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A parabolic quasi-variational inequality arising in hydraulics

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A Parabolic Quasi-Variational Inequality Arising in Hydraulics.

AVNER FRIEDMAN - ROBERT JENSEN (*)

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Introduction.

Consider a compressible fluid in a vertical underground pipe filled with porous medium. The potential $u(x, t)$ and the level of fluid in the pipe, $\varphi(t)$, satisfy a system of equations. This system is essentially equivalent to a quasi-variational inequality for the pair (w, φ) where $u = x - w_x$. The purpose of this paper is to solve the quasi-variational inequality. The main result is that there exists a unique classical solution (w, φ), and $\varphi \in C^\infty$.

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This quasi-variational inequality has the special feature that the derivative $\dot{\varphi}(t)$ of the free boundary $x = \varphi(t)$ enters into the non-homogeneous term of the differential equation; more specifically,

$$-w_{xx} + w_t = -1 - \dot{\varphi}(t) \quad \text{if } w(x, t) > 0 ,$$

$$0 \geqslant -1 - \dot{\varphi}(t) \quad \text{if } w(x, t) = 0 .$$

In Section 1 we describe in detail the physical problem. In Section 2 we introduce the quasi-variational inequality and show that solving it is equivalent to solving the original physical problem. Another useful formulation of the quasi-variational inequality is given in Section 3.

In Section 4 we prove a uniqueness theorem.

Sections 5-9 are devoted to the proof of existence of a solution. The method consists of:

(i) Introducing a finite difference approximation to the original problem. Each approximation is a system of quasi-variational inequalities in one space variable.

(ii) Proving that the system has a unique solution.

(iii) Proving the convergence of the solutions of the approximating models to a solution of the original quasi-variational inequality.

Section 5 deals with item (i). A similar finite-difference approximation was used in [7] to solve some variational inequalities.

In Section 6 we prove a general existence theorem for a certain type of quasi-variational inequality in one space variable. This is used in Section 7 to establish item (ii).

In Section 8 we derive a priori estimates for the solution of the finite-difference approximations. Finally, in Section 9 we establish item (iii).

Some additional properties of the solution are given in Section 10. In particular it is proved that if $w_{xx}(x, 0) - 1$ changes sign a finite number of times, say m, then there exist points $0 < t_1 < t_2 < \dots < t_k < T$, where $k \leqslant m$, such that $\varphi(t)$ is monotone in each interval (t_i, t_{i+1}) and in the intervals $(0, t_1)$ and (t_k, T), where $(0, T)$ is the t-interval where the solution exists. An analogous result for the Stefan problem was proved in [11], [4], [7].

In Section 11 we study the behavior of the solution as $t \to \infty$. It is proved that, as $t \to \infty$, $\big(w(x, t), \varphi(t)\big)$ converges to a stationary solution $\big(w^*(x), b^*\big)$, where $w^*(x) = (x - b^*)^2/2$.

1. – The physical problem.

Consider the following problem: Find a function $u(x, t)$ and a curve $x = \varphi(t)$, $\varphi \geqslant 0$, with $\varphi(0) = b$ given, such that

$$u_t - u_{xx} = 0 \quad \text{if } 0 < x < \varphi(t),\ 0 < t < T, \tag{1.1}$$

$$u_x(0, t) = -l(t) \quad \text{if } 0 < t < T, \tag{1.2}$$

$$u(x, 0) = g(x) \quad \text{if } 0 < x < b, \tag{1.3}$$

$$u(x, t) = x \quad \text{if } x = \varphi(t),\ 0 < t < T, \tag{1.4}$$

$$u_t = u_x^2 - u_x \quad \text{if } x = \varphi(t),\ 0 < t < T. \tag{1.5}$$

Here $l(t)$ and $g(x)$ are given functions, and b, T are given positive numbers. It is assumed throughout this paper that $l(t)$ is continuous and $g(x)$ is continuously differentiable.

This problem represents a physical model which arises when compressible fluid is moving in an underground vertical pipe, and the interior of the pipe consists of porous medium. The variable x represents the height, and $x = 0$ is the bottom of the pipe. The variable t represents the time. The porous medium is assumed to be uniform throughout the pipe. The function $u(x, t)$ represents the piezometric head and $-u_x(x, t)$ is the velocity of the fluid. Thus the condition (1.2) means that the fluid is moving through the bottom of the pipe upward $\big($if $l(t) \geqslant 0\big)$ or downward $\big($if $l(t) \leqslant 0\big)$ at the rate $|l(t)|$.

Since the piezometric head u is the sum $p + x$ where p is the inner pressure and x comes from the gravity, we should have $u(x, t) > x$ if $x < \varphi(t)$. Taking $t = 0$, we arrive at the physical conditions

$$g(x) > x \quad \text{if } 0 \leqslant x < b,\ g(b) = 0. \tag{1.6}$$

This condition will be assumed throughout this paper. We shall also always assume that

$$l(t) > -1 \quad \text{if } 0 \leqslant t \leqslant T. \tag{1.7}$$

The problem formulated above is derived $\big($for $l(t) \equiv 0\big)$ in Bear [1], for instance, in the case of several space variables.

The system (1.1)-(1.5) is a free boundary problem with $x = \varphi(t)$ as the free boundary.

2. – Reduction to quasi-variational inequality.

If (i) u_x, u_t are continuous for $0\leqslant x\leqslant \varphi(t)$, $0<t<T$; (ii) u, φ form a solution of (1.1)-(1.5), and (iii) $\varphi'(t)$ is continuous in t and $\varphi(t)>0$ for $t\in(0, T]$, then we call (u, φ) a *classical solution* of (1.1)-(1.5).

LEMMA 2.1. *Let* (u, φ) *be a classical solution of* (1.1)-(1.5). *Then*

$$u(x, t)>x \quad \textit{if } 0<x<\varphi(t),\ 0<t\leqslant T, \tag{2.1}$$

and, for all $t\in(0, T]$,

$$1-u_x(x, t)>0 \quad \textit{if } x=\varphi(t). \tag{2.2}$$

PROOF. Let

$$\Omega=\{(x, t);\ 0<x<\varphi(t),\ 0<t<T\}.$$

Consider the function

$$v(x, t)=u(x, t)-x$$

in Ω. It satisfies

$$v_t-v_{xx}=0 \quad \text{in } \Omega.$$

We claim that $v\geqslant 0$ in Ω. Indeed, otherwise v must take its negative minimum at a point $(\bar{x}, \bar{t})$ of the parabolic boundary of Ω. We cannot have $(\bar{x}, \bar{t})=(\bar{x}, 0)$, for $v(\bar{x}, 0)=g(\bar{x})-\bar{x}>0$, by (1.6). We also cannot have $\bar{x}=\varphi(\bar{t})$ for

$$v=u-x=0 \quad \text{on } x=\varphi(t), \text{ by (1.4)}.$$

Thus we must have $\bar{x}=0$, $0<\bar{t}\leqslant T$ and, consequently,

$$v_x(0, \bar{t})\geqslant 0.$$

Since, by (1.7), $v_x(0, \bar{t})=-l(\bar{t})-1<0$, we get a contradiction.

We have thus proved that $v\geqslant 0$ in Ω. Since $v_t-v_{xx}=0$ in Ω, the strong maximum principle gives $v>0$ in Ω; hence (2.1).

Now, v is a solution of the heat equation in Ω, $v>0$ in Ω, and $v=0$ on $x=\varphi(t)$. Since $\varphi\in C^1$, the inside strong sphere property is satisfied at each point $(\varphi(t), t)$. By a version of the strong maximum principle (see [3]) it follows that

$$v_x < 0 \quad \text{at } x = \varphi(t)\,,\ 0 < t \leqslant T\,,$$

i.e., (2.2) holds.

LEMMA 2.2. *Let u, φ satisfy the conditions* (i), (iii) *in the definition of a classical solution of* (1.1)-(1.5), *and let* (1.1)-(1.4) *hold. Then the condition* (1.5) *is equivalent to the condition*

$$u_x(x, t) = -\varphi'(t) \quad \text{at } x = \varphi(t),\ 0 < t \leqslant T\,. \tag{2.4}$$

PROOF. Suppose (1.5) holds. Differentiating $u(\varphi(t), t) = \varphi(t)$ we get

$$u_x\varphi' + u_t = \varphi'\,. \tag{2.5}$$

Substituting u_t from (1.5) we get

$$(1 - u_x)\varphi' = u_x(1 - u_x)\,.$$

By Lemma 2.1, $1 - u_x > 0$ for $x = \varphi(t)$, $t \in (0, T]$. Hence

$$\varphi'(t) = u_x(\varphi(t), t)\,.$$

Conversely, if (2.4) holds then by substituting φ' from (2.4) into (2.5) the relation (1.5) follows.

COROLLARY 2.3. *If (u, φ) is a classical solution of* (1.1)-(1.5) (*or* (1.1)-(1.4) *and* (2.4)) *then*

$$1 + \varphi'(t) > 0 \quad \text{for all } t \in (0, T]\,. \tag{2.6}$$

Indeed, this follows from (2.4) and (2.2).

REMARK. Notice that if the condition (1.4) is replaced by the condition $u(\varphi(t), t) = 0$ then, together with (1.1)-(1.3), (2.4), we have the Stefan free boundary problem [3].

Let u be a solution of (1.1)-(1.5) and let R be a positive number, sufficiently large, such that

$$\varphi(t) < R \quad \text{if } 0 \leqslant t \leqslant T\,.$$

Let

$$Q_R = \{(x, t);\, 0 < x < R,\, 0 < t < T\}$$

and introduce the functions

$$\tilde{u}(x, t) = \begin{cases} u(x, t) & \text{if } 0 \leqslant x < \varphi(t), \\ x & \text{if } \varphi(t) \leqslant x \leqslant R, \end{cases} \tag{2.7}$$

$$w(x, t) = \int_x^R [\tilde{u}(\zeta, t) - \zeta]\, d\zeta \quad \text{if } 0 \leqslant x \leqslant R,\ 0 \leqslant t \leqslant T, \tag{2.8}$$

$$h(x) = \begin{cases} \int_x^b [g(\zeta) - \zeta]\, d\zeta & \text{if } 0 \leqslant x < b, \\ 0 & \text{if } b \leqslant x \leqslant R. \end{cases} \tag{2.9}$$

Observe that the condition (1.6) implies that

$$h(x) > 0 \quad \text{if } 0 < x < b, \tag{2.10}$$

$$h'(x) < 0 \quad \text{if } 0 < x < b, \quad h'(b) = 0. \tag{2.11}$$

Consider the system of equations

$$w_t - w_{xx} = -\varphi'(t) \quad \text{if } 0 < x < \varphi(t),\ 0 < t \leqslant T, \tag{2.12}$$

$$w_t - w_{xx} \geqslant -1 - \varphi'(t) \quad \text{if } \varphi(t) < x < R,\ 0 < t \leqslant T, \tag{2.13}$$

$$w(R, t) = 0 \quad \text{if } 0 < t < T, \tag{2.14}$$

$$w(x, 0) = h(x) \quad \text{if } 0 < x < R, \tag{2.15}$$

$$w_t(0, t) = l(t) - \varphi'(t) \quad \text{if } 0 < t < T. \tag{2.16}$$

The last equation is, of course, equivalent to

$$w(0, t) = h(0) + b + \int_0^t l(s)\, ds - \varphi(t). \tag{2.17}$$

THEOREM 2.4. *If* (u, φ) *is a classical solution of* (1.1)-(1.5) *then* (w, φ) *is a solution of* (2.12)-(2.16).

PROOF. We calculate

$$(2.18)\qquad w_x(x,t)=\begin{cases}x-u(x,t) & \text{if } x<\varphi(t)\,,\\ 0 & \text{if } x>\varphi(t)\,,\end{cases}$$

$$(2.19)\qquad w_{xx}(x,t)=\begin{cases}1-u_x(x,t) & \text{if } x<\varphi(t)\,,\\ 0 & \text{if } x>\varphi(t)\,,\end{cases}$$

and

$$(2.20)\qquad w_t(x,t)=\begin{cases}\int\limits_x^R u_t(\zeta,t)\,d\zeta & \text{if } x<\varphi(t)\,,\\ 0 & \text{if } x>\varphi(t)\,,\end{cases}$$

since $u(\zeta,t)-\zeta=0$ if $\zeta=\varphi(t)$.

Consequently, if $x<\varphi(t)$,

$$\begin{aligned}w_t-w_{xx}&=\int\limits_x^R u_t(\zeta,t)\,d\zeta-\big(1-u_x(x,t)\big)\\&=\int\limits_x^{\varphi} u_{\zeta\zeta}(\zeta,t)\,d\zeta-\big(1-u_x(x,t)\big)\\&=u_x\big(\varphi(t),t\big)-1=-\varphi'(t)-1\end{aligned}$$

by (2.4). If $x>\varphi(t)$ then $w\equiv 0$ and

$$w_t-w_{xx}=0\geqslant -1-\varphi'(t)$$

by Corollary 2.3.

The equations (2.14), (2.15) are obvious. To prove (2.16), take $x=0$ in (2.19):

$$w_{xx}(0,t)=1-x_x(0,t)=1+l(t)\,.$$

Substituting into (2.12), (2.16) follows. This completes the proof of the theorem.

Since $-1-\varphi'(t)\leqslant 0$, we can rewrite (2.12), (2.13) in the equivalent form

$$(2.21)\qquad \int\limits_0^R[w_t(x,t)-w_{xx}(x,t)][z(x)-w(x,t)]\,dx\geqslant\int\limits_0^R[-1-\varphi'(t)][z(x)-w(x,t)]\,dx$$

for any $z\in L^2(0,R)$, $z\geqslant 0$ a.e.

The conditions (2.14)-(2.16) determine w on the parabolic boundary of Q_R Using (2.1) we see that

$$\begin{cases} w(x,t) > 0 & \text{if } 0 \leqslant x < \varphi(t)\,, \\ w(x,t) = 0 & \text{if } \varphi(t) < x < 0\,, \end{cases} \tag{2.22}$$

and $w_x(x,t) < 0$ if $0 < x < \varphi(t)$.

If $\varphi(t)$ is a known function, then the system (2.21), (2.14)-(2.16) together with the condition

$$w(x,t) \geqslant 0 \quad \text{in } Q_R \tag{2.23}$$

form a variational inequality for w and, if $w_x \leqslant 0$, then the curve $x = \psi(t)$ given by

$$\psi(t) = \inf\{x;\, w(x,t) = 0\}$$

is the free boundary. If $\psi(t)$ coincides with $\varphi(t)$ then the pair u, φ will form a solution of (2.21), (2.22) and (2.14)-(2.16). Thus, we may consider the system (2.21), (2.22) and (2.14)-(2.16) as a *quasi-variational inequality.* Quasi-variational inequalities of a similar nature have been considered in [5], [6]; however the methods of the present paper are entirely different from the methods of [5], [6]; see Sec. 3 for more details.

