ANNALI DELLA
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DORIAN GOLDFELD

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An Asymptotic Formula Relating the Siegel Zero and the Class Number of Quadratic Fields.

DORIAN GOLDFELD (*)

§ 1. – For a fundamental discriminant d, let $h(d) = \sum_{a,b,c} 1$ denote the number of reduced, primitive, inequivalent binary quadratic forms $ax^2 + bxy + cy^2$ with $d = b^2 - 4ac$. In view of the correspondence

$$ax^2 + bxy + cy^2 \leftrightarrow \left[a, \frac{b + \sqrt{d}}{2}\right] \qquad (Z\text{-module}),$$

$h(d)$ is also the narrow class number of the quadratic field $Q(\sqrt{d})$.

If, for a real primitive character $\chi (\operatorname{mod} d)$

$$L(1, \chi) \ll (\log|d|)^{-1}, \tag{1}$$

then $L(s, \chi)$ will have a real zero β in the interval $0 < 1 - \beta \ll (\log|d|)^{-1}$. This can be seen, by simply considering the following integral (see [1])

$$1 \ll \int_{2-i\infty}^{1+i\infty} \zeta(s + \beta') L(s + \beta', \chi) \frac{x^s}{s(s+1)(s+2)} ds = \tag{2}$$

$$= L(1, \chi) \frac{x^{1-\beta'}}{(1-\beta')(2-\beta')(3-\beta')} + \frac{1}{2} \zeta(\beta') L(\beta', \chi) + O\left(|d| x^{-\frac{1}{2}}\right)$$

after shifting the line of integration to $\operatorname{Re}(s) = -\frac{1}{2}$. Now, if $1 - \beta' = c_1 (\log|d|)^{-1}$ for suitable c_1, and $L(s, \chi) \neq 0$ for $\beta' \leqslant s \leqslant 1$ then $\zeta(\beta') L(\beta', \chi) < 0$. Choosing $x = |d|^3$, say, it follows from (2) that $L(1, \chi) \gg (\log|d|)^{-1}$, which contradicts (1) unless $L(s, \chi)$ has a zero in the interval. The precise location

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of this zero can be given as follows

THEOREM 1. *Let $d<0$. If the Siegel zero β exists, then*

$$1-\beta=\frac{6}{\pi^2}\{L(1,\chi)+O(1-\beta)^2(\log|d|)^3\}\Big\{\sum_{a,b,c}\frac{1}{a}+O\big(L(1,\chi)\log|d|\big)\Big\}^{-1},$$

$$1-\beta=\big(L(1,\chi)+O(1-\beta)^2(\log|d|)^3\big)L'(1,\chi)^{-1},$$

$$L'(1,\chi)=\frac{-\pi}{\sqrt{|d|}}\sum_{m=1}^{|d|}\chi(m)\log\Gamma\left(\frac{m}{|d|}\right)+\frac{h(d)\pi(\gamma+\log 2\pi)}{\sqrt{|d|}},$$

where γ is Euler's constant, and all other constants occuring in the O-symbols are effectively computable.

When $d>0$, let $\omega=(-b+\sqrt{d})/2a$, $\bar{\omega}=(-b-\sqrt{d})/2a$, p_n/q_n (with $q_1=1$) be the principal convergents to ω, and $\|q_n\omega\|=|q_n\omega-p_n|$. If ε is the fundamental unit of $Q(\sqrt{d})$, define M to be 0 if $a\varepsilon/\sqrt{d}<1$, and otherwise the unique integer satisfying $q_M\leqslant a\varepsilon/\sqrt{d}<q_{M+1}$.

