ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

LUIS A. CAFFARELLI

Surfaces of minimum capacity for a knot

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, nº 4 (1975), p. 497-505
<http://www.numdam.org/item?id=ASNSP_1975_4_2_4_497_0>

Surfaces of Minimum Capacity for a Knot.

LUIS A. CAFFARELLI (*)

In [2] G. C. Evans constructed the surface of minimum capacity spanned by a prescribed Jordan curve $\Gamma \subset \boldsymbol{R}^3$, Γ of capacity 0, with the pair $(\boldsymbol{R}^3, \Gamma)$ homeomorphic to $(\boldsymbol{R}^3, C)$, where C is a circle.

Evans considers a double valued harmonic function f in $\boldsymbol{R}^3 \sim \Gamma$ such that if we loop once around Γ we pass from one value of f to the other, and at infinity takes alternatively the values -1 and 1. The sum of the two values of f at a given point is identically 0 and the set of points where the two values of f are 0 defines the surface of minimum capacity.

In this paper, we construct a multiple valued harmonic function in $\boldsymbol{R}^3$ minus a general closed curve of zero capacity given its asymptotic behavior at infinity in each leaf, and in particular the surface of minimum capacity. But, in general, Evans' surface of minimum capacity does not yield an absolute minimum.

Hence, the family of surfaces competing for the minimum is characterized by a local condition.

I would like to thank Professor H. Lewy for his many suggestions.

1. – In this paragraph, we construct a multiple valued function associated with a given family of single-valued harmonic functions.

In order to do that, we use the method of balayage, whose properties we collect in the following lemma.

LEMMA 1. *Let be $\Omega \subset \boldsymbol{R}^n$ an open domain, φ_0 a continuous superharmonic function, $B \subset \Omega$, a closed ball. The function φ_1 defined by*

$$\varphi_1/\Omega \sim B = \varphi_0$$
$$\varphi_1/B = \int_{\partial B} P(X, Y)\varphi_0(Y)\, d\sigma(Y)$$

(*) University of Minnesota.
Pervenuto alla Redazione il 24 Gennaio 1975.

(where $P(X, Y)$ is the corresponding Poisson kernel) verifies:

a) φ_1 *is continuous and superharmonic.*

b) $\varphi_1 \leqslant \varphi_0$.

If now, we repeat the process with a sequence of balls B_n, and if a neighborhood, U, of a point Y, verifies $U \subset B_{n_k}$ for some subsequence n_k, the sequence φ_n converges in U to a harmonic function or diverges to $-\infty$. We also need the following lemma.

LEMMA 2. *Given K_1, K_2, two compact sets in $\boldsymbol{R}^n$, $K_1 \subset K_2^0$, there exists a compact set K such that*

a) $K_1 \subset K^0$, $K \subset K_2^0$ *(K^0 means the interior of K).*

b) K has smooth boundary (in fact analytic) and if U_K denotes the conductor potential of K, $(\partial/\partial\nu) U_K$ never vanishes (where $(\partial/\partial\nu) U$ denotes the outer normal derivative of U in ∂K).

PROOF. Let $\bar{K}$ be a finite union of closed subes, such that $K_1 \subset (\bar{K})^0$, $\bar{K} \subset K_2^0$. Let U_K be the conductor potential of $\bar{K}$ and S_α the equipotential surface $S_\alpha = \{X, U_{\bar{K}}(X) = \alpha\}$. Then, $\forall \alpha > \beta_0$, ($\beta_0 < 1$, a constant), $S_\alpha \subset K_2^0$, and, $\partial\bar{K}$ being regular, $\bar{K} \subset S_\alpha^0 = \{X; U_{\bar{K}}(X) > \alpha\}$. Also, except for a finite number of α's (See Kellog [7], pg. 276), we can choose S_{α_0} such that $(\partial/\partial\nu) U_{\bar{K}}$ does not vanish on S_{α_0}. Then, our compact set K is $K = \{X: U_{\bar{K}}(X) \geqslant \alpha_0\}$ and its conductor potential $U_K = (1/\alpha_0) U_{\bar{K}}$ in $\boldsymbol{R}^n \sim K$. Q.E.D.