By a classical solution (w, φ) of the quasi-variational inequality (2.21), (2.22), (2.14)-(2.16) we mean a solution such that w_x, w_t, w_{xx} are continuous for $0 \leqslant x \leqslant \varphi(t)$, $0 < t \leqslant T$; w, w_x are continuous in $\bar{Q}_R$, $\varphi(t)$ is continuous for $0 \leqslant t \leqslant T$, $\varphi'(t)$ is continuous for $0 < t \leqslant T$, and $0 < \varphi(t) < R$ for $t \in [0, T]$.

The next theorem is a converse to Theorem 2.4.

THEOREM 2.5. *Let* (w, φ) *be a classical solution of the quasi-variational inequality* (2.21), (2.22), (2.14)-(2.16). *If*

$$u(x,t) = x - w_x(x,t)\,, \tag{2.24}$$

then (u, φ) *is a classical solution of* (1.1)-(1.5).

PROOF. The verification of (1.1), (1.3) is immediate. Since w takes the minimum 0 in Q_R at $x = \varphi(t)$, $w_x(\varphi(t), t) = 0$. Hence (2.24) yields (1.4). Next, from (2.19),

$$\begin{aligned} 1 - u_x(0,t) = w_{xx}(0,t) &= w_t(x, 0) + 1 + \varphi'(t) \\ &= \big(l(t) - \varphi'(t)\big) + 1 + \varphi'(t) = 1 + l(t)\,, \end{aligned}$$

and (1.2) follows. It remains to prove (1.5) or, equivalently (by Lemma (2.2), (2.4).

Let ζ be a C^∞ function with support in Q_R. Let $\Omega_n = \{(x, t); 0 < x < \varphi(t) - 1/n, 0 < t < T\}$. By integration by parts,

$$(2.25) \qquad 0 = \lim_{n\to\infty} \iint_{\Omega_n} (u_{xx} - u_t)\zeta\, dx\, dt = \int_0^T u_x\zeta \Big|_{x=\varphi(t)} dt + \int_0^T u\zeta\varphi' \Big|_{x=\varphi(t)} dt + I$$

where

$$I = \iint_\Omega (-u_x\zeta_x + u\zeta_t)\, dx\, dt .$$

By (2.18), (2.19) and (2.12),

$$I = \iint_\Omega [(w_{xx} - 1)\zeta_x + (x - w_x)\zeta_t]\, dx\, dt = \iint_\Omega [(w_t + \varphi')\zeta_x + (x - w_x)\zeta_t]\, dx\, dt .$$

Since

$$\iint_\Omega (w_t\zeta_x - w_x\zeta_t)\, dx\, dt = \iint_\Omega (-w\zeta_{xt} + w\zeta_{xt})\, dx\, dt = 0 ,$$

we get

$$I = \iint_\Omega (\varphi'\zeta_x + x\zeta_t)\, dx\, dt = \int_0^T \varphi'\zeta \Big|_{x=\varphi(t)} dt - \int_0^T x\zeta\varphi' \Big|_{x=\varphi(t)} dt .$$

Substituting this into (2.25), and recalling that $u = x$ on $x = \varphi(t)$, we obtain

$$\int_0^T [u_x(\varphi(t), t) + \varphi'(t)]\zeta(\varphi(t), t)\, dt = 0 .$$

Since ζ is arbitrary, (2.4) follows.

From now on we shall study only the quasi-variational inequality (2.21), (2.22), (2.14)-(2.16).

3. – Another formulation of the quasi-variational inequality.

If we set

$$(3.1) \qquad v(x, t) = w(x, t) + \varphi(t) ,$$

then the quasi-variational inequality (2.21), (2.22), (2.14)-(2.16) reduces to

$$\int_0^R [v_t(x,t) - v_{xx}(x,t)][z(x) - v(x,t)]\,dx \geqslant \int_0^R (-1)[z(x) - v(x,t)]\,dx\,, \tag{3.2}$$

$$0 < t < T, \text{ for any } z \in L^2(0,R),\ z(x) \geqslant \varphi(t)\,,$$

$$v(x,t) \geqslant \varphi(t) \quad \text{in } Q_R\,, \tag{3.3}$$

$$v(R,t) = \varphi(t) \quad \text{if } 0 < t < T\,, \tag{3.4}$$

$$v(x,0) = h(x) + b \quad \text{if } 0 < x < R\,, \tag{3.5}$$

$$v_t(0,t) = l(t) \quad \text{if } 0 < t < T\,, \tag{3.6}$$

$$v(x,t) > \varphi(t) \quad \text{if and only if } 0 < x < \varphi(t)\,. \tag{3.7}$$

Notice that (3.2) means that

$$v_t - v_{xx} = -1 \quad \text{if } v(x,t) > \varphi(t)\,, \tag{3.8}$$

and

$$1 + \varphi'(t) \geqslant 0 \tag{3.9}$$

when we take $v(x,t) \equiv \varphi(t)$ (provided of course $\varphi(t) < R$ so that $v(x,t) \equiv \varphi(t)$ holds in some x-interval).

Since $v - \varphi$ takes minimum 0 in Q: at the points where $x = \varphi(t)$, we also have

$$v = \varphi\,, \quad v_x = 0 \quad \text{if } x = \varphi(t),\ 0 < t < T\,. \tag{3.10}$$

Let

$$K(\varphi(t)) = \{z \in L^2(0,R),\ z \geqslant \varphi(t)\}\,.$$

For a given $\varphi(t)$, consider the variational inequality: Find $v(x,t)$ such that (3.2) holds for all $z \in K(\varphi(t))$, $v(x,t) \in K(\varphi(t))$, and v satisfies (3.4)-(3.6). If $\varphi(t)$ is smooth, say in C^1, then the methods of Bensoussan and Friedman [2] can be applied to establish the existence of a unique solution v. Let

$$\psi(t) = \inf\{x;\ v(x,t) = \varphi(t)\}\,.$$

We write $\psi = W\varphi$. Then, the quasi-variational inequality (3.2)-(3.7) has a solution (v, φ) if and only if $W\varphi = \varphi$, i.e., φ is a fixed point of W.

One is thus tempted to solve (3.2)-(3.7) by establishing the existence of a fixed point for W. However, the following difficulties arise:

(i) If W is to be defined for C^1 curves φ, then one must establish that $\psi = W\varphi$ is also a C^1 curve.

(ii) One must show that W satisfies conditions which ensure the existence of a fixed point. Here there is a difficulty proving that W is continuous.

(iii) In [5] [6] quasi-variational inequalities were solved using Tartar's fixed point theorem [10] which does not require the continuity of the mapping W. That theorem, however, requires that W is a monotone operator. In the present problem W does not appear to be monotone.

We shall adopt a different approach to the solution of the quasi-variational inequality. This approach is based on finite-difference approximations in the t-variable of the quasi-variational inequality. The scheme will be described in Section 5. The subsequent sections will be devoted to establishing the convergence of the scheme.

4. – Uniqueness of solutions.

DEFINITION. A pair $(w(x,t), \varphi(t))$ is said to form a *solution* of (2.21), (2.22), (2.14)-(2.16) if (i) $\varphi(t)$ is continuous for $0 \leqslant t \leqslant T$, $\varphi'(t)$ is continuous for $0 < t \leqslant T$, $0 < \varphi(t) < R$ for all $0 \leqslant t \leqslant T$; (ii) w is continuous in $\bar{Q}_R$, and w_t, w_x, w_{xx} belong to $L^2(Q_R)$, and (iii) the relations (2.21), (2.22) and (2.14)-(2.16) hold.

Similarly one defines the concept of a solution $(v(x,t), \varphi(t))$ of (3.2)-(3.7) by requiring: (i) $\varphi(t)$ be continuous for $0 \leqslant t \leqslant T$; (ii) v is continuous in $\bar{Q}_R$ and v_t, v_x, v_{xx} belong to $L^2(Q_R)$, and (iii) (3.2)-(3.7) are satisfied. Note that the derivative $\varphi'(t)$ of $\varphi(t)$ does enter into the quasi-variational inequality (3.2)-(3.7).

THEOREM 4.1. *There exists at most one solution* (w, φ) *of the quasi-variational inequality* (2.21), (2.22), (2.14)-(2.16) *such that* w_x *is continuous for* $0 < x \leqslant \varphi(t)$, $0 < t < T$.

We shall need the following lemma.

LEMMA 4.2. *If* (w, φ) *is a solution of* (2.21), (2.22), (2.14)-(2.16) *with* w_x *continuous for* $0 < x \leqslant \varphi(t)$, $0 < t < T$, *then* $w_x < 0$ *if* $0 < x < \varphi(t)$, $0 < t < T$.

PROOF. Consider the function $\zeta = v_x$ where v is defined by (3.1). Then

$$\zeta_t - \zeta_{xx} = 0 \qquad \text{in } \Omega \equiv \{(x,t); 0 < x < \varphi(t), 0 < t < T\}$$

and $\zeta = w_x = 0$ on $x = \varphi(t)$ (since $w \geqslant 0$ in Q_R, $w = 0$ on $x = \varphi(t)$). Further, $\zeta_x(0, t) = h'(x) \leqslant 0$ by (2.11). We claim that ζ cannot take a positive maximum in $\bar{\Omega}$. Indeed, otherwise, by what we have already shown, that maximum must be assumed at a point $(0, \bar{t})$, $0 < \bar{t} \leqslant T$. Consequently,

$$v_{xx}(0, \bar{t}) = \zeta_x(0, \bar{t}) < 0 .$$

Since however

$$v_{xx} = 1 + v_t = 1 + \varphi'(t) \geqslant 0 \qquad \text{at } (0, t) ,$$

we get a contradiction. We have thus proved that $\zeta \leqslant 0$ in Ω. By the strong maximum principle it then follows that $\zeta < 0$ in Ω, i.e., $w_x = v_x = \zeta < 0$ in Ω.

Proof of Theorem 4.1. Let (w, φ) and $(\hat{w}, \hat{\varphi})$ be two solutions of the quasi-variational inequality, with w_x continuous for $0 < x \leqslant \varphi(t)$ and $\hat{w}_x$ continuous for $0 < x \leqslant \hat{\varphi}(t)$. Define

$$v = \varphi + w , \qquad \hat{v} = \hat{\varphi} + \hat{w}$$

$$\Omega_0 = \{(x, t) ; 0 < x < \min\{\varphi(t), \hat{\varphi}(t)\} , \qquad 0 < t < T .$$

The function $z = \hat{v} - v$ satisfies

$$z_t - z_{xx} = 0 \qquad \text{in } \Omega_0 .$$

We claim that $z \leqslant 0$ in Ω_0. Suppose otherwise, then z attains its positive maximum in $\bar{\Omega}_0$ at some point $(\bar{x}, \bar{t})$ of the parabolic boundary of Ω_0.

Since $\hat{v} = v$ when either $t = 0$ or $x = 0$, we must have $\bar{x} = \min(\hat{\varphi}(t), \varphi(t))$. We claim:

$$\hat{\varphi}(\bar{t}) > \varphi(\bar{t}) . \tag{4.1}$$

Indeed, if $\hat{\varphi}(\bar{t}) < \varphi(\bar{t})$, then

$$z(\bar{x}, \bar{t}) = \hat{\varphi}(\bar{t}) < \varphi(\bar{t}) = v(\varphi(\bar{t}), \bar{t}) \qquad (\bar{x} = \hat{\varphi}(\bar{t})) .$$

But since, by Lemma 4.2, $v_x(x, \bar{t}) < 0$ when $0 < x < \varphi(\bar{t})$, $\hat{v}(\bar{x}, \bar{t}) < v(\hat{\varphi}(\bar{t}), \bar{t})$ i.e., $z(\bar{x}, \bar{t}) < 0$; a contradiction. If $\hat{\varphi}(\bar{t}) = \varphi(\bar{t})$ then $z(\bar{x}, \bar{t}) = 0$ and we again obtain a contradiction to $z(\bar{x}, \bar{t}) > 0$. Thus (4.1) holds.

Since z takes its maximum in $\bar{\Omega}_0$ at $(\varphi(\bar{t}), \bar{t})$, we have $z_x(\varphi(\bar{t}), \bar{t}) \geqslant 0$, i.e.,

$$z_x(\varphi(\bar{t}), \bar{t}(\geqslant v_x(\varphi(\bar{t}), \bar{t}) = 0 .$$

But this is impossible since $\hat{v}_x(x, t) < 0$ if $0 < x < \hat{\varphi}(\bar{t})$ and, by (4.1), $\varphi(\bar{t}) < \hat{\varphi}(\bar{t})$. This contradiction completes the proof that $z \leqslant 0$ in Ω_0. Similarly one shows that $z \geqslant 0$ in Ω_0. It follows that $\hat{\varphi}(t) = \varphi(t)$ and $\hat{w} = w$.

5. – The finite difference approximations.

We introduce a finite difference scheme for (2.21), (2.22), (2.14)-(2.16).

For any positive integer n, divide $(0, T)$ into n intervals of equal length T/n and let

$$\alpha = n/T\,, \qquad t_i = i/\alpha \qquad (i = 0, 1, \dots, n)\,.$$

The free boundary $x = \varphi(t)$ will be replaced by a polygonal curve with vertices (b_i, t_i) where $b_i = \varphi(t_i)$, $b_0 = b$. Writing

$$w^i(x) = w(x, t_i)\,,$$

the variational inequality (2.21) is replaced by a sequence of variational inequalities

$$(5.1) \qquad (-w^i_{xx} + \alpha w^i, z - w^i) \geqslant \big(-1 + \alpha(w^{i-1} + b_{i-1}) - \alpha b_i, z - w^i\big)$$
$$\text{for any } z \in L^2(0, R),\ z \geqslant 0 \ \ (1 \leqslant i \leqslant n)$$

where (,) denotes the scalar product in $L^2(0, R)$.