THEOREM 2. *Let $d>0$. If the Siegel zero β exists, then*

$$1-\beta=\frac{6}{\pi^2}\{L(1,\chi)+O(1-\beta)^2(\log|d|)^3\}\Big\{\sum_{a,b,c}\Big(\frac{1}{a}+\frac{2}{\pi\sqrt{d}}Q\Big)+O\big(L(1,\chi)\log d\big)\Big\}^{-1},$$

$$Q=\sum_{m=1}^{M-1}\frac{1}{q_m\|q_m\omega\|}\left[\text{Arctan}\left(\frac{q_{m+1}^2\|q_m\omega\|}{q_m}\right)-\text{Arctan}\,(q_m\|q_m\omega\|)\right]+$$
$$+\frac{1}{q_M\|q_M\omega\|}\left[\text{Arctan}\left(\frac{a^2\varepsilon^2\|q_M\omega\|}{dq_M}\right)-\text{Arctan}\,(q_M\|q_M\omega\|)\right]$$

and all constants occurring in O-symbols are effectively computable.

§ 2. – Following a suggestion of Gallagher, a simple proof of Theorem (1) can be given by use of Kronecker's limit formula. Let, for $z=x+iy$

$$(3)\qquad f(z,s)=y^s\sideset{}{'}\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}|m+nz|^{-2s}=$$
$$=2y^s\zeta(2s)+2\sqrt{\pi}\,y^{1-s}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)+$$
$$+4\frac{\pi\sqrt{y}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos(2\pi xn)\int_0^{\infty}\exp\left(-n\pi y\left(t+\frac{1}{t}\right)\right)t^{s-\frac{1}{2}}\frac{dt}{t},$$

a result which easily follows from [2].

Now, by a classical theorem of Dirichlet (for $d<0$)

$$\zeta(s)L(s,\chi)=\frac{1}{2}\sum_{a,b,c}2^s|d|^{-s/2}f(z,s)\,,\qquad \left(z=\frac{b+\sqrt{d}}{2a}\right), \tag{4}$$

and by equations (3) and (4)

$$\begin{aligned}\lim_{s\to1}\left(\zeta(s)L(s,\chi)-\frac{L(1,\chi)}{s-1}\right)&=\gamma L(1,\chi)+L'(1,\chi)=\\&=\frac{\pi^2}{6}\sum_{a,b,c}\frac{1}{a}+O\big(L(1,\chi)\log|d|\big)\,.\end{aligned} \tag{5}$$

On the other hand, the Taylor series expansion for L about β gives

$$0=L(1,\chi)+(\beta-1)L'(1,\chi)+O(1-\beta)^2(\log|d|)^3\,. \tag{6}$$

The first part of Theorem 1 now follows from (5) and (6). The second part follows from (6) and the formula for $L'(1,\chi)$ (see [2], p. 110).

§ **3.** - In the case $d>0$, let

$$x=\frac{\bar{\omega}+\omega u^2}{u^2+1}\,,\qquad y=\frac{\sqrt{d}}{a}\,\frac{u}{u^2+1}\,.$$

Following Hecke, Siegel [3]

$$\zeta(s)L(s,\chi)=\sum_{a,b,c}\frac{\Gamma(s)}{\Gamma^2(s/2)}\,d^{-s/2}\int_{\eta}^{\eta\varepsilon^2}f(z,s)\,\frac{du}{u} \tag{7}$$

where $\eta>0$ is arbitrary. Henceforth, we take $\eta=a/\sqrt{d}$.

Now, $f(z,s)$ is invariant under a unimodular transformation

$$z\to z^*=\frac{\alpha z+\lambda}{\theta z+\delta}\,, \tag{8}$$

$$y^*=\frac{y}{(\theta x+\delta)^2+(\theta y)^2}\,.$$

It follows that for

$$q_n\leqslant y^{-\frac{1}{2}}\leqslant q_{n+1} \tag{9}$$

and the choice $\theta = q_n$, $\delta = -p_n$ that

$$1 \ll y^* \ll \sqrt{d}. \tag{10}$$

The condition (9) can be expressed

$$Q_n \leqslant u \leqslant Q_{n+1}$$

$$Q_n = \frac{1}{2}\left|\frac{\sqrt{d}}{a} q_n^2 + \left(\frac{d}{a^2} q_n^4 - 4\right)^{\frac{1}{2}}\right|, \qquad (1 \leqslant n \leqslant M),$$

$Q_{M+1} = a\varepsilon^2/\sqrt{d}$.

