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Global properties of components of solutions of non-linear second order ordinary differential equations on the half-line

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Global Properties of Components of Solutions of Non-Linear Second Order Ordinary Differential Equations on the Half-Line.

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1. – Introduction and abstract setting.

Suppose that $p > 0$ and q are continuous real-valued functions on $(0, \infty)$ such that the symmetric operator τ defined by

$$\tau u(x) = -\big(p(x)u'(x)\big)' + q(x)u(x) \qquad \text{for } x > 0 \tag{1.1}$$

on $C_0^\infty(0, \infty)$ has a unique self-adjoint extension S in $L^2(0, \infty)$ whose domain is $\{u \in W_2^2(0, \infty) \colon u(0) = 0\}$ and such that S is bounded below. Let Q be a lower bound for the essential spectrum of S. Some global properties of the set of non-trivial real solutions of the non-linear eigenvalue problem

$$\tau u(x) + f\big(x, u(x), u'(x), \lambda\big) = \lambda u(x) \qquad \text{for } x > 0 \tag{1.2}$$

$$u(0) = 0 \tag{1.3}$$

are established in the general case where the essential spectrum of S is non-empty.

Several authors (for example [1] to [6]) have treated the corresponding two-point boundary-value problem for regular second order ordinary differential equations on a finite interval which may be taken to be $[0, 1]$. There the appropriate setting is the scale of Banach spaces $C^k\big([0, 1]\big)$ of k-times continuously differentiable functions ($k = 0, 1, 2, \ldots$) and the boundary condition (1.3) must be replaced by

$$\alpha u(0) + \beta u'(0) = \delta u(1) + \gamma u'(1) = 0$$

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where $(\alpha^2+\beta^2)(\delta^2+\gamma^2)\neq 0$. Then compactness is readily available both in the embeddings of $C^{k+1}([0,1])$ in $C^k([0,1])$ for $k=0,1,2,\dots$ and for the resolvent of the linear differential operator τ which is even compact as a mapping from $C^0([0,1])$ into $C^1([0,1])$. Here, since we are assuming that τ is in the limit-point case at infinity, it is appropriate to seek solutions of the equation (1.2) in the usual Sobolev spaces of square integrable functions on $(0,\infty)$ together with the boundary condition (1.3). Then the continuous embedding of $W_k^2(0,\infty)$ in $L^2(0,\infty)(\equiv W_0^2(0,\infty))$ is not compact for any $k=0,1,2,\dots$ and, in general, τ does not have a compact resolvent. Nevertheless, we find that the abstract approach devised for the case of bounded intervals can be modified so as to apply in cases where compactness is not available. Following the method laid down by Rabinowitz for the compact case, it is shown in Stuart [8] that the topological degree theory for k-set contractions ($k<1$) developed by Nussbaum [7] can be used to this end. The results are obtained by showing that, given appropriate behaviour of the function f, the system (1.2), (1.3) is equivalent to an operator equation of the form,

$$u=\lambda Au+H(u,\lambda)\quad \text{for } (u,\lambda)\in B\times\boldsymbol{R} \tag{1.4}$$

where $\boldsymbol{R}$ denotes the real numbers, B denotes a real Banach space with norm $\|\ \|$ and the operators A and H have the following properties.

P1) $A: B\to B$ is a bounded linear operator.

P2) $H: B\times\boldsymbol{R}\to B$ is a bounded, continuous (non-linear) operator such that

$$\frac{\|H(u,\lambda)\|}{\|u\|}\to 0\qquad \text{as } \|u\|\to 0$$

uniformly for λ in bounded intervals and such that $H(u,\cdot): \boldsymbol{R}\to B$ is uniformly continuous on bounded intervals for u in bounded subsets of B.

P3) The mapping $G: B\times\boldsymbol{R}\to B$ defined by

$$G(u,\lambda)=\lambda Au+H(u,\lambda)$$

is such that $G(\cdot,\lambda): B\to B$ is a $k(\lambda)$-set contraction where $k:\boldsymbol{R}\to[0,\infty)$ is a known continuous mapping.

Alternatively the hypotheses P2) can be replaced by the following one.

P2)′ $H: B\times\boldsymbol{R}\to B$ is a continuous compact mapping such that

$$\frac{\|H(u,\lambda)\|}{\|u\|}\to 0\qquad \text{as } \|u\|\to 0$$

uniformly for λ in bounded intervals.

Under these hypotheses, it is clear that $(0, \lambda) \in B \times \boldsymbol{R}$ is a solution of (1.4) for all $\lambda \in \boldsymbol{R}$, and such solutions are called trivial. Let $\mathcal{T}$ denote the subset of $B \times \boldsymbol{R}$ consisting of all non-trivial solutions of (1.4) and let

$$\mathcal{T}' = \mathcal{T} \cup \{(0, \lambda) \in B \times \boldsymbol{R} \colon \lambda \text{ is a characteristic value of } A\}.$$

It is readily shown that, if $k(\lambda) < 1$ and $(0, \lambda)$ is in the closure of $\mathcal{T}$ with respect to the $B \times \boldsymbol{R}$ topology, then indeed $(0, \lambda) \in \mathcal{T}'$. In other words, in the range $k(\lambda) < 1$, bifurcation from the curve of trivial solutions of (1.4) can take place only at characteristic values of A. Let $\mathcal{C}_\mu$ denote the component (maximal connected subset) of $\mathcal{T}'$ (endowed with the induced topology from $B \times \boldsymbol{R}$) containing $(0, \mu) \in B \times \boldsymbol{R}$. Then $\mathcal{C}_\mu$ is nonempty if and only if μ is a characteristic value of A.

THEOREM 1.1. *Let μ be a characteristic value of A of odd algebraic multiplicity with $k(\mu) < 1$. Then $\mathcal{C}_\mu$ has at least one of the following properties.*

1) $\mathcal{C}_\mu$ *is an unbounded subset of* $B \times \boldsymbol{R}$.

2) $\sup\{k(\lambda) \colon (u, \lambda) \in \mathcal{C}_\mu\} \geq 1$.

3) $\mathcal{C}_\mu$ *contains an element* $(0, \hat{\mu}) \in \mathcal{T}'$ *where* $\hat{\mu} \neq \mu$.

In the case where we can choose $k(\lambda) \equiv 0$ for all $\lambda \in \boldsymbol{R}$ (corresponding to $G(\cdot, \lambda) \colon B \to B$ being compact for each $\lambda \in \boldsymbol{R}$) this result coincides with Theorem 1.3 of Rabinowitz [5]. The case where the mapping H has the special form $H(u, \lambda) = \lambda \mathcal{H}(u)$ for some $\mathcal{H} \colon B \to B$ is established as Theorem 1.6 of Stuart [8]. However, given either P2) or P2)′, we see that, for a bounded sequence $\{(u_n, \lambda_n)\}$ in $B \times \boldsymbol{R}$ with $\lambda_n \to \lambda$ as $n \to \infty$,

$$\alpha(G(u_n, \lambda_n)) \leq \alpha(G(u_n, \lambda)) + \alpha(G(u_n, \lambda_n) - G(u_n, \lambda)) = k(\lambda)\alpha(u_n)$$

where $\alpha(x_n)$ denotes the set measure of non-compactness of a bounded sequence $\{x_n\}$ in B as defined in Stuart [8]. Hence, for a bounded sequence $\{(u_n, \lambda_n)\}$ in $B \times \boldsymbol{R}$ such that $u_n = G(u_n, \lambda_n)$ for all n and $\sup\{k(\lambda_n) \colon n = 1, 2, \ldots\} < 1$, we see that $\{(u_n, \lambda_n)\}$ contains a convergent subsequence. With this noted, the proofs of Lemma 1.7 and Theorem 1.6 of Stuart [8] immediately yield the result in the above generality.