Let us consider now a compact set $\Gamma \subset \boldsymbol{R}^n$, such that $\boldsymbol{R}^n \sim \Gamma$ is connected and the capacity of Γ is zero.

Instead of the k-covering space used by Evans in [3], we must consider the covering space $H_{G'}$, associated with a subgroup G' of the fundamental group G of $\boldsymbol{R}^n \sim \Gamma$ (See Hu[6], pg. 93). In fact, let us call Π the natural projection $\Pi: H_{G'} \to \boldsymbol{R}^n \sim \Gamma$, then Π is locally a homeomorphism and defines an Euclidean-structure on $H_{G'}$, hence the concept of harmonicity (Super-harmonicity and sub-harmonicity) is well defined in any open subset of $H_{G'}$.

More than that, suppose that B is a closed ball in $\boldsymbol{R}^n \sim \Gamma$, then its inverse image by Π is a family of disjoint closed balls and if some of them are included in the domain of definition of a superharmonic function φ, lemma 1 still applies.

We want to prove the following theorem.

THEOREM 1. *Let G' and Γ be as above, that is Γ compact and of capacity 0, G' a subgroup of the fundamental group of $\boldsymbol{R}^n \sim \Gamma$. Suppose that $\boldsymbol{R}^n \sim \Gamma$ is* *connected and K_1, K_2 are two compact sets of $\boldsymbol{R}^n$ such that*

a) $\Gamma \subset K_1$, $\boldsymbol{R}^n \sim K_1$ *is connected and simply connected.*

b) $K_1 \subset K_2^0$.

Then we can prove:

i) If $\{h_g\}_{g \in G/G'}$ is a family of harmonic functions in $\boldsymbol{R}^n$, uniformly bounded in K_2, there exists a unique harmonic function φ in $H_{G'}$, such that

a) φ is bounded in $\Pi^{-1}(K_1)$ and

b) $\lim\limits_{\substack{X \in H_g \\ X \to \infty}} |\varphi(X) - h^i(\Pi(X))| = 0$,

$(\Pi^{-1}(\boldsymbol{R}^n \sim K_1) = (R^n \sim K_1) \times G/G'$; so $X \to \infty$, $X \in H_g$, means $\Pi(X) \to \infty$, $X \in (\boldsymbol{R}^n \sim K_1) \times \{g\}$.)

ii) Conversely, given φ, harmonic in $H_{G'}$, and bounded in $\Pi^{-1}(K_2)$, there exist functions h_g, such that φ is the function constructed in part i).

iii) If the h_g's are constants, $h_g = \lambda_g$, and $\sum |\lambda_g| < \infty$ then

$$\sum_{X \in \Pi^{-1}(X_0)} |\varphi(X) \leqslant \infty \qquad \forall X_0$$

and

$$\sum_{X \in \Pi^{-1}(X_0)} \varphi(X) \equiv \sum \lambda^i .$$

REMARK. If G/G' has only k elements, the preceeding theorem establishes a correspondence between the sets of k single valued harmonic functions and the harmonic functions on $H_{G'}$, bounded in a neighborhood of Γ.

PROOF. Let K be as in lemma 2, then, on K, all the derivatives of h_g of order m are bounded by $C(m) M \operatorname{dist}(K, \complement K_2)^{-m}$, where M is the constant that bounds $|h_g|$ in K_2. Hence, if h_g^* is the solution of the Dirichlet problem: $\Delta h_g^* = 0$ in $R^n \sim K$, $h_g^* \to 0$ at ∞ and $h_g^* / \partial K = h_g$, the outer normal derivative of h_g^* on ∂K is bounded by a constant

$$C(M, K, K_2)$$

(See Courant-Hilbert [1], pg. 335), therefore the function

$$\gamma_{g,\lambda} \begin{cases} h_g - h_g^* + \lambda U_K & \text{in } \boldsymbol{R}^n \sim K \\ \lambda & \text{in } K . \end{cases}$$

is superharmonic (sub-harmonic) as soon as λ_0(resp. λ_1), is taken big enough, (small enough) to make $(\partial/\partial\nu)\gamma_{g,\lambda}$ strictly negative (positive) on ∂K. This choice of λ, depends only on M, K, K_2.