Consequently, the integral in (7) can be written as a sum of integrals

$$\zeta(s) L(s, \chi) = \frac{\Gamma(s)}{\Gamma^2(s/2)} d^{-s/2} \sum_{a,b,c} \left[\int_{a/\sqrt{d}}^{Q_1} f(z, s) \frac{du}{u} + \sum_{m=1}^{M} \int_{Q_m}^{Q_{m+1}} f(z^*, s) \frac{du}{u} \right]. \tag{11}$$

We now get from (3), (10), and (11) that

$$\lim_{s \to 1} \left(\zeta(s) L(s, \chi) - \frac{L(1, \chi)}{s-1} \right) = \gamma L(1, \chi) + L'(1, \chi) = \tag{12}$$

$$= \frac{\pi}{3\sqrt{d}} \sum_{m=0}^{M} I_m + R_1 + R_2$$

where

$$I_0 = \int_{a/\sqrt{d}}^{Q_1} y \frac{du}{u}, \qquad I_m = \int_{Q_m}^{Q_{m+1}} y^* \frac{du}{u}, \qquad (1 \leqslant m \leqslant M)$$

and

$$\begin{cases} R_1 \ll d^{-\frac{1}{2}} \sum_{a,b,c} \log d \int_{a/\sqrt{d}}^{\varepsilon^2 a/\sqrt{d}} \frac{du}{u} \ll L(1, \chi) \log d, \\ R_2 \ll d^{-\frac{1}{2}} \sum_{a,b,c} \int_{a/\sqrt{d}}^{\varepsilon^2 a/\sqrt{d}} \frac{du}{u} \ll L(1, \chi). \end{cases} \tag{13}$$

Since $x = \omega - \sqrt{d}/a(u^2 + 1)$, it follows from (8) that

$$y^* = \frac{\sqrt{d}}{a} \|q_m \omega\|^{-2} \frac{u}{u^2 + B_m^2},$$

$$B_m = \frac{\sqrt{d}}{a} \frac{q_m}{\|q_m \omega\|} - 1.$$

Therefore (for $1 \leqslant m \leqslant M$)

$$I_0 = \frac{\sqrt{d}}{a}\left[\operatorname{Arctan}(Q_1) - \operatorname{Arctan}(a/d)\right],$$

$$I_m = \frac{\sqrt{d}}{a}\|q_m\omega\|^{-2}B_m^{-1}\left[\operatorname{Arctan}\left(\frac{Q_{m+1}}{B_m}\right) - \operatorname{Arctan}\left(\frac{Q_m}{B_m}\right)\right].$$

We get

$$(14)\quad \begin{cases} \displaystyle\sum_{a,b,c} I_0 = \frac{\pi\sqrt{d}}{2}\sum_{a,b,c}\frac{1}{a} + O(h(d)), \\ \displaystyle\sum_{a,b,c}\sum_{m=1}^{M-1} I_m = \sum_{a,b,c}\sum_{m=1}^{M-1}\frac{1}{q_m\|q_m\omega\|}\left[\operatorname{Arctan}\left(\frac{q_{m+1}^2\|q_m\omega\|}{q_m}\right) - \right. \\ \qquad\qquad \left. - \operatorname{Arctan}\left(q_m\|q_m\omega\|\right)\right] + O(h(d)), \\ \displaystyle\sum_{a,b,c} I_M = \sum_{a,b,c}\frac{1}{q_M\|q_M\omega\|}\left[\operatorname{Arctan}\left(\frac{a^2\varepsilon^2\|q_M\omega\|}{dq_M}\right) - \operatorname{Arctan}\left(q_M\|q_M\omega\|\right)\right] + \\ \qquad\qquad + O(1). \end{cases}$$

Theorem (2) now follows from (6), (12), (13) and (14).

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