ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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On an initial-boundary value problem for the equation $w_t = w_{xx} - xw_y$

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, nº 2 (1975), p. 219-263

<http://www.numdam.org/item?id=ASNSP_1975_4_2_2_219_0>

On an Initial-Boundary Value Problem for the equation $w_t = w_{xx} - xw_y$ (*).

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Summary. – *The following problem is considered: to find a function $w(x, y, t)$ in the region $\Omega \equiv \{0 < t < T,\ y > 0,\ -\infty < x < +\infty\}$ which is a solution of the equation $w_t = w_{xx} - xw_y$, with the boundary conditions: $w(x, y, 0) = f(x, y)$, where f is a given function defined for $-\infty < x < +\infty$, $y \geqslant 0$ and $w(x, 0, t) = 0$ for $x > 0$, $0 \leqslant t < T$. We prove the existence and uniqueness of a solution w, which is a continuously differentiable function of t, with values in a certain Banach space. The uniqueness assertion follows from a suitable a priori estimate of the solution; the existence theorem is proved by means of the semigroup theory, as an application of the Hille-Yosida theorem. Then the main part of the paper contains the study of the stationary equation: $u_{xx} - xu_y - ku = f$ $(k > 0)$. This is a forward-backward parabolic equation, and it is studied via a Wiener-Hopf technique. This procedure requires the study of the associated parabolic equation of evolution: $u_{xx} - |x| u_y - ku = f$ $(k > 0)$ and of a certain integral equation of a Wiener-Hopf type.*

1. – Introduction.

In this paper a boundary problem for the simple-looking equation

$$w_t = w_{xx} - xw_y \tag{1.1}$$

is considered in the region $\Omega \equiv \{0 < t < T,\ y > 0,\ -\infty < x < +\infty\}$; the solution w satisfies the initial condition $w(x, y, 0) = f(x, y)$ and the boundary condition $w(x, 0, t) = h(x, t)$ only for $x > 0$ (and every $0 \leqslant t < T$).

(*) This research was partly developed at the University of California at Berkeley (supported by a scholarschip of the italian C.N.R.) and partly at the Politecnico di Milano (Italy) (supported by the « contratto di ricerca sulle Equazioni Funzionali » from the C.N.R.

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Pervenuto alla Redazione il 4 Dicembre 1974.

Such a problem arises when studying a randomly accelerated particle moving on the half-line $y > 0$ (see [1] for the analogous stationary problem); then eq. (1.1) is the backward equation for the vector Markov process $(r(t), v(t))$, where $r(t)$ is the position of the particle at time t, $v(t)$ is its velocity.

We can notice also that eq. (1.1) is a linearized model of the (onedimensional) kinetic equation, when the collision term is simply approximated by the differential operator $\partial^2/\partial x^2$. In this case the function w has the meaning of the (deviation from the equilibrium of the) distribution function of the molecules at the time t, position y, and velocity x; the gas fills the half-space $y > 0$, has the initial distribution f, and has a distribution h on the boundary $y = 0$, but this distribution is assigned only for the emerging molecules $(x > 0)$.

From a mathematical point of wiew, (1.1) is an equation of the kind

$$w_t = Lw \equiv \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} + \sum_{i=1}^{n} a_i w_{x_i} + aw \tag{1.2}$$

where rank $\|a_{ij}\| \leqslant n-1$. Such equations (sometime called in the literature ultraparabolic) have been considered especially in connection with the Cauchy problem: see, e.g., Kolmogorov [4], Weber [13], Il'in [5], Oleinik [6] and others. Conversely, there are few papers dealing with mixed (initial and boundary) problems: see [2], [11]; see also [7] recalled below. On the other hand, we note that boundary problems for the so-called elliptic-parabolic equations (i.e., (second order) equations with nonnegative characteristic form) which are obviously a wider class then (1.2), have been largely considered by various authors (see the book [7] for a detailed bibliography): in particular Kohn and Nirenberg [3] give an existence and uniqueness theorem for the first boundary value problem in Sobolev spaces; but some restrictions imposed to the boundaries make their results inapplicable, e.g., to the problem outlined above for eq. (1.1), as well as for the most interesting problems arising in the theory of brownian motion or in kinetic theory.

When discussing eq. (1.1), our aim is to find solutions $w(t)$ which are continuously differentiable functions of t, with values in a certain Banach space; w, as a vector in this space, has generalized derivatives w_x, w_{xx} and w_y, which are square integrable with suitable weights. In such a class, a uniqueness theorem will be easily proved: it will follow from an a priori estimate of the solution. An existence theorem will be proved by means of the semigroup theory, as an application of the Hille-Yosida theorem. Thus, we have to discuss first the stationary equation,

$$u_{xx} - xu_y - ku = f \qquad (k > 0)\,. \tag{1.3}$$

This is a forward-backward parabolic equation, and we will treat it by the same technique already used in [10] (see also [8]), which involves the discussion of an integral equation of the Wiener-Hopf type.

In Sect. 2 we list some functional spaces and state some trace theorems. In Sect. 3 we discuss an evolution equation associated to (1.3), i.e.

$$u_{xx} - |x|u_y - ku = f \qquad (k > 0)\,. \tag{1.4}$$

The results obtained in this section are preliminary to the discussion of eq. (1.3) in Sect. 4. Finally, in Sect. 5, the results previously obtained are applied to the study of eq. (1.1).

2. – Function spaces and trace theorems.

Let us define first some functional spaces used in the following. R^2 will denote the whole plane and (when this plane is referred to an orthogonal cartesian system) X_+ is the half-plane $x > 0$, Y_+ the half-plane $y > 0$. R denotes the real line, $R_{+(-)}$ the positive (negative) half-line.

$C_0^\infty(R^2)$ denotes, as usual, the space of infinitely differentiable functions, defined on R^2, with compact support. $C_0^\infty(\bar{G})$, where G is an open subset of R^2, is the space of the restrictions to $\bar{G}$ (the closure of G) of functions belonging to $C_0^\infty(R^2)$.

$Z(G)$ is the set of complex-valued functions u, defined on G, such that u, u_x, u_{xx}, xu_y (the derivatives of u are taken in the sense of distributions) belong to $L^2(G)$.

We can make two simple *remarks*: i) $Z(G)$, equipped with the norm,

$$\|u\|_{Z(G)} = \left\{ \iint_G (x^2|u_y|^2 + |u_{xx}|^2 + |u_x|^2 + |u|^2)\, dx\, dy \right\}^{1/2} \tag{2.1}$$

is a (complete) Banach space: ii) $C_0^\infty(\bar{G})$ is dense in $Z(G)$, provided G is sufficiently smooth; the cases of main interest for us will be $G = X_+$ or $G = Y_+$.

$W(Y_+)$ is a subset of $Z(Y_+)$; it is defined by the functions u with the properties

i) $u \in Z(Y_+)$

ii) $\displaystyle \inf_{h>0} \operatorname*{ess\,sup}_{0<y<h} \int_{-\infty}^{+\infty} x^2 |u|^2\, dx < \infty\,.$

$W_+(Y_+)$ is quite analogous to the previous one; it is defined by functions u such that

i) $u \in Z(Y_+)$

ii) $\inf_{h>0} \operatorname{ess\,sup}_{0<y<h} \int_0^\infty x^2 |u|^2 \, dx < \infty$.

For $W(Y_+)$ and $W_+(Y_+)$ we can make *remarks* analogous to those made for Z, i.e.: i) $W(Y_+)[W_+(Y_+)]$ is a (complete) Banach space (equipped with the norm

$$\|u\|^2_{W(Y_+)} = \|u\|^2_{Z(Y_+)} + \inf_{h>0} \operatorname{ess\,sup}_{0<y<h} \int_{-\infty}^{+\infty} x^2 |u|^2 \, dx \tag{2.2}$$

[and an analogous definition for $\|\cdot\|_{W_+(Y_+)}$]) and ii) $C_0^\infty(\overline{Y}_+)$ is dense in $W(Y_+)[W_+(Y_+)]$.

We will use also the notation $W_+(X_+ \cap Y_+)$ for the set of functions u defined in the first quadrant of R^2, belonging to $Z(X_+ \cap Y_+)$, with the same property ii) in the definition of $W_+(Y_+)$.

We will consider the usual Sobolev spaces $H^s(I)$ (I is an open subset of R; s real) and the space

$\mathcal{H}^1(I)$: the set of functions φ, defined on I, such that the following norm

$$\|\varphi\|_{\mathcal{H}^1(I)} = \left\{ \int_I [(|x| + x^2)|\varphi|^2 + |x||\varphi'|^2] \, dx \right\}^{1/2} \tag{2.3}$$

is finite.

Remark. Let us point out that $\mathcal{H}^1(R_+) \subset L^2(R_+)$; the assertion may be deduced from the following inequality

$$\int_0^\infty |\varphi|^2 \, dx \leqslant 2 \Big(\int_0^\infty x|\varphi|^2 \, dx \Big)^{1/2} \Big(\int_0^\infty x|\varphi'|^2 \, dx \Big)^{1/2}$$

which may be proved by integration by parts. Similarly, we have also $\mathcal{H}^1(R) \subset L^2(R)$.

Now we can state the following theorems (trace theorems) which characterize the traces of functions u of class Z or W on the coordinate axes.

Theorem 2.1. *The following operator*

$$C_0^\infty(\overline{X}_+) \ni u \to u(0, \cdot) \equiv [\text{restriction of } u \text{ to the line } x = 0] \tag{2.4}$$

is a densely defined bounded operator from $Z(X_+)$ *into* $H^{1/2}(R)$. *Let*

$$\gamma_1 \colon Z(X_+) \to H^{1/2}(R)$$

be its (uniquely defined) linear bounded extension. Then the range of γ_1 *is exactly* $H^{1/2}(R)$ *and* γ_1 *is an isomorphism between* $Z(X_+)/(\mathrm{Kernel}\,\gamma_1)$ *and* $H^{1/2}(R)$.

THEOREM 2.2. *The operator*

$$C_0^\infty(\overline{X}_+) \ni u \to u_x(0,\cdot) \equiv [\text{restriction of } u_x \text{ to the line } x=0] \tag{2.5}$$

is a densely defined bounded operator from $Z(X_+)$ *into* $H^{1/6}(R)$. *Let*

$$\gamma_2 \colon Z(X_+) \to H^{1/6}(R)$$

be its (uniquely defined) linear bounded extension. Then the range of γ_2 *is exactly* $H^{1/6}(R)$ *and* γ_2 *is an isomorphism between* $Z(X_+)/(\mathrm{Kernel}\,\gamma_2)$ *and* $H^{1/6}(R)$.

The boundedness of (2.4) and (2.5) will follow from Lemma 2.3 below. The other assertion (about the invertibility) will follow from Ths. 3.2 and 3.3 of Sec. 3.

LEMMA 2.1. *Let* v *be a complex-valued function belonging to* $C^2(R_+)$ *and such that* xv *and* v'' *belong to* $L^2(R_+)$. *Then the following inequality holds*

$$|v(0)| \leqslant C_1\Big(\int_0^\infty x^2|v|^2\,dx\Big)^{1/4}\Big(\int_0^\infty |v''|^2\,dx\Big)^{1/4} \tag{2.6}$$

where the constant C_1 *(which is the best possible) is given by formula* (2.13) *below*.

PROOF. The proof of this lemma can be achieved by a standard procedure. We give here only few accounts of the calculations.

Let v be real-valued (for convenience). If one is not interested in finding the best possible constant, an inequality like (2.6) can be showed directly by putting $g = v'' - xv$ and representing v in the form

$$v(x) = v(0)\,v_1(x) + Hg(x)\,; \tag{2.7}$$

here v_1 is the solution of the homogeneous equation $v_1'' - xv_1 = 0$ such that $xv_1 \in L^2(R_+)$ and so normalized that $v_1(0) = 1$: we get

$$v_1(x) = \pi^{-1}3^{1/6}\Gamma(\tfrac{2}{3})\,x^{1/2}K_{1/3}(\tfrac{2}{3}x^{3/2})$$

Hg is the solution of the non-homogeneous equation vanishing at the origin: H is the integral operator whose kernel is given by

$$H(x, y) = -\tfrac{2}{3}(xy)^{1/2} I_{1/3}(\tfrac{2}{3}x^{3/2}) K_{1/3}(\tfrac{2}{3}y^{3/2}) \qquad \text{for } x < y;$$

change x with y to obtain the expression of $H(x, y)$ for $x > y$. $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of order ν. By Schwartz inequality one easily proves that

$$\|Hg\|_{L^2(R_+;x^2)} \leq \text{const} \|g\|_{L^2(R_+)} \tag{2.8}$$

where $\|\cdot\|_{L^2(R_+;x^2)}$ means $\left(\int_0^\infty (\cdot)^2 x^2\, dx\right)^{1/2}$. Then, from the representation (2.7) and (2.8) we have

$$|v(0)| \|v_1\|_{L^2(R_+;x^2)} \leq \text{const}\left(\|v\|_{L^2(R_+;x^2)} + \|g\|_{L^2(R_+)}\right);$$

thus we get, since $g = v'' - xv$,

$$|v(0)| \leq \text{const}\left(\|v\|_{L^2(R_+;x^2)} + \|v''\|_{L^2(R_+)}\right). \tag{2.9}$$

From (2.9) one obtains an inequality like (2.6) via standard arguments of dimensional analysis.