The conditions in (2.22) become

$$(5.2) \qquad w^i(x) > 0 \quad \text{if } 0 < x < b_i,\ w^i(x) = 0 \text{ if } b_i < x < R\,.$$

The conditions (2.14)-(2.16), when finite-differenced, become

$$(5.3) \qquad w^i(R) = 0\,,$$

$$(5.4) \qquad w_0(x) = h(x)\,,$$

$$(5.5) \qquad w^i(0) = w^{i-1}(0) - (b_i - b_{i-1}) + \frac{l(t_i)}{\alpha}\,.$$

Later on we shall solve the system (5.1)-(5.5) under the additional condition that

$$(5.6) \qquad w^i \in W^{2,p}(0, R) \quad \text{for any } 1 \leqslant p < \infty\,.$$

In order to solve this system, we first consider a general quasi-variational inequality in one dimension:

$$(5.7)\qquad (-\tilde{w}_{xx}+\alpha\tilde{w}, z-\tilde{w})\geqslant(-1+\alpha G-\alpha\tilde{b}, z-\tilde{w}) \quad \text{for any } z\in L^2(0,R),\ z\geqslant 0\,,$$

$$(5.8)\qquad \tilde{w}(R)=0\,,$$

$$(5.9)\qquad \tilde{w}(0)=\max\{G(0)-\tilde{b}+\gamma/\alpha, 0\}\,,$$

$$(5.10)\qquad \tilde{w}(x)>0 \quad \text{if } 0<x<\tilde{b}\,, \qquad \tilde{w}(x)=0 \quad \text{if } \tilde{b}<x<R\,,$$

$$(5.11)\qquad \tilde{w}\in W^{2,p}(0,R) \quad \text{for any } 1\leqslant p<\infty$$

where $G=G(x)$ is a given function and γ is a given number. In the special case of (5.1)-(5.6)

$$(5.12)\qquad G(x)=w^{i-1}(x)+b_{i-1}\,, \qquad \gamma=l(t_i)\,.$$

We shall assume that

$$(5.13)\qquad \begin{cases} G(0)>0\,, \\ G'(x)<0 \quad \text{if } x<G(x)\,, \\ G'(x)=0 \quad \text{if } x\geqslant G(x)\,, \end{cases}$$

$$(5.14)\qquad 1+\gamma>0\,.$$

It will be shown later on that $w_x^{i-1}(x)<0$ if $0<x<b_{i-1}$ (for $i=0$ this follows from (2.11)). Consequently, $G'(x)<0$ if $x<b_{i-1}$ and $G'(x)=0$ if $x>b_{i-1}$. Since

$$x<b_{i-1} \quad \text{implies } G(x)>b_{i-1}>x\,,$$
$$x>b_{i-1} \quad \text{implies } G(x)=b_{i-1}<x\,,$$

the last two conditions in (5.13) hold when $G(x)$ is defined as in (5.12).

The condition (5.14) is a consequence of (1.7) and the definition of γ in (5.12).

Later on we shall prove that when G, γ are defined by (5.12), $\tilde{w}(0)>0$. Hence (5.9) reduces to

$$(5.15)\qquad \tilde{w}=G(0)-\tilde{b}+\gamma/\alpha\,,$$

i.e., to (5.5).

6. – Existence of unique solution for (5.7)-(5.11).

This section is entirely devoted to proving the following theorem:

THEOREM 6.1. *There exists a unique solution* $(\tilde{w}(x), \tilde{b})$ *for the quasi-variational inequality* (5.7)-(5.11), *and* $\tilde{w}_x(x) < 0$ *if* $0 < x < \tilde{b}$.

For any $\varepsilon > 0$, let $\beta_\varepsilon(t)$ be a C^∞ function satisfying:

$$\beta'_\varepsilon(t) \geqslant 0\,, \quad \beta_\varepsilon(t) \leqslant 0\,,$$
$$\beta_\varepsilon(t) \to 0 \qquad \text{if } \varepsilon \to 0,\ t > 0$$
$$\beta_\varepsilon(t) \to -\infty \qquad \text{if } \varepsilon \to 0,\ t < 0\,,$$
$$-K \leqslant \beta_\varepsilon(0) \leqslant 0 \qquad \text{for some constant } K \text{ independent of } \varepsilon\,.$$

Given a number $c \in [0, R]$ consider the variational inequality

$$\text{(6.1)} \qquad \begin{cases} (-w_{xx} + \alpha w, z - w) \geqslant (-1 + \alpha G - \alpha c, z - w) \\ \qquad\qquad\qquad\qquad \text{for any } z \in L^2(0, R),\ z \geqslant 0\,, \\ w(x) \geqslant 0 \qquad \text{if } 0 \leqslant x \leqslant R\,, \\ w(R) = 0\,, \\ w(0) = \max\left\{G(0) - c + \dfrac{\gamma}{\alpha}, 0\right\}, \\ w \in W^{2,p}(0, R) \qquad \text{for any } 1 \leqslant p < \infty\,. \end{cases}$$

Denote by w^ε the solution of

$$\text{(6.2)} \qquad -w^\varepsilon_{xx} + \alpha w^\varepsilon + \beta_\varepsilon(w^\varepsilon) = -1 + \alpha G(x) - \alpha c \qquad (0 < x < R)\,,$$

$$\text{(6.3)} \qquad w^\varepsilon(R) = 0\,,$$

$$\text{(6.4)} \qquad \alpha w^\varepsilon(0) + \beta_\varepsilon(w^\varepsilon(0)) = \alpha\left(G(0) - c + \frac{\gamma}{\alpha}\right).$$

(By a monotonicity argument, $w^\varepsilon(0)$ can be uniquely solved from (6.4), and (6.2), (6.3) has a unique solution $w^\varepsilon(x)$ when $w^\varepsilon(0)$ is given.)

If we multiply (6.4) by $\beta_\varepsilon(w^\varepsilon(0))$ and use the conditions on β_ε, we find that

$$\text{(6.5)} \qquad H \leqslant \beta_\varepsilon(w^\varepsilon(0)) \leqslant 0$$

where H is a constant independent of ε. Hence, as $\varepsilon\downarrow 0$ through some sequence $\{\varepsilon'\}$,

$$w(0) = \lim w^{\varepsilon'}(0) \text{ exists}.$$

Observe that $w(0)\geqslant 0$ for otherwise $\beta_\varepsilon(w^\varepsilon(0)) \to -\infty$, which is impossible by (6.5). Now, if $w(0) > 0$ then from (6.4) we obtain, when $\varepsilon = \varepsilon' \to 0$,

$$w(0) = G(0) - c + \frac{\gamma}{\alpha}.$$

If $w(0) = 0$ then from (6.4) we obtain, when $\varepsilon = \varepsilon' \to 0$,

$$G(0) - c + \frac{\gamma}{\alpha} \leqslant 0\ ;$$

thus, in both cases,

$$w(0) = \max\left\{G(0) - c + \frac{\gamma}{\alpha}, 0\right\}.$$

With $w^\varepsilon(0)$ solved from (6.4), we can treat (6.2), (6.3), when $\varepsilon = \varepsilon' \to 0$, by standard methods. We thus deduce that

$$w^\varepsilon \to w \text{ weakly in } W^{2,p}(0, R) \quad (\text{for any } 1 \leqslant p < \infty) \tag{6.6}$$

as $\varepsilon = \varepsilon'' \to 0$ ($\{\varepsilon''\}$ is a subsequence of $\{\varepsilon'\}$), and w is a solution of (6.1). Since further the system (6.1) has a unique solution, (6.6) holds when $\varepsilon\downarrow 0$ (instead of just $\varepsilon = \varepsilon''\downarrow 0$).

LEMMA 6.2. $w_x \leqslant 0$.

PROOF. Suppose first that $w(0) > 0$. The function $\zeta = w_x^\varepsilon$ satisfies

$$-\zeta_{xx} + \alpha\zeta + \beta_\varepsilon'(w^\varepsilon)\zeta = \alpha G'(x) \leqslant 0 \quad \text{in } (0, R).$$

Also, since $w(0) > 0$,

$$\zeta_x(0) = w_{xx}^\varepsilon(0) = \alpha w^\varepsilon(0) + 1 - \alpha G(0) + \alpha c = 1 + \gamma > 0,$$

by (5.14), if ε is sufficiently small (so that $\beta_\varepsilon(w_\varepsilon(0)) = 0$). Next,

$$\zeta_x(R) = w_{xx}^\varepsilon(R) = 1 - \alpha G(R) + \alpha c + \beta_\varepsilon(0) < 0$$

if $\beta_\varepsilon(0)$ is chosen so that

$$\beta_\varepsilon(0) \leqslant -1 + \alpha G(R) - \alpha c .$$

We can now apply the maximum principle to conclude that $\zeta(x)$ cannot take a positive maximum in $[0, R]$ at the end-points. Hence $\zeta(x) \leqslant 0$ in $(0, R)$. Taking $\varepsilon \downarrow 0$ we get $w_x \leqslant 0$ in the interval $(0, R)$.

So far we have assumed that $w(0) > 0$. If $w(0) = 0$, then

$$-1 + \alpha G(x) - \alpha c \leqslant -1 + \alpha G(0) - \alpha c = \\ = \alpha \left(G(0) - c + \frac{\gamma}{\alpha} \right) - (1 + \gamma) \leqslant -(1 + \gamma) < 0$$

by (5.14). Consequently $w(x) \equiv 0$ is a solution of (6.1), and the assertion $w_x \leqslant 0$ follows.

Let

$$\tilde{c} = \inf\{x; w(x) = 0\} .$$

Denote by W the mapping $c \to \tilde{c}$, i.e., $\tilde{c} = Wc$. We would like to prove that W has a fixed point, by showing that W is a continuous mapping. Since there seems to be some difficulty in trying to prove that W is continuous, we modify our procedure as follows.

For any small $\mu > 0$, consider the variational inequality

$$(6.7) \quad \begin{cases} (-w_{xx} + \alpha w, z - w) \geqslant (-1 + \alpha G - \alpha c - \mu x, z - w) \\ \qquad\qquad\qquad\qquad \text{for any } z \in L^2(0, R),\ z \geqslant 0 \text{ a.e.} , \\ w(x) \geqslant 0 \qquad \text{if } 0 \leqslant x \leqslant R , \\ w(R) = 0 , \\ w(0) = \max\left\{G(0) - c + \dfrac{\gamma}{\alpha}, 0\right\} , \\ w \,\varepsilon\!\in W^{2,p}(0, R) \qquad \text{for any } 1 \leqslant p < \infty . \end{cases}$$

As in the case $\mu = 0$, for any $c \in [0, R]$, there is a unique solution w of (6.7), and $w_x \leqslant 0$.

Denote by W_μ the mapping $c \to \tilde{c}$ where

$$\tilde{c} = \inf\{x; w(x) = 0\{ .$$

We shall prove:

LEMMA 6.3. *W_μ is a continuous function.*

PROOF. If c increases then, for the corresponding solution of (6.7), $w(0)$ decreases. Since also $-1+\alpha G-\alpha c-\mu x$ decreases when c increases, a standard comparison theorem for variational inequalities shows that $w(x)$ decreases when c increases. We conclude that

$$\text{if } c_1 \leqslant c \leqslant c_2 \quad \text{then } W_\mu c_2 \leqslant W_\mu c \leqslant W_\mu c_1 . \tag{6.8}$$

Therefore, in order to prove that W_μ is continuous it suffices to show that

$$\text{if } c_n \downarrow c \quad \text{then } W_\mu c_n \uparrow W_\mu c , \tag{6.9}$$

$$\text{if } c_n \uparrow c \quad \text{then } W_\mu c_n \downarrow W_\mu c . \tag{6.10}$$

To prove (6.9), notice that

$$W_\mu c_n \leqslant W_\mu c; \quad \text{hence } \lim W_\mu c_n \leqslant W_\mu c . \tag{6.11}$$

On the other hand, the solution w_n of (6.7) corresponding to $c = c_n$ satisfies

$$w_n(W_\mu c_n) = 0 . \tag{6.12}$$

It is easy to show that, as $n \to \infty$,

$$w_n \to w \quad \text{uniformly} ,$$

where w is the solution of (6.7) corresponding to c. Taking $n \to \infty$ in (6.12) we then obtain $w(\lim W_\mu c_n) = 0$. Hence $W_\mu c \leqslant \lim W_\mu c_n$. In conjunction with (6.11), the assertion (6.9) now follows.

To prove (6.10), suppose the assertion is false. Then there exists a $\delta > 0$ such that

$$W_\mu c_n \geqslant W_\mu c + \delta \quad \text{if } n \text{ is sufficiently large} . \tag{6.13}$$

Let ζ be a smooth function with support in the interval $W_\mu c < x < W_\mu c + \delta$. From the variational inequality for w_n we get

$$\int(-w_{nxx} + \alpha w_n)\zeta\, dx = \int(-1 + \alpha G - \alpha c - \mu x)\zeta\, dx . \tag{6.14}$$

As $n \to \infty$, $w_n \to w$ weakly in $W^{2,p}(0, R)$. Hence the left-hand side of (6.14) converges to

$$\int(-w_{xx} + \alpha w)\zeta\, dx = 0 , \quad \text{since } w = 0 \text{ on supp } \zeta .$$

It follows that

$$\int(-1+\alpha G-\alpha c-\mu x)\zeta\,dx=0\,.$$

Since ζ is arbitrary,

$$-1+\alpha G(x)-\alpha c-\mu x=0 \quad \text{if } W_\mu c<x<W_\mu c+\delta\,.$$

Since $\mu>0$ and $G'(x)\leqslant 0$, the left-hand side is strictly monotone decreasing; a contradiction. This completes the proof of (6.10).

The function $c\to W_\mu c$ maps $[0,R]$ into itself and, by Lemma 6.3, it is continuous. By a very special case of Brower's fixed point theorem it follows that W_μ has a fixed point $c=c_\mu$. Denote the corresponding solution w of (6.7) by w_μ. Thus w_μ is a solution of (6.7) for $c=c_\mu$ and, in addition,

$$\text{(6.15)}\qquad \begin{cases} w(x)>0 & \text{if } 0<x<c\,,\\ w(x)=0 & \text{if } c<x<R \qquad \text{for } c=c_\mu,\ w=w_\mu\,. \end{cases}$$

LEMMA 6.4. *There is a unique solution $(w_\mu(x), c_\mu)$ of the quasi-variational problem* (6.7), (6.15).

PROOF. Existence was already proved. To prove uniqueness, suppose $(\overline{w}_\mu, \overline{c}_\mu)$ is another solution. We may take $\overline{c}_\mu>c_\mu$. Now, w_μ and $\overline{w}_\mu$ are solutions of variational inequalities (6.7) with $c=c_\mu$ and $c=\overline{c}_\mu$. Since $\overline{c}_\mu>c_\mu$, (6.8) implies that $\overline{w}_\mu(x)\leqslant w_\mu(x)$. This is impossible since

$$\overline{w}_\mu(c_\mu)>0\,,\qquad w_\mu(c_\mu)=0\,.$$

We return to the quasi-variational inequalities (6.1), (6.15). The proof of Lemma 6.4 shows that there is at most one solution of (6.1), (6.15).

Now take $\mu\downarrow 0$ through a sequence $\mu=\mu'$, so that

$$\text{(6.16)}\qquad c_\mu\to\tilde{c}\,,\quad w_\mu\to w \qquad \text{weakly in } w^{2,p}(0,R) \text{ for any } 1\leqslant p<\infty\,.$$

It is easily seen that w is a solution of the variational inequality (6.1) with $c=\tilde{c}$. Let

$$\tilde{b}=\inf\{x;\, w(x)=0\}\,.$$

If we prove that $\tilde{b}=\tilde{c}$ then $(w(x),\tilde{b})$ is a solution of the quasi-variational inequality (6.1), (6.15).