For applications to differential equations, it is useful to interpret Theorem 1.1 in the following setting. Let B now be a real Hilbert space with norm $\|\ \|$ and let $S \colon \mathcal{D}(S) \to B$ be an unbounded self-adjoint operator in B which is bounded below, Q being a lower bound for the essential spectrum of the complexification of S. Let H, with norm $|||\ |||$, denote the real Hilbert space obtained from the domain of S equipped with the graph topology.

Then H is continuously (but in general not compactly) embedded in B. Consider the non-linear eigenvalue problem set by

$$Su + M(u, \lambda) = \lambda u \quad \text{for } (u, \lambda) \in H \times \boldsymbol{R} \tag{1.5}$$

where M has the following properties.

P4) $M: H \times \boldsymbol{R} \to B$ is a bounded, continuous mapping such that

$$\frac{\|M(u, \lambda)\|}{|||u|||} \to 0 \qquad \text{as } |||u||| \to 0$$

uniformly for λ in bounded intervals and such that $M(u, \cdot)\colon \boldsymbol{R} \to B$ is uniformly continuous on bounded intervals for u in bounded subsets of H.

Let $\mathcal{S}$ denote the set of all non-trivial solutions of (1.5) in $H \times \boldsymbol{R}$ and let $\mathcal{S}' = \mathcal{S}' \cup \{(0, \lambda) \in H \times \boldsymbol{R}\colon \lambda \text{ is an eigenvalue of } S\}$.

THEOREM 1.2. *Suppose that $\mu < Q$ is an eigenvalue of odd multiplicity of S and that, in addition to satisfying* P4), *the mapping M is such that $M(\cdot, \lambda)\colon H \to B$ is compact for each $\lambda \in \boldsymbol{R}$. Let $\mathcal{C}_\mu$ denote the component of $\mathcal{S}'$ (endowed with the topology induced from $H \times \boldsymbol{R}$) containing $(0, \mu)$. Then $\mathcal{C}_\mu$ has at least one of the following properties.*

1) *$\mathcal{C}_\mu$ is an unbounded subset of $H \times \boldsymbol{R}$.*

2) $\sup\{\lambda\colon (u, \lambda) \in \mathcal{C}_\mu\} \geqslant Q$.

3) *$\mathcal{C}_\mu$ contains an element $(0, \hat{\mu}) \in \mathcal{S}'$ where $\hat{\mu} \neq \mu$.*

This result is deduced from Theorem 1.1 by following the method which is set out in detail for a special case in Stuart [9].

Given any two of the properties 1), 2), 3) of Theorem 1.2, an example can be provided in which the hypotheses of Theorem 1.2 are all satisfied and yet there is a component which has neither of the two selected properties. See [14].

In the case of a simple eigenvalue of S we can say more.

THEOREM 1.3. *Suppose that $\mu < Q$ is an eigenvalue of multiplicity* 1 *of S. Also, in addition to* P4), *suppose that the partial derivatives M_λ, M_u and $M_{\lambda u}$ exist and are continuous in an open neighbourhood of $(0, \mu)$ in $H \times \boldsymbol{R}$. Then it follows from* P4) *that $M_u(0, \lambda) = 0$ for all $\lambda \in \boldsymbol{R}$ and we assume that*

$M_{\lambda u}(0, \lambda) = 0$ *for all* $\lambda \in \boldsymbol{R}$. *Then there exist an open neighbourhood* U *of* $(0, \mu)$ *in* $H \times \boldsymbol{R}$, *an open interval* $(-\delta, \delta)$ *and continuous functions*

$$\varphi\colon (-\delta, \delta) \to \boldsymbol{R} \quad \text{and} \quad w\colon (-\delta, \delta) \to H$$

such that

$$\varphi(0) = 0 \quad \text{and} \quad w(0) = 0$$

and

$$\mathcal{C}_\mu \cap U = \{(\alpha u + \alpha w(\alpha), \mu + \varphi(\alpha))\colon \alpha \in (-\delta, \delta)\}$$

where $u \in H$, $|||u||| = 1$ *and* $Su = \mu u$. (*Here* M *is regarded as a continuous mapping from* $H \times \boldsymbol{R}$ *in* B *and the derivatives are taken in the Fréchet sense.*)

That $\mathcal{C}_\mu$ has the local structure indicated above follows immediately from Theorem 1.4 of Crandall and Rabinowitz [10]. In this case we set

$$\mathcal{C}_\mu^+ = \mathcal{D}_\mu^+ \cup \{(0, \mu)\} \quad \text{and} \quad \mathcal{C}_\mu^- = \mathcal{D}_\mu^- \cup \{(0, \mu)\}$$

where $\mathcal{D}_\mu^+$ and $\mathcal{D}_\mu^-$ denote the components of $\mathcal{S}' \setminus \{(0, \mu)\}$ containing $\{(\alpha u + \alpha w(\alpha), \mu + \varphi(\alpha))\colon \alpha \in (0, \delta)\}$ and $\{(\alpha u + \alpha w(\alpha), \mu + \varphi(\alpha))\colon \alpha \in (-\delta, 0)\}$ respectively. It is not claimed that $\mathcal{C}_\mu^+$ and $\mathcal{C}_\mu^-$ are in general distinct, although in the applications to ordinary differential equations considered below this will be seen to be so. Using the method introduced by Rabinowitz in Theorem 1.27 of [5], we can deduce the following global results about the subsets $\mathcal{C}_\mu^+$ and $\mathcal{C}_\mu^-$.

THEOREM 1.4. *Suppose that all the hypotheses of Theorem* 1.3 *are satisfied and that, in addition,* $M(\cdot, \lambda)\colon H \to R$ *is a compact mapping for each* $\lambda \in \boldsymbol{R}$. *Then the set* $\mathcal{C}_\mu^+$ *has at least one of the following properties.*

1) $\mathcal{C}_\mu^+$ *is an unbounded subset of* $H \times \boldsymbol{R}$.

2) $\sup\{\lambda\colon (u, \lambda) \in \mathcal{C}_\mu^+\} \geqslant Q$.

3) $\mathcal{C}_\mu^+$ *contains an element* $(0, \hat{\mu}) \in \mathcal{S}'$ *where* $\hat{\mu} \neq \mu$.

4) *There exists an element* $(u, \lambda) \in \mathcal{C}_\mu^+ \cap \mathcal{S}$ *such that* $(-u, \lambda) \in \mathcal{C}_\mu^+$.

The above conclusion remains valid when $\mathcal{C}_\mu^+$ *is replaced by* $\mathcal{C}_\mu^-$.

To ensure that the linearisation of (1.2) about the trivial solution $(0, \lambda)$ is

$$\tau u(x) = \lambda u(x) \quad \text{for } x > 0\,,$$

it is assumed that the function f which determines the non-linearity is continuous and that $f(x, r, s, \lambda) = o\big((r^2+s^2)^{\frac{1}{2}}\big)$ as $r^2+s^2 \to 0$, uniformly for λ in bounded intervals and $x \geqslant 0$. This assumption is stated formally as A1) in Section 2 after the notation for the function spaces to be used and the hypotheses concerning the linear differential operation τ have been introduced. It is shown in Section 2 that under the assumption A1) the nodal properties of solutions of (1.2) are preserved on connected sets of non-trivial solutions in $W_2^2(0, \infty) \times \boldsymbol{R}$. The assumption A1) is sufficient (*) to guarantee that the Nemytskii operator induced by f maps $W_2^2(0, \infty)$ continuously into $L^2(0, \infty)$. However, as may be seen by considering the map $f(x, r, s, \lambda) = r^2$ for all $x \geqslant 0$ and $r, s, \lambda \in \boldsymbol{R}$, the assumption A1) is not sufficient to imply that the Nemytskii operator is a compact mapping from $W_2^2(0, \infty)$ into $L^2(0, \infty)$ (indeed it does not even ensure boundedness (**)). Compactness is implied by requiring that, in addition to A1), we have also that $f(x, r, s, \lambda) \to 0$ as $x \to \infty$, uniformly for r, s, λ in bounded intervals. This additional requirement of «spatial decay» in the non-linear term is the content of A2). In Theorems 2.8 and 2.9 the results for (1.2) under the assumption A2) are presented. Then, in Section 3, this special case is used to treat (1.2) when only the weaker assumption A1) is made, provided that f is ultimately almost sign preserving in the sense of (2.19). This restriction is essentially used to establish the compactness of any closed bounded subset U of solutions of (1.2) such that $\sup\{\lambda\colon (u, \lambda) \in U\} < Q$, which is the content of Corollary 2.4. The main results for (1.2) under these assumptions are then given in Theorems 3.3 and 3.7 and it is indicated how, in the usual way, additional structure on f enables the properties of the components to be further specified. Such considerations yield additional results such as Theorem 3.7 on the multiplicity of solutions of (1.2) for prescribed values of λ. Many more details of this kind of analysis are given in [13]. A survey of some related material is given in [15].