The collection of functions

$$\{\gamma_{g,\lambda_0}\}_{g\in G/G'}$$

being coincident in K, defines a continuous superharmonic function $\varphi^0(X)$ in $H_{G'}$.

In the same way $\{\gamma_{g,\lambda_1}\}$ defines a continuous subharmonic function $\varphi_0(X)$.

We now consider a covering of $\boldsymbol{R}^n \sim \Gamma$ by a denumerable family of open balls, B_n, such that $\bar{B}_n \cap \Gamma = \emptyset$ and we renumber them, repeating each one infinitely many times, $\{B_n'\}$. Then, by lemma 1, we form a monotone sequence of superharmonic functions $\varphi^n(X)$, by the method of balayage applied to each one of the balls composing $\Pi^{-1}(B_n')$. We have $\varphi^n(x) \searrow \varphi(x) \geqslant \varphi_0(x)$ and φ is the desired harmonic function. To show uniqueness, we note that if φ_1 and φ_2 are two such solutions, $\varphi_1 - \varphi_2$ is bounded in $\Pi^{-1}(K_1)$ (by hypothesis), and $\varphi_1 - \varphi_2 \to 0$ as $x \to \infty$ in each leaf; therefore $\varphi_1 - \varphi_2$ is uniformly bounded in all of $H_{G'}$. If B is a closed ball such that $B \cap \Gamma = \emptyset$, $\nabla(\varphi_1 - \varphi_2)$ is, hence, bounded in $\Pi^{-1}(B)$. But, then,

$$\psi(X_0) = \sup_{X\in\Pi^{-1}(X_0)} (\varphi_1 - \varphi_2)(X)$$

is a bounded, continuous, sub-harmonic function in $\boldsymbol{R}^n \sim \Gamma$, tending to 0 at infinity. Since Γ has 0 capacity, ψ is negative. To prove ii) we invert the process: Since φ is bounded in $\Pi^{-1}(K_2)$, all the derivatives of φ are uniformly bounded in $[\Pi^{-1}(\partial K)] \cap H_g$, for a fixed g, and we can construct, for a leaf H_g, super and sub-harmonic functions h_g^0 and h_{0_g}, with

$$h_{0_g} \leqslant h_g^0$$

and

$$\lim_{\substack{X\in H_g \\ X\to\infty}} |h_{0_g} - \varphi| = \lim_{\substack{X\in H_g \\ X\to\infty}} |h_g^0 - \varphi| = 0 .$$

The function h_g is obtained by the method of balayage.

Finally, as to iii), we just notice that restricting ourselves to the case $\lambda_g \leqslant 0$, $\forall g$, if we choose K to be a ball $B \supset S$, of radius R, and begin the process with the super-harmonic functions

$$h_g^0 = \begin{cases} \lambda_g \left(1 - \left(\dfrac{R}{|x|}\right)^{n-2}\right) & \text{outside of } B , \\ 0 & \text{in } B , \end{cases}$$

then $h_g^0 \leqslant 0$, $\sum \lambda_g \leqslant \sum h_g^0 \leqslant 0$ and $\sum_g (h_g - \lambda_g) \to 0$ as $x \to \infty$; this three properties are preserved when the method of balayage is applied.

2. – We study now, the case in which Γ is a tame knot, that is, there is a homeomorphism: $\Delta: \boldsymbol{R}^3 \to \boldsymbol{R}^3$, mapping Γ into a poligonal curve Γ^*. It is known that $G/[G, G] \sim Z$ and hence, there is a unique subgroup G', such that $G/G' \sim Z_2$.

$H_{G'}$ consists of two leaves, and hence, there is a unique bounded double valued harmonic function φ (a bounded harmonic function in $H_{G'}$) such that $\varphi \to -1$ in H_{-1} as $x \to \infty$, $\varphi \to +1$ in H_{+1} as $x \to \infty$. To apply the technique of Evans, the comparison surfaces must be cutting surfaces for $H_{G'}$ and it is also necessary to apply Green's formula.