Since $w_\mu(c_\mu)=0$ and since $w_\mu \to w$ uniformly in $x\in[0,R]$ as $\mu=\mu''\downarrow 0$ (for a suitable subsequence μ'' of μ') we have $w(\tilde{c})=0$. Hence $\tilde{b}\leqslant\tilde{c}$. To prove the converse inequality, we suppose that $\tilde{b}<\tilde{c}$ and derive a contradiction. Since $\tilde{b}<\tilde{c}$, there exists a $\delta>0$ such that

$$\tilde{b}+\delta\leqslant c_\mu \quad \text{for all } \mu=\mu'' \text{ sufficiently small}.$$

Let χ be a smooth function with support in $(\tilde{b},\tilde{b}+\delta)$. From the quasi-variational inequality for (w_μ, c_μ), $\mu=\mu''$, we then have:

$$\int(-w_{\mu xx}+\alpha w_\mu)\chi\,dx=\int(-1+\alpha G(x)-\alpha c_\mu-\mu x)\chi\,dx. \tag{6.17}$$

From (6.16) we conclude that, as $\mu=\mu''\to 0$, the left-hand side of (6.17) converges to

$$\int(-w_{xx}+\alpha w)\chi\,dx=0, \quad \text{since } w=0 \text{ on supp}\,\chi.$$

Hence

$$\int_{\tilde{b}}^{\tilde{b}+\delta}(-1+\alpha G(x)-\alpha\tilde{c})\chi(x)\,dx=0.$$

Since χ is arbitrary,

$$-1+\alpha G(x)-\alpha\tilde{c}=0 \quad \text{if } \tilde{b}<x<\tilde{b}+\delta. \tag{6.18}$$

Now, since $w(x)=0$ for $\tilde{b}<x<\tilde{b}+\delta$, the variational inequality for w gives

$$-1+\alpha G(x)-\alpha\tilde{b}\leqslant 0 \quad \text{if } \tilde{b}<x<\tilde{b}+\delta.$$

Since $\tilde{c}>\tilde{b}$ we then obtain

$$-1+\alpha G(x)-\alpha\tilde{c}<0 \quad \text{if } \tilde{b}<x<\tilde{b}+\delta,$$

thus contradicting (6.18). Thus completes the proof that $\tilde{c}=\tilde{b}$.

Since (6.1), (6.15) coincide with (5.7)-(5.11), we have thus completed the proof of existence and uniqueness of a solution of (5.7)-(5.11). Lemma 6.2 shows that $w_x\leqslant 0$ where $(w,\tilde{b})$ is the solution of (5.7)-(5.11). Applying the strong maximum principle to w_x in the interval $0<x<\tilde{b}$, we conclude that $w_x<0$ in this interval. This completes the proof of Theorem 6.1.

REMARK. In the proof of Theorem 6.1 we have actually not used all the conditions in (5.13); we have just used the condition that $G'(x)\leqslant 0$.

7. – Solution of the finite difference approximations.

Consider the system of variational inequalities consisting of (5.1)-(5.4), (5.6) and

$$w^i = \max\left\{0,\ w^{i-1}(0) - (b_i - b_{i-1}) + \frac{l(t_i)}{\alpha}\right\}. \tag{7.1}$$

Applying Theorem 6.1 step-by-step we conclude that this system has a unique solution (w^1, b_1), (w^2, b_2), ..., (w^n, b_n) and

$$w^i_x(x) < 0 \quad \text{if } 0 < x < b_i. \tag{7.2}$$

Suppose $b_i > 0$ if $0 \leqslant i \leqslant j$. Then (7.1) implies (5.5) for $0 \leqslant i \leqslant j$. Adding the equations (5.5) for $0 \leqslant i \leqslant j$, we get

$$w^j(0) = w^0(0) - b_j + b_0 + \frac{1}{\alpha}[l(t_1) + \dots + l(t_j)]. \tag{7.3}$$

We shall now assume that

$$h(0) + b + \min_{0 \leqslant t \leqslant T} \int_0^T l(s)\,ds > 0 \tag{7.4}$$

and prove that

$$b_i > 0 \quad \text{if } 0 \leqslant i \leqslant n \tag{7.5}$$

provided n is sufficiently large.

Since $w_0(x) = h(x)$, (7.5) is valid for $i = 0$. Suppose (7.5) is true for $0 \leqslant i \leqslant j$. We shall prove that it is also true for $i = j + 1$. Indeed, otherwise we have $b_{j+1} = 0$, $w^{i+1}(0) = 0$, and (7.1) gives

$$w^j(0) + b_j + \frac{1}{\alpha} l(t_{j+1}) \leqslant 0.$$

Substituting $w^j(0)$ from (7.3), we then get a contradiction to (7.4), provided n is sufficiently large. This completes the proof of (7.5).

From (7.5) it follows that (7.1) is equivalent to (5.5). Furthermore, (7.3) holds for all $1 \leqslant j \leqslant n$. Hence

$$0 \leqslant w^j(0) + b_j \leqslant h(0) + b + C_0 T \equiv M \qquad \left(C_0 = \max_{0 \leqslant t \leqslant T} |l(t)|\right). \tag{7.6}$$

Using (7.2), the inequalities

$$0 \leqslant w^j(x) + b_j \leqslant M \qquad (0 \leqslant j \leqslant n) \tag{7.7}$$

follow.

We have proved the following theorem.

THEOREM 7.1. *If* (7.4) *holds then for any n sufficiently large, say $n \geqslant n_0$, there exists a unique solution* (w_0, b_0), $(w^1, b_1), \ldots, (w^n, b_n)$ *of the system of quasi-variational inequalities* (5.1)-(5.6), *and* $0 < b_j < R$ *for* $0 \leqslant j \leqslant n$, *provided* $R > M$ *where M is defined in* (7.6). *Further,* (7.2), (7.3) *and* (7.7) *hold.*

If $l(t) \geqslant 0$ then we can take $n_0 = 1$.

8. – A priori estimates.

In this section we assume that $n \geqslant n_0$ where n_0 is defined in Theorem 7.1. We shall derive estimates on the solutions (w^i, b_i), independently of n.

We shall need the following comparison lemma.

LEMMA 8.1. *Let $\hat{G}(x)$ be a function satisfying* (5.13), *let $\hat{\gamma}$ be a constant, and denote by $(\hat{w}, \hat{b})$ the corresponding solution of the quasi-variational inequality* (5.7)-(5.12) (*when G, γ are replaced by $\hat{G}$, $\hat{\gamma}$ respectively*). *If $\hat{G} \geqslant G$, $\hat{\gamma} \geqslant \gamma$ then $\hat{b} \geqslant b$.*

PROOF. Suppose $\hat{b} < b$. Denote by w^* the solution of the variational inequality (6.7) corresponding to $c = \hat{b}$. By comparison, $w^*(x) \geqslant w(x)$. Next observe that, with $\hat{b}$ fixed, the functions w^* and $\hat{w}$ are solutions of variational inequalities and, since $\hat{G} \geqslant G$, $\hat{\gamma} \geqslant \gamma$, a standard comparison theorem gives $\hat{w}(x) \geqslant w^*(x)$. Hence $\hat{w}(x) \geqslant w(x)$ and, in particular, $\hat{w}(\hat{b}) \geqslant w(\hat{b})$. Since $\hat{w}(\hat{b}) = 0$, we then have $w(\hat{b}) \leqslant 0$, $\hat{b} < b$, which is impossible.

Let

$$K = \max\left\{\max_{0 \leqslant x \leqslant b} |1 - h''(x)|, \max_{0 \leqslant t \leqslant T} |l(t)|\right\}. \tag{8.1}$$

Let us use the (standard) shorter notation:

$$-w^i_{xx} + \alpha w^i + \beta(w) \ni -1 + \alpha(w^{i-1} + b_{i-1}) - \alpha b_i \qquad (1 \leqslant i \leqslant n) \tag{8.2}$$

for the variational inequality (5.1), where $\beta(t)$ is the monotone graph $\beta(0) = (-\infty, 0)$, $\beta(t) = 0$ if $t > 0$. Notice that the function $w_0 \equiv h$ satisfies the variational inequality

$$-w^0_{xx} + \alpha w_0 + \beta(w_0) \ni -1 + \alpha(w_0 + b_0) + (1 - w^0_{xx}) - \alpha b_0 . \tag{8.3}$$

LEMMA 8.2. *For all* $1 \leqslant i \leqslant n$,

$$b_i - b_{i-1} \leqslant \frac{\alpha}{K}, \tag{8.4}$$

$$\big(w^i(x) + b_i\big) - \big(w^{i-1}(x) - b_{-i-1}\big) \leqslant \frac{K}{\alpha} \qquad \textit{if } 0 \leqslant x \leqslant R. \tag{8.5}$$

PROOF. We first prove (8.4) in case $i = 1$. We may assume that $b_1 > b_0$. Then, by (8.2) (with $i = 1$) and (8.3),

$$-(w^1 - w_0)_{xx} + \alpha(w^1 - w_0) = -1 + w^0_{xx} + \alpha(b_0 - b_1) \qquad \text{if } 0 < x < b_0. \tag{8.6}$$

We also have

$$(w^1 - w^0)(0) = b_0 - b_1 + \frac{l(t_1)}{\alpha}. \tag{8.7}$$

Suppose the right-hand side of (8.6) is < 0 for all $x \in (0, b_0)$. Then $w^1 - w_0$ cannot take a positive maximum in $0 \leqslant x \leqslant b_0$ at an interior point. If also the right-hand side of (8.7) is < 0 then, since $(w^1 - w_0)(b_0) = w^1(b_0) \geqslant 0$, the function $w^1 - w_0$ attains its strict maximum in $0 \leqslant x \leqslant b_0$ at $x = b_0$. Consequently,

$$\frac{d}{dx}[w^1(x) - w^0(x)] > 0 \qquad \text{at } x = b_0.$$

Since however $w^0_x(b_0) = 0$, $w^1_x(b_1) \leqslant 0$, this is impossible. We have thus proved that either

$$-1 + w^0_{xx}(x) + \alpha(b_0 - b_1) \geqslant 0 \qquad \text{for some } x \in (0, b_0),$$

or

$$b_0 - b_1 + \frac{l(t_1)}{\alpha} > 0.$$

This readily completes the proof of (8.4) in case $i = 1$.

To prove (8.5) for $i = 1$, suppose first that $b_1 \geqslant b_0$. Then (by (8.2) with $i = 1$ and (8.3)) the function $\zeta = (w^1 + b_1) - (w_0 + b_0)$ satisfies

$$-\zeta_{xx} + \alpha\zeta = \begin{cases} -1 + w^0_{xx} & \text{if } 0 < x < b_0, \\ -1 & \text{if } b_0 < x < b_1. \end{cases} \tag{8.8}$$

Hence

$$-\zeta_{xx}+\alpha\zeta\leqslant K \qquad \text{if } 0<x<b_1 .$$

Since also

$$\zeta(0)=w^1(0)-\big(w^0(0)+b_0\big)-b_1=\frac{l(t_1)}{\alpha}\leqslant\frac{K}{\alpha},$$
$$\zeta(b_1)=b_1-b_0\leqslant\frac{K}{\alpha},$$

we can apply the maximum principle to $\zeta-K/\alpha$ to conclude that $\zeta-K/\alpha\leqslant 0$ if $0<x<b_1$. This gives (8.5) for $i=1$, in case $b_1\geqslant b_0$.

If $b_1<b_0$ then $\zeta(x)\leqslant 0$ if $b_1<x<R$. Next,

$$-\zeta_{xx}+\alpha\zeta=-1+w^0_{xx}\leqslant K \qquad \text{in } (0,b_1),$$

and

$$\zeta(0)\leqslant\frac{K}{\alpha}, \qquad \zeta(b_1)\leqslant 0 .$$

Applying the maximum principle to $\zeta-K/\alpha$, we conclude that $\zeta-K/\alpha\leqslant 0$ in $(0,b_1)$. This completes the proof of (8.5) for $i=1$.

We now proceed by induction and assume that

$$\text{(8.9)} \qquad \big(w^i(x)+b_i)\big)-w^{i-1}(x)+b_{i-1}\big)\leqslant\frac{K}{\alpha}, \qquad b_i-b_{i-1}\leqslant\frac{K}{\alpha}.$$

We shall prove the same inequality with i replaced by $i+1$, i.e.,

$$\text{(8.10)} \qquad b_{i+1}-b_i\leqslant\frac{K}{\alpha},$$

$$\text{(8.11)} \qquad \big(w^{i+1}(x)+b_{i+1}\big)-\big(w^i(x)+b_i\big)\leqslant\frac{K}{\alpha}.$$

To prove (8.10) it suffices to consider the case where $b_{i+1}>b_i$. By (8.2),

$$\text{(8.12)} \qquad -(w^{i+1}-w^i)_{xx}+\alpha(w^{i+1}-w^i)=$$
$$=\alpha(w^i+b_i)-\alpha(w^{i-1}+b_{i-1})-\alpha b_{i+1}-\alpha b_i \qquad \text{if } 0<x<b_i,$$

$$\text{(8.13)} \qquad w^{i+1}(0)-w^i(0)=b_i-b_{i+1}+\frac{l(t_{i+1})}{\alpha}.$$

The argument following (8.6), (8.7) can now be applied to show that either the right-hand side of (8.12) is $\geqslant 0$ for at least one value of $x \in (0, b_i)$ or else the right-hand side of (8.13) is $\geqslant 0$. In either case, after using (8.9), we get (8.10).

To prove (8.11), suppose first that $b_{i+1} \geqslant b_i$ and consider the function

$$\zeta = (w^{i+1} + b_{i+1}) - (w^i + b_i)\,. \tag{8.14}$$

Using (8.2) we find that

$$-\zeta_{xx} + \alpha\zeta = \begin{cases} \alpha(w^i + b_i) - \alpha(w^{i-1} + b_{i-1}) & \text{if } 0 < x < b_i\,, \\ -1 & \text{if } b_i < x < b_{i+1}\,. \end{cases} \tag{8.15}$$

After making use of (8.9) we get

$$-\zeta_{xx} + \alpha\zeta \leqslant K \qquad \text{if } 0 < x < b_{i+1}\,.$$

We also have

$$\zeta(0) = \frac{l(t_{i+1})}{\alpha} \leqslant \frac{K}{\alpha}\,,$$

$$\zeta(b_{i+1}) = b_{i+1} - b_i \leqslant \frac{K}{\alpha}\,, \qquad \text{by (8.10)}\,.$$

Applying the maximum principle to $\zeta - K/\alpha$, we conclude that $\zeta - K/\alpha \leqslant 0$ if $0 < x < b_{i+1}$, i.e., (8.11) holds.