2. – Preliminaries.

Let $\boldsymbol{R}$ denote the real numbers and L^2 denote the real Hilbert space of all real-valued measurable «functions» u such that

$$\|u\|^2 = \int_0^\infty u(x)^2\,dx < \infty\,.$$

(*) Indeed, as the proof of Proposition 2.2(a) shows, $f(x, r, s, \lambda) = O\big((r^2+s^2)^{\frac{1}{2}}\big)$ as $r^2+s^2 \to 0$ uniformly for λ in bounded intervals and $x \geqslant 0$ is sufficient to ensure this.

(**) See the remark following Proposition 2.2.

The real Hilbert space of all elements $u \in L^2$ such that $u'' \in L^2$ will be denoted by W_2^2 where the norm in W_2^2 is given by

$$|||u|||^2 = \|u\|^2 + \|u''\|^2 .$$

Here u' denotes the derivative of u in the sense of distributions. For $u \in W_2^2$, both u and u' can be taken to be continuous functions on $[0, \infty)$ and we set $H_2 = \{u \in W_2^2 \colon u(0) = 0\}$. Thus H_2 is a real Hilbert space with respect to the norm $||| \; |||$. For future reference we note that

$$\{\|u\|^2 + \|u'\|^2\}^{\frac{1}{2}} \leqslant C|||u||| \tag{2.1}$$

and

$$\{u(x)^2 + u'(x)^2\}^{\frac{1}{2}} \leqslant C|||u||| \tag{2.2}$$

for all $u \in H_2$ and all $x > 0$, where here and henceforth C denotes some, but not always the same, positive constant. The following result is easily established.

PROPOSITION 2.1. *Given $\varepsilon > 0$ and $u \in H_2$, there exist an $X \geqslant 0$ and an open neighbourhood U of u in H_2 such that*

$$\{v(x)^2 + v'(x)^2\}^{\frac{1}{2}} \leqslant \varepsilon \quad \textit{for all } v \in U \textit{ and all } x > X .$$

Henceforth, we shall assume that the linear differential operator τ has the following properties.

The coefficient p is a continuous non-negative function on $(0, \infty)$ such that

$$0 < p_1 \leqslant \liminf_{x\to\infty} p(x) \leqslant \limsup_{x\to\infty} p(x) \leqslant p_2 < \infty . \tag{2.3}$$

The coefficient q is continuous on $(0, \infty)$ and

$$-\infty < Q = \liminf_{x\to\infty} q(x) \leqslant \limsup_{x\to\infty} q(x) < \infty . \tag{2.4}$$

It is further assumed that the minimal symmetric operator τ defined in (1.1) by p and q has a unique self-adjoint extension S whose domain equipped with the graph topology coincides (up to norm equivalence) with the real Hilbert space H_2. Also the essential spectrum of the complexification of S is assumed to lie entirely in the interval $[Q, \infty)$. Then the spectrum in $(-\infty, Q)$ consists of simple eigenvalues which are indexed

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots < Q . \tag{2.5}$$

The eigenfunction corresponding to λ_n is infinitely differentiable, belongs to H_2 and has exactly $n-1$ zeros in $(0,\infty)$.

Numerous sufficient conditions on p and q are known which will ensure the above properties of the operator τ, and we do not select one in particular here. Certainly τ has all these properties provided that p, p' and q are all continuous and bounded on $(0,\infty)$ with $\inf p(x)>0$. However p and q need not be bounded on $(0,\infty)$. Indeed, in the important special case $p(x)\equiv 1$ and $q(x)=-x^{-1}$ for $x>0$, τ has all the properties listed above where $Q=0$ and the spectrum of S is $\{-1/4n^2: n=1,2,\dots\}\cup[0,\infty)$. On the other hand, there are cases (e.g. $p(x)\equiv 1$ and $q(x)=0$ for $x>0$) in which the above conditions are satisfied but the spectrum of S lies entirely in the interval $[Q,\infty)$, and for such cases the theory presented below is vacuous. The problem of determining when there are eigenvalues in $(-\infty,Q)$ is well understood. See for example [12].

Turning now to the non-linear term in (1.2), we state formally the assumptions described in the introduction.

A1) The function $f\colon[0,\infty)\times\boldsymbol{R}^3\to\boldsymbol{R}$ is continuous and given any bounded interval J and any $\varepsilon>0$ there exists an $\eta>0$ such that

$$\max\left\{\frac{|f(x,r,s,\lambda)|}{(r^2+s^2)^{\frac12}}: x\geqslant 0,\ (r^2+s^2)^{\frac12}\leqslant\eta \text{ and } \lambda\in J\right\}\leqslant\varepsilon\,. \tag{2.6}$$

The following stronger condition requires that f also has spatial decay.

A2) The function f satisfies A1) and, in addition, given any bounded interval J and any $\varepsilon>0$, there exists an $X\geqslant 0$ such that

$$\max\{|f(x,r,s,\lambda)|: x\geqslant X \text{ and } r,s,\lambda\in J\}\leqslant\varepsilon\,. \tag{2.7}$$

A consequence of A2) which will be useful later is that, given a bounded subset $K\subset\boldsymbol{R}^3$ and an $\varepsilon>0$, there exists an $X\geqslant 0$ such that

$$\max\left\{\frac{|f(x,r,s,\lambda)|}{(r^2+s^2)^{\frac12}}: x\geqslant X \text{ and } (r,s,\lambda)\in K\right\}\leqslant\varepsilon\,. \tag{2.8}$$

We note that if $f(x,r,s,\lambda)=\psi(x)h(r,s,\lambda)$ for $x\geqslant 0$ and $r,s,\lambda\in\boldsymbol{R}$ where

a) $\psi\colon[0,\infty)\to\boldsymbol{R}$ is continuous and $\psi(x)\to 0$ as $x\to\infty$ and

b) $h\colon\boldsymbol{R}^3\to\boldsymbol{R}$ is continuous and $h(r,s,\lambda)=o\big((r^2+s^2)^{\frac12}\big)$ uniformly for λ in bounded intervals,

then f satisfies the condition A2).

In the following discussion of the Nemytskii operator induced by f it should be noted that the condition (2.6) is instrumental not only in guaranteeing that the Nemytskii operator is $o(|||u|||)$ as $|||u|||\to 0$ but even in ensuring the H_2 is taken continuously into L^2. In the absence of (2.6), growth estimates would be required to ensure this.

PROPOSITION 2.2.

(*a*) *Suppose that f satisfies the condition* A1). *Then the Nemytskii operator defined by*

$$F(u,\lambda)(x)=f(x,u(x),u'(x),\lambda)\qquad \text{for } x>0 \tag{2.9}$$

is a continuous mapping from $H_2\times\boldsymbol{R}$ into L^2. Furthermore

$$\frac{\|F(u,\lambda)\|}{|||u|||}\to 0 \qquad as\ |||u|||\to 0\,,$$

uniformly for λ in bounded intervals.