So, we will restrict ourselves to the following family of surfaces:

Definition 1. S belongs to the family $\mathcal{F}(\Gamma)$ if

a) S is a compact set of $\boldsymbol{R}^3$, $\Gamma \subset S$, $\Gamma \subset \overline{(S \sim \Gamma)}$,

b) $\boldsymbol{R}^3 \sim S$ is connected,

c) if U is an open neighborhood of Γ, $S \sim U$ can be decomposed into $V_1 \cup V_2$ where

- c_1) V_2 is a closed set of 2 dimensional Hausdorff measure 0.
- c_2) V_1 is locally homeomorphic to a plane. (Given $x \in V_1$, there exists $B(x, \varrho(x))$ and a homeomorphism $\theta: (B(x, \varrho(x)) \to \Omega \subset \boldsymbol{R}^3$ such that $\theta(V_1) = \Omega \cap \Pi$ where Π is a plane.

d) S is locally oriented in the sense that given $x \in S \sim \Gamma$, there exists $B(x, \varrho(x))$ such that

- d_1) $B(x, \varrho(x)) \sim S = \bigcup_1^n D_k$, (where D_k are the connected components) $n < \infty$, $n \geqslant 2$
- d_2) $x \in \partial D_k$, $\forall_k$ and
- d_3) A sign ($+1$ or -1) can be defined on the D_k such that if $\partial D_k \cap \partial D_k \cap V_1 \neq \emptyset$, sign $(D_k) \neq$ sign $(D_{k'})$.

The subfamily of surfaces such that V_1 is locally regular in the sense of Kellogg [7] pg. 105, will be denoted $\mathcal{F}'(\Gamma)$.

Remark. *a*) If an orientation as in (d_3) exists it is uniquely determined by the sign of one domain (let us say D_1).

To see this, we consider $\Delta_1 = \bigcup D_{k_1}$ and $\Delta_2 = B(x, \varrho(x)) \sim (S \cup \Delta_1)$, where D_{k_1} are the domains whose sign becomes defined in accordance with d_3

by that of D_1 (but not necessarily equal). Then, if $\Delta_2 \neq \emptyset$ we consider a family of parallel segments joining points of a small disk in Δ_1 with a small disk in Δ_2. Each segment intersects $\partial\Delta_2 \cap \partial\Delta_1$ and the set of these intersections has 2-dimensional Hausdorff measure different from 0, because its projection on a suitable plane is a disk. So, there is a point of $V_1 \cap \partial\Delta_1 \cap \partial\Delta_2$ that contradicts our assumptions.

In particular, this shows that given two balls, the assignments of ± 1 to D_k can be defined in such a way as to be coincident in their intersection.

EXAMPLES. *a*) Given Γ, we can construct a surface $S \in \mathcal{F}(\Gamma)$ without self intersections (that is $S \sim \Gamma = V_1$). See Fox ([5], pg. 158). We project the polygonal knot on a plane Π in such a way that the projection has only a finite number of double points. This divides the plane into two regions Δ_1 and Δ_2 such that two connected components of Δ_1 (or Δ_2) have only double points as common boundary. If Δ_1 is the bounded region, we span it with the surface, twisting it along the knot at the double points of the projection. For a general tame knot Γ, we use the homeomorphism between Γ and a polygon.

b) Given Γ, and supposing that the projection of Γ on a sphere of center at P has at most isolated multiple points (but not double segments) the union of segments AP with $A \in \Gamma$, is a surface of the family $\mathcal{F}$.

When we consider the ball $B(x, \varrho(x))$, as in Def. 1, *c*) or *d*) the union of the subdomains D_k of positive sign will be denoted by D^+ and the union of the negative ones by D^-. Of course, this definition is local. In what follows the roles of D^+, D^- are interchangeable.

THEOREM 2. *If $S \in \mathcal{F}(\Gamma)$, S is a cutting surface of Γ. (That is, any polygon that loops Γ once, intersects S).*

PROOF. Let $\sigma(\tau)$ be a polygonial curve of $\boldsymbol{R}^3$ and $\{I_k\}_{k=1}^n$ the τ-intervals of $[0, 1]$ such that σ is linear on I_k, $T_k = \sigma(I_k)$ $\left(\Gamma \cap \bigcup T_k = \emptyset\right)$.

Then, if we consider a small tubular neighborhood U_k of T_k, by the preceeding remarks, we can give a sign to the domains of $U_k \sim S$ in such a way that

a) It is coherent with the orientation of S around any point of $T_k \cap S$.

b) They are coincident in $U_k \cap U_{k+1}$.