It remains to establish (8.11) in case $b_{i+1} < b_i$. In this case, the function ζ defined in (8.14) is $\leqslant 0$ if $b_{i+1} < x < R$. On the other hand

$$-\zeta_{xx} + \alpha\zeta = \alpha(w^i + b_i) - \alpha(w^{i-1} + b_{i-1}) \leqslant K \qquad \text{if } 0 < x < b_{i+1}\,,$$

and

$$\zeta(0) \leqslant \frac{K}{\alpha}\,, \qquad \zeta(b_{i+1}) \leqslant 0\,.$$

It follows, by the maximum principle, that $\zeta - K/\alpha \leqslant 0$ if $0 < x < b_{i+1}$. This completes the proof of (8.11).

LEMMA 8.3. *For all* $1 \leqslant i \leqslant n$,

$$b_i - b_{i-1} \geqslant -\frac{K}{\alpha}\,, \tag{8.16}$$

$$\big(w^i(x) + b_i\big) - \big(w^{i-1}(x) + b_{i-1}\big) \geqslant -\frac{K}{\alpha} \qquad \textit{if } 0 \leqslant x \leqslant R\,. \tag{8.17}$$

PROOF. To prove (8.16) for $i = 1$, it suffices to consider the case where $b_1 < b_0$. Then (cf. (8.6), (8.7))

$$-(w^1 - w_0)_{xx} + \alpha(w^1 - w_0) = -1 + w^0_{xx} - \alpha(b_1 - b_0) \quad \text{if } 0 < x < b_1, \tag{8.18}$$

$$(w^1 - w_0)(0) = \frac{l(t_1)}{\alpha} - (b_1 - b_0). \tag{8.19}$$

If the right-hand side of (8.18) is > 0 for all $x \in (0, b_1)$ and if the right-hand side of (8.19) is > 0, then the minimum of $w^1 - x_0$ in $[0, b_1]$ must occur at the boundary point b_1. Consequently, $(w^1 - w_0)_x(b_1) < 0$, which is impossible. Hence either the right-hand side of (8.19) is $\leqslant 0$ or the right-hand side of (8.18) is $\geqslant 0$ for some $x \in (0, b_1)$. In either case we obtain the inequality $(b_1 - b_0) \geqslant -K/\alpha$.

In what follows we shall need the inequality

$$-1 + \alpha\big(w^{i-1}(x) + b_{i-1}\big) - \alpha b_i \leqslant 0 \quad \text{if } b_i \leqslant x \leqslant R. \tag{8.20}$$

This inequality follows from the variational inequality (8.2) (since $w^i \equiv 0$ in (b_i, R)).

To prove (8.17) for $i = 1$ consider the function

$$\zeta = (w^1 + b_1) - (w_0 + b_0).$$

If $b_1 \leqslant b_0$ then

$$-\zeta_{xx} + \alpha\zeta = \begin{cases} -1 + w^0_{xx} & \text{if } 0 < x < b_1, \\ -1 + w^0_{xx} + [1 - \alpha w_0 - \alpha(b_0 - b_1)] & \text{if } b_1 < x < b_0. \end{cases}$$

The expression in brackets is $\geqslant 0$, by (8.20) with $i = 1$. Hence

$$-\zeta_{xx} + \alpha\zeta \geqslant -K \quad \text{if } 0 < x < b_0.$$

Next,

$$\zeta(0) = \frac{l(t_1)}{\alpha}, \qquad \zeta(b_0) = b_1 - b_2$$

and, consequently,

$$\zeta(0) \geqslant -\frac{K}{\alpha}, \qquad \zeta(b_0) \geqslant -\frac{1}{\alpha}.$$

Applying the maximum principle, we conclude that $\zeta + K/\alpha \geqslant 0$ if $0 < x < b_0$, and (8.17) for $i = 1$ follows.

We next have to consider the case where $b_1 > b_0$. Then $\zeta(x) \geqslant 0$ in (b_0, R). In $(0, b_0)$,

$$-\zeta_{xx} + \alpha\zeta = -1 + w^0_{xx} \geqslant -K .$$

Also

$$\zeta(0) \geqslant -K/\alpha , \qquad \zeta(b_0) \geqslant 0 .$$

The maximum principle yields $\zeta + K/\alpha \geqslant 0$ if $0 < x < b_0$, and (8.17) for $i = 1$ follows.

We proceed to prove (8.16), (8.17) by induction. Suppose

$$\text{(8.21)} \qquad b_i - b_{i-1} \geqslant -\frac{K}{\alpha} , \qquad (w^i + b_i) - (w^{i-1} + b_{i-1}) \geqslant -\frac{K}{\alpha} .$$

We wish to prove that

$$\text{(8.22)} \qquad b_{i+1} - b_i \geqslant -\frac{K}{\alpha} ,$$

$$\text{(8.23)} \qquad (w^{i+1} + b_{i+1}) - (w^i + b_i) \geqslant -\frac{K}{\alpha} .$$

To prove (8.22), we may assume that $b_{i+1} < b_i$. Then (8.12) holds for $0 < x < b_{i+1}$. By the argument following (8.18), (8.19) we deduce that either the right-hand side of (8.12) must be $\leqslant 0$ for some $x \in (0, b_{i+1})$, or the right-hand side of (8.13) must be >0. In either case we obtain, upon exploiting (8.21), the inequality (8.22).

To prove (8.23) suppose first $b_{i+1} < b_i$. Then the function ζ, defined by (8.14), satisfies

$$-\zeta_{xx} + \alpha\zeta = \begin{cases} \alpha(w^i + b_i) - \alpha(w^{i-1} + b_{i-1}) & \text{if } 0 < x < b_{i+1} , \\ [1 - \alpha(w^{i-1} + b_{i-1}) + \alpha b_i] + \alpha(b_{i+1} - b_i) & \text{if } b_{i+1} < x < b_i . \end{cases}$$

By (8.20),

$$-[1 - \alpha(w^i + b_i) + \alpha b_{i+1}] \leqslant 0 \qquad \text{if } b_{i+1} < x < R .$$

Hence

$$-\zeta_{xx} + \alpha\zeta \geqslant \alpha(w^i + b_i) - \alpha(w^{i-1} + b_{i-1}) \qquad \text{if } b_{i+1} < x < R .$$

We thus conclude that

$$-\zeta_{xx} + \alpha\zeta \geqslant \alpha(w^i + b_i) - \alpha(w^{i-1} + b_{i-1}) \qquad \text{if } 0 < x < b_i .$$

Using (8.21) we then have

$$-\zeta_{xx}+\alpha\zeta\geqslant-\frac{K}{\alpha} \qquad \text{if } 0<x<b_i\,,$$

$$\zeta(0)\geqslant-\frac{K}{\alpha}\,, \qquad \zeta(b_i)=b_{i+1}-b_i\geqslant-K\,.$$

By the maximum principle it follows that $\zeta+K/\alpha\geqslant 0$ in $(0,b_i)$, i.e., (8.23) holds.

It remains to prove (8.23) in case $b_{i+1}>b_i$. In this case the function ζ, defined by (8.14), satisfies

$$\zeta(x)\geqslant 0 \qquad \text{if } b_i\leqslant x\leqslant R\,.$$

Thus it remains to estimate it for $0<x<b_i$. Since ζ satisfies (8.15), we get, after using (8.21),

$$-\zeta_{xx}+\alpha\zeta\geqslant-K \qquad \text{if } 0<x<b_i\,.$$

Also,

$$\zeta(0)\geqslant-\frac{K}{\alpha}\,, \qquad \zeta(b_i)\geqslant 0\,.$$

It follows that $\zeta+K/\alpha\geqslant 0$ in $(0,b_i)$, and (8.23) follows.

Combining Lemma 8.2, 8.3 we have:

LEMMA 8.4. *For all* $1\leqslant i\leqslant n$,

$$|b_i-b_{i-1}|\leqslant\frac{K}{\alpha}\,, \tag{8.24}$$

$$\big|\big(w^i(x)+b_i\big)-\big(w^{i-1}(x)+b_i\big)\big|\leqslant\frac{K}{\alpha} \qquad \text{if } 0\leqslant x\leqslant R\,, \tag{8.25}$$

where K is the constant given by (8.1).

Noting that

$$-w^i_{xx}=\alpha(w^{i+1}+b_{i+1})-\alpha(w^i+b_i) \qquad \text{in } (0,b_i)\,,$$

and using Lemma 8.4, we get:

LEMMA 8.5. *For all* $1\leqslant i\leqslant n$,

$$|w^i_{xx}(x)|\leqslant K \qquad \textit{if } 0\leqslant x\leqslant R\,. \tag{8.26}$$

Lemmas 8.4, 8.5 will be crucial to the convergence proof of Section 9. We shall now prove some lemmas concerning the monotonic behavior of b_i and $w^i + b_i$ with respect to i, under the additional assumptions: either

$$h''(x)\leqslant 1 \quad \text{for } 0\leqslant x< b\,, \qquad l(t)\leqslant 0 \quad \text{for } 0\leqslant t\leqslant T\,, \tag{8.27}$$

or

$$h''(x)\geqslant 1 \quad \text{for } 0< x< b\,, \qquad l(t)\geqslant 0 \quad \text{for } 0\leqslant t\leqslant T\,. \tag{8.28}$$

LEMMA 8.6. *If* (8.27) *holds then, for all* $1\leqslant i\leqslant n$,

$$b_i\leqslant b_{i-1} \tag{8.29}$$

and, more generally,

$$w^i(x)+b_i\leqslant w^{i-1}(x)+b_{i-1} \quad \text{if } 0\leqslant x\leqslant R\,. \tag{8.30}$$

LEMMA 8.7. *If* (8.28) *holds then, for all* $1\leqslant i\leqslant n$,

$$b_i\geqslant b_{i-1} \tag{8.31}$$

and, more generally,

$$w^i(x)+b_i\geqslant w^{i-1}(x)+b_{i-1} \quad \text{if } 0\leqslant x\leqslant R\,. \tag{8.32}$$

PROOF OF LEMMA 8.6. The pairs (w^0, b_0) and (w^1, b_1) satisfy quasi-variational inequalities; the variational inequality for w^0 is (8.3). Since $1-w^0_{xx}\geqslant 0$ and $l(t_1)\leqslant 0$, Lemma 8.1 can be applied to deduce that $b_0\geqslant b_1$. To prove (8.30) for $i=1$ consider the function

$$\zeta=(w^1+b_1)-(w_0+b_0)\,.$$

It satisfies

$$-\zeta_{xx}+\alpha\zeta=-1+w^0_{xx}\leqslant 0 \quad \text{if } 0<x<b_1\,,$$

$$\zeta(0)=\frac{l(t_1)}{\alpha}\leqslant 0\,, \qquad \zeta(b_1)=b_1-b_0-w^0(b_1)\leqslant 0\,.$$

Hence, by the maximum principle, $\zeta\leqslant 0$ in $0\leqslant x\leqslant b_1$, and (8.30) is proven for $i=1$.

We now proceed by induction. We assume that

$$w^i + b_i \leqslant w^{i-1} + b_{i-1}, \qquad b_i \leqslant b_{i-1}. \tag{8.33}$$

Since

$$\begin{cases} w^{i+1}(0) = w^i(0) + b_i + \dfrac{l(t_{i+1})}{\alpha} - b_{i+1}, \\ w^i(0) \quad = w^{i-1}(0) + b_{i-1} + \dfrac{l(t_i)}{\alpha} - b_i, \end{cases} \tag{8.34}$$

we have

$$w^{i+1}(0) = w^{i-1}(0) + b_{i-1} + \frac{l(t_i) + l(t_{i+1})}{\alpha} - b_{i+1}. \tag{8.35}$$

Using (8.33) and (8.34), (8.35), we are in a position to apply Lemma 8.1 to (w^i, b_i) and (w^{i+1}, b_{i+1}). We then conclude that

$$b_{i+1} \leqslant b_i.$$

Consider the function

$$\zeta = (w^{i+1} + b_{i+1}) - (w^i + b_i).$$

It satisfies

$$-\zeta_{xx} + \alpha\zeta = \alpha[(w^i + b_i) - (w^{i-1} + b_{i-1})] \quad \text{in } 0 < x < b_{i+1}.$$

Also

$$\zeta(0) \quad = \frac{1}{\alpha} l(t_{i+1}) \leqslant 0.$$

$$\zeta(b_{i+1}) = b_{i+1} - b_i - w^i(b_{i+1}) \leqslant 0.$$

The maximum then implies that $\zeta \leqslant 0$ in $0 < x < b_{i+1}$. Consequently, $w^{i+1} + b_{i+1} \leqslant w^i + b_i$ if $0 \leqslant x \leqslant R$. We have thus completed the proof of Lemma 8.6.

The proof of Lemma 8.7 is similar, and is omitted.

9. – Convergence of the finite-difference scheme.

We shall now write the solution (w^i, b_i) of (5.1)-(5.6) as $(w^{n,i}, b_{n,i})$. Let

$$\sigma(t) = [\alpha t], \qquad \alpha = n/T,$$

$$\tau(t) = \alpha\left(t - \frac{\sigma(t)}{\alpha}\right).$$

Thus, if $t_{n,i-1} < t < t_{n,i}$ where $t_{n,i} = iT/n$ then

$$\sigma(t) = \alpha t_{n,i}\,, \qquad \tau(t) = \frac{t - t_{n,i-1}}{t_{n,i} - t_{n,i-1}}\,.$$

We piece together the functions $w^{n,i}$ and points $b_{n,i}$ linearly:

$$\begin{aligned} w^n(x, t) &\equiv \big(1 - \tau(t)\big)\, w^{n,\sigma(t)}(x) + \tau(t)\, w^{n,\sigma(t)+1}(x)\,, \\ \varphi^n(t) &\equiv \big(1 - \tau(t)\big)\, b_{n,\sigma(t)} + \tau(t)\, b_{n,\sigma(t)+1}\,. \end{aligned}$$

LEMMA 9.1. *For any subsequence* $\{n'\}$ *of* $\{n\}$ *there exists a subsequence* $\{n''\} \subset \{n'\}$ *such that*

(i) *if* $n = n'' \to \infty$ *then*

$w^n \to w$, $w^n_x \to w_x$ *uniformly for* $0 \leqslant x \leqslant R$, $0 \leqslant t \leqslant T$,

$\varphi^n \to \varphi$ *uniformly for* $0 \leqslant t \leqslant T$,

$w^n_{xx} \to w_{xx}$, $w^n_t \to w_t$ *weakly in* $L^p(0, R) \times (0, T)\big)$ *for any* $1 \leqslant p < \infty$;

(ii) $w \geqslant 0$, $w_x \leqslant 0$, $w_t \in L^\infty\big((0, R) \times (0, T)\big)$,

$w \in L^\infty\big((0, T); W^{2,\infty}(0, R)\big)$,

$\varphi(t)$ *is uniformly Lipschitz continuous in* $[0, T]$ *and* $-1 \leqslant \varphi'(t) \leqslant K$ *a.e.*;

(iii) w *satisfies the variational inequality*

$\big(- w_{xx}(\cdot, t) + w_t(\cdot, t), z - w(\cdot, t)\big) \geqslant \big(-1 - \varphi'(t), z - w(\cdot, t)\big)$

for any $z \in L^2(0, R)$, $z \geqslant 0$, *for a.e.* $t \in (0, T)$;

(iv) *for any* $t \in (0, T)$,

$w(x, t) > 0$ *if and only if* $x < \varphi(t)$.