(*b*) *Suppose that f satisfies* A2). *Then, in addition to the properties in* (*a*), *we have that $F\colon H_2\times\boldsymbol{R}\to L^2$ is a compact mapping and also that the mapping $F(u,\cdot)\colon \boldsymbol{R}\to L^2$ is uniformly continuous on bounded intervals for u in bounded subsets of H_2.*

The proof is straight forward and is given in [13].

REMARK. In connection with the above results, it should be noted that A1) is not sufficient to guarantee that the Nemytskii operator F takes bounded subsets of $H_2\times\boldsymbol{R}$ into bounded subsets of L^2. For example, set

$$\begin{aligned} f(x,r,s,\lambda)&=r^2 &&\text{for } x\geqslant 0,\ |r|\leqslant 1,\ s,\lambda\in\boldsymbol{R}\\ &r^2\exp[x(r^2-1)] &&\text{for } x\geqslant 0,\ |r|>1,\ s,\lambda\in\boldsymbol{R}.\end{aligned}$$

Let $u\colon\boldsymbol{R}\to\boldsymbol{R}$ be an infinitely differentiable function with support contained in $[1,4]$ and such that $u(x)=2$ for $x\in[2,3]$. Then, for each positive integer n, let

$$u_n(x)=u(x-n)\qquad\text{for } x\geqslant 0\,.$$

Thus $u_n\in H_2$ and $|||u_n|||^2=\int_{-\infty}^{\infty}\{u(x)^2+u''(x)^2\}\,dx<\infty$ for all n. Also f satisfies A1). However $\|F(u_n,\lambda)\|\to\infty$ as $n\to\infty$.

To ensure the boundedness of F it is enough to supplement A1) by the following assumption.

AB) Given any bounded interval J,

$$\sup\{|f(x, r, s, \lambda)|: x \geqslant 0 \text{ and } r, s, \lambda \in J\} < \infty.$$

It is a pleasant feature of the approximation procedure used in Section 3 that it yields global results about (1.2) without requiring that the non-linearity induces a bounded Nemytskii operator from the graph space of τ into L^2. Setting $f(x, r, s, \lambda) = r^2$ for all $x \geqslant 0$ and $r, s, \lambda \in \boldsymbol{R}$, and considering $F(u_n, \lambda)$ where $\{u_n\}$ is the sequence in H_2 defined above, shows that A1) and AB) are not sufficient hypotheses of f to ensure that $F: H_2 \times \boldsymbol{R} \to L^2$ is compact. Of course, when the stronger assumption A2) is made, AB) is automatically satisfied and the boundedness follows « a fortiori » from the compactness as established in Proposition 2.2 (*b*).

We give hypotheses on f which will ensure that the Nemytskii operator (2.9) has the smoothness properties required in Theorem 1.3. Partial differentiation with respect to the i-th variable is denoted by ∂_i for $i = 1, 2, 3, 4$.

AS) The function $f: [0, \infty) \times \boldsymbol{R}^3$ and the partial derivatives $\partial_2 f$, $\partial_3 f$, $\partial_4 f$, $\partial_{42} f$ and $\partial_{43} f$ are all continuous on $[0, \infty) \times \boldsymbol{R}^3$. Furthermore $\partial_2 f$, $\partial_3 f$, $\partial_{42} f$ and $\partial_{43} f$ all tend to 0 as $(r^2 + s^2)^{\frac{1}{2}} \to 0$ uniformly for $x \geqslant 0$ and λ in bounded intervals.

Note that if f satisfies AS) then both f and $\partial_4 f$ satisfy A1).

PROPOSITION 2.3. *Suppose that f satisfies* AS). *Then, for the Nemytskii operator F defined by* (2.9), *all the* (*Fréchet*) *partial derivatives F_λ, F_u and $F_{\lambda u}$ exist and are continuous. Also $F_u(0, \lambda) = 0$ and $F_{\lambda u}(0, \lambda) = 0$ for all $\lambda \in \boldsymbol{R}$.*

A proof is given in [13].

We can now proceed with a discussion of (1.2) in the space $H_2 \times \boldsymbol{R}$. Henceforth $\mathcal{S}$ will denote the set of all non-trivial solutions of (1.2) in $H_2 \times (-\infty, Q)$ and

$$\mathcal{S}' = \mathcal{S} \cup \{(0, \lambda_n) \in H_2 \times (-\infty, Q): \lambda_n \text{ is an eigenvalue of } S\}.$$

The proof of Lemma 2.9 in Rabinowitz [6] shows that, provided f satisfies A1) and $(u, \lambda) \in \mathcal{S}$, the zeros of u in $[0\ \infty)$ are all simple. Let $\mathcal{C}_n$ denote the component of $\mathcal{S}'$ (endowed with the topology induced from $H_2 \times \boldsymbol{R}$) containing $(0\ \lambda_n)$.

In order to discuss the nodal properties of solutions we introduce a mapping N defined on $\mathcal{S}'$ as follows. For $(u\ \lambda) \in \mathcal{S}$, $N((u\ \lambda))$ is the number of zeros (possibly infinite) of u in $(0, \infty)$. On $\mathcal{S}' \setminus \mathcal{S}$ we set $N((0\ \lambda_n)) = n - 1$.

When f has the special form $f(x, r, s, \lambda) = rh(x, r, s, \lambda)$ for $x \geqslant 0$ and $r, s, \lambda \in \boldsymbol{R}$ where h: $[0, \infty) \times \boldsymbol{R}^3 \to \boldsymbol{R}$ is continuous and $h(x, 0, 0, \lambda) = 0$ for all $x \geqslant 0$ and all $\lambda \in \boldsymbol{R}$, it can be shown simply, using the variational characterisation of the eigenvalues of linear second order ordinary differential operators which are essentially self-adjoint with respect to the given boundary condition, that $N((u, \lambda)) = n - 1$ for all $(u, \lambda) \in \mathrm{C}_n$. This method was used in Stuart [8] and [9]. To treat the general case in which f satisfies only A1), we must use a different approach. Both for this and for subsequent proofs, it is convenient to introduce the following definition.

A sequence $\{f_k\}$ of functions is said to satisfy the condition (E) provided that:

a) the functions f_k satisfy A1) uniformly in k in the sense that, given any $\varepsilon > 0$ and any bounded interval J, there exists an $\eta > 0$ (independent of k) such that

$$\max\left\{\frac{|f_k(x, r, s, \lambda)|}{(r^2 + s^2)^{\frac{1}{2}}} : x \geqslant 0, (r^2 + s^2)^{\frac{1}{2}} \leqslant \eta, \lambda \in J\right\} \leqslant \varepsilon \qquad \text{for all } k;$$

b) $f_k \to f$ as $k \to \infty$ uniformly on compact subsets of $[0, \infty) \times \boldsymbol{R}^3$;

c) given any $\varepsilon > 0$ and any bounded interval J, there exists an $X \geqslant 0$ (independent of k) such that

$$|f_k(x, r, s, \lambda)| \leqslant \varepsilon \quad \text{whenever} \quad f_k(x, r, s, \lambda)r < 0 \qquad \text{and } x \geqslant X,\ r, s, \lambda \in J\,.$$

LEMMA 2.3. *Suppose that $\{f_k\}$ satisfies (E). Let $\{(u_k, \mu_k)\}$ be a bounded sequence in $H_2 \times \boldsymbol{R}$ such that* $\sup \mu_k < Q$ *and such that*

$$\text{(2.10))} \qquad \begin{gathered} \tau u_k(x) + F_k(u_k, \mu_k)(x) = \mu_k u_k(x) \qquad \text{for } x > 0 \\ u_k(0) = 0\,, \qquad u_k \not\equiv 0\,, \end{gathered}$$

where F_k: $H_2 \times \boldsymbol{R} \to L^2$ is the Nemytskii operator induced by f_k.