In order to get the best possible constant appearing in (2.9) one has to look for the maximum of the functional

$$J(v) = v^2(0)\left[\int_0^\infty x^2 v^2\, dx + \int_0^\infty v''^2\, dx\right]^{-1}.$$

This problem has a unique solution; for, let V be the set of functions v belonging to $C^2(R_+)$ and such that $\int_0^\infty x^2 v^2\, dx$ and $\int_0^\infty v''^2\, dx < \infty$ and X the set of $v \in V$ with $v(0) = 1$. V is an Hilbert space (with obvious definition of the scalar product) and X is a convex subset of V; X is also closed (thanks to (2.9)). Now the problem to find the maximum of J is equivalent to project the origin of V on X; thus, this problem has a unique solution.

The maximizing function is the solution (in V) of the following problem (Euler equation):

$$v^{(4)} + x^2 v = 0, \qquad v''(0) = 0; \tag{2.10}$$

let us put $\lambda = \max J(v)$; λ is then given by $v(0)/v'''(0)$. The most general solution v of the differential equation appearing in (2.10) is given by

$$v(x) = \mathrm{Re}\left[a\int_{\Gamma} z^{1/2}\exp(-xz)H^{(1)}_{1/6}(z^3/3)\,dz\right].$$

Here $H^{(1)}(z)$ is the first Bessel function of the third kind (Hankel's function); a is an arbitrary complex constant; Γ is the path in the complex z-plane sketched in the figure.

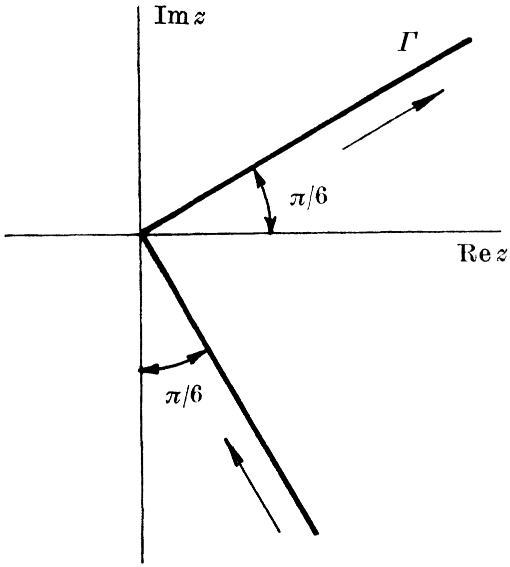

Figure 1.

Now, imposing the condition $v''(0) = 0$ and calculating λ gives

$$\lambda = -\frac{\mathrm{Im}\,I(5/2)\,\mathrm{Re}\,I(1/2) - \mathrm{Im}\,I(1/2)\,\mathrm{Re}\,I(5/2)}{\mathrm{Im}\,I(5/2)\,\mathrm{Re}\,I(7/2) - \mathrm{Im}\,I(7/2)\,\mathrm{Re}\,I(5/2)} \tag{2.11}$$

where we put

$$I(\nu) = \int_{\Gamma} z^{\nu} H^{(1)}_{1/6}(z^3/3)\,dz. \tag{2.12}$$

These numbers can be explicitly calculated, and we get

$$\lambda = \frac{\Gamma^2(1/3)\,\Gamma^2(1/6)}{8\pi^2}.$$

The constant C_1 appearing in (2.6) is related to λ by

$$C_1 = (2\lambda)^{1/2} = \frac{\Gamma(1/3)\Gamma(1/6)}{2\pi} . \tag{2.13}$$

LEMMA 2.2. *Let v be as in lemma 2.1; then the following inequality holds*

$$|v'(0)| \leqslant C_2\Big(\int_0^\infty x^2|v|^2\,dx\Big)^{1/12}\Big(\int_0^\infty |v''|^2\,dx\Big)^{5/12} \tag{2.14}$$

where the constant C_2 (which is the best possible) is given by formula (2.15) *below.*

PROOF. It is quite analogous to the previous one. To calculate C_2 one has now to solve the problem (in the same class as before)

$$v^{(4)} + x^2 v = 0 , \qquad v'''(0) = 0$$

and calculate $\mu = -v'(0)/v''(0)$. One finds

$$\mu = \frac{\mathrm{Re}\, I(3/2)\,\mathrm{Im}\, I(7/2) - \mathrm{Re}\, I(7/2)\,\mathrm{Im}\, I(3/2)}{\mathrm{Re}\, I(5/2)\,\mathrm{Im}\, I(7/2) - \mathrm{Re}\, I(7/2)\,\mathrm{Im}\, I(5/2)} \tag{2.14'}$$

where the numbers $I(\nu)$ are given by (2.12). We get

$$\mu = \frac{3^{1/6}\Gamma(1/6)}{2\pi^{1/2}} .$$

The constant C_2 is then given by

$$C_2 = 6^{1/2}\, 5^{-5/12}\, \mu^{1/2} = \frac{3^{7/12}\Gamma^{1/2}(1/6)}{5^{5/12}\,\pi^{1/4}} . \tag{2.15}$$

LEMMA 2.3. *Let $u \in C_0^\infty(\overline{X}_+)$; let $v(x, \eta)$ be the Fourier transform (with respect to the variable y) of $u(x, y)$; then we have*

$$\int_{-\infty}^{+\infty} |\eta|\,|v(0, \eta)|^2\,d\eta \leqslant C_1^2\Big(\iint_{X_+} x^2|u_y|^2\,dx\,dy\Big)^{1/2}\cdot\Big(\iint_{X_+} |u_{xx}|^2\,dx\,dy\Big)^{1/2} \tag{2.16}$$

$$\int_{-\infty}^{+\infty} |\eta|^{1/3}|v_x(0, \eta)|^2\,d\eta \leqslant C_2^2\Big(\iint_{X_+} x^2|u_y|^2\,dx\,dy\Big)^{1/6}\cdot\Big(\iint_{X_+} |u_{xx}|^2\,dx\,dy\Big)^{5/6} \tag{2.17}$$

where C_1 and C_2 are the constants defined in (2.13) *and* (2.15).

PROOF. Let us apply inequality (2.6) to the function $v(\cdot, \eta)$; square both members of the inequality, multiply by $|\eta|$ and integrate over η; by applying Hölder's inequality one gets

$$\int_{-\infty}^{+\infty} |\eta||v(0,\eta)|^2 d\eta \leq C_1^2 \Big(\int_{-\infty}^{+\infty} d\eta \int_0^{\infty} x^2 |\eta v|^2 dx \Big)^{1/2} \cdot \Big(\int_{-\infty}^{+\infty} d\eta \int_0^{\infty} |v_{xx}|^2 dx \Big)^{1/2}.$$

Now, from Parseval's formula, one gets (2.16). (2.17) is proved in a similar way.

THEOREM 2.3. *The operator*

$$(2.18) \qquad C_0^\infty(\overline{Y}_+) \ni u \to [\text{restriction of } u \text{ to the half-line } y=0,\ x>0]$$

is a densely defined bounded operator from $W_+(Y_+)$ into $\mathcal{H}^1(R_+)$.
Let

$$\gamma_+ : W_+(Y_+) \to \mathcal{H}^1(R_+)$$

be its (uniquely defined) linear bounded extension. Then the range of γ_+ is exactly $\mathcal{H}^1(R_+)$ and γ_+ is an isomorphism between $W_+(Y_+)/(\text{Kernel } \gamma_+)$ and $\mathcal{H}^1(R_+)$.

The boundedness of (2.18) will follow from Lemma 2.4 below. The other assertion will follow from Thm. 4.3.

LEMMA 2.4. *Let $u \in C_0^\infty(\overline{Y}_+)$; let h be the restriction of u to the x-axis. Then the following inequalities hold*

$$(2.19) \qquad \int_{-\infty}^{+\infty} |x||h(x)|^2 dx \leq 2 \Big(\iint_{Y_+} x^2 |u_y|^2 dx\, dy \Big)^{1/2} \cdot \Big(\iint_{Y_+} |u|^2 dx\, dy \Big)^{1/2}$$

$$(2.20) \qquad \int_{-\infty}^{+\infty} |x||h'(x)|^2 dx \leq \text{const} \Big(\iint_{Y_+} x^2 |u_y|^2 dx\, dy \Big)^{1/2} \cdot \Big(\iint_{Y_+} |u_{xx}|^2 dx\, dy \Big)^{1/2}.$$

PROOF. (2.19) immediately follows from the identity

$$\int_{-\infty}^{+\infty} |x||h(x)|^2 dx = -\iint_{Y_+} |x| \frac{\partial}{\partial y}(u\bar{u})\, dx\, dy = 2\,\text{Re} \iint_{Y_+} |x| u_y \bar{u}\, dx\, dy$$

by applying Schwartz inequality.

To prove (2.20) we need a result that will be proved later. For the moment, let us suppose $u_x(0,\cdot)=0$; by integration by parts, we get

$$\operatorname{Re}\int_0^\infty dy\int_0^\infty x u_{xx}\bar{u}_y\,dx = -\operatorname{Re}\int_0^\infty dy\int_0^\infty x u_x\bar{u}_{yx}dx - \operatorname{Re}\int_0^\infty dy\int_0^\infty u_x\bar{u}_y\,dx =$$

$$=\frac{1}{2}\int_0^\infty x|h'(x)|^2\,dx - \operatorname{Re}\int_0^\infty\int_0^\infty\left(\frac{u_x}{x}\right)(x\bar{u}_y)\,dx\,dy\,.$$

By using now Schwartz and Hardy's inequalities, we get

$$\int_0^\infty x|h'(x)|^2\,dx \leqslant 2\Big(\int_0^\infty\int_0^\infty x^2|u_y|^2\,dx\,dy\Big)^{1/2}\Big(\int_0^\infty\int_0^\infty |u_{xx}|^2\,dx\,dy\Big)^{1/2} +$$

$$+\,2\Big(\int_0^\infty\int_0^\infty x^{-2}|u_x|^2\,dx\,dy\Big)^{1/2}\Big(\int_0^\infty\int_0^\infty x^2|u_y|^2\,dx\,dy\Big)^{1/2}\leqslant$$

$$\leqslant 6\Big(\int_0^\infty\int_0^\infty x^2|u_y|^2\,dx\,dy\Big)^{1/2}\Big(\int_0^\infty\int_0^\infty |u_{xx}|^2\,dx\,dy\Big)^{1/2}.$$

From this inequality we immediately derive (2.21) if $u_x(0,\cdot)=0$; for the general case, the proof will be given in the corollary to Th. 3.3.

3. – The forward equation.

This section, which has an auxiliary character in view of the next one, contains some results for the parabolic equation (1.4), which may be of some interest by themselves. First, in the subsection 3.1, we will find solutions of (1.4) defined in all R^2; then, for the associated homogeneous equation, we will consider the first (and second) boundary value problem in X_+ (subsect. 3.2) and the Cauchy problem in the half-plane Y_+ (subsect. 3.3). Let us notice that we are dealing with a parabolic equation of evolution type degenerating on a line and the problems we will speak about are the classical ones.

Let us write eq. (1.4) in the standard form

$$u_y = -Au - f\,. \tag{3.1}$$

Here A is the linear second-order differential operator $-(1/|x|)(\partial^2/\partial x^2 - k)$ $(k > 0)$, acting in some Hilbert space H. In [9], where a class of equations similar to (3.1) were considered, we choosed as H the space $L^2(R; |x|)$, i.e., the space of square summable functions with the weight $|x|$. This choice is, in some sense, « natural », since A is symmetric in $L^2(R; |x|)$; but it turns not to be the good one in view of the applications of eq. (3.1) to the nonstationary equation (1.1).