PROOF. From Lemmas 8.4, 8.5, we get

$$\left|\frac{d\varphi^n}{dt}\right| \leqslant K\,, \tag{9.1}$$

$$\left|\frac{\partial}{\partial t} w^n\right| \leqslant K\,, \tag{9.2}$$

$$\left|\frac{\partial^2}{\partial x^2} w^n\right| \leqslant K\,. \tag{9.3}$$

It is shown in [7] that (9.2), (9.3) imply that

$$|w_x^n(x, t) - w_x^n(\bar{x}, \bar{t})| \leqslant C\big(|x - \bar{x}|^{\frac{1}{2}} + |t - \bar{t}|^{\frac{1}{4}}\big) \tag{9.4}$$

where C is a constant independent of n. The assertions (i), (ii) now follow from the estimates (9.1)-(9.4).

To prove (iii), let t be any number not of the form i/α and set $j = \sigma(t)$. Thus

$$\frac{j}{\alpha} < t < \frac{j+1}{\alpha}, \qquad \sigma(t) = j \ \ (j = j(n)) . \tag{9.5}$$

For any $z \in L^2(0, R)$, $z \geqslant 0$, consider the expression

$$I_n = \int_0^R [- w_{xx}^n(\xi, t) + w_t^n(\xi, t)][z(\xi) - w^n(\xi, t)]\, d\xi$$

for $n = n''$. We can write

$$\begin{aligned} I_n = &\int_0^R (- w_{xx}^{n,j+1} + w_t^n)(\xi, t)[z(\xi) - w^n(\xi, t)]\, d\xi \\ &+ \int_0^R \big(1 - \tau(t)\big)(- w_{xx}^{n,j} + w_{xx}^{n,j+1})(\xi, t)[z(\xi) - w^n(\xi, t)]\, d\xi \equiv I_n^1 + I_n^2 . \end{aligned} \tag{9.6}$$

Since

$$w^{n,j}(x) - w(x, j/\alpha) \to 0 , \qquad w^{n,j+1}(x) - w\big(x, (j+1)/\alpha\big) \to 0$$

uniformly in $x \in (0, R)$ as $n = n'' \to \infty$, and w is continuous in $(x, t) \in \in [0, R] \times [0, T]$,

$$w^{n,j}(x) \to w(x, t) , \qquad w^{n,j+1}(x) \to w(x, t) \quad \text{uniformly in } x \in (0, R), \text{ as } n = n'' \to 0 . \tag{9.7}$$

Notice next that since

$$\operatorname*{ess\,sup}_{0<t<T} |w(\cdot . t)|_{W^{2,p}(0,R)} \leqslant C \qquad (C \text{ constant})$$

and, since $w(x, t)$ is continuous in $(x, t) \in [0, R] \times [0, T]$,

$$|w(\cdot, t)|_{W^{2,p}(0,R)} \leqslant C \qquad \text{for all } t \in [0, T] . \tag{9.8}$$

From (9.7), (9.8) we conclude that if $n = n'' \to \infty$,

$$w^{n,j+1} - w^{n,j} \to 0 \quad \text{weakly in } W^{2,p}(0, R). \tag{9.9}$$

Using this and the fact that, as $n = n'' \to \infty$,

$$w^n \to w \quad \text{uniformly in } [0, R] \times [0, T], \tag{9.10}$$

we easily deduce that

$$I_n^2 \to 0 \quad \text{if } n = n'' \to \infty. \tag{9.11}$$

Next,

$$\begin{aligned} I_n^1 &= \int_0^R (-w_{xx}^{n,j+1} + w_t^n)(\xi, t)[z(\xi) - w^{n,j+1}(\xi, t)]\, d\xi \\ &\quad + \int_0^R (1 - \tau(t))(-w_{xx}^{n,j+1} + w_t^n)(\xi, t)(w^{n,j+1} - w^{n,j})(\xi, t)\, d\xi \\ &\equiv J_n^1 + J_n^2. \end{aligned} \tag{9.12}$$

Using (9.2), (9.3) and (9.4), we find that

$$J_n^2 \to 0 \quad \text{if } n = n'' \to \infty. \tag{9.13}$$

From the variational inequality satisfied by $w^{n,j+1}$ we have

$$\begin{aligned} J_n^1 &\geq \int_0^R \big(-1 - \dot\varphi^n(t)\big)\big(z(\xi) - w^{n,j+1}(\xi, t)\big)\, d\xi \\ &= \int_0^R \big(-1 - \dot\varphi^n(t)\big)\big(z(\xi) - w^n(\xi, t)\big)\, d\xi \\ &\quad + \int_0^R \big(1 - \tau(t)\big)\big(-1 - \dot\varphi^n(t)\big)(w^{n,j+1} - w^{n,j})(\xi, t)\, d\xi \\ &= H_n^1 + H_n^2. \end{aligned} \tag{9.14}$$

Using (9.7), (9.1) we find that

$$H_n^2 \to 0 \quad \text{if } n = n'' \to \infty. \tag{9.15}$$

Since $\dot{\varphi}^n \to \varphi$ weakly in $L^p(0, T)$ we also get, as $n = n'' \to \infty$,

$$(9.16) \qquad H_n^1 \to \int_0^R \big(-1 - \dot{\varphi}(t)\big)\big(z(\xi) - w(\xi, t)\big)\, d\xi \,.$$

Combining (9.6) with (9.11)-(9.16), the assertion (iii) follows.

We shall now prove (iv). Taking $n = n'' \to \infty$ in the relation $w^n\big(\varphi^n(t), t\big) = 0$ we get $w\big(\varphi(t), t\big) = 0$. Since $w_x(x, t) \leqslant 0$, we see that if $w(x, t) > 0$ then $x < \varphi(t)$. We shall now prove the converse, namely, $w > 0$ in the region

$$D = D = \big\{(x, t);\, 0 \leqslant x < \varphi(t),\, 0 < t \leqslant T\big\}\,.$$

Let G be an open set with $\bar{G} \subset D$. For t fixed, let z be any function in $L^2(0, R)$ such that the support of $z(x) - w^n(x, t)$ is in

$$G_t = \{x;\, (x, t) \in G\}.$$

If $n = n''$ is sufficiently large then $w^n(x, t) > 0$ if $x \in G_t$ and therefore (9.14) holds with « $=$ » instead of « $\geqslant$ ». Taking $n = n'' \to \infty$ we obtain the equality

$$\int_0^R (-w_{xx} + w_t)(\xi, t)\big(z(\xi) - w(\xi, t)\big)\, d\xi = \int_0^R \big(-1 - \dot{\varphi}(t)\big)\big(z(\xi) - w(\xi, t)\big)\, d\xi \,.$$

Since z is arbitrary, this implies that

$$(9.17) \qquad -w_{xx} + w_t = -1 - \varphi'(t) \quad \text{in } D\,.$$

Recall that the $w^{n,j}(x)$ vanish in some interval $R - \eta \leqslant x \leqslant R$, where η is positive and independent of i, n. Hence $w(x, t) = 0$ if $R - \eta \leqslant x \leqslant R$. Taking $z(x)$ with support in $(R - \eta, R)$ in (iii) we find that

$$(9.18) \qquad 1 + \varphi'(t) \geqslant 0 \quad \text{for a.e. } t \in (0, T)\,.$$

Hence

$$-w_{xx} + w_t \leqslant 0 \quad \text{in } D\,.$$

Since $w \geqslant 0$ and $w \not\equiv 0$ on the parabolic boundary of D, the strong maximum principle gives: $w > 0$ in D. This completes the proof of the lemma.

LEMMA 9.2. *If w, φ are as in Lemma 9.1, then $\varphi(t) > 0$ and $\varphi(t)$ is continuously differentiable in $(0, T]$, and w_{xx}, w_t are continuous for $0 < x \leqslant \varphi(t)$, $0 < t \leqslant T$.*

PROOF. From (7.3) we obtain

$$w(0, t) + \varphi(t) = h(0) + b + \int_0^t l(s)\, ds .$$

In view of (7.4), the right-hand side is positive for all $0 \leqslant t \leqslant T$. Consequently $\varphi(t) > 0$ for $0 \leqslant t \leqslant T$.

Introduce the function

$$v = w + \varphi .$$

Then $v_t - v_{xx} = -1$ if $0 < x < \varphi(t)$, $0 < t \leqslant T$. Consequently

$$-(v_x)_{xx} + (v_x)_t = 0 \qquad \text{if } 0 < x < \varphi(t),\ 0 < t \leqslant T .$$

Since w_x is continuous for $0 < x \leqslant \varphi(t)$, $0 < t \leqslant T$, the same is true of v_x. Finally, $\varphi(t)$ is Lipschitz continuous and

$$v_x = 0 \qquad \text{on } x = \varphi(t) . \tag{9.19}$$

By a well known result for the heat equation (see [3], [4], [11]) $(v_x)_x$ is then continuous up to the boundary $x = \varphi(t)$. Consequently also

$$v_t \text{ is continuous} \qquad \text{if } 0 < x \leqslant \varphi(t),\ 0 < t \leqslant T . \tag{9.20}$$

Notice that

$$|\varphi(t) - \varphi(s)| \leqslant K|t - s| . \tag{9.21}$$

We can now write, for a positive number h with $t + h \leqslant T$,

$$\begin{aligned} \varphi(t+h) - \varphi(t) &= v\big(\varphi(t+h), t+h\big) - v(\varphi(t), t) \\ &= \big[v\big(\varphi(t+h), t+h\big) - v\big(\varphi(t+h) - \delta, t+h\big)\big] \\ &\quad + \big[v\big(\varphi(t+h) - \delta, t+h\big) - v\big(\varphi(t+h) - \delta, t\big)\big] \\ &\quad + \big[v\big(\varphi(t+h) - \delta, t\big) - v(\varphi(t), t)\big] \\ &\equiv J_1 + J_2 + J_3 . \end{aligned} \tag{9.22}$$

In view of (9.21), if $\delta = Lh$ where $L > K$, then the closed line segment connecting $(\varphi(t+h)-\delta, t+h))$ to $(\varphi(t+h)-\delta, t)$ lies in the set $0 < x < \varphi(t)$, $0 < t \leqslant T$. Applying the mean value theorem, we find, upon using (9.19), that

$$\frac{J_1}{h} \to 0 , \qquad \frac{J_3}{h} \to 0 .$$

Using (9.20) we also have

$$\frac{J_2}{h} = v_t(\varphi(t+h)-\delta, t+\theta h) \to v_t(\varphi(t), t) \qquad (0 < \theta < 1)$$

as $h \to 0$. Hence

$$\lim_{h \downarrow 0} \frac{\varphi(t+h)-\varphi(t)}{h} = v_t(\varphi(t), t) .$$

The same assertion can be proved when $h \uparrow 0$. We conclude that

$$\varphi'(t) = v_t(\varphi(t), t) . \tag{9.23}$$

Since the right-hand side is a continuous function, the same is true of $\varphi'(t)$. This completes the proof of the lemma.

COROLLARY 9.3. *The pair* (w, φ) *in Lemma* 9.1 *is a classical solution of the quasi-variational inequality* (2.21), (2.22), (2.14)-(2.16).

This follows from Lemma 9.2 and from the definition of a classical solution.

COROLLARY 9.4. *The assertion of Lemma* 8.1 *holds for the full sequence* $\{n\}$ *(not just for* $\{n''\}$*).*

Indeed this follows from Corollary 9.3 and the uniqueness of the classical solution (Theorem 4.1).

We now state the main result of this paper.

THEOREM 9.5. *Let* (1.6), (1.7), (7.4) *hold. Then there exists a unique classical solution* (w, φ) *of the quasi-variational inequality* (2.21), (2.22), (2.14)-(2.16). *Furthermore,* $\varphi(t) \in C^\infty(0, T]$ *and* w *is infinitely differentiable for* $0 < x \leqslant \varphi(t)$, $0 < t \leqslant T$.

PROOF. Existence and uniqueness follow from Corollary 9.3 and Theorem 4.1. To prove that $\varphi \in C^\infty$ we shall use a method due to Schaeffer [9]. We make the transformation

$$y = x - \varphi(t) , \qquad \hat{u}(y, t) = u(x, t) - \varphi(t)$$

in a neighborhood of the free boundary where $u = x - w_x$. The heat equation for u becomes

$$-\hat{u}_{yy} + \hat{u}_t - \hat{u}_y\dot{\varphi} = -\dot{\varphi} \qquad \big(-\varphi(t) < y < 0,\, 0 < t < T\big)\,. \tag{9.24}$$

Now, a careful look at the proof that $(v_x)_x$ is continuous up to the boundary $x = \varphi(t)$ shows that $(v_x)_x$ is actually Hölder continuous. The same is therefore true of u_x. Next, by Lemma 2.5,

$$u_x\big(\varphi(t), t\big) = -\dot{\varphi}(t)\,. \tag{9.25}$$

Hence $\dot{\varphi}(t)$ is Hölder continuous, say $\dot{\varphi} \in C_\alpha$ for some $0 < \alpha < 1$. The coefficients in (9.24) are therefore in C_α.

Using the Schauder estimates [8] for $\hat{u}$ in some region $-\eta < y < 0$, $0 < t \leqslant T$, we deduce that $\hat{u}_y$ is in $C_{\alpha+\frac{1}{2}}$, i.e., u_x is in $C_{\alpha+\frac{1}{2}}$. But then, by (9.25), also $\dot{\varphi}$ is in $C_{\alpha+\frac{1}{2}}$. Notice that (9.25) implies

$$\hat{u}_y(0, t) = -\dot{\varphi}(t)\,. \tag{9.26}$$

Working now with the relations (9.24), (9.26), we can establish step-by-step that $\dot{\varphi} \in C_{\alpha+m/2}$, for $m = 1, 2, \ldots$. This implies that $\varphi \in C^\infty$ and then also that w is in C^∞ for $0 < x \leqslant \varphi(t)$, $0 < t \leqslant T$.