(*a*) *Then there exist positive constants β and X (independent of k) such that*

$$u_k(x)^2 + u_k'(x)^2 \leqslant \exp[-\beta x]$$

and

$$u_k(x)u_k'(x) < 0$$

for all $x \geqslant X$ and all k.

(*b*) *Furthermore $\{(u_k, \mu_k)\}$ contains a subsequence which converges strongly in $H_2 \times \boldsymbol{R}$ as $k \to \infty$.*

PROOF. Suppose that $|||u_k||| \leqslant M$.

(a) Since $u_k \in H_2$ we have that $F_k(u_k, \mu_k) \in L^2$ and consequently integration of (2.10), after multiplication by u_k, yields

$$p(x)u_k'(x)u_k(x) + \int_x^\infty p(y)u_k'(y)^2\,dy + \int_x^\infty (q(y)-\mu_k)u_k(y)^2\,dy + \int_x^\infty F_k(u_k, \mu_k)(y)u_k(y)\,dy = 0 \quad \text{for all } x \geqslant 0\,. \tag{2.11}$$

Let $g_k\colon [0, \infty)\times \boldsymbol{R}^3 \to \boldsymbol{R}$ be defined by

$$g_k(x, r, s, \lambda) = \begin{cases} 0 & \text{if } f_k(x, r, s, \lambda)r \geqslant 0 \\ f_k(x, r, s, \lambda) & \text{if } f_k(x, r, s, \lambda)r < 0\,. \end{cases}$$

It follows from (E) that given any $\varepsilon > 0$ and any bounded interval J, there exists an $X \geqslant 0$ such that

$$\max\{|g_k(x, r, s, \lambda)|\colon\ x \geqslant X,\ r, s, \lambda \in J\} \leqslant \varepsilon \quad \text{for all } k\,. \tag{2.12}$$

Then, combining (2.12) with $(E)(a)$, we see that, given any $\varepsilon > 0$, there exists an $X \geqslant 0$ such that

$$\left|\int_X^\infty g_k(x, u_k(x), u_k'(x), \mu_k)u_k(x)\,dx\right| \leqslant \varepsilon \int_X^\infty \{u_k(x)^2 + u_k'(x)^2\}^{\frac12}|u_k(x)|\,dx \leqslant \varepsilon\int_X^\infty \left(u_k(x)^2 + \tfrac12 u_k'(x)^2\right)dx \quad \text{for all } k\,. \tag{2.13}$$

Hence, using (2.3) and (2.4), it follows from (2.11) and (2.13) that there exists an $X \geqslant 0$ (independent of k) such that

$$0 < \tfrac12\int_x^\infty p(y)u_k'(y)^2\,dy + \tfrac12\int_x^\infty (q(y)-\mu_k)u_k(y)^2\,dy \leqslant -p(x)u_k'(x)u_k(x) \tag{2.14}$$

for all $x \geqslant X$ and all k, where we also have that $q(y) - \mu_k \geqslant \delta > 0$ for all $y \geqslant X$ and all k. This establishes the second statement in (a).

Now multiplying (2.10) by $p(x)u_k'(x)$ yields,

$$-[(p(x)u_k'(x))^2]' + 2p(x)u_k'(x)F_k(u_k, u_k)(x) \geqslant -\tfrac12 p_1\delta[u_k(x)^2]' \tag{2.15}$$

for all $x \geqslant X$ and all k, where X has also been chosen such that $p(x) \geqslant \frac{1}{2}p_1 > 0$ for all $x \geqslant X$. But, for $x \geqslant X$ and all k we have that

$$2p(x)u_k'(x)F_k(u_k, \mu_k)(x) \leqslant 2p(x)|u_k'(x)|\,|g_k(x, u_k(x), u_k'(x), \mu_k)|$$

and so, given any $\varepsilon > 0$ there exists $Y \geqslant X$ such that

$$\begin{aligned}(2.16)\qquad 2\int_x^\infty p(y)u_k'(y)F_k(u_k, \mu_k)(y)\,dy &\leqslant \varepsilon\int_x^\infty p(y)\{\tfrac{1}{2}u_k(y)^2 + u'(y)^2\}\,dy \leqslant \\ &\leqslant -p(x)\,u_k'(x)\,u_k(x) \quad \text{for all } x \geqslant Y \text{ and all } k\,,\end{aligned}$$

where we have used (2.14) and chosen ε to be sufficiently small.

Combining (2.15) and (2.16) yields

$$\begin{aligned}[p(x)u_k'(x)]^2 &\geqslant p(x)u_k'(x)u_k(x) + \tfrac{1}{2}p_1\delta u_k(x)^2 \geqslant \\ &\geqslant \tfrac{1}{2}p_1\delta u_k(x)^2 - p_2\gamma u_k'(x)^2 - p_2\gamma^{-1}u_k(k)^2\end{aligned}$$

for all $\gamma > 0$, $x \geqslant Y$ and all k, where Y has now been chosen sufficiently large so as to ensure that $p(x) \leqslant 2p_2$ for all $x \geqslant Y$. Choosing $\gamma = 4p_2(p_1\delta)^{-1}$ and $\beta = p_1\delta\{16p_2^2(1 + p_1\delta)\}^{-\frac{1}{2}}$, this gives

$$(2.17)\qquad u_k'(x)^2 \geqslant \beta^2 u_k(x)^2 \quad \text{for all } x \geqslant Y \text{ and all } k\,.$$

Solving (2.17) shows that

$$\begin{aligned}(2.18)\qquad |u_k(x)| &\leqslant |u_k(Y)|\exp\{-\beta(x - Y)\} \\ &\leqslant M\exp\{-\beta(x - Y)\} \quad \text{for all } x \geqslant Y \text{ and all } k\,.\end{aligned}$$

Returning to (2.14) and recalling that $2p_2 \geqslant p(x) \geqslant \frac{1}{2}p_1$ for all $x \geqslant Y$, we have that

$$\int_x^\infty u_k'(y)^2\,dy \leqslant C\exp\{-\beta(x - Y)\} \quad \text{for all } x \geqslant Y \text{ and all } k\,.$$

However, since $u_k \in H_2$,

$$u_k'(x)^2 = 2\int_x^\infty u_k'(y)u_k''(y)\,dy \leqslant 2\Big\{\int_x^\infty u_k'(y)^2\,dy\Big\}^{\frac{1}{2}}\Big\{\int_x^\infty u_k''(y)^2\,dy\Big\}^{\frac{1}{2}} \leqslant C\exp\{-\tfrac{1}{2}\beta(x - Y)\}$$
$$\text{for all } x \geqslant Y \text{ and all } k\,.$$

(*b*) Using part (*a*) and (*E*), it is easy to see that, for each positive integer n, there exists an $X_n \geqslant 0$ such that

$$\int_{X_n}^{\infty} |F_k(u_k, \lambda_k)(x)|^2 \, dx < 2^{-n} \qquad \text{for all } k .$$

Also the compactness of the embeddings for Sobolev spaces over bounded domains and the uniform continuity of f on bounded subsets of $[0, \infty) \times \boldsymbol{R}^3$, together imply that a subsequence $\{(u_{k_i}, \lambda_{k_i})\}$ can be selected such that

$$\int_0^{X_n} |u_{k_i}(x) - u_{k_j}(x)|^2 + |F_{k_i}(u_{k_i}, \lambda_{k_i})(x) - F_{k_j}(u_{k_j}, \lambda_{k_j})(x)|^2 \, dx < 2^{-n}$$
$$\text{for all } k_i, k_j > K_n .$$

Hence, by extracting a suitable diagonal subsequence we see that $\{(u_k, \lambda_k)\}$ contains a subsequence $\{(u_{k'}, \lambda_{k'})\}$ such that

$$\|(S - \xi)(u_{k'} - u_{l'})\| = \|(\lambda_{k'} - \xi)u_{k'} - (\lambda_{l'} - \xi)u_{l'} - F_{k'}(u_{k'}, \lambda_{k'}) + F_{l'}(u_{l'}, \lambda_{l'})\| \to 0$$

as $k', l' \to \infty$, where $\xi < Q$ has been chosen so that $u \mapsto (S - \xi)u$ is a linear homeomorphism of H_2 onto L^2. Therefore we have that $|||u_{k'} - u_{l'}||| \to 0$ as $k', l' \to \infty$ and the proof is complete.