Now we will take for H the space $L^2(R; x^2)$ (our solutions u must have the property that xu_y is square summable). Unfortunately, in this space, A is not symmetric nor dissipative; and this fact will give some complications in the discussion of the Cauchy problem for eq. (3.1) (see subsect. 3.3).

3.1. *Solutions defined in the whole plane.*

THEOREM 3.1. *Let $f \in L^2(R^2)$ be a given function; there exists only one solution u to eq.* (1.4) *belonging to $Z(R^2)$. This solution satisfies the inequality*

$$(3.2) \qquad \iint_{R^2} (x^2|u_y|^2 + |u_{xx}|^2 + k|u_x|^2 + k^2|u|^2)\, dx\, dy \leqslant \text{const} \iint_{R^2} |f|^2\, dx\, dy\,.$$

Moreover, if $f = 0$ for $y < 0$, this solution vanishes on the lower half-plane and its restriction to the upper half-plane belongs to $W(Y_+)$ and satisfies the inequality

$$(3.3) \qquad \int_{-\infty}^{+\infty} x^2 |u(x, y)|^2\, dx \leqslant \text{const}\, y \iint_{Y_+} |f|^2\, dx\, dy\,. \text{ (*)}$$

PROOF. *Uniqueness.* It follows from the inequality

$$(3.4) \qquad \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\bar{y}} (k^2|u|^2 + 2k|u_x|^2)\, dy \leqslant \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\bar{y}} |f|^2\, dy$$

which can be derived a priori. For, let $u \in C_0^\infty(R^2)$ and put $f = u_{xx} - ku - |x|u_y$. By taking the square of the modulus of f and neglecting the nonnegative term $\left|u_{xx} - |x|u_y\right|^2$ we get

$$k^2|u|^2 - 2k \operatorname{Re}\left(u_{xx} - |x|u_y\right)\bar{u} \leqslant |f|^2.$$

(*) All along this paper, const means an absolute constant; sometimes we will write const(k) to indicate a constant depending on k.

Integrating now over the half-plane $y<\bar{y}$, using integration by parts, gives

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{\bar{y}} (k^2|u|^2 + 2k|u_x|^2)\, dy \leqslant \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\bar{y}} |f|^2\, dy - k\int_{-\infty}^{+\infty} |x||u(x,\bar{y})|^2\, dx;$$

from this, we derive immediately (3.4). This inequality, established for $C_0^\infty(R^2)$ functions, holds in $Z(R^2)$, owing to density arguments.

Existence. Elementary calculations show that, if a solution $u\in Z(R^2)$ of eq. (1.4) exists, its Fourier transform (with respect to y), v say, can be represented in the following way

$$v(\cdot,\eta) = V(\eta)g(\cdot,\eta) \tag{3.5}$$

where $g(x,\cdot)$ is the Fourier transform of $f(x,\cdot)$ and $V(\eta)$ is a one-parameter family of integral operators whose Kernels are

$$E(x,x';\eta) = W^{-1}(\eta)\begin{cases} v_0(x,\eta)\,v_0(-x',\eta) & \text{for } x\geqslant x', \\ v_0(x',\eta)\,v_0(-x,\eta) & \text{for } x\leqslant x'. \end{cases} \tag{3.6}$$

Here $v_0(x,\eta)$ is that solution of equation $v_{xx}-(k+i\eta|x|)\ v=0$ which vanishes for $x\to+\infty$, and $W(\eta)$ is the wronskian of the two independent solutions $v_0(x,\eta)$ and $v_0(-x,\eta)$.

Let us put

$$v_0(x,\eta) = \begin{cases} v_+(x,\eta) & \text{for } x\geqslant 0 \\ v_-(x,\eta) & \text{for } x\leqslant 0 \end{cases} \tag{3.7}$$

and

$$z = z(x) = |\eta|^{-2/3}(k+i\eta|x|)\ . \tag{3.8}$$

Then, if $\eta\geqslant 0$, so that $0\leqslant\arg(z)<\pi/2$, we have

$$v_+(x,\eta) = z^{1/2}H^{(1)}_{1/3}(\tfrac{2}{3}z^{3/2}) \tag{3.9}$$

$$v_-(x,\eta) = z^{1/2}\{a(\eta)\,H^{(1)}_{1/3}(\tfrac{2}{3}z^{3/2}) + b(\eta)\,H^{(2)}_{1/3}(\tfrac{2}{3}z^{3/2})\} \tag{3.10}$$

where a and b are given by

$$a(\eta) = -\frac{\pi}{6i}\frac{k^{3/2}}{|\eta|}(H^{(1)}_{1/3}H^{(2)}_{-2/3} + H^{(2)}_{1/3}H^{(1)}_{-2/3})\left(\frac{2}{3}\frac{k^{3/2}}{|\eta|}\right), \tag{3.11}$$

$$b(\eta) = \frac{\pi}{3i}\frac{k^{3/2}}{|\eta|}(H^{(1)}_{1/3}H^{(1)}_{-2/3})\left(\frac{2}{3}\frac{k^{3/2}}{|\eta|}\right). \tag{3.12}$$

In (3.11) and (3.12) the notation means that the arguments of the Bessel functions $H^{(1)(2)}$ is $\frac{2}{3}(k^{3/2}/|\eta|)$.

If $\eta \leqslant 0$, and $-\pi/2 < \arg(z) \leqslant 0$, we have

$$v_+(x, \eta) = z^{1/2} H^{(2)}_{1/3}(\tfrac{2}{3} z^{3/2}) \tag{3.9 bis}$$

$$v_-(x, \eta) = z^{1/2} \bar{b}(\eta) H^{(1)}_{1/3}(\tfrac{2}{3} z^{3/2}) + \bar{a}(\eta) H^{(2)}_{1/3}(\tfrac{2}{3} z^{3/2}) . \tag{3.10 bis}$$

Finally, the wronskian W is given by:

$$W(\eta) = \frac{6}{\pi} |\eta|^{1/3} \cdot \begin{cases} b(\eta) & \text{if } \eta \geqslant 0 , \\ \overline{b(\eta)} & \text{if } \eta \leqslant 0 . \end{cases} \tag{3.13}$$

We need now the following lemma:

LEMMA 3.1. *Let $V(\eta)$ be the previously defined family of integral operators. For every $\varphi \in L^2(R)$ and every real η we have*:

$$\| V(\eta) \varphi \|_{L^2(R)} \leqslant \text{const}\, k^{-1} \| \varphi \|_{L^2(R)} \tag{3.15}$$

$$\| V(\eta) \varphi \|_{L^2(R; x^2)} \leqslant \text{const}\, |\eta|^{-1} \| \varphi \|_{L^2(R)} . \tag{3.16}$$

We can now complete the proof of Th. 3.1. From (3.5), taking account of (3.16), we get

$$\eta^2 \int_{-\infty}^{+\infty} x^2 |v(x, \eta)|^2 \, dx \leqslant \text{const} \int_{-\infty}^{+\infty} |g(x, \eta)|^2 \, dx .$$

By integrating now over η and taking account of Plancherel theorem for Fourier transforms we obtain

$$\iint_{R^2} x^2 |u_y|^2 \, dx \, dy \leqslant \text{const} \iint_{R^2} |f|^2 \, dx \, dy . \tag{3.17}$$

From (3.15) we have also

$$k^2 \int_{-\infty}^{+\infty} |v(x, \eta)|^2 \, dx \leqslant \text{const} \int_{-\infty}^{+\infty} |g(x, \eta)|^2 \, dx$$

from which we get

$$k^2 \iint_{R^2} |u|^2 \, dx \, dy \leqslant \text{const} \iint_{R^2} |f|^2 \, dx \, dy . \tag{3.18}$$

Then, let us write

$$u(x, y) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \exp[iy\eta]\, v(x, \eta)\, d\eta .$$

If φ is a test function, we can write, by changing the order of integrations (all the variables run from $-\infty$ to $+\infty$)

$$\iint \varphi_{xx}(x, y)\, u(x, y)\, dx\, dy = (2\pi)^{-1/2}\int d\eta \int dx\, v(x, \eta)\, \frac{\partial^2}{\partial x^2}\int \exp[iy\eta]\varphi(x, y)\, dy .$$

By integration by parts, taking account that the function $(x, \eta) \to \to \int \exp[iy\eta]\varphi(x, y)\, dy$ has compact support with respect to x and is fast decreasing with respect to η, one has

$$\iint \varphi_{xx}(x, y)\, u(x, y)\, dx\, dy = (2\pi)^{-1/2}\iint \varphi(x, y)\, dx\, dy \cdot \int v_{xx}(x, \eta) \exp[iy\eta]\, d\eta .$$

Thus u is twice differentiable (with respect to x) in the sense of distributions and u_{xx} is given by $(2\pi)^{-1/2}\int \exp[iy\eta] v_{xx}\, d\eta$. Then u solves eq. (1.4) and, from (3.17) and (3.18), inequality (3.2) is easily derived. Let now be $f = 0$ for $y < 0$. The previously found solution u, which vanishes for $y < 0$ (remember (3.14)), satisfies the equation $|x|u_y + ku = F$, where $F = u_{xx} - f$; then it may be represented as follows:

$$u(x, y) = \int_0^y |x|^{-1} F(x, t) \exp[-k|x|^{-1}(y - t)]\, dt .$$

Thus, by Schwartz inequality, we get

$$|u|^2 \leqslant (2k|x|)^{-1}\left[1 - \exp\left(-\frac{2ky}{|x|}\right)\right]\int_0^y |F(x, t)|^2\, dt \leqslant x^{-2}\, y \int_0^y |F(x, t)|^2\, dt .$$

Then we have

$$\int_{-\infty}^{+\infty} x^2 |u|^2\, dx \leqslant y \iint_{Y_+} |F|^2\, dx\, dt .$$

Now inequality (3.3) follows, since $F = u_{xx} - f$ and the norm of u_{xx} can be estimated by the norm of f. Theorem 3.1 is completely proved.

LEMMA 3.2. *Let $E(x, x'; \eta)$ be defined by* (3.6) *and subsequent formulae. The following majorizations hold*:

$$|E(x, x'; \eta)| \leqslant \text{const}\, k^{-1/2} \exp\left[-\tfrac{2}{3} k^{1/2} \big||x| - |x'|\big|\right] \tag{3.19}$$

$$|E(x, x'; \eta)| \leqslant \text{const}\, |\eta|^{-1/3} (1 + |\eta| |xx'|^{3/2})^{-1/6} \cdot \\ \cdot \exp\left[-\frac{\sqrt{2}}{3} |\eta|^{1/2} \big||x|^{3/2} - |x'|^{3/2}\big|\right] \tag{3.20}$$

for every x, x', η reals.

LEMMA 3.3. *The functions*

$$0 < t \to t^{2/3} \int_0^\infty t'^{-1/3} (1 + tt')^{-1/6} \exp\left(-|t - t'|\right) dt'$$

$$0 < t \to \int_0^\infty t'^{1/3} (1 + tt')^{-1/6} \exp\left(-|t - t'|\right) dt'$$

are bounded by positive constants.

PROOF OF LEMMA 3.1. Let us write out

$$V(\eta)\varphi(x) = \int_{-\infty}^{+\infty} E(x, x'; \eta)\varphi(x')\, dx'.$$

Then, by Schwartz inequality, and changing the order of integration, we get

$$\|V(\eta)\varphi\|^2_{L^2(R)} \leqslant \int_{-\infty}^{+\infty} |\varphi(x')|^2 dx' \int_{-\infty}^{+\infty} |E(x, x'; \eta)|\, dx \int_{-\infty}^{+\infty} |E(x, x''; \eta)|\, dx''.$$

If we estimate the integrals in dx and dx'' by making use of (3.19) we get easily (3.15).

With a similar procedure we have

$$\|V(\eta)\varphi\|^2_{L^2(R; x^2)} \leqslant \int_{-\infty}^{+\infty} |\varphi(x')|^2 dx' \int_{-\infty}^{+\infty} x^2 |E(x, x'; \eta)|\, dx \int_{-\infty}^{-\infty} |E(x, x''; \eta)|\, dx''.$$

Let us make use now of estimate (3.20); let us put $|x|^{3/2}|\eta|^{1/2} = t$ and

$|x''|^{3/2}|\eta|^{1/2} = t''$; we obtain

$$\|V(\eta)\varphi\|^2_{L^2(R;x^2)} \leqslant \text{const}\,|\eta|^{-2}\int_{-\infty}^{+\infty}|\varphi(x')|^2\,dx'\cdot$$

$$\cdot\int_0^\infty t(1+|x'|^{3/2}|\eta|^{1/2}t)^{-1/6}\exp\left[-\frac{\sqrt{2}}{3}\left|t-|x'|^{3/2}|\eta|^{1/2}\right|\right]dt\cdot$$

$$\cdot\int_0^\infty t''^{-1/3}(1+t''t)^{-1/6}\exp\left[-\frac{\sqrt{2}}{3}|t-t''|\right]dt''$$

By estimating integrals in dt and dt'' by the aid of lemma 3.3 we get (3.16).