Also, once we fix the orientation in a neighborhood of $\sigma(0)$, it becomes uniquely determined.

So we may define the parity of the number of intersections of the closed polygonal curve $\sigma(t)$ as $p_S(\sigma)=0$ if the orientation of U_1 and U_m are coincident in $U_1 \cap U_m$, and $p_S(\sigma)=1$ if not.

Now, if we consider a continuous deformation $\sigma_\alpha(\alpha\in[0,1])$, the set of α's such that $p_S(\sigma_\alpha)=0$ (resp. $p_S(\sigma_\alpha)=1$) is open. Hence p_S is a group homomorphism.

$$p_S(\sigma): G \to Z_2 .$$

Therefore, to show that S is a cutting surface is equivalent to show that $p_S(\sigma)$ is not identically 0.

But, for any $S\in\mathcal{F}(\Gamma)$, we can find a σ such that $p_S(\sigma)=1$, in the following way: $R^3\sim S$ is open and connected and in a neighborhood of a point of V_1, the surface S is two sided; so, joining points of both sides of S, by a segment, and by a polygon in $R^n\sim S$, we obtain our σ.

REMARK. The set $V_1\neq\emptyset$, because $S\sim\Gamma\neq\emptyset$, and if $x\in V_2$, there are at least two different domains in $B(x,\varrho)\sim S$ (See d_1 of definition 1). Therefore, if we consider a family of parallel segments joining points of the two domains we obtain a portion of $S\sim\Gamma$ of non-zero 2-dimensional Hausdorff measure.

THEOREM 3. *Let be φ the double valued harmonic function, (the harmonic function in $H_{G'}$ with $G/G'=Z_2$) verifying*

$$\lim_{\substack{x\to\infty\\ x\in H_i}} \varphi = i \qquad (i=1 \quad \text{or } -1) .$$

And let be $S^*=\{x:\varphi(x)=0\}$. Then $\Gamma\cup S^*\in\mathcal{F}'(\Gamma)$ and the capacity of S^* is a minimum among the surfaces of $\mathcal{F}'(\Gamma)$. The conductor potential of S^* is $U(x_0)=1-\max\limits_{x\in\Pi^{-1}(x_0)}\varphi(x)$.

PROOF. Evidently S^* is a set in $H_{G'}$, but as $\sum\limits_{x\in\Pi^{-1}(x_0)}\varphi(x)\equiv 0$, the two values of φ must vanish simultaneously. Therefore S^* is a well defined set in $\boldsymbol{R}^3\sim\Gamma$, the only possible accumulation points of S^* but not on S^* being points of Γ.

On the other hand, looping once around Γ, we pass continuously from a negative value of φ to a positive one. (because $\sum\varphi(x)=0$). Hence the image by the homeomorphism Δ between Γ and a polygon Γ', of any small circle around Γ', intersects S^*. Therefore $\Gamma\subset\overline{(S^*)}$ and $S^*\cup\Gamma$ is compact. To prove *b*) of Def. 1, we notice that if there exists a bounded connected component D of $\boldsymbol{R}^3\sim S^*\cup\Gamma$, $u(x_0)=1-\max\varphi(x)$ would be harmonic on D with boundary value 1 (recall that the capacity of (Γ) is zero), but then φ

would be constant. To prove *c*) we put

$$V_2 = (S^* \sim U) \cap \{\operatorname{grad} \varphi = 0\}$$

So that $V_1 = (S^* \sim U) \sim V_2$, V_1 is then locally analytic and V_2 is locally contained in a finite number of arcs and points (for instance, we can decompose $V_2 = \bigcup_{n \geqslant 2} H_n$), where H_n is the set where the first nonvanishing derivative of one of the branches of φ is of order n (since $\sum_{x \in \Pi^{+1}(x_0)} \varphi(x) = 0$, the set is the same for both branches).

Now, φ being harmonic, if $x \in H_n$, in a neighborhood of x, H_n is a point or a differentiable arc. On the other hand, V_2 being compactly contained in the domain of harmonicity of φ, $H_n = \emptyset$ for n large enough.