COROLLARY 2.4. *Suppose that f satisfies* A1) *and that, given any $\varepsilon > 0$ and any bounded interval J, there exists an $X \geqslant 0$ such that*

$$(2.19) \qquad |f(x, r, s, \lambda)| \leqslant \varepsilon \quad \textit{whenever} \quad f(x, r, s, \lambda) r < 0 , \qquad x \geqslant X \textit{ and } r, s, \lambda \in J$$

Let U be a closed bounded subset of $\mathcal{S}'$ such that $\sup\{\lambda\colon (u, \lambda) \in U\} < Q$. *Then U is compact and there exists a $Y \geqslant 0$ such that $u(x)u'(x) < 0$ for all $x \geqslant Y$ and all $(u, \lambda) \in U \cap S$.*

PROOF. This follows immediately from Lemma 2.3(*b*) by choosing $f_k = f$ for all k.

LEMMA 2.5. *Suppose that f satisfies* A1). *Then there exists an open neighbourhood U of $(0, \lambda_n)$ in $H_2 \times \mathbf{R}$ such that $N\big((u, \lambda)\big) = n - 1$ for all $(u, \lambda) \in U \cap \mathcal{S}'$.*

PROOF. Suppose that there exists a sequence (v_k, μ_k) in $\mathcal{S}$ converging to $(0, \lambda_n)$ in $H_2 \times \boldsymbol{R}$ and such that $N((v_k, \mu_k)) \neq n-1$ for all k. Setting $w_k(x) = \||v_k\||^{-1} v_k(x)$ for $x \geqslant 0$, we have that

$$\tau w_k(x) + g_k\big(x, w_k(x), w_k'(x), \mu_k\big) = \mu_k w_k(x) \qquad \text{for } x > 0$$
$$w_k(0) = 0$$
$$\||w_k\|| = 1$$

for all k, where $g_k \colon [0, \infty) \times \boldsymbol{R}^3 \to \boldsymbol{R}$ is defined by

$$g_k(x, r, s, \lambda) = \||v_k\||^{-1} f(x, \||v_k\|| r, \||v_k\|| s, \lambda) \qquad \text{for } x \geqslant 0, r, s, \lambda \in \boldsymbol{R}.$$

It is easily checked that $\{g_k\}$ satisfies the condition (E) and that indeed $g_k \to 0$ as $k \to \infty$ uniformly on compact subsets of $[0, \infty) \times \boldsymbol{R}^3$. Hence, using Lemma 2.3$(b)$, we may assume that (w_k, μ_k) contains a subsequence converging strongly to an element (w, μ) in $H_2 \times \boldsymbol{R}$. But then we have $\mu = \lambda_n$ and

$$\tau w(x) = \lambda_n w(x) \qquad \text{for } x > 0$$
$$w(0) = 0$$
$$\||w\|| = 1,$$

showing that $N((w, \lambda_n)) = n-1$.

Furthermore there exists $X > 0$ such that all of the zeros of w and w_k lie in the interval $[0, X)$ for all k. Therefore there exists an open neighbourhood V of w in H_2 such that v has exactly $n-1$ zeros (all of which are simple) in $[0, X)$ for all $v \in V$. However $\{w_k\}$ lies eventually in V but w_k does not have exactly $n-1$ zeros in $[0, X)$ for any k.

This contradiction establishes the result.

LEMMA 2.6. *Suppose that f satisfies* A1). *Then $N((u, \lambda)) = n-1$ for all $(u, \lambda) \in \mathcal{C}_n$.*

PROOF. In view of Lemma 2.5, it is sufficient to prove that, given $(u, \lambda) \in \mathcal{S}$ with $N((u, \lambda)) < \infty$ there exists an open neighbourhood V of (u, λ) in $H_2 \times \boldsymbol{R}$ such that $N((v, \mu)) = N((u, \lambda))$ for all $(v, \mu) \in V \cap \mathcal{S}$. With this in mind, choose $(u, \lambda) \in \mathcal{S}$ with $N((u, \lambda)) < \infty$. Then using (2.3), (2.4) and Proposition 2.1, we see that there exist an $X \geqslant 0$ and an open neighbourhood V

of (u, λ) in $H_2 \times \boldsymbol{R}$ such that

$$\left|\int_x^\infty F(v, \mu)(y)\, v(y)\, dy\right| \leqslant$$

$$\leqslant \max\left\{\frac{|f(x, v(y), v'(y), \mu)|}{(v(y)^2 + v'(y)^2)^{\frac{1}{2}}} : y \geqslant x,\ (v, \mu) \in V\right\} \int_x^\infty \{v(y)^2 + v'(y)^2\}^{\frac{1}{2}} |v(y)|\, dy \leqslant$$

$$\leqslant \frac{1}{2}\int_x^\infty \bigl(q(y) - \mu\bigr) v(y)^2\, dy + \frac{1}{2}\int_x^\infty p(y)\, v'(y)^2\, dy \qquad \text{for all } x \geqslant X \text{ and all } (v, \mu) \in V,$$

where X also has the property that $q(y) - \mu \geqslant \delta > 0$ and $p(y) \geqslant \frac{1}{2}p_1$ for all $y \geqslant X$ and $(v, \mu) \in V$. If $(v, \mu) \in V \cap \mathcal{S}$, this yields

$$0 < \frac{1}{2}\int_x^\infty \bigl(q(y) - \mu\bigr) v(y)^2\, dy + \frac{1}{2}\int_x^\infty p(y)\, v'(y)^2\, dy \leqslant -p(x)\, v'(x)\, v(x) \qquad \text{for all } x \geqslant X .$$

Thus we have shown that there exists an open neighbourhood V of (u, λ) in $H_2 \times \boldsymbol{R}$ such that all the zeros of v lie in $[0, X)$ whenever $(v, \mu) \in V \cap \mathcal{S}$. Choosing V small enough, we can ensure that v has the same number of zeros as u in $[0, X)$ for all $(v, \mu) \in V$ and so the proof is complete.

For smooth enough f, we can say more.