Proof of Lemma 3.2. Let us recall a bound for Hankel functions:

$$|H_\nu^{(1)(2)}(z)| \leqslant \text{const}\,|z|^{-\nu}(1+|z|)^{|\nu|-1/2}\exp(\mp\,\text{Im}\,z)\,.$$

Then we see that $|a(\eta)|$ and $|b(\eta)|$ are bounded by positive constants (independent on k):

$$|a(\eta)| \leqslant \text{const} \qquad |b(\eta)| \leqslant \text{const}\,.$$

Now a bound for v_+ and v_-, and thus for v_0, is easily derived

$$|v_+(x,\eta)| \leqslant \text{const}\,(1+|z|)^{-1/4}\exp\left[-\tfrac{2}{3}|\text{Im}\,(z^{3/2})|\right]$$
$$|v_-(x,\eta)| \leqslant \text{const}\,(1+|z|)^{-1/4}\exp\left[\ \tfrac{2}{3}|\text{Im}\,(z^{3/2})|\right]$$

where z is given by (3.8). Then we get, for E, the following bounds:

i) $x' > x > 0$; $\quad E(x,x';\eta) = W^{-1}(\eta)\,v_+(x',\eta)\,v_-(-x,\eta)$

$$|E(x,x';\eta)| \leqslant \text{const}\,|\eta|^{-1/3}(1+|z|)^{-1/4}(1+|z'|)^{-1/4}\cdot$$
$$\cdot\exp\left[-\tfrac{2}{3}\left(|\text{Im}\,(z'^{3/2})|-|\text{Im}\,(z^{3/2})|\right)\right]$$

where we put $z' = z(x')$.

ii) $0 > x' > x$; $\quad E(x,x';\eta) = W^{-1}(\eta)\,v_-(x',\eta)\,v_+(-x,\eta)$

and we still get a bound like the previous one

iii) $x' > 0 > x$; $\quad E(x,x';\eta) = W^{-1}(\eta)\,v_+(x',\eta)\,v_+(-x,\eta)$

$$|E(x,x';\eta)| \leqslant \text{const}\,|\eta|^{-1/3}(1+|z|)^{-1/4}(1+|z'|)^{-1/4}\cdot$$
$$\cdot\exp\left[-\tfrac{2}{3}\left(|\text{Im}\,(z^{3/2})|+|\text{Im}\,(z'^{3/2})|\right)\right].$$

Similar bounds can be written when $x'<x$; summarizing the results gives

$$(3.21)\qquad xx'>0 \qquad |E(x,x';\eta)|\leqslant \text{const}\,|\eta|^{-1/3}(1+|z|)^{-1/4}\cdot \\ \cdot(1+|z'|)^{-1/4}\exp\left[-\tfrac{2}{3}\big||\mathrm{Im}\,z^{3/2}|-|\mathrm{Im}\,z'^{3/2}|\big|\right]$$

$$(3.22)\qquad xx'<0 \qquad |E(x,x';\eta)|\leqslant \text{const}\,|\eta|^{-1/3}(1+|z|)^{-1/4}\cdot \\ \cdot(1+|z'|)^{-1/4}\exp\left[-\tfrac{2}{3}\left(|\mathrm{Im}\,z^{3/2}|+|\mathrm{Im}\,z'^{3/2}|\right)\right]$$

and, finally, we can use, in any case, the bound (3.21).

Now, from the definition of z, we have

$$(3.23)\qquad (1+|z|)^{-1/4}=(1+|\eta|^{-2/3}(k^2+x^2\eta^2)^{1/2})^{-1/4}\leqslant\begin{cases} k^{-1/4}|\eta|^{1/6} \\ (1+|\eta|^{1/3}|x|)^{-1/4}\end{cases}$$

Let us now provide an estimate for $\big||\mathrm{Im}\,z^{3/2}|-|\mathrm{Im}\,z'^{3/2}|\big|$. First, by putting $\theta=\arg(k+i\eta|x|)$, $|\theta|<\pi/2$, we have

$$(3.24)\qquad |\mathrm{Im}\,(k+i\eta|x|)^{3/2}|=|k+i\eta|x||^{3/2}|\sin(3/2\theta)|= \\ =\frac{1}{\sqrt{2}}\left[|x\eta|(k+(k^2+x^2\eta^2)^{1/2})^{1/2}+k((k^2+x^2\eta^2)^{1/2}-k)^{1/2}\right]\geqslant \\ \geqslant\frac{1}{\sqrt{2}}|x\eta|(k^2+\eta^2x^2)^{1/4}.$$

Now let us suppose, for the moment, $x'>x$ (so that $|\mathrm{Im}z'^{3/2}|>|\mathrm{Im}\,z^{3/2}|$); we have

$$|\mathrm{Im}\,z'^{3/2}|-|\mathrm{Im}\,z^{3/2}|= \\ =\frac{1}{\sqrt{2}}\left[|x'|(k+(k^2+x'^2\eta^2)^{1/2})^{1/2}+k((k^2+\eta^2x'^2)^{1/2}-k)^{1/2}\right]-\frac{1}{\sqrt{2}}[\ldots]\geqslant \\ \geqslant\frac{1}{\sqrt{2}}\left[|x'|(k+(k^2+x'^2\eta^2)^{1/2})^{1/2}-|x|(k+(k^2+x^2\eta^2)^{1/2})^{1/2}\right].$$

Now we can see that the bracked expression, as a function of k, attains its minimum value at the origin (for every η) and, as a function of η, it attains its minimum value still at the origin (for every k). Similar conclusions can be deduced in case $x'<x$. Thus we got

$$(3.25)\qquad \big||\mathrm{Im}\,z^{3/2}|-|\mathrm{Im}\,z'^{3/2}|\big|\geqslant\begin{cases} k^{1/2}\big||x|-|x'|\big| \\ \dfrac{1}{\sqrt{2}}|\eta|^{1/2}\big||x|^{3/2}-|x'|^{32}\big|.\end{cases}$$

Now, from (3.21), by using (3.23) and (3.25), bounds (3.19) and (3.20) can be derived without difficulty.

PROOF OF LEMMA 3.3. Let us notice that, since $1 + tt' > tt'$, we can write

$$t^{2/3} t'^{-1/3}(1 + tt')^{-1/6} < t^{1/2} t'^{-1/2}.$$

Then the first integral appearing in the statement of the lemma is majorized by

$$t^{1/2}\int_0^\infty t'^{-1/2} \exp(-|t-t'|)\, dt' = 2t^{1/2}\Big[e^{-t}\int_0^{\sqrt{t}} e^{\tau^2}\, d\tau + e^{t}\int_{\sqrt{t}}^\infty e^{-\tau^2}\, d\tau\Big] \leqslant \text{const} \cdot$$

The other assertion of the lemma is a special case of lemma (3.4) in [9].

3.2. *Boundary value problems in* X_+.

THEOREM 3.2. *Let* $\psi \in H^{1/2}(R)$ *be a given function. There exists a unique function* u *such that*

(3.26) $\quad u \in Z(X_+)$

(3.27) $\quad u_{xx} - x u_y - ku = 0 \qquad (k > 0)$

(3.28) $\quad u(0, \cdot) = \psi \qquad$ *(in the sense of traces)*

Such a function can be represented by the formula

$$u(x, y) = \int_{-\infty}^{y} \psi(t) \cdot A(x, y - t)\, dt \tag{3.29}$$

where $\hat{A}(x, \eta)$, *the Fourier transform (with respect to* y*) of* $A(x, y)$, *is given by*

$$\hat{A}(x, \eta) = \frac{z^{1/2} H^{(1)}_{1/3}(\frac{2}{3} z^{3/2})}{z_0^{1/2} H^{(1)}_{1/3}(\frac{2}{3} z_0^{3/2})} \quad \text{for } \eta \geqslant 0 \tag{3.30}$$

change $H^{(1)}_{1/3}$ *with* $H^{(2)}_{1/3}$ $\qquad$ *for* $\eta \leqslant 0$

we put

$$z = z(x) = |\eta|^{-2/3}(k + i\eta x)\,, \qquad z_0 = z(0) = |\eta|^{-2/3} k\,. \tag{3.31}$$

Moreover, the following bounds on u hold

$$\operatorname*{ess\,sup}_{x>0}\int_{-\infty}^{+\infty}|u(x,y)|^2\,dy\leqslant\int_{-\infty}^{+\infty}|\psi|^2\,dy \tag{3.32}$$

$$\iint_{X_+}(x^2|u_y|^2+|u_{xx}|^2+k|u_x|^2+k^2|u|^2)\,dx\,dy\leqslant\text{const}\int_{-\infty}^{+\infty}|D^{1/2}\psi|^2\,dy \tag{3.33}$$

where $D^{1/2}\psi$ is the fractional derivative (order $\frac{1}{2}$) of ψ.

PROOF. The uniqueness assertion follows from inequality (3.32) which can be deduced a priori from the fact that, if u is a well-behaved (e.g., u is of class Z) solution of eq. (3.27), the function $0<x\to\int_{-\infty}^{+\infty}|u(x,y)|^2dy$ is convex; if $u\in Z(X_+)$, it is bounded too; for $0<x\to\int_{-\infty}^{+\infty}|u(x,y)|^2dy$ belongs to $L^1(R_+)$ together with its derivative; thus this function takes its maximum value at the boundary.

Representation (3.29) can be easily checked by straightforward calculations; for, by taking the Fourier transform with respect to y, and denoting by $\hat{}$ the transformed functions, we get

$$\hat{u}(x,\eta)=\hat{A}(x,\eta)\hat{\psi}(\eta) \tag{3.34}$$

where $\hat{A}$ is given by (3.30). Now, we can estimate $|\hat{A}(x,\eta)|$ very roughly in the following way

$$|\hat{A}(x,\eta)|\leqslant\text{const}\exp\left(-\frac{\sqrt{2}}{3}\,x^{3/2}\,|\eta|^{1/2}\right). \tag{3.35}$$

To get (3.35), remember (3.24) and usual bounds for Hankel functions recalled in the proof of lemma 3.2.

From representation (3.29) by taking account of (3.35), one can show that u is a C^∞-solution of eq. (3.27). Let us show that $u\in Z(X_+)$. From (3.34) and (3.35) we obtain

$$\int_0^\infty x^2|\hat{u}|^2\,dx\leqslant\text{const}\,|\hat{\psi}(\eta)|^2\int_0^\infty x^2\exp\left(-\frac{2\sqrt{2}}{3}\,x^{3/2}\,|\eta|^{1/2}\right)dx=\text{const}\,|\eta|^{-1}\,|\hat{\psi}(\eta)|^2.$$

Thus we have

$$\int_{-\infty}^{+\infty} d\eta \int_0^{\infty} x^2 \, |\eta \hat{u}|^2 \, dx \leqslant \text{const} \int_{-\infty}^{+\infty} |\eta| \, |\hat{\varphi}(\eta)|^2 \, d\eta$$

and, from Plancherel theorem,

$$\iint_{X_+} x^2 |u_y|^2 \, dx \, dy \leqslant \text{const} \int_{-\infty}^{+\infty} |D^{1/2} \varphi|^2 \, dy \,. \tag{3.36}$$

Now, taking account of (3.36), directly from the equation $u_{xx} - ku = xu_y$, we get (3.33).

Besides the problem (3.26), (3.27), (3.28), we will need to consider also the following one: to find u such that (3.26) and (3.27) hold and

$$u_x(0, \cdot) = \varphi \quad \text{in the sense of traces.} \tag{3.37}$$

Since the treatment of this problem is completely analogous to the previous one, we limit ourselves to the statement of the results in the following theorem.