COROLLARY 9.6. *Under the assumptions of Theorem* 9.5, *the solution* (w, φ) *of* (2.21), (2.22), (2.14)-(2.16) *satisfies the inequalities*:

$$|\dot{\varphi}(t)| \leqslant K\,, \tag{9.27}$$

$$|w_t(x, t) + \dot{\varphi}(t)| \leqslant K \tag{9.28}$$

for $0 < t \leqslant T$, $0 \leqslant x \leqslant R$ $\big(x \neq \varphi(t)\big)$, *where* K *is the constant given by* (8.1).

Indeed, this follows immediately from (8.24), (8.25) and the assertion (i) of Lemma 9.1.

10. – Further properties of the solution.

THEOREM 10.1. (i) *Let the conditions of Theorem* 9.5 *hold, and let* (8.27) *hold. Then* $\varphi(t)$ *and each of the polygonal approximations* $\varphi^n(t)$ (*for* $n \geqslant n_0$) *is monotone decreasing in* t, *and* (8.30) *holds.*

(ii) *Let the conditions of Theorem* 9.5 *hold, and let* (8.28) *hold. Then* $\varphi(t)$ *and each of the polygonal approximations* $\varphi^n(t)$ (*for* $n \geqslant n_0$) *is monotone increasing in* t, *and* (8.32) *holds.*

This follows from Lemmas 8.6, 8.7.

THEOREM 10.2. *Let the conditions of Theorem* 9.5 *hold. If* $l(t)\not\equiv 0$ *in any interval, then* $\varphi(t)\not\equiv$ const. *in any interval.*

Thus, in particular, under the additional assumption (8.27) or (8.28), $\varphi(t)$ is strictly monotone.

PROOF. Suppose $\varphi(t)\equiv$ const. $= x_0$ for $t_0 < t < t_1$: Consider the function

$$\hat{w} = w - \tfrac{1}{2}(x - x_0)^2 .$$

It satisfies

$$\begin{aligned} &\hat{w}_t - \hat{w}_{xx} = 0 && \text{if } 0 < x < x_0,\ t_0 < t < t_1 , \\ &\hat{w}(x_0, t) = \hat{w}_x(x_0, t) = 0 && \text{if } t_0 < t < t_1 . \end{aligned}$$

By the Cauchy-Kowalewski theorem it follows that $\hat{w}(x, t)\equiv 0$ if $0 < x < x_0$, $0 < t < t_0$. In particular,

$$0 = \hat{w}_t(0, t) = l(t) - \varphi'(t) = l(t) \qquad (t_0 < t < t_1) ,$$

thus contradicting our assumption that $l(t)\not\equiv 0$ in any interval.

THEOREM 10.3. *Let the conditions of Theorem* 9.5 *hold and suppose that* $l(t)\equiv 0$. *If* $\varphi(t)\equiv$ const. $= x_0$ *in some interval* $t_0 < t < t_1$ *then* $\varphi(t)\equiv x_0$ *for all* $t_0 < t < T$.

PROOF. The argument given in the proof of Theorem 10.2 shows that $w(x, t) = (x - x_0)^2/2$ for $t_0 < t < t_1$. If we define

$$w(x, t) = \tfrac{1}{2}(x - x_0)^2 , \qquad \hat{\varphi}(t)\equiv x_0 \text{ for } t_0 < t < T$$

then $(\hat{w}, \hat{\varphi})$ is a classical solution of the same quasi-variational inequality as (w, φ), for $t > t_0$. By uniqueness, the two solutions must coincide. Hence $\varphi(t)\equiv x_0$ for $t_0 < t < T$.

REMARK. From Theorems 10.1, 10.3 we see that in case (8.27) or (8.28) holds either $\varphi(t)$ is strictly monotone for all $t\in(0, T)$ or there exist a $t_0\in(0, T)$ such that $\varphi(t)$ is strictly monotone in $(0, t_0)$ and constant in (t_0, T).

We shall now assume that $h''(x) - 1$ changes sign a finite number of times; more precisely,

$$(10.1)\qquad \begin{cases} 0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < \alpha_{m+1} = b , \\ h''(\alpha_i) - 1 = 0 , \quad h''(x) - 1 \neq 0 \text{ if } 0 < x < R,\ x \neq \alpha_i\ (1 \leqslant i \leqslant m) , \\ \operatorname{sgn}(h''(x) - 1) \neq \operatorname{sgn}(h''(y) - 1) \quad \text{if } \alpha_{i-1} < x < \alpha_i,\ \alpha_i < y < \alpha_{i+1} . \end{cases}$$

THEOREM 10.4. *Assume that* $l(t) \equiv 0$ *and that* (1.6), (10.1) *hold. Then* (i) $\varphi(t)$ *is piecewise monotone and the direction of monotonicity changes at most* m *times*; (ii) *for any* $n \geqslant 1$, *the polygonal approximation* $\varphi^n(t)$ *is piecewise monotone, and the direction of monotonicity changes at most* m *times.*

Notice that (i) is actually a consequency of (ii).

PROOF. A proof of (i) can be given by exploiting Theorem 10.1. We introduce the curves Γ_i passing through $(\alpha_i, 0)$ and along which $v_t(x, t) = 0$; the continuity of these curves was studied in [4]. Let us assume for definiteness that $h''(x) - 1 > 0$ if $\alpha_m < x < b$. Let δ be a small positive number to be determined later. We shall prove that $\varphi(t)$ is increasing for $0 \leqslant t \leqslant \delta$.

Denote by t^* the value of t at the point where Γ_m first intersects the free boundary ($t^* = T$ if no such point exists). We take $\delta < t^*$ such that $v_t(x_0, t) > 0$ for some $x_0 > \alpha_m$ and $0 \leqslant t \leqslant \delta$. Denote by γ the vertical segment $x = x_0$, $0 \leqslant t \leqslant \delta$. Then v, φ satisfy a variational inequality in the rectangle $x_0 < x < R$, $0 < t < \delta$. Since $v_t(x_0, t) > 0$ on γ, Theorem 10.1 can be applied to deduce that $\varphi(t)$ is increasing for $0 < t < \delta$.

The above argument can be repeated step-by-step, as long as $t < t^*$. We thus find that $\varphi(t)$ is increasing for $0 \leqslant t \leqslant t^*$.

In the region bounded by Γ_m, Γ_{m-1} and $t = 0$ the maximum principle can be applied to v_t to deduce that $v_t < 0$. Hence $v_t(x, t^*) < 0$ for $x_1 < x < \varphi(t^*)$ where $(x_1, t^*) \in \Gamma_{m-1}$. We can now proceed as before and show that $\varphi(t)$ is decreasing for $t^* < t < t^{**}$, where t^{**} is the value of t where Γ_{m-1} intersects the free boundary. Proceeding in this way step-by-step, the proof of (i) is completed.

Since some of the Γ_i may coincide after some time, the total number of changes in the direction of monotonicity is actually $\leqslant m$.

We shall now give a proof of (ii) based on the methods of [7]. Let

$$u^i = (w^i + b_i) - (w^{i-1} + b_{i-1}) \quad \text{for } 1 \leqslant i \leqslant n,\ u_0 = -1 + h'',$$

$$J_i = \text{number of sign changes of } u^i,\ 0 \leqslant i \leqslant n,$$

$$\hat{b}_i = \min\{b_i, b_{i-t}\}.$$

It is obvious that the number of sign changes of u^i in $(0, \hat{b}_i)$ is also J_i. Consider the four cases:

(1) $b_i > b_{i-1}$ and u^{i-1} ends up positive in $(0, \hat{b}_i)$;

(2) $b_i < b_{i-1}$ and u^{i-1} ends up negative in $(0, \hat{b}_i)$;

(3) $b_i > b_{i-1}$ and u^{i-1} ends up negative in $(0, \hat{b}_i)$;

(4) $b_i < b_{i-1}$ and u^{i-1} ends up positive in $(0, \hat{b}_i)$.

The function u^i satisfies

$$(10.2)\qquad -u^i_{xx}+\alpha u^i=\alpha u^{i-1}\quad \text{in } (0,\hat{b}_i)\,.$$

We shall prove that if (1) holds then

$$(10.3)\qquad J_i\leqslant J_{i-1}\,.$$

Since $u^i(\hat{b}_i)>0$ and $u^i_x(\hat{b}_i)<0$, we can (using (10.2)) apply Lemma 15.2 of [7] to conclude that

$$(10.4)\qquad J_i\leqslant J_{i-1}+1\,.$$

We now consider two cases:

(A) u^i starts out positive;

(B) u^i starts out negative.

Suppose (A) holds. If (10.3) is false then, in view of (10.4),

$$J_i=J_{i-1}+1\,.$$

But then u^{-1} must start out negative. Let $(0,a)$ be the largest interval such that $u^i\geqslant 0$ in $(0,a)$, and let $(0,a')$ be the largest interval such that $u^{i-1}\leqslant 0$ in $(0,a')$. Applying the maximum principle to u^i in $(0,a')$ we get

$$a'\leqslant a\qquad\text{and}\qquad u^i(a')>0\,.$$

Now let

$$K_i=\text{number of sign changes of } u^i \text{ in } (a,\hat{b}_i)\,.$$

Since $u^i>0$ in $(0,a']$,

$$K_i=J_i\,.$$

On the other hand, by the same Lemma 15.2 of [7] applied to $u^i(x)$ for $a'<x<\hat{b}_i$ we get

$$K_i\leqslant J_{i-1}=J_i-1\,,$$

a contradiction.

Similarly one can prove (10.3) in case (B). We have thus completed the proof of (10.3) in case (1) holds. Similarly one can show that if (2) holds then (10.3) holds and, furthermore,

$$(10.5)\qquad J_i\leqslant J_{i-1}-1,\ \text{if } u^i \text{ and } u^{i-1} \text{ end up with opposite signs}\,.$$

The same type of arguments also shows that if (3) or (4) holds then

$$J_i \leqslant J_{i-1} - 1 .$$

Combining all the assertions for cases (1)-(4) it follows that

$$\text{if } \operatorname{sgn}(b_i - b_{i-1}) \neq \operatorname{sgn}(b_{i-1} - b_{i-2}) \text{ then } J_i \leqslant J_{i-1} - 1 .$$

This clearly implies the assertion (ii) of Theorem 10.4.

We conclude this section with some comparison results.

THEOREM 10.5. *Let* (w, φ) *and* $(\hat{w}, \hat{\varphi})$ *be the solutions of the quasi-variational inequalities* (2.21), (2.22), (2.14)-(2.16) *corresponding to* l, h *and to* $\hat{l}$, $\hat{h}$ *respectively. If* $\hat{l} \leqslant l$, $\hat{h} \leqslant h$ *then* $\hat{v} \leqslant v$, $\hat{\varphi} \leqslant \varphi$ *where* $v = \varphi + w$, $\hat{v} = \hat{\varphi} + \hat{w}$.

The proof is similar to the proof of Theorem 4.1.

COROLLARY 10.6. (i) *If, for some* $b \leqslant \beta \leqslant R$,

$$h(x) \leqslant \tfrac{1}{2}(x - \beta)^2 \text{ for } 0 \leqslant x \leqslant b , \qquad l(t) \leqslant 0 \text{ for } 0 \leqslant t \leqslant T ,$$

then

$$w(x, t) \leqslant \tfrac{1}{2}(x - \beta)^2 , \qquad \varphi(t) \leqslant \beta \;\; (0 \leqslant x \leqslant R, 0 \leqslant t \leqslant T) .$$

(ii) *If, for some* $0 < \gamma \leqslant b$,

$$h(x) \geqslant \tfrac{1}{2}(x - \gamma)^2 \quad \text{for } 0 \leqslant x \leqslant \gamma , \qquad l(t) \geqslant 0 \quad \text{for } 0 \leqslant t \leqslant T ,$$

then

$$w(x, t) \geqslant \tfrac{1}{2}(x - \gamma)^2 , \qquad \varphi(t) \geqslant \gamma \;\; (0 \leqslant x \leqslant R, 0 \leqslant t \leqslant T) .$$

Since $w \equiv \frac{1}{2}(x - c)^2$, $\varphi \equiv c$ is a solution of the quasi-variational inequality with $l \equiv 0$, $h(x) = \frac{1}{2}(x - c)^2$, Corollary 10.6 follows immediately from Theorem 10.5.

11. – Asymptotic behavior of the solution.

We shall consider the asymptotic behavior of $\big(w(x, t), \varphi(t)\big)$ as $t \to \infty$: It is assumed that $l(t)$ is continuous for all $t \geqslant 0$, that (1.7) holds for all $t \geqslant 0$, and that

$$\beta \leqslant h(0) + b + \int_0^t l(s)\, ds \leqslant B \quad \text{if } 0 \leqslant t < \infty \;\; (B, \beta \text{ positive constants}), \tag{11.1}$$

$$\sup_{0 \leqslant t < \infty} |l(t)| < \infty . \tag{11.2}$$

Then there exists a unique classical solution (w, φ) of the quasi-variational inequality (2.21), (2.22), (2.14)-(2.16) for $T = \infty$, and, from (7.3),

$$w(0, t) + \varphi(t) = h(0) + b + \int_0^t l(s)\, ds . \tag{11.3}$$

By Taylor's formula and the fact that $w = w_x = 0$ at $x = \varphi(t)$,

$$w(0, t) = \tfrac{1}{2} w_{xx}(\theta_0 \varphi(t), t)(\varphi(t))^2 \qquad (0 < \theta_0 < 1) .$$

Substituting this into (11.3) and recalling that $|w_{xx}| \leqslant K$, we get

$$\varphi(t)(1 + K\varphi(t)) \geqslant h(0) + b + \int_0^t l(s)\, ds .$$

Using (1.11) we then find that

$$\varphi(t) > \min\left(\frac{1}{K}, \frac{\beta}{2}\right) \qquad (0 \leqslant t < \infty) . \tag{11.4}$$

From (11.4) and the second inequality of (11.1) we also get

$$\varphi(t) \leqslant B \qquad (0 \leqslant t < \infty) . \tag{11.5}$$

Thus:

THEOREM 11.1. *If* (11.1), (11.2) *hold, then* $\varphi(t)$ *satisfies* (11.4), (11.5). Set

$$\Omega_s = \{(x, t);\, 0 < x < \varphi(t),\, 0 < t < s\} .$$

We shall need the conditions:

$$\sup_{0<t<\infty} |\dot{l}(t)| < \infty , \qquad \int_0^\infty |l(t)|\, dt < \infty . \tag{11.6}$$

LEMMA 11.2. *If* (11.1), (11.2), (11.6) *hold, then*

$$\iint_{\Omega_\infty} v_{xt}^2(x, t)\, dx\, dt + \sup_{0<t<\infty} \int_0^{\varphi(t)} v_x^2(x, t)\, dx + \int_0^\infty \dot{\varphi}^2(t)\, dt < \infty . \tag{11.7}$$