LEMMA 2.7. (Smoothness) *Suppose that f satisfies* AS). *Then there exist an open neihgbourhood U of $(0, \lambda_n)$ in $H_2 \times \boldsymbol{R}$, an open interval $(-\delta, \delta)$ and continuous functions*

$$\varphi\colon (-\delta, \delta) \to \boldsymbol{R} \qquad \textit{and} \qquad w\colon (-\delta, \delta) \to H_2$$

such that

$$\varphi(0) = 0\,, \qquad w(0) = 0$$

and

$$\mathcal{C}_n \cap U = \{\bigl(\alpha u_n + \alpha w(\alpha), \lambda_n + \varphi(\alpha)\bigr)\ \alpha \in (-\delta, \delta)\}$$

where u_n is the unique eigenfunction with

$$u_n \in H_2\,, \qquad |||u_n||| = 1\,, \qquad u_n'(0) > 0 \qquad \textit{and} \qquad Su_n = \lambda_n u_n\,.$$

Furthermore,

$$N((u,\lambda)) = n-1 \quad \textit{and} \quad u'(0) > 0 \qquad \textit{for all } (u,\lambda) \in \mathcal{D}_\mu^+$$

and

$$N((u,\lambda)) = n-1 \quad \textit{and} \quad u'(0) < 0 \qquad \textit{for all } (u,\lambda) \in \mathcal{D}_\mu^-$$

where $\mathcal{D}_n^+$ *and* $\mathcal{D}_n^-$ *denote the components of* $\mathcal{S}'\setminus\{(0,\lambda_n)\}$ *containing the sets* $\{(\alpha u_n + \alpha w, \lambda_n + \varphi(\alpha)) \colon \alpha \in (0,\delta)\}$ *and* $\{(\alpha u_n + \alpha w(\alpha), \lambda_n + \varphi(\alpha) \colon \alpha \in (-\delta, 0)\}$ *respectively.*

PROOF. Note that $(0, \lambda_n)$ is the only trivial solution of (1.2) in $\mathcal{C}_n$. This follows from Lemmas 2.5 and 2.6. Also, given any $(u,\lambda) \in \mathcal{S}$, $u'(0) \neq 0$ and so there exists an open neighbourhood V of (u,λ) in $H_2 \times (-\infty, Q)$ such that $v'(0)u'(0) > 0$ for all $(v,\mu) \in V$. The lemma now follows from Proposition 2.3 and Theorem 1.3.

Henceforth, whenever f satisfies AS), we shall denote $\mathcal{D}_n^+ \cup \{(0,\lambda_n)\}$ and $\mathcal{D}_n^- \cap \{(0,\lambda_n)\}$ by $\mathcal{C}_n^+$ and $\mathcal{C}_n^-$ respectively. Then $\mathcal{C}_n = \mathcal{C}_n^+ \cup \mathcal{C}_n^-$ and $C_n^+ \cap \mathcal{C}_n^- = \{(0,\lambda_n)\}$.

We end this section by stating the global results for (1.2) in the case of spatial decay in f.

THEOREM 2.8. (Spatial decay). *Suppose that f satisfies* A2). *Then the component C_n of $\mathcal{S}'$ has at least one of the following properties.*

1) $\mathcal{C}_n$ *is an unbounded subset of* $H_2 \times \boldsymbol{R}$.

2) $\sup\{\lambda \colon (u,\lambda) \in \mathcal{C}_n\} = Q$.

Furthermore $N((u,\lambda)) = n-1$ *for all* $(u,\lambda) \in \mathcal{C}_n$.

PROOF. The assumptions made concerning τ together with Proposition 2.2(*b*) are sufficient to guarantee that Theorem 1.2 is applicable to (1.2) in $H_2 \times \boldsymbol{R}$. Then Lemma 2.6 shows that $\mathcal{C}_n \cap \mathcal{C}_m = \varphi$ for $m \neq n$ and so we have the desired result.

THEOREM 2.9. (Spatial decay and smoothness). *Suppose that f satisfies* A2) *and* AS). *Then the conclusions of Theorem* 2.8 *remain valid when the component $\mathcal{C}_n$ of $\mathcal{S}'$ is replaced by either of the subsets $\mathcal{C}_n^+$ or $\mathcal{C}_n^-$.*

3. – The general case.

We maintain the notation introduced in the previous section together with the assumptions concerning the linear differential operator τ made there. For each positive integer k, set

$$\theta_k(x) = \begin{cases} 1 & \text{for } 0 \leqslant x \leqslant k \\ k+1-x & \text{for } k < x < k+1 \\ 0 & \text{for } k+1 \leqslant x \end{cases}$$

Then, assuming that f satisfies A1), the Theorem 2.8 applies to the problem

$$\begin{aligned} \tau u(x) + \theta_k(x) f\big(x, u(x), u'(x), \lambda\big) &= \lambda u(x) \quad \text{for } x > 0 \\ u(0) &= 0\,, \end{aligned} \tag{3.1}$$

for each k. Our aim is to show that the set obtained by « taking the limit » as $k \to \infty$ of the components of solutions of (3.1) discussed in Theorem 2.8 contains a component with the desired properties.

LEMMA 3.1. *Suppose that f satisfies* A1) *and* (2.19). *Let V be an open bounded subset of $H_2 \times \boldsymbol{R}$ such that $(0, \lambda_n) \in V$ and* $\sup\{\lambda\colon (u, \lambda) \in V\} < Q$. *Then $\mathcal{S}' \cap \partial V \neq \varphi$. ($\partial V$ denotes the boundary of V in $H_2 \times \boldsymbol{R}$.)*

PROOF. It follows from Theorem 2.8 that there exists a sequence $\{(u_k, \mu_k)\}$ in ∂V such that (u_k, μ_k) is a non-trivial solution of (3.1), u_k having exactly $n-1$ zeros in $(0, \infty)$. But it is easily checked that the sequence $\{f_k\}$, defined by

$$f_k(x, r, s, \lambda) = \theta_k(x) f(x, r, s, \lambda) \qquad \text{for } x > 0 \text{ and } r, s, \lambda \in R\,,$$

satisfies (E). Hence, according to Lemma 2.3, there exists a subsequence $\{(u_{k'}, \mu_{k'})\}$ converging strongly in $H_2 \times \boldsymbol{R}$ to an element $(u, \mu) \in H_2 \times \boldsymbol{R}$. Clearly (u, μ) satisfies (1.2) and $(u, \mu) \in \partial V$. If $(u, \lambda) \in \mathcal{S}$ the proof is complete and so we suppose that $|||u_{k'}||| \to 0$ as $k' \to \infty$ and set

$$w_{k'}(x) = |||u_{k'}|||^{-1} u_{k'}(x) \qquad \text{for } x \geqslant 0\,.$$

Then $(w_{k'}, \mu_{k'}) \in H_2 \times \boldsymbol{R}$ and is a solution of

$$\begin{aligned} \tau u(x) + g_{k'}\big(x, u(x), u'(x), \lambda\big) &= \lambda u(x) \quad \text{for } x > 0 \\ u(0) &= 0 \end{aligned}$$

with $|||w_k||| = 1$, where g_k is defined by,

$$g_k(x, r, s, \lambda) = |||u_k|||^{-1} f_k\big(x, |||u_k|||r, |||u_k|||s, \lambda\big)$$

for $x > 0$ and $r, s, \lambda \in \boldsymbol{R}$.

However $\{g_k\}$ also satisfies (E) and indeed $g_k \to 0$ as $k \to \infty$ uniformly on compact subsets of $[0, \infty) \times \boldsymbol{R}^3$. Hence, again referring to Lemma 2.3, we can select the subsequence $\{(u_{k'}, \mu_{k'})\}$ such that $\{(w_{k'}, \mu_{k'})\}$ converges strongly in $H_2 \times \boldsymbol{R}$ to an element (w, μ) where

$$\tau w(x) = \mu w(x) \quad \text{for } x > 0$$

$$w(0) = 0 \quad \text{and} \quad |||w||| = 1 .$$

This shows that μ is an eigenvalue of S and so $(0, \mu) \in \mathcal{S}'$, completing the proof of the lemma.

As usual we find it convenient to employ the following result proved in Whyburn [11].

LEMMA 2.2. *Suppose that T_1 and T_2 are closed subsets of a compact metric space T such that there is no connected subset of T which intersects both T_1 and T_2. Then there exist disjoint compact sets K_1 and K_2 such that*

$$T = K_1 \cup K_2 , \qquad T_1 \subset K_1 \qquad \textit{and} \qquad T_2 \subset K_2 .$$

THEOREM 3.3. *Suppose that f satisfies* A1) *and* (2.19). *Then the component $\mathcal{C}_n$ of $\mathcal{S}'$ has at least one of the following properties.*

1) *$\mathcal{C}_n$ is an unbounded subset of $H_2 \times \boldsymbol{R}$*

2) $\sup\{\lambda\colon (u, \lambda) \in \mathcal{C}_n\} = Q$.