THEOREM 3.3. *Let $\varphi \in H^{1/6}(R)$ be a given function. There exists a unique solution u to problem* (3.26), (3.27), (3.37); *this solution can be represented by the formula*

$$u(x, y) = \int_{-\infty}^{y} \varphi(t) \, B(x, y - t) \, dt \tag{3.38}$$

where $\hat{B}(x, \eta)$, the Fourier transform of $B(x, y)$ is given by

$$\hat{B}(x, \eta) = -\, i |\eta|^{-1/3} \frac{z^{1/2} H^{(1)}_{1/3}(\frac{2}{3} z^{3/2})}{z_0 H^{(1)}_{-2/3}(\frac{2}{3} z_0^{3/2})} \qquad \text{for} \qquad \eta \geqslant 0$$

take the complex coniugate of the above expression for $\eta \leqslant 0$ *and z, z_0 are the same as in Th.* 3.2. *The following bounds on u hold*

$$\operatorname*{ess\,sup}_{x>0} \int_{-\infty}^{+\infty} |u_x(x, y)|^2 \, dy \leqslant \int_{-\infty}^{+\infty} |\varphi|^2 \, dy \tag{3.40}$$

$$\iint_{X_+} \left(x^2 |u_y|^2 + |u_{xx}|^2 + k |u_x|^2 + k^2 |u|^2 \right) dx \, dy \leqslant \text{const} \int_{-\infty}^{+\infty} |D^{1/6} \varphi|^2 \, dy \,. \tag{3.41}$$

In carrying out the proof, the following estimate is usefull

$$|\hat{B}(x,\eta)| \leqslant \text{const}\,(k^3+\eta^2)^{-1/6}\,|\hat{A}(x,\eta)|\,. \tag{3.42}$$

We can complete now the proof of inequality (2.20). More precisely, we want to prove the following statement:

COROLLARY. *Let u be a given function belonging to $C_0^\infty(\overline{Y}_+)$; let h be its restriction to the x-axis; thus inequality* (2.20) *is valid.*

PROOF. Let us put $\varphi(y)=u_x(0,y)$ for $y>0$ and $\varphi(y)=0$ for $y<0$. Then $\varphi\in H^s(R)$ for $s<\frac{1}{2}$, in particular for $s=\frac{1}{6}$. Let now w be a solution of the equation $w_{xx}-x\,w_y=0$ for $x>0$ such that: w_{xx}, $xw_y\in L^2(X_+)$, $\sup_{x>0}\int_{-\infty}^{+\infty}|u_x(x,y)|^2dy<\infty$ and satisfying the condition: $w_x(0,\cdot)=\varphi$.

Such a solution exists and is unique, thanks to a slight modification of Th. 3.3; obviously, this solution vanishes for $y=0$. Now, let us put $v=u-w$ in Y_+; then it will be $v_x(0,\cdot)=0$ and $v(\cdot,0)=h$. Now, we can apply to v inequality (2.20), which we already proved for functions whose x-derivative vanishes on the y-axis; we get

$$\int_0^\infty x|h'|^2\,dx\leqslant 6\Big(\int_0^\infty\!\!\int_0^\infty x^2|v_y|^2\,dx\,dy\Big)^{1/2}\Big(\int_0^\infty\!\!\int_0^\infty|v_{xx}|^2\,dx\,dy\Big)^{1/2}\leqslant$$
$$\leqslant 6\Big(\int_0^\infty\!\!\int_0^\infty x^2|u_y|^2\,dx\,dy+\int_0^\infty\!\!\int_0^\infty x^2|w_y|^2\,dx\,dy\Big)^{1/2}\cdot\Big(\int_0^\infty\!\!\int_0^\infty|u_{xx}|^2\,dx\,dy+\int_0^\infty\!\!\int_0^\infty|w_{xx}|^2\,dx\,dy\Big)^{1/2}.$$

Now we can estimate the norms of xw_y and w_{xx} by the norm of $D^{1/6}\varphi$ (thanks to inequality (3.41) with $k=0$) and $D^{1/6}\varphi$ by means of the norms of xu_y and u_{xx} (thanks to inequality (2.17)). The same proof can be carried out for $x<0$ and so (2.20) is completely proved.

LEMMA 3.4. *The restrictions of the solutions to pbs.* (3.26) (3.27), (3.28) *and* (3.26), (3.27), (3.37) *to the first quadrant $X_+\cap Y_+$ belong to $W_+(X_+\cap Y_+)$.*

PROOF. The function $0\leqslant y\to\int_0^\infty x^2|u(x,y)|^2dx$ is bounded. For (let us carry out the proof for the first problem) we have from the representation (3.34),

$$\int_0^\infty x^2|u(x,y)|^2\,dx\leqslant\int_0^\infty x^2\,dx\Big(\int_{-\infty}^{+\infty}|\hat{u}(x,\eta)|\,d\eta\Big)^2=\int_0^\infty x^2\,dx\Big(\int_{-\infty}^{+\infty}|\hat{A}(x,\eta)|\,|\hat{\varphi}(\eta)|\,d\eta\Big)^2.$$

Now, let $0<\alpha<\frac{1}{2}$; by Schwartz inequality, and by using estimate (3.35), we obtain

$$\int_0^\infty x^2|u(x,y)|^2\,dx \leqslant \int_0^\infty x^2\,dx \int_{-\infty}^{+\infty} |\eta|^{2\alpha}|\hat{\varphi}(\eta)|^2|\hat{A}(x,\eta)|\,d\eta \int_{-\infty}^{+\infty} |\eta_1|^{-2\alpha}|\hat{A}(x,\eta)|\,d\eta \leqslant$$

$$\leqslant \operatorname{const}\int_{-\infty}^{+\infty} |\eta|^{2\alpha}|\hat{\varphi}(\eta)|^2\,d\eta \int_{-\infty}^{+\infty}|\eta_1|^{-2\alpha}\,d\eta_1 \cdot \int_0^\infty x^2 \exp\left[-\frac{\sqrt{2}}{3}x^{3/2}\big(|\eta|^{1/2}+|\eta_1|^{1/2}\big)\right]dx\,.$$

The integral in dx may be estimated by omothety, and equals $\operatorname{const}\big(|\eta|^{1/2}+|\eta_1|^{1/2}\big)^{-2}$. Now the same we can do for the integral in $d\eta_1$ and we obtain finally

$$\int_0^\infty x^2|u(x,y)|^2\,dx \leqslant \operatorname{const}\int_{-\infty}^{+\infty}|\hat{\varphi}(\eta)|^2\,d\eta\,. \tag{3.43}$$

An analogous procedure for the second problem gives us

$$\int_0^\infty x^2|u(x,y)|^2\,dx \leqslant \operatorname{const}\int_{-\infty}^{+\infty}(k^3+\eta^2)^{-1/3}|\hat{\varphi}(\eta)|^2\,d\eta\,. \tag{3.44}$$

3.3. *Initial value problem in Y_+.*

Let us consider now the Cauchy problem for the evolution equation

$$u_{xx}-|x|\,u_y-ku=0 \qquad (k>0) \tag{3.45}$$

in the half-plane Y_+. Because of the symmetry properties of eq. (3.45), which is invariant under transformations $x\to -x$, we will consider separately the cases that the initial value, assigned at $y=0$, is an even or an odd function. There are some reasons for a separate treatment: i) the calculations will be more clearly understandable, ii) in Sec. 4, when the results of this section will be used, we will need only even solutions of eq. (3.45); iii) finally, the results in the two cases are different.

a) Even solutions.

THEOREM 3.4. *Let $h\in\mathcal{H}^1(R_+)$ be a given function; there exists only one function u belonging to $W_+(X_+\cap Y_+)$, solution to eq.* (3.45) *and verifying the* *conditions* $u(\cdot, 0) = h$ *and* $u_x(0, \cdot) = 0$ *in the sense of traces. Such a function satisfies the inequality*

$$\|u\|_{W_+(X_+ \cap Y_+)} \leqslant \text{const}\,(k)\,\|h\|_{\mathcal{H}^1(R_+)}\,. \tag{3.46}$$

The proof of Th. 3.4 will be easily deduced from the following lemma:

LEMMA 3.5. *Let* $h \in \mathcal{H}^1(R_+)$ *be a given function; there exists a function* w *defined in* $X_+ \cap Y_+$ *with the properties*

$$w_{xx} - x w_y = 0 \tag{3.47}$$

$$w(\cdot, 0) = h \tag{3.48}$$

$$w_x(0, \cdot) = 0 \tag{3.49}$$

$$\sup_{y>0} \int_0^\infty (1 + x^2)|w(x, y)|^2\, dx \leqslant \text{const}\, \|h\|^2_{\mathcal{H}^1(R_+)} \tag{3.50}$$

$$\int_0^\infty \int_0^\infty \left(x^2 |w_y|^2 + |w_{xx}|^2\right) dx\, dy \leqslant \text{const} \int_0^\infty x |h'(x)|^2\, dx\,. \tag{3.51}$$

Proof of TH. 3.4: let us introduce a new unknown $\Phi = u - w \exp(-ky)$, where w is the function considered in lemma 3.5. It is clear that $w \cdot \exp(-ky)$ belongs to $W_+(X_+ \cap Y_+)$ and satisfies the same conditions as w at the boundaries. Then in order u to be a solution of pb. (3.47) ... (3.49), it is sufficient that Φ must be a solution, belonging to $W_+(X_+ \cap Y_+)$, of the equation

$$\Phi_{xx} - x\Phi_y - k\Phi = k(1 - x) \exp(-ky)\, w \tag{3.52}$$

with homogeneous boundary conditions. Because of (3.50), we see that the right member of (3.52) is an $L^2(X_+ \cap Y_+)$ function; then, after having continued this function into the whole plane as an even function of x vanishing for $y < 0$, we can apply Th. 3.1. We deduce that the function Φ exists and, from (3.2), (3.3), it satisfies the inequalities

$$\int_0^\infty \int_0^\infty \left(x^2 |\Phi_y|^2 + |\Phi_{xx}|^2 + k|\Phi_x|^2 + k^2|\Phi|^2\right) dx\, dy \leqslant$$

$$\leqslant \text{const}\, k^2 \int_0^\infty \int_0^\infty (1 + x^2) \exp(-2ky)|w|^2\, dx\, dy$$

$$\int_0^\infty x^2 |\Phi|^2\, dx \leqslant \text{const}\, k^2 y \int_0^\infty \int_0^\infty (1 + x^2) \exp(-2ky)|w|^2\, dx\, dy\,.$$

From these, by taking account of (3.50) and (3.51), we obtain

$$\int_0^\infty\int_0^\infty \left(x^2|u_y|^2+|u_{xx}|^2+k|u_x|^2+k^2|u|^2\right)dx\,dy\leq \tag{3.53}$$
$$\leq \text{const}\left[\int_0^\infty x|h'(x)|^2\,dx+k\|h\|^2_{\mathcal{H}^1(R_+)}\right]$$

$$\int_0^\infty x^2|u(x,y)|^2\,dx\leq \text{const}\left(ky+\exp(-2ky)\right)\|h\|^2_{\mathcal{H}^1(R_+)}\,. \tag{3.54}$$

Then we proved (3.46), by showing also the dependence on k of the constant appearing there.

The uniqueness of the solution may be derived in the same way as we did in the proof of Th. 3.1. For, by following the same procedure there used to get inequality (3.4), we can obtain, by integrating on the first quadrant and putting $f=0$,

$$\int_0^\infty\int_0^\infty \left(k|u|^2+2|u_x|^2\right)dx\,dy\leq\int_0^\infty x|h|^2\,dx\,. \tag{3.55}$$

This a priori inequality implies uniqueness for our problem. Theorem 3.4 is proved.

To prove lemma 3.5, we need the following lemma.

LEMMA 3.6. *Let* $\alpha\leq\frac{1}{2}$, $\nu>-\alpha-\frac{1}{2}$; *the operator represented by*

$$(\mathfrak{H}_{\nu,\alpha}f)(x)=\int_0^\infty (xy)^\alpha J_\nu(xy)f(y)\,dy \tag{3.56}$$

is a bounded one-to-one operator of $L^2(R_+)$ *into itself, whose range is*

$$\mathcal{B}\equiv\{f\in L^2(R_+);\ \varrho\to|\varrho|^{1/2-\alpha}\mathcal{M}f(\tfrac{1}{2}+i\varrho)\in L^2(R)\}; \tag{3.57}$$

the inverse operator (which is unbounded) is represented by

$$(\mathfrak{H}_{\nu,\alpha}^{-1}f)(x)=\int_0^\infty (xy)^{1-\alpha}J_\nu(xy)f(y)\,dy; \tag{3.58}$$

the integrals appearing in (3.56) *and* (3.58) *converge in L^2-norm. Moreover we have*

$$\int_0^\infty |\mathfrak{H}_{\nu,\alpha} f|^2 dx \leqslant \text{const} \int_0^\infty |f|^2 dx \tag{3.59}$$

$$\int_{-\infty}^{+\infty} (1+\varrho^2)^{1/2-\alpha} |(\mathcal{M}\mathfrak{H}_{\nu,\alpha} f)(\tfrac{1}{2}+i\varrho)|^2 d\varrho \leqslant \text{const} \int_0^\infty |f|^2 dx \tag{3.59'}$$

$$\int_0^\infty |\mathfrak{H}_{\nu,\alpha}^{-1} f|^2 dx \leqslant \text{const} \int_{-\infty}^{+\infty} (1+\varrho^2)^{1/2-\alpha} |(\mathcal{M} f)(\tfrac{1}{2}+i\varrho)|^2 d\varrho\,. \tag{3.60}$$

Here $\mathcal{M}f$ is the Mellin transform of f and J_ν is the first Bessel function of order ν.