PROOF. Differentiating the relation

$$v_x(\varphi(t), t) = 0$$

and using the relations $v_{xx} = v_t + 1$, $v_t = \dot{\varphi}$ for $x = \varphi(t)$, we get

$$- v_{xt} = (\dot{\varphi}^2 + \dot{\varphi}) \qquad \text{at } x = \varphi(t) . \tag{11.8}$$

The function v_t satisfies the heat equation in Ω_∞ and $v_t = l$ on $x = 0$, $v_t = \dot{\varphi}$ on $x = \varphi(t)$. Since $|\dot{\varphi}| \leqslant K$ (by Corollary 9.6), the maximum principle implies that

$$|v_t| \leqslant \text{const. in } \Omega_\infty .$$

Since $\dot{l}$ is a bounded function and (by (11.4)) v_t satisfies the heat equation in a strip $0 < x < \alpha_0$, $0 < t < \infty$ (where $\alpha_0 > 0$), it follows by the standard parabolic theory that

$$\sup_{0<t<\infty} |v_{tx}(0, t)| < \infty .$$

Hence

$$\int_0^\infty |v_{xt}(0, t)\, l(t)| dt < \infty . \tag{11.9}$$

Multiplying the equation $- v_{xxt} + v_{tt} = 0$ by v_t and integrating over Ω_T, we get

$$0 = - \iint_{\Omega_T} v_{xxt} v_t \, dx\, dt + \iint_{\Omega_T} v_{tt} v_t \, dx\, dt \equiv - I + J . \tag{11.10}$$

Integrating by parts and using (11.8), we find that

$$- I = \iint_{\Omega_T} v_{xt}^2 \, dx\, dt + \int_0^T (\dot{\varphi}^2 + \dot{\varphi}) \dot{\varphi}\, dt + \int_0^T v_{xt}(0, t)\, l(t)\, dt .$$

We also have, by integration by parts,

$$J = \tfrac{1}{2} \int_0^{\varphi(T)} v_t^2(x, t)\, dx - \tfrac{1}{2} \int_0^b v_t^2(x, 0)\, dx - \tfrac{1}{2} \int_0^T v_t^2(\varphi(t), t)\, \dot{\varphi}(t)\, dt$$

and $v_t(\varphi(t), t) = \dot\varphi(t)$. Substituting these expressions for I, J into (11.10), using (11.9), and noting that

$$\int_0^T (\dot\varphi^2 + \dot\varphi)\dot\varphi\, dt - \tfrac{1}{2}\int_0^T \dot\varphi^3\, dt = \tfrac{1}{2}\int_0^T \dot\varphi^2\, dt + \tfrac{1}{2}\int_0^T \dot\varphi^2(1+\dot\varphi)\, dt$$
$$\geqslant \tfrac{1}{2}\int_0^T \dot\varphi^2\, dt ,$$

since $1 + \dot\varphi \geqslant 0$, the assertion (11.7) readily follows.

We shall need the condition:

$$l(t) \to 0 \quad \text{if } t \to \infty . \tag{11.11}$$

THEOREM 11.3. *Let $l(t)$ satisfy the conditions* (11.1), (11.2), (11.6) *and* (11.11). *Then, as $t \to \infty$,*

$$\varphi(t) \to \gamma , \tag{11.12}$$

$$\dot\varphi(t) \to 0 \tag{11.13}$$

where

$$\gamma + \tfrac{1}{2}\gamma^2 = h(0) + b + \int_0^\infty l(t)\, dt , \tag{11.14}$$

and, uniformly in $x \in [0, R]$,

$$\lim_{t\to\infty} w(x, t) = \begin{cases} \frac{1}{2}(x-\gamma)^2 & \text{if } 0 \leqslant x \leqslant \gamma , \\ 0 & \text{if } \gamma < x \leqslant R . \end{cases} \tag{11.15}$$

Finally, for any $x \neq \gamma$,

$$\lim_{t\to\infty} w_t(x, t) = 0 , \tag{11.16}$$

$$\lim_{t\to\infty} w_{xx}(x, t) = \begin{cases} 1 & \text{if } 0 < x < \gamma , \\ 0 & \text{if } \gamma < x < R . \end{cases} \tag{11.17}$$

PROOF. From (11.7) it follows that there is a sequence $t_m \uparrow \infty$ such that

$$\int_0^{\varphi(t_m)} v_{xt}^2(x, t_m)\, dx \to 0 \quad \text{if } m \to \infty .$$

Since $v_t(x, 0) = l(t) \to 0$, we also have

$$\int_0^{\varphi(t_m)} v_t^2(x, t_m)\, dx \to 0 \quad \text{if } m \to \infty .$$

Using the Sobolev inequality, we deduce that

$$\sup_{0<x<\varphi(t_m)} |v_t(x, t_m)| \to 0 \qquad \text{if } m \to \infty .$$

Since $v_{xx} - 1 = v_t$, we get

$$\sup_{0<x<\varphi(t_m)} |v_{xx}(x, t_m) - 1| \to 0 \qquad \text{if } m \to \infty . \tag{11.18}$$

Set

$$K_m = \max\left\{ \sup_{0<x<\varphi(t_m)} |v_{xx}(x, t_m) - 1|, \sup_{t_m<t<\infty} |l(t)| \right\} .$$

Then, by Corollary 9.6,

$$|\dot\varphi(t)| \leqslant K_m , \qquad |v_t(x, t)| \leqslant K_m$$

if $t > t_m$, $x \in [0, R]$, $x \neq \varphi(t)$. Since, by (11.18) and the assumption (11.11), $K_m \to 0$ if $m \to \infty$, the assertions (11.13), (11.16) follow.

From (11.16) we conclude that

$$\sup_{0<x<\varphi(t)} |w_{xx}(x, t) - 1| = \sup_{0<x<\varphi(t)} |v_t(x, t)| \to 0 \qquad \text{if } t \to \infty . \tag{11.19}$$

Since

$$w\big(\varphi(t), t\big) = w_x\big(\varphi(t), t\big) = 0 ,$$

it follows that

$$\sup_{0<x<\varphi(t)} |w(x, t) - \tfrac{1}{2}\big(x - \varphi(t)\big)^2| \to 0 \qquad \text{if } t \to \infty . \tag{11.20}$$

In particular,

$$w(0, t) - \tfrac{1}{2}\varphi^2(t) \to 0 \quad \text{if } t \to \infty .$$

Recalling (11.3), we get

$$\varphi(t) + \tfrac{1}{2}\varphi^2(t) \to h(0) + b + \int_0^\infty l(s)\, ds . \tag{11.21}$$

Now, there is a unique non-negative solution γ of the quadratic equation (11.14). Since $\varphi(t) \geqslant 0$, the relation (11.21) implies that $\varphi(t) \to \gamma$ as $t \to \infty$, i.e., (11.12) holds. Finally, the assertions (11.15), (11.17) follow from (11.20), (11.19) respectively.

REMARK 1. We give another proof of (11.12), (11.15) (in case $l \equiv 0$) which does not depend on the estimates (9.27), (9.28). The proof is based on (11.18) (which was obtained using the estimate (11.7)). We may assume that $\gamma_0 = \lim \varphi(t_m)$ exists. From (11.18) it follows that there exist numbers b_m such that

$$b_m > \varphi(t_m), \quad w(x, t_m) < \tfrac{1}{2}(x - b_m)^2 \quad \text{if } 0 < x < \varphi(t_m),\ b_m \to \gamma_0 .$$

Since $l \equiv 0$, Corollary 10.6 yields

$$\varphi(t) \leqslant b_m, \quad w(x, t) \leqslant \tfrac{1}{2}(x - b_m)^2 \quad \text{if } 0 < x < \varphi(t),\ t > t_m . \tag{11.22}$$

Similarly,

$$\varphi(t) \geqslant \tilde{b}_m, \quad w(x, t) \geqslant \tfrac{1}{2}(x - \tilde{b}_m)^2 \quad \text{if } 0 < x < \tilde{b}_m,\ t > t_m , \tag{11.23}$$

where $\tilde{b}_m \to \gamma_0$ if $m \to \infty$. The assertions (11.12), (11.15) clearly follow from (11.22), (11.23).

REMARK 2. If $l(t) \geqslant 0$, $\int\limits_0^\infty l(t)\,dt = \infty$, then $\varphi(t)$ may increase to ∞. One can obtain bounds on the growth of $\varphi(t)$, in terms of the growth of $\int\limits_0^t l(s)\,ds$. For instance, if

$$\int\limits_0^t l(s)\,ds \leqslant C_0 t^m \quad (C_0,\ m \text{ positive constants}) \tag{11.24}$$

then

$$\varphi(t) \leqslant C t^{(m+1)/3} . \tag{11.25}$$

To prove it, we begin with the equality

$$\iint\limits_{\Omega_t} x(w_t - w_{xx})\,dx\,dt = \iint\limits_{\Omega_t} \big(-1 - \dot{\varphi}(t)\big)\,x\,dx\,dt .$$

After integrating by parts on the left, and evaluating the integral on the right, we obtain the identity

$$\tfrac{1}{6}\varphi^3(t) + \tfrac{1}{2}\int_0^t \varphi^2(s)\,ds + \int_0^t \varphi(s)\,ds = \tfrac{1}{6}l^3 - \int_0^{\varphi(t)} xw(x,t)\,dx + \int_0^b xh(x)\,dx + \big(h(0)+b\big)\,t + \int_0^t\Big(\int_0^s l(s')\,ds'\Big)\,ds\,.$$

Observing that $w(x,t) > 0$ and using (11.24), the assertion (11.25) immediately follows.

Added in proof.

We outline another approach to the main existence and uniqueness results of this paper.

1. – If we apply Green's formula to $u - x$ and the Neuman function of the heat equation in $x > 0$, and use (1.1)-(1.4), (2.4), we can obtain a nonlinear Volterra integral equation for $u_x(\varphi(t), t)$, as in the case of the Stefan problem [3]. We can then prove the existence and uniqueness of a (regular) local solution.

2. – In order to complete the proof of global existence and uniqueness, it suffices (as in the Stefan problem) to establish the a priori estimate

$$(*)\qquad |u_x(x,s)| \leqslant A_s\,,\qquad 0 < x < \varphi(s)\,,\qquad A_s = \max\{\max_{0\leqslant x\leqslant b}|u_x(x,0)|,\ \max_{0\leqslant t\leqslant s}|u_x(0,t)|\}\,.$$

First proof of (*). If u_x takes a positive maximum in $0 \leqslant x \leqslant \varphi(t)$, $0 \leqslant t \leqslant s$ at $x = \varphi(\bar t)$ then, by the maximum principle, $u_{xx} > 0$, i.e., $u_t > 0$ at $x = \varphi(\bar t)$ (we use here the continuity of u_{xx} up to $x = \varphi$, which is proved as in Theorem 9.5). Differentiating $u(\varphi(t), t) = \varphi(t)$ at $t = \bar t$ we then get $u_x\dot\varphi < \dot\varphi$, or $\dot\varphi(1+\dot\varphi) > 0$. Since (by Corollary 2.3) $1 + \dot\varphi > 0$, $\dot\varphi > 0$ at $\bar t$, i.e., $u_x < 0$ at $x = \varphi(\bar t)$; a contradiction. Similarly u_x cannot take a negative minimum on $x = \varphi$, and (*) follows by the maximum principle for u_x.

3. – *If g' and l change sign a finite number of times then $\dot\varphi$ changes sign a finite number of times.* (This contains part (i) of Theorem 10.4). The proof is similar to the proof of Theorem 10.4. Moreover, it can be given without using finite differences, i.e. the special case given in Theorem 10.1 can be proved without finite differences. Thus, if $\dot\varphi(t) > 0$ for $0 < t < \sigma$ and $u_x \geqslant 0$, $u_x \not\equiv 0$ on the remaining parabolic boundary, then $\dot\varphi(\sigma) = -u_x(\varphi(\sigma), \sigma)$ cannot vanish; for, otherwise, the relation $d[u(\varphi(t), t) - \varphi(t)]/dt = 0$ at $t = \sigma$ gives $u_{xx} = 0$ at $x = \varphi(\sigma)$, whereas $u_{xx} < 0$ at $x = \varphi(\sigma)$ by the maximum principle.

4. – *Second proof of* (*). If $\dot\varphi(t) \geqslant 0$ for $0 < t < s$ then $\tilde u = -A_s(x - \varphi(t)) + \varphi(t)$ satisfies $\tilde u_t - \tilde u_{xx} = (1 + A_s)\,\dot\varphi \geqslant 0$ and, by the maximum principle, $\tilde u \geqslant u$, so that

$(\tilde u - u)_x \leqslant 0$ at $x = \varphi$, i.e., $\dot\varphi(t) = -u_x(\varphi(t), t) \leqslant -\tilde u_x(\varphi(t), t) = A_s$:

Similarly, if $\dot{\varphi} \leqslant 0$ and $A_s < 1$ then we compare u with $A_s(x - \varphi(t)) + \varphi(t)$ and deduce that $\dot{\varphi}(t) \geqslant -\min(1, A_s)$. From these two cases we obtain (*) in case $\dot{\varphi}$ changes sign a finite number of times. Now approximate l, g by polynomials l_n, g_n. By **3**, the corresponding $\dot{\varphi} = \dot{\varphi}_n$ changes sign a finite number of times. Applying (*) for the case of l_n, g_n and taking $n \to \infty$, the proof of (*) is complete for general l, g.

REFERENCES

[1] J. BEAR, *Dynamics of Fluids in Porous Media*, American Elsevier Publishing Company, New York, 1972.

[2] A. BENSOUSSAN - A. FRIEDMAN, *Nonlinear variational inequalities and differential games with stopping times*, J. Funct. Analys., **16** (1974), 305-352.

[3] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[4] A. FRIEDMAN, *Parabolic variational inequalities in one space dimension and smoothness of the free boundary*, J. Funct. Analys., **18** (1975), 151-176.

[5] A. FRIEDMAN, *A parabolic system of quasi-variational inequalities*, to appear.

[6] A. FRIEDMAN - D. KINDERLEHRER, *A class of parabolic quasi-variational inequalities*, J. Diff. Eqs., to appear.

[7] R. JENSEN, *Finite difference approximations to the free boundary of a parabolic variational inequality*, to appear.

[8] O. A. LADYZHENSKAJA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Soc. Translations, vol. 23, 1968, Providence, R. I.

[9] D. SCHAEFFER, *A new proof of infinite differentiability of the free boundary in the Stefan problem*, to appear.

[10] L. TARTAR, *Inequations quasi variationelles abstraite*, C. R. Acad. Sci. Paris, **278** (1974), 1193-1196.

[11] P. VAN MOERBEKE, *An optimal stopping problem for linear reward*, Acta Math., **132** (1974), 1-41.