Furthermore, $N((u, \lambda)) = n - 1$ for all $(u, \lambda) \in \mathcal{C}_n$.

PROOF. Let U be an open bounded subset of $H_2 \times \boldsymbol{R}$ such that $(0, \lambda_n) \in U$ and $\sup\{\lambda\colon (u, \lambda) \in U\} < Q$. It follows from Corollary 2.4 that $T \equiv \overline{U} \cap \mathcal{S}'$ is a compact metric space where $\overline{U}$ denotes the closure of U in $H_2 \times \boldsymbol{R}$. Also $T_1 = \mathcal{S}' \cap \partial U$ and $T_2 = \{(0, \lambda_n)\}$ are closed subspaces of T. Suppose that there is not a connected subset of T which intersects both T_1 and T_2. Then, by Lemma 3.2 there exist disjoint compact sets K_1 and K_2 such that $T = K_1 \cup K_2$, $T_1 \subset K_1$ and $T_2 \subset K_2$. Let $V = \{(u, \lambda) \in H_2 \times \boldsymbol{R}\colon |||u - v|||^2 + |\lambda - \mu|^2 < (1/16)\gamma^2$ for some $(v, \mu) \in K_2\}$ where $\gamma = \operatorname{dist}(K_1, K_2 \cup \partial U) > 0$. Then V is an open bounded subset of $H_2 \times \boldsymbol{R}$ such that $(0, \lambda_n) \in V$, $\sup\{\lambda\colon (u, \lambda) \in V\} < Q$ and $\mathcal{S}' \cap \partial V = \emptyset$. This contradicts Lemma 3.1 and so we have that $\mathcal{C}_n \cap \partial U \neq \emptyset$. Recalling Lemma 2.6, we see that the proof is complete.

REMARK 1. The hypotheses A1) and (2.19) are satisfied by f provided that

$$f(x, r, s, \lambda) = f_1(x, r, s, \lambda) + f_2(x, r, s, \lambda) \quad \text{for } x \geqslant 0 \text{ and } r, s, \lambda \in \boldsymbol{R},$$

where f_1 satisfies A2) and f_2 satisfies A1) and $f_2(x, r, s, \lambda)r \geqslant 0$ for all $x \geqslant 0$ and $r, s, \lambda \in \boldsymbol{R}$. In particular, the function f_2 has the desired properties provided that it is in the form

$$f_2(x, r, s, \lambda) = rh(x, r, s, \lambda) \quad \text{for } x \geqslant 0 \text{ and } r, s, \lambda \in \boldsymbol{R}$$

where $h\colon [0, \infty) \times \boldsymbol{R}^3 \to \boldsymbol{R}$ is a non-negative continuous function such that

$$h(x, 0, 0, \lambda) = 0 \quad \text{for all } x \geqslant 0 \text{ and all } \lambda \in \boldsymbol{R}.$$

REMARK 2. The hypotheses of Theorem 3.3 do not imply that the Nemytskii operator induced by the non-linearity in (1.2) is necessarily a bounded mapping from H_2 into L^2. However it is often the case that this Nemytskii operator is indeed a bounded continuous mapping of H_2 with some topology even weaker than that induced by $|||\ |||$. For example, if f is independent of s and, given a bounded interval J,

$$\sup\{|f(x, r, s, \lambda)|\colon x \geqslant 0 \text{ and } r, \lambda \in J\} < \infty,$$

then, as in the remark following Proposition 2.2, we have that $F(\cdot, \lambda)\colon H_2 \to L^2$ is a bounded and continuous mapping for each $\lambda \in \boldsymbol{R}$ where we can consider H_2 to have the topology induced by the norm

$$\|u\|_2 = \left(\int_0^\infty \{u(x)^2 + u'(x)^2\}\, dx\right)^{\frac{1}{2}}.$$

Consequently, under these assumptions, the property 1) in Theorem 3.3 can be replaced by

$$\sup\{\|u\|_2 + |\lambda|\colon (u, \lambda) \in \mathcal{C}_n\} = \infty.$$

Let us consider now the improvements available when f is smooth.

LEMMA 3.5. *Suppose that f satisfies* AS) *and* (2.19). *Let V be an open bounded subset of $H_2 \times \boldsymbol{R}$ such that $(0, \lambda_n) \in V$ and* $\sup\{\lambda\colon (u, \lambda) \in V\} < Q$. *Then*

$$\mathcal{S}'_+ \cap \partial V \neq \emptyset \quad \textit{and} \quad \mathcal{S}'_- \cap \partial V \neq \emptyset$$

where

$$\mathcal{S}'_+ = \mathcal{S}' \setminus \{(u, \lambda) \in \mathcal{S}:\ u'(0) < 0\} \quad \textit{and} \quad \mathcal{S}'_- = \mathcal{S}' \setminus \{(u, \lambda) \in \mathcal{S}:\ u'(0) > 0\}.$$

PROOF. Given that f satisfies AS), it follows from Theorem 2.9 that the sequence $\{(u_k, \mu_k)\}$ in ∂V chosen at the beginning of Lemma 3.1 can be selected so as to have the additional property that $u_k \in \mathcal{S}'_+$ for all k. As in Lemma 3.1, it then follows that it contains a subsequence converging to an element (u, λ) in $\mathcal{S}'$. However $\mathcal{S}'_+$ is clearly a closed subset of $\mathcal{S}'$ and so indeed $(u, \lambda) \in \mathcal{S}'_+$.

Similarly we see that $\mathcal{S}'_- \cap \partial V \neq \varphi$.

THEOREM 3.6. *Suppose that f satisfies* AS) *and* (2.19). *Then the conclusions of Theorem* 3.3 *remain valid when the component $\mathcal{C}_n$ of $\mathcal{S}'$ is replaced by either of its subsets $\mathcal{C}_n^+$ or $\mathcal{C}_n^-$.*

PROOF. Replacing S' by S'_+ of S'_- respectively in the proof of Theorem 3.3 and using Lemma 3.5 instead of Lemma 3.1, proves the result.

REMARK 3. As in the case of a compact interval, [1]-[6], the properties of the components can be further investigated if more is assumed about the structure of f. For example, comparison arguments can be used to find bounds on the range of λ covered by components and « a priori » L^2-bounds can be found under appropriate hypotheses. Details are given in [13]. Here we give only one result which can be obtained from such considerations.

THEOREM 3.7. *Suppose that*

$$f(x, r, s, \lambda) = r g(x, r, \lambda) \quad \textit{for all } x \geq 0 \textit{ and } r, s, \lambda \in \boldsymbol{R}$$

where $g\colon [0, \infty) \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is a continuous function such that

(*a*) *$g(x, r, \lambda) \to 0$ as $r \to 0$ uniformly for λ in bounded intervals,*

(*b*) *$g(x, r, \lambda) \geq 0$ for all $x \geq 0$, $r \in \boldsymbol{R}$ and $\lambda \in (\alpha, \beta)$,*

(*c*) *$g(x, r, \lambda) \to \infty$ as $|r| \to \infty$ uniformly for $x \geq 0$ and λ in compact subsets of (α, β).*

In addition, suppose that f satisfies AS) *and that $\alpha < \lambda_n < \lambda_m < \beta$. Then, for each fixed $\lambda \in (\lambda_{n+i}, \lambda_{n+i+1}]$ where $i = 0, 1, \ldots, m - n - 1$, the equation* (1.2) *has at least $2(i+1)$ distinct non-trivial solutions $\{u_j^\pm : j = 0, 1, \ldots, i\}$* in H_2 *where u_j has exactly $n + j - 1$ zeros in $(0, \infty)$, $u_j^{+\prime}(0) > 0$ and $u_j^{-\prime}(0) < 0$.*

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