Remark i). As it is known from the theory of Mellin transform, if $f \in L^2(R_+)$, its Mellin transform, $(\mathcal{M}f)(s)$, is a square summable function on the line $s = \frac{1}{2} + i\varrho$, $-\infty < \varrho < +\infty$, and the Parseval formula holds:

$$\int_0^\infty |f|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |(\mathcal{M}f)(\tfrac{1}{2}+i\varrho)|^2 d\varrho\,.$$

As it is easy to see, the Mellin transform $\varrho \to (\mathcal{M}f)(\frac{1}{2}+i\varrho)$ is the Fourier transform of the function $\tau \to \exp(\tau/2) f(e^\tau)$ so that the condition that f belongs to $\mathcal{B}$ means that $\tau \to \exp(\tau/2) f(e^\tau)$ belongs to $H^{1/2-\alpha}(R)$.

Remark ii). The case $\alpha = \frac{1}{2}$ is the well-known case of Hankel transforms; the set $\mathcal{B}$ coincides with the whole space $L^2(R_+)$ and $\mathfrak{H}_{\nu,1/2} = \mathfrak{H}_{\nu,1/2}^{-1}$ is a unitary operator of $L^2(R_+)$.

Proof of lemma 3.5. Let us represent the requested function w as a superposition of separated variables solutions of eq. (3.47):

$$w(x,y) = \int_0^\infty \exp\left(-\frac{9}{4} t^2 y\right) W(x,t)\, dt \tag{3.61}$$

where $W(\cdot, t)$ is a bounded continuously differentiable solution of the equation $W'' + \frac{9}{4}t^2x\ W = 0$ satisfying the condition $W'(0, t) = 0$; we have

$$W(x, t) = t^{5/6}\, b(t)\, x^{1/2}\, J_{-1/3}(tx^{3/2}) \tag{3.62}$$

where b is an arbitrary function. Requiring that the condition at $y = 0$ to be satisfied, gives

$$h(x) = x^{1/2}\int_0^\infty t^{5/6}\, b(t)\, J_{-1/3}(tx^{3/2})\, dt \tag{3.63}$$

and putting

$$\xi = x^{3/2}, \qquad l(\xi) = x^{3/4}h(x) \tag{3.64}$$

we can write formally this condition as follows

$$l(\xi) = \mathfrak{H}^{-1}_{-1/3,1/6}\{b(t); \xi\}\,; \tag{3.65}$$

this notation means that l is the $\mathfrak{H}^{-1}_{-1/3,1/6}$ transform of b. Now, since $\int_0^\infty x^2|h|^2 dx < \infty$ by hypothesis, then $l \in L^2(R_+)$ and (3.65) has a unique solution $b \in \mathfrak{B}$ (thanks to lemma 3.6) given by

$$b(t) = \mathfrak{H}_{-1/3,1/6}\{l(\xi); t\}\,; \tag{3.66}$$

thus we can write the function w as follows

$$w(\xi^{2/3}, y) = \xi^{-1/2}\left(\mathfrak{H}^{-1}_{-1/3,1/6}\circ\exp\left[-\frac{9}{4}(\cdot)^2 y\right]\circ\mathfrak{H}_{-1/3,1/6}\, l\right)(\xi)\,. \tag{3.67}$$

Then the function $\xi \to \xi^{1/2}w(\xi^{2/3}, y)$ is given by the composition of three operators applied to l: the operator $\mathfrak{H}_{-1/3,1/6}$ which acts from $L^2(R_+)$ to $\mathfrak{B}$; the multiplication by $\exp[-\frac{9}{4}(\cdot)^2 y]$, which is a continuous operation from $\mathfrak{B}$ to $\mathfrak{B}$ (see below) and, finally, $\mathfrak{H}^{-1}_{-1/3,1/6}$ which acts from $\mathfrak{B}$ to $L^2(R_+)$. Thus the composition of these three operators is defined on all $L^2(R_+)$ and it is also continuous; for, let $y > 0$ be fixed; by taking account of (3.60), we get

$$\int_0^\infty \xi |w(\xi^{2/3}, y)|^2 d\xi \leqslant \text{const} \int_{-\infty}^{+\infty} (1+\varrho^2)^{1/3} \left| \mathcal{M}\left\{ \exp\left(-\frac{9}{4}t^2 y\right) b(t);\ \frac{1}{2}+i\varrho \right\} \right|^2 d\varrho \leqslant$$

$$(*) \qquad \leqslant \text{const} \int_{-\infty}^{+\infty} (1+\varrho^2)^{1/3} \left| \mathcal{M}\left\{ b(t);\ \frac{1}{2}+i\varrho \right\} \right|^2 d\varrho \leqslant$$

$$(\text{from } (3.59')) \qquad \leqslant \text{const} \int_0^\infty |l(\xi)|^2 d\xi \,.$$

Then, by taking account of the change of variables (3.64), we get

$$(3.68) \qquad \sup_{y>0} \int_0^\infty x^2 |w(x, y)|^2 dx \leqslant \text{const} \int_0^\infty x^2 |h|^2 dx \,.$$

Let us consider now w_x; from (3.67) we obtain

$$(3.69) \qquad w_x(\xi^{2/3}, y) = -\frac{3}{2} \xi^{-1/6} \mathfrak{H}^{-1}_{2/3,1/6}\left\{ \exp\left(-\frac{9}{4}t^2 y\right) t b(t);\ \xi \right\}.$$

First, let us show that

$$(3.70) \qquad t b(t) = \mathfrak{H}_{2/3,1/6}\{\tfrac{1}{2}\xi^{-1} l(\xi) - l'(\xi);\ t\} \,.$$

For, by integration by parts, taking account that

$$J_{-1/3}(x) = J'_{2/3}(x) + \frac{2}{3x} J_{2/3}(x)$$

we get

$$b(t) = \int_0^\infty (\xi t)^{1/6} J_{-1/3}(\xi t)\, l(\xi)\, d\xi =$$

$$= \frac{1}{t} \int_0^\infty (\xi t)^{1/6} J_{2/3}(\xi t) [\tfrac{1}{2} \xi^{-1} l(\xi) - l'(\xi)]\, d\xi + \frac{1}{t} [(\xi t)^{1/6} J_{2/3}(\xi t)\, l(\xi)]_0^\infty \,.$$

(*) Remember the remark i) after lemma 3.6 and the fact that the multiplication by the function $\tau \to \exp(-9/4\, y\, e^{2\tau})$ is a continuous operation in $H^{1/3}(R)$; for it is continuous in $L^2(R)$ and in $H^1(R)$, because of the uniform boundedness of the above written function and of its derivative.

The finite part vanishes (it can be checked by easy calculations; in [12], Sect. 2.2, calculations of this type are extensively done in a more general situation); thus we get (3.70). Now, since $l'(\xi)-\frac{1}{2}\xi^{-1}l(\xi)=\frac{2}{3}x^{1/4}h'(x)$, the condition $x\to x^{1/2}h'(x)\in L^2(R_+)$ means $\xi\to l'(\xi)-\frac{1}{2}\xi^{-1}l(\xi)\in L^2(R_+)$; thus the function $t\to tb(t)$ belongs to $\mathcal{B}$. With calculations and considerations analogous to those previously done to prove (3.68) we get

$$\int_0^\infty \xi^{1/3}|w_x(\xi^{2/3},y)|^2\,d\xi\leqslant \qquad \text{(from (3.60))}$$

$$\text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}|\mathcal{M}\{\exp(-\tfrac{9}{4}t^2y)\,tb(t);\ \tfrac{1}{2}+i\varrho\}|^2\,d\varrho\leqslant$$

$$\text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}|\mathcal{M}\{tb(t);\ \tfrac{1}{2}+i\varrho\}|^2\,d\varrho\leqslant \qquad \text{(from (3.59'))}$$

$$\text{const}\int_0^\infty |l'(\xi)-\tfrac{1}{2}\xi^{-1}l(\xi)|^2\,d\xi\,.$$

Taking account of the change of variables (3.64), we obtain

$$\sup_{y>0}\int_0^\infty x|w_x(x,y)|^2\,dx\leqslant\text{const}\int_0^\infty x|h'|^2\,dx\,. \tag{3.71}$$

Besides (3.68) and (3.71) we can obtain also the following estimate

$$\sup_{y>0}\int_0^\infty x|w(x,y)|^2\,dx\leqslant\int_0^\infty x|h|^2\,dx\,. \tag{3.72}$$

To get (3.72), let us rewrite condition (3.63) in the following way:

$$h(x)=x^{1/2}\int_0^\infty t^{1/2}\tilde{b}(t)\,J_{-1/3}(tx^{3/2})\,dt$$

where we put $\tilde{b}(t)=t^{1/3}b(t)$; putting also $\tilde{l}(\xi)=\xi^{-1/3}l(\xi)$, the same condition writes

$$\tilde{l}(\xi)=\mathfrak{H}^{-1}_{-1/3,1/2}\{\tilde{b}(t);\ \xi\}\,.$$

Then we can proceed as before, when we deduced (3.68); the calculations are even simpler, since we are dealing now with the case $\alpha = \frac{1}{2}$ of lemma 3.6, which is the well-known case of Hankel transforms.

Finally, from (3.71) and (3.72) we deduce (remeber the remark after the definition of the space $\mathcal{H}^1(I)$ in Sec. 2):

$$\sup_{y>0}\int_0^\infty |w(x, y)|^2\,dx \leqslant \text{const}\Big(\int_0^\infty x|h'|^2\,dx\Big)^{1/2}\Big(\int_0^\infty x|h|^2\,dx\Big)^{1/2}. \tag{3.73}$$

Now, from (3.68) and (3.37), we deduce (3.50). Let us consider now w_y; from (3.67) we have

$$w_y(\xi^{2/3}, y) = -\tfrac{9}{4}\xi^{-1/2}\,\mathfrak{H}^{-1}_{-1/3,-1/6}\{\exp(-\tfrac{9}{4}t^2y)\,t^2 b(t);\, \xi\}\,. \tag{3.74}$$

Then, by applying inequality (3.60), we get

$$\int_0^\infty \xi|w_y(\xi^{2/3}, y)|^2\,d\xi \leqslant \text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}\,|\mathcal{M}\{\exp(-\tfrac{9}{4}t^2y)\,t^2 b(t);\, \tfrac{1}{2}+i\varrho\}|^2\,d\varrho =$$

$$\text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}\,d\varrho\int_0^\infty t^{1/2+i\varrho}\exp(-\tfrac{9}{4}t^2y)\,tb(t)\,dt\,\cdot$$

$$\cdot\int_0^\infty t_1^{1/2-i\varrho}\exp(-\tfrac{9}{4}t_1^2y)\,t_1 b(t_1)\,dt_1\,.$$

By integrating over y from 0 to ∞ we obtain

$$\int_0^\infty dy\int_0^\infty \xi|w_y(\xi^{2/3}, y)|^2 d\xi \leqslant \text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}d\varrho\,\cdot$$

$$\cdot\int_0^\infty t^{1/2+i\varrho}\,tb(t)\,dt\int_0^\infty t_1^{1/2-i\varrho}\,t_1\overline{b(t_1)}(t^2+t_1^2)^{-1}dt_1\,.$$

By putting, in the first integral, $t_1 = t\tau$ and changing the order of integrations, we get

$$\ldots \leqslant \text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}\,d\varrho\int_0^\infty \tau^{1/2-i\varrho}(1+\tau^2)^{-1}\,d\tau\int_0^\infty tb(t)(t\tau)\overline{b(t\tau)}\,dt\,.$$

Finally, by applying the convolution theorem for Mellin transforms, we get

$$\ldots \leqslant \text{const}\int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}d\varrho\int_0^{\infty}\tau^{1/2-i\varrho}(1+\tau^2)^{-1}d\tau\cdot\int_{-\infty}^{+\infty}\tau^{-1/2-i\sigma}|\mathcal{M}\{tb(t);\tfrac12+i\sigma\}|^2 d\sigma$$

(integrate first on τ, then on ϱ, finally on σ)

$$= \text{const}\int_{-\infty}^{+\infty}|\mathcal{M}\{tb(t);\tfrac12+i\sigma\}|^2 d\sigma\int_{-\infty}^{+\infty}\frac{(1+\varrho^2)^{1/3}}{\operatorname{Ch}\big(\pi/2(\varrho+\sigma)\big)}\,d\varrho \leqslant$$

$$\text{const}\int_{-\infty}^{+\infty}(1+\sigma^2)^{1/3}|\mathcal{M}\{tb(t);\tfrac12+i\sigma\}|^2 d\sigma \leqslant \text{const}\int_0^{\infty}|l'(\xi)-\tfrac12\xi^{-1}l(\xi)|^2 d\xi\,.$$

Thus, by returning to the old variables, we proved

$$\int_0^{\infty}\int_0^{\infty}x^2|w_y|^2\,dx\,dy \leqslant \text{const}\int_0^{\infty}x|h'|^2\,dx\,. \tag{3.75}$$

Then (3.51) is also proved and this complete the proof of lemma 3.5.

