{1}{1-\varrho} \quad \text{for } \frac{1}{2} \leq \varrho < 1 \text{ and } j = 1, 2, 3 . \tag{14}$$

The constant C_3 depends on $\bar{\theta}$, N and C only. Due to the analyticity of the functions $g_j(w)$ in Q, the inequality (14) holds, with the same constant C_3, for all θ_0, $|\theta_0| \leq \bar{\theta}$. Therefore, (see [11], esp. p. 322) the vector $\mathfrak{x}(e^{i\theta})$ belongs to class $C^{1,\beta}$ for $|\theta| \leq \bar{\theta}$. Here β is an arbitrary number subject to the restriction $0 < \beta < 1$. Thus the first part of our theorem ($C^{1,\beta}$-regularity of the trace as consequence of the C^2-character of T) is proved.

Assume next that the supporting surface T belongs to class $C^{2,\alpha}$, $0 < \alpha \leq 1$. Using facts already known we find

$$\psi_x(x, y) = x\psi_{xx}(0, 0) + y\psi_{xy}(0, 0) + O\big((x^2 + y^2)^{(1+\alpha)/2}\big)$$
$$\psi_y(x, y) = x\psi_{xy}(0, 0) + y\psi_{yy}(0, 0) + O\big((x^2 + y^2)^{(1+\alpha)/2}\big)$$

and

$$x(e^{i\theta}) = (\theta - \theta_0) x_\theta(e^{i\theta_0}) + O\big(|\theta - \theta_0|^{1+\beta}\big)$$
$$y(e^{i\theta}) = (\theta - \theta_0) y_\theta(e^{i\theta_0}) + O\big(|\theta - \theta_0|^{1+\beta}\big) .$$

Let us first consider the case $\alpha < 1$. We then choose $\beta = \alpha$ and have

$$\psi_x\big(x(e^{i\theta}), y(e^{i\theta})\big) = [x_\theta(e^{i\theta_0})\psi_{xx}(0, 0) + y_\theta(e^{i\theta_0})\psi_{xy}(0, 0)](\theta - \theta_0) + O\big(|\theta - \theta_0|^{1+\alpha}\big)$$

etc. and finally

$$x_\varrho(e^{i\theta}) - x_\varrho(e^{i\theta_0}) = X(\theta_0)(\theta - \theta_0) + O\big(|\theta - \theta_0|^{1+\alpha}\big)$$

where

$$X(\theta_0) = z_\varrho(e^{i\theta_0})[x_\theta(e^{i\theta_0})\psi_{xx}(0, 0) + y_\theta(e^{i\theta_0})\psi_{xy}(0, 0)]$$

and similar relations for the differences $y_\varrho(e^{i\theta}) - y_\varrho(e^{i\theta_0})$ and $z_\theta(e^{i\theta}) - z_\theta(e^{i\theta_0})$.

From these relations it can be concluded as in [11], esp. p. 324, that the three functions $g_j(w)$ satisfy the conditions

$$(15) \qquad |g_j''(\varrho e^{i\theta})| \leqslant \mathrm{C}_4 \quad \text{for } \tfrac{1}{2} \leqslant \varrho < 1, \ |\theta| \leqslant \bar{\theta}.$$

Therefore, the earlier conclusions concerning the radial limits of $g_j'(\varrho e^{i\theta})$ and the first derivatives of the vector $\mathfrak{x}(w)$ apply now also to the radial limits of $g_j''(\varrho e^{i\theta})$ and the second derivatives of $\mathfrak{x}(w)$.

Differentiation of the relation (12_1) for a « good » value θ_0 in an interval $|\theta_0| \leqslant \bar{\bar{\theta}} < \bar{\theta}$ leads to

$$\begin{aligned} x_{\varrho\theta}(e^{i\theta}) - x_{\varrho\theta}(e^{i\theta_0}) &= [\psi_{xx}(x(e^{i\theta}), y(e^{i\theta})) - \psi_{xx}(0, 0)]\, x_\theta(e^{i\theta_0}) z_\varrho(e^{i\theta_0}) + \\ &\quad + [\psi_{xy}(x(e^{i\theta}), y(e^{i\theta})) - \psi_{xy}(0, 0)]\, y_\theta(e^{i\theta_0}) + O(|\theta - \theta_0|) = \\ &= O(|\theta - \theta_0|^\alpha). \end{aligned}$$

From the identities (12_2) and (13) we obtain similarly

$$y_{\varrho\theta}(e^{i\theta}) - y_{\theta\theta}(e^{i\theta_0}) = O(|\theta - \theta_0|^\alpha)$$

and

$$z_{\theta\theta}(e^{i\theta}) - z_{\theta\theta}(e^{i\theta_0}) = O(|\theta - \theta_0|^\alpha).$$

Considering that

$$\begin{aligned} \varrho x_{\varrho\theta}(w) &= \operatorname{Re}\{iw^2 g_1''(w) + iw g_1'(w)\} \\ \varrho y_{\varrho\theta}(w) &= \operatorname{Re}\{iw^2 g_2''(w) + iw g_2'(w)\} \\ z_{\theta\theta}(w) &= \operatorname{Re}\{-w^2 g_3''(w) - w g_3'(w)\} \end{aligned}$$

it follows that the functions $g_j(w)$ satisfy inequalities

$$(16) \qquad |g_j''(\varrho e^{i\theta})| \leqslant \mathrm{C}_5 (1 - \varrho)^{\alpha - 1} \quad \text{for } \tfrac{1}{2} \leqslant \varrho < 1, \ |\theta| \leqslant \bar{\bar{\theta}}$$

and further, in a fashion patterned after the proofs in [11], [12], and by now familiar to us, that the vector $\mathfrak{x}(e^{i\theta})$ belongs to class $C^{2,\alpha}$ for $|\theta| \leqslant \bar{\bar{\theta}}$.

If $\alpha = 1$, we can choose β arbitrarily and conclude that $\mathfrak{x}(e^{i\theta})$ belongs to class $C^{2,\beta}$ in $|\theta| \leqslant \bar{\bar{\theta}}$ for any $\beta \in (0, 1)$.

The higher cases $m = 3, 4, \ldots$ can be treated analogously by repeated differentiation of the identities (12), (13).

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