Finally the orientation of the point *d*) is given by the sign $\tilde{\varphi}$, where $\tilde{\varphi}$ is one of the branches of φ in a neighborhood of the point.

To prove that S^* is the surface of minimum capacity, we will sketch the method of Evans (a detailed proof can be found in Evans [4]).

Notice first that U is a harmonic function in $R^3 \sim (S^* \cap \Gamma)$ and takes the values 1 on S^* and 0 at ∞, hence u is the conductor potential of $S^* \cup \Gamma$.

Let now $S \in \mathcal{F}'(\Gamma)$. S being a cutting surface, it divides φ in two branches φ_1 and φ_2, and if $D(u, v)$ denotes the Dirichlet integral

$$D(u, v) = \iiint (\operatorname{grad} u, \operatorname{grad} v)\, dx\,,$$

we have, formally at least,

$$D(u, u) = D(\varphi_1, \varphi_1) = D(\varphi_2, \varphi_2)$$

and, if ω is the conductor potential of S

$$D(\omega, u) = \iiint (\operatorname{grad} \omega) \operatorname{grad} u = -\iint_S \frac{\partial}{\partial \nu} u\, d\sigma =$$

(the integral being taken over both sides of the surface, the normal being inward directed)

$$= \iint_\Sigma \frac{\partial u}{\partial \nu} \cdot d\sigma = 4\Pi C(S^*)$$

(Σ is a sphere of large radius, the normal is directed inward). Hence,

$$0 \leqslant D(u - \omega, u - \omega) = D(u, u) + D(\omega, \omega) - 2D(u, \omega) = 4\Pi\big(C(S) - C(S^*)\big)\,.$$

To perform this computation, Evans removes a small torus around Γ and then lets it shrink to Γ, utilizing the behavior of u or of φ upon approach to Γ.

In order to close this section we want to make some comments on the topological character of the surface S^*.

We notice first that the surface is not necessarily an absolute minimum. Consider for instance the knot (k an integer)

$$u_\varepsilon(\theta) = \begin{cases} x_1(\theta) = \cos k\theta(1+\varepsilon\cos\theta) \\ x_2(\theta) = \sin k\theta(1+\varepsilon\cos\theta) \\ x_3(\theta) = \varepsilon\sin\theta \end{cases}$$

and the surface

$$S_\varepsilon(\theta, s) = u_\sigma'(\theta) \qquad (0 \leqslant \theta \leqslant 2\pi)\ \ 0 \leqslant s \leqslant \varepsilon\,.$$

If k is odd, any vertical line intersecting the disk $D = \{x_1^2 + x_2^2 \leqslant (1-\varepsilon)^2,$ $x_3 = 0\}$, loops u_ε an odd number of times and therefore must intersect our surface S^*. But then, $C(S^*) \geqslant C(D)$ (Capacity decreases under projection). On the other hand, when $\varepsilon \to 0$, $C(S_\varepsilon) \to 0$. If k is even, the surface S_ε belongs to our family $\mathcal{F}(S)$ and the above reasoning shows that for $k = 2$ and ε small enough, the surface of minimal capacity cannot be homeomorphic to a disc. In general, we do not know if the surface of minimum capacity is of the type of S_ε. More than that, if we project the knot on the plane $x_3 = 0$, we can construct the locally two sided surface of example b, which for $k > 2$ is of different type than S_ε, and whose capacity tends to 0 with ε.

REFERENCES

[1] Courant - Hilbert, *Methods of Mathematical Physics*, vol. II.
[2] G. C. Evans, *Surfaces of Minimum Capacity*, Proc. Nat. Acad. Sci., **26** (1940), 664-667.
[3] G. C. Evans, *Lectures on Multiple Valued Functions in Space*, Univ. Calif. Publ. Math., N.S. I (1951), 281-340.
[4] G. C. Evans, *Kellogg's Uniqueness Theorem and Applications*, Courant Anniversary Volume (1948), 95-104.
[5] R. H. Fox, *A Quick Trip through Knot Theory, Topology of 3-Manifolds*, Proc. Univ. of Georgia (1962).
[6] T. S. Hu, *Homotopy Theory*, Academic Press (1959).
[7] O. D. Kellogg, *Foundations of Potential Theory*, Berlin (1929).