PROOF of lemma 3.6: the proof is an application, with some modifications, of Th. C given in [9], sec. 2.3; we refer to [9] for details.

Let $P(s)$ be defined by

$$P(s) = 2^{s+\alpha-1}\,\frac{\Gamma(\nu/2+s/2+\alpha/2)}{\Gamma(\nu/2-s/2-\alpha/2+1)}\,. \tag{3.76}$$

On the line $\operatorname{Re} s = \frac12$ we have

$$P(\tfrac12+i\varrho) = 2^{i\varrho+\alpha-1/2}\,\frac{\Gamma\big(\nu/2+\alpha/2+1/4+i(\varrho/2)\big)}{\Gamma\big(\nu/2-\alpha/2+3/4-i(\varrho/2)\big)}\,. \tag{3.77}$$

By taking account of the asymptotic behaviour of Γ function, we get the bounds

$$c_1 \leqslant \big(1+|\varrho|\big)^{1/2-\alpha}|P(\tfrac12+i\varrho)| \leqslant c_2 \tag{3.78}$$

where c_1 and c_2 are positive constants.

Now, given $f\in L^2(R_+)$, let g be defined by

$$(\mathcal{M}g)(s) = P(s)(\mathcal{M}f)(1-s)\,. \tag{3.79}$$

Clearly, $g \in L^2(R_+)$ and $\|g\|_{L^2(R_+)} \leqslant c_2 \|f\|_{L^2(R_+)}$. Moreover, $g \in \mathcal{B}$, for

$$(3.80) \qquad (1+|\varrho|)^{1/2-\alpha}(\mathcal{M}g)(\tfrac{1}{2}+i\varrho) = (1+|\varrho|)^{1/2-\alpha}P(\tfrac{1}{2}+i\varrho)(\mathcal{M}f)(\tfrac{1}{2}+i\varrho)$$

and $\varrho \to (1+|\varrho|)^{1/2-\alpha}P(\frac{1}{2}+i\varrho)$ is bounded. Then the correspondence $f \to g$ established by (3.79) defines an operator $f \to \mathfrak{H}_{\nu,\alpha} f$ from $L^2(R_+)$ into $\mathcal{B}$ which is bounded in $L^2(R_+)$. The representation (3.56) follows by taking the inverse Mellin transform of (3.79): it is easy to see, by the calculus of residues, that

$$(3.81) \qquad \int_0^\infty x^{s-1+\alpha} J_\nu(x)\,dx = P(s)\,.$$

Inequality (3.59) follows from (3.80) and (3.68). Conversely, let $g \in \mathcal{B}$ and f be defined by

$$(3.82) \qquad (\mathcal{M}f)(s) = Q(s)(\mathcal{M}g)(1-s)$$

where

$$(3.83) \qquad Q(s) = 1/P(1-s)\,.$$

For $s = \frac{1}{2} + i\varrho$ we can write

$$(3.84) \qquad (\mathcal{M}f)(\tfrac{1}{2}+i\varrho) = (1+|\varrho|)^{\alpha-1/2}Q(\tfrac{1}{2}+i\varrho)(1+|\varrho|)^{1/2-\alpha}(\mathcal{M}g)(\tfrac{1}{2}-i\varrho)\,.$$

Then, since $\varrho \to (1+|\varrho|)^{\alpha-1/2}Q(\frac{1}{2}+i\varrho)$ is bounded, $(\mathcal{M}f)(\frac{1}{2}+i\varrho)$ is $L^2(R)$ and $f \in L^2(R_+)$. The correspondence $g \to f$ established by (3.82) defines an operator from $\mathcal{B}$ into $L^2(R_+)$; this operator is the inverse of $\mathfrak{H}_{\nu,\alpha}$ by reason of (3.83), and inequality (3.60) follows from (3.78). The representation (3.58) also follows by taking the inverse Mellin transform of (3.82).

b) Odd solutions.

THEOREM 3.5. *Let $h \in \mathcal{K}^1(R_+)$ be a given function, there exists at most one function $u \in W_+(X_+ \cap Y_+)$, solution to eq. (3.45) and verifying the conditions $u(\cdot, 0) = h$ and $u(0, \cdot) = 0$ in the sense of traces. Such a solution actually exists, provided that h verifies the following condition:*

$$\int_0^\varepsilon \frac{|h(x)|^2}{x}\,dx < \infty \qquad \textit{for some } \varepsilon > 0\,.$$

To prove the theorem, we need the following lemma.

LEMMA 3.7. *Let* $\alpha \leqslant \frac{1}{2}$, $\nu = -\alpha - \frac{1}{2}$; *the operator represented by formula* (3.56) *is a one-to-one operator from the set* $\mathcal{A}$ *defined by*

$$\mathcal{A} \equiv \{f \in L^2(R_+);\ \varrho \to |\varrho|^{-1}(\mathcal{M}f)(\tfrac{1}{2} + i\varrho) \in L^2(R)\} \tag{3.85}$$

to the set $\mathcal{B}$ *given by* (3.57). *For the inverse operator, the same representation* (3.58) *holds. Moreover we have*

$$\int_{-\infty}^{+\infty} (1 + \varrho^2)^{1/2-\alpha} |(\mathcal{M}\mathfrak{H}_{\nu,\alpha} f)(\tfrac{1}{2} + i\varrho)|^2 d\varrho \leqslant \text{const} \int_{-\infty}^{+\infty} (1 + \varrho^{-2}) |(\mathcal{M}f)(\tfrac{1}{2} + i\varrho)|^2 d\varrho \tag{3.86}$$

$$\int_{-\infty}^{+\infty} (1 + \varrho^{-2}) |(\mathcal{M}\mathfrak{H}_{\nu,\alpha}^{-1} f)(\tfrac{1}{2} + i\varrho)|^2 d\varrho \leqslant \text{const} \int_{-\infty}^{+\infty} (1 + \varrho^2)^{1/2-\alpha} |(\mathcal{M}f)(\tfrac{1}{2} + i\varrho)|^2 d\varrho\,. \tag{3.87}$$

REMARK. Let $\varepsilon > 0$ and $g(t) = t^{\varepsilon - 1/2}$ for $0 < t < 1$ and $g(t) = 0$ for $t > 1$; then $g \in L^2(R_+)$ and $(\mathcal{M}g)(s) = (s - \frac{1}{2} + \varepsilon)^{-1}$. By applying the convolution theorem for Mellin transforms, we get

$$(s - \tfrac{1}{2} + \varepsilon)^{-1} (\mathcal{M}f)(s) = \mathcal{M}\Big[\int_0^\infty g(t) f(\xi/t)\, dt/t\Big](s) = \mathcal{M}\Big[\int_0^1 t^{-3/2+\varepsilon} f(\xi/t)\, dt\Big](s)\,.$$

Thus the condition that $L^2(R_+) \ni f$ belongs to $\mathcal{A}$ means that the one-parameter family of functions $\xi \to \int_0^1 t^{-3/2+\varepsilon} f(\xi/t)\, dt$ be bounded in $L^2(R_+)$, i.e.,

$$\xi \to \xi^{-1/2+\varepsilon} \int_\xi^\infty \tau^{-1/2-\varepsilon} f(\tau)\, d\tau$$

must be uniformly bounded in $L^2(R_+)$.

PROOF OF TH. 3.5. This proof follows closely the proof of Th. 3.4. The proof of uniqueness is the same as in Th. 3.4; the proof of the existence can be reduced, as we did in the case *a*) to the determination of a solution w to eq. (3.47) satisfying the same boundary conditions as the requested solution u.

In order to determinate w, let us look for a representation like (3.61) where now $W(\cdot, t)$, having to satisfy the condition $W(0, t) = 0$, is given by

$$W(x, t) = t^{5/6} a(t) x^{1/2} J_{1/3}(tx^{3/2}) \tag{3.88}$$

where a is an arbitrary function. Imposing the condition at $y=0$, and using the same change of variables (3.64), gives

$$l(\xi)=\mathfrak{H}^{-1}_{1/3,1/6}\{a(t);\,\xi\}\,; \tag{3.89}$$

eq. (3.89) has a unique solution $a\in\mathcal{B}$ and we can write the function w given by (3.61) as follows

$$w(\xi^{2/3},y)=\xi^{-1/2}\big(\mathfrak{H}^{-1}_{1/3,1/6}\circ\exp\left[-\tfrac{9}{4}(\cdot)^2y\right]\circ\mathfrak{H}_{1/3,1/6}\,l\big)(\xi)\,. \tag{3.90}$$

From (3.90), an inequality like (3.68) can be derived as we did in a.

Let us consider now w_x; from the representation of w we get

$$w_x(x,y)=\tfrac{3}{2}x\int_0^\infty\exp\left(-\tfrac{9}{4}t^2y\right)t^{11/6}\,a(t)\,J_{-2/3}(tx^{3/2})\,dt\,.$$

Thus we can write, with the change of variables (3.64)

$$w_x(\xi^{2/3},y)=\tfrac{3}{2}\xi^{-1/6}\,\mathfrak{H}^{-1}_{-2/3,1/6}\{\exp\left(-\tfrac{9}{4}t^2y\right)t\,a(t);\,\xi\}\,. \tag{3.91}$$

Let us consider now the function $t\to ta(t)$. As we did in a) in deriving (3.70) we get now

$$ta(t)=\mathfrak{H}_{-2/3,1/6}\{\tfrac{1}{2}\xi^{-1}l(\xi)-l'(\xi);\,t\}\,. \tag{3.92}$$

Thus, in order $t\to ta(t)$ to belong to the set $\mathcal{B}$, the function $\xi\to\frac{1}{2}\xi^{-1}l(\xi)-$ $-l'(\xi)$ must belong to the set $\mathcal{A}$ defined by (3.85). This means (remember the remark after lemma 3.7), that

$$\xi\to\xi^{\varepsilon-1/2}\int_\xi^\infty\tau^{-1/2-\varepsilon}\left[\tfrac{1}{2}\tau^{-1}l(\tau)-l'(\tau)\right]d\tau$$

be uniformley bounded in $L^2(R_+)$. Returning to the old variables, we have

$$\int_0^\infty\xi^{2\varepsilon-1}\left|\int_\xi^\infty\tau^{-\frac{1}{2}-\varepsilon}\left[\tfrac{1}{2}\tau^{-1}l(\tau)-l'(\tau)\right]d\tau\right|^2d\xi=\tfrac{2}{3}\int_0^\infty x^{3\varepsilon-1}\left|\int_x^\infty t^{-\frac{3}{2}\varepsilon}h'(t)\,dt\right|^2dx\leqslant$$

(by an integration by parts)

$$\tfrac{4}{3}\int_0^\infty x^{-1}|h(x)|^2\,dx+3\varepsilon^2\int_0^\infty x^{3\varepsilon-1}\left|\int_x^\infty t^{-\frac{3}{2}\varepsilon-1}h(t)\,dt\right|^2dx\,.$$

The second term may be estimated by an extension of Hardy's inequality (see [15], formula (9.9.10)) and we get

$$\ldots \leqslant \tfrac{2}{3}\int_0^\infty x^{-1}|h(x)|^2\,dx\,.$$

Then, by hypothesis, this integral is finite and we have, from (3.86)

$$(3.94)\qquad \int_{-\infty}^{+\infty}(1+\varrho^2)^{1/3}|\mathcal{M}\{ta(t);\tfrac{1}{2}+i\varrho\}|^2 d\varrho \leqslant$$

$$\operatorname{const}\int_{-\infty}^{+\infty}(1+\varrho^{-2})|\mathcal{M}\{l'(\xi)-\tfrac{1}{2}\xi^{-1}l(\xi);\tfrac{1}{2}+i\varrho\}|^2 d\varrho =$$

(from the Parseval formula for Mellin transform)

$$\operatorname{const}\Big[\int_0^\infty |l'(\xi)-\tfrac{1}{2}\xi^{-1}l(\xi)|^2 d\xi + \int_0^\infty \xi^{-1}\Big|\int_\xi^\infty t^{-1/2}[l'(t)-\tfrac{1}{2}t^{-1}l(t)]\,dt\Big|^2 d\xi\Big] \leqslant$$

$$\operatorname{const}\Big[\int_0^\infty x|h'(x)|^2\,dx + \int_0^\infty x^{-1}|h(x)|^2\,dx\Big]\,.$$

Now, from (3.91), we see that also the function $\xi \to \xi^{1/6} w_x(\xi^{2/3}, y)$ belongs to the set $\mathcal{A}$, and, from (3.87) and (3.94), we get the estimate

$$(3.95)\qquad \int_0^\infty |w(x,y)|^2 x^{-1}\,dx + \int_0^\infty x|w_x(x,y)|^2\,dx \leqslant$$

$$\operatorname{const}\Big[\int_0^\infty x^{-1}|h(x)|^2\,dx + \int_0^\infty x|h'(x)|^2\,dx\Big]\,.$$

Now an estimate of $\sup\limits_{y>0}\int_0^\infty |w(x,y)|^2\,dx$ follows from the estimates of $\sup\limits_{y>0}\int_0^\infty x^{-1}|w|^2\,dx$ and $\sup\limits_{y>0}\int_0^\infty x^2|w|^2\,dx$. Finally, let us consider w_y; directly from the representation (3.90) we get

$$(3.96)\qquad w_y(\xi^{2/3}, y) = \tfrac{9}{4}\xi^{1/2}\mathfrak{H}^{-1}_{1/3,1/6}\{\exp(-\tfrac{9}{4}t^2 y)\,t^2 a(t);\,\xi\}\,.$$

By the same calculations as in a), but taking account now of (3.94), we get the estimate

$$\int_0^\infty \int_0^\infty x^2 |w_y|^2 \, dx\, dy \leqslant \text{const} \left[\int_0^\infty x |h'|^2 dx + \int_0^\infty x^{-1} |h|^2 dx \right]. \tag{3.97}$$

By the knowledge of the function w, we can now complete the proof of Th. 3.5, as we did for Th. 3.4.

PROOF OF LEMMA 3.7. The proof follows the same line as for lemma 3.6 (see also the proof of Th. 2.4 in [12]). Let $P(s)$ be the function defined by (3.76); we have now

$$P(\tfrac{1}{2} + i\varrho) = 2^{i\varrho + \alpha - 1/2} \frac{\Gamma(i\varrho/2)}{\Gamma(\frac{1}{2} - \alpha - i\varrho/2)}$$

and so we get the bounds

$$c_1 \leqslant \frac{(1 + |\varrho|)^{1/2-\alpha}}{1 + |\varrho|^{-1}} |P(\tfrac{1}{2} + i\varrho)| \leqslant c_2$$

where c_1 and c_2 are positive constants.

Now, let $f \in \mathcal{A}$, let g be defined by

$$(\mathcal{M}g)(s) = P(s)(\mathcal{M}f)(1 - s). \tag{3.98}$$

We say that $g \in \mathcal{B}$, for

$$(1 + |\varrho|)^{1/2-\alpha} (\mathcal{M}g)(\tfrac{1}{2} + i\varrho) = \frac{(1 + |\varrho|)^{1/2-\alpha}}{1 + |\varrho|^{-1}} P(\tfrac{1}{2} + i\varrho)(\mathcal{M}f)(\tfrac{1}{2} + i\varrho) \cdot (1 + |\varrho|^{-1})$$

and the function $\varrho \to (1 + |\varrho|^{-1})^{-1}(1 + |\varrho|)^{1/2-\alpha} P(\frac{1}{2} + i\varrho)$ is upperly bounded. Conversely, let $g \in \mathcal{B}$ and f be defined by

$$(\mathcal{M}f)(s) = Q(s)(\mathcal{M}g)(1 - s) \tag{3.99}$$

where $Q(s) = 1/P(1 - s)$. Then $f \in \mathcal{A}$, since we can write

$$(1 + |\varrho|^{-1}((\mathcal{M}f)(\tfrac{1}{2} + i\varrho) = \frac{1 + |\varrho|^{-1}}{(1 + |\varrho|)^{1/2-\alpha}} Q(\tfrac{1}{2} + i\varrho)(1 + |\varrho|)^{1/2-\alpha} (\mathcal{M}g)(\tfrac{1}{2} - i\varrho).$$

Then the correspondence $f \leftrightarrow g$ established by (3.98) and (3.99) defines two operators, one inverse of the other, between the sets $\mathcal{A}$ and $\mathcal{B}$. The representations and the inequalities (3.86) and (3.87) follow as in lemma 3.6.

4. – The forward-backward equation.

In this section we want to discuss eq. (1.3), that we rewrite here,

$$u_{xx} - xu_y - ku = f \qquad (k > 0) \tag{4.1}$$

in the half-plane $y > 0$. The problem is to find a function

$$u \in W_+(Y_+) \tag{4.2}$$

which is a solution of eq. (4.1) and such that

$$u(x, 0) = h(x) \quad \text{ for } x > 0 \,. \tag{4.3}$$

Here f and h are given functions, $f \in L^2(Y_+)$ and $h \in \mathcal{H}^1(R_+)$.

THEOREM 4.1. *Let $f \in L^2(Y_+)$ and $h \in \mathcal{H}^1(R_+)$ be given; there exists only one solution u to pb.* (4.1), (4.2), (4.3). *Such a solution satisfies the inequalities*

$$\iint\limits_{Y_+} (k^2|u|^2 + 2k|u_x|^2)\, dx\, dy \leqslant \iint\limits_{Y_+} |f|^2\, dx\, dy + k\int\limits_0^\infty x|h|^2\, dx \tag{4.4}$$

$$\|u\|_{W_+(Y_+)} \leqslant \mathrm{const}\,(k)\left[\|f\|_{L^2(Y_+)} + \|h\|_{\mathcal{H}^1(R_+)}\right] . \tag{4.5}$$

The exact dependence of const(k) on k will be established in the proof of the theorem.

Proof of the uniqueness: let u be a $C_0^\infty(\overline{Y}_+)$ function; let us put $f = u_{xx} - xu_y - ku$; taking the square of the modulus of f and suppressing the non-negative term $|u_{xx} - xu_y|^2$ gives

$$k^2|u|^2 - 2k\,\mathrm{Re}\,(u_{xx} - xu_y)\bar{u} \leqslant |f|^2 .$$

Integrating now over Y_+ gives, by integration by parts,

$$\iint\limits_{Y_+} (k^2|u|^2 + 2k|u_x|^2)\, dx\, dy \leqslant \iint\limits_{Y_+} |f|^2\, dx\, dy + k\int\limits_{-\infty}^{+\infty} x|u(x, 0)|^2\, dx \leqslant$$

$$\iint\limits_{Y_+} |f|^2\, dx\, dy + k\int\limits_0^\infty x|u(x, 0)|^2\, dx .$$

This inequality, established for C_0^∞-functions, can be extended to functions belonging to $W_+(Y_+)$; then we obtain (4.4) as an a priori inequality, from which the uniqueness of the solution to pb. (4.1), (4.2), (4.3) may be derived.

The proof of the existence will follow from Ths. 4.2 and 4.3 of the next two subsections.

4.1. *The non-homogeneous equation.*

THEOREM 4.2. *For every $f \in L^2(R^2)$, there exists a (unique) solution u of eq. (4.1) belonging to $Z(R^2)$. This solution verifies an inequality of the form*

$$\iint_{R^2} \left(x^2|u_y|^2 + |u_{xx}|^2 + k|u_x|^2 + k^2|u|^2\right) dx\,dy \leqslant \text{const} \iint_{R^2} |f|^2\,dx\,dy\,. \tag{4.6}$$

Moreover, if $f = 0$ for $y < 0$, the restriction of u to Y_+ belongs to $W_+(Y_+)$ and satisfies the inequality

$$\int_0^\infty x^2|u(x, y)|^2\,dx < \text{const}\,(y + k^{-3/2}) \iint_{Y_+} |f|^2\,dx\,dy \tag{4.7}$$

PROOF. Standard calculations show that, if a solution u exists, its Fourier transform (with respect to y), $\hat{u}$ say, may be represented in the following form:

$$\hat{u}(x, \eta) = \begin{cases} \hat{B}(x, \eta)\,\hat{u}_x(0, \eta) + V(\eta)\cdot f(|x|, \eta) & \text{if } x > 0 \\ -\bar{\hat{B}}(-x, \eta)\,\hat{u}_x(0, \eta) + V^*(\eta)\cdot f(-|x|, \eta) & \text{if } x < 0 \end{cases} \tag{4.8}$$

where $V(\eta)$ is the family of operators defined in subsection 3.1 and $\hat{B}$ is given by (3.39); the bar means, as usual, complex conjugate and the star means adjoint. Moreover $\hat{u}_x(0, \eta)$ is given by:

$$\hat{u}_x(0, \eta) = \frac{\mathfrak{L}(\eta)}{2\,\mathrm{Re}\,B(0, \eta)} \tag{4.9}$$

where we put

$$\mathfrak{L}(\eta) = \Big(\int_{-\infty}^{0} \bar{E}(x', 0; \eta) + \bar{E}(-x', 0; \eta) - \\ - \int_0^\infty E(0, x'; \eta) + E(0, -x'; \eta)\Big) \hat{f}(x', \eta)\,dx'. \tag{4.10}$$

Let us prove that the function u defined by (4.8) actually belongs to $Z(R^2)$. $\mathfrak{L}(\eta)$ may be estimated by Schwartz inequality as follows:

$$|\mathfrak{L}(\eta)|^2 \leqslant \text{const}\,(k^3+\eta^2)^{-1/6}\|\hat{f}(\cdot,\eta)\|^2_{L^2(\mathrm{R})}. \tag{4.11}$$

To get (4.11) one has only to use the following estimate for $E(x,0;\eta)$:

$$|E(x,0;\eta| \leqslant \text{const}\,(k^3+\eta^2)^{1/12}(k^2+\eta^2x^2)^{-1/8}\cdot \cdot\exp\left\{-\frac{\sqrt{2}}{3}|x|(k^2+x^2\eta^2)^{1/4}\right\}. \tag{4.12}$$

(4.12) is easily derived from (3.21) by making use of (3.23) and (3.24). Thus we have

$$|\mathfrak{L}(\eta)|^2 \leqslant 4\|\hat{f}(\cdot,\eta)\|^2_{L^2(R)}\cdot\int_0^\infty |E(x,0;\eta)|^2\,dx \leqslant$$

$$\leqslant \text{const}\,\|\hat{f}(\cdot,\eta\|^2_{L^2(R)}(k^3+\eta^2)^{1/6}\int_0^\infty (k^3+\eta^2x^2)^{-1/4}\cdot \cdot\exp\left\{-\frac{2\sqrt{2}}{3}x(k^2+\eta^2x^2)^{1/4}\right\}dx\,.$$

It is not difficult to see that the last integral is bounded by $\text{const}\,(k^3+\eta^2)^{-1/3}$; thus we get (4.11). Moreover, by using usual estimates for Hankel functions, we get

$$|\mathrm{Re}\,\hat{B}(0,\eta)| \geqslant \text{const}\,(k^3+\eta^2)^{-1/6}\,. \tag{4.13}$$

Thus we have an estimate for $\hat{u}_x(0,\eta)$:

$$(k^3+\eta^2)^{1/6}|\hat{u}_x(0,\eta)|^2 \leqslant \text{const}\,\|\hat{f}(\cdot,\eta)\|_{\mathrm{L}^2(R)}\,. \tag{4.14}$$

Now, from the representation (4.8), remembering what was proved in subsections 3.1 and 3.2, we obtain an inequality like (4.6).

Finally, if $f=0$ for $y<0$, inequality (4.7) can be obtained, for instance, by taking the representation (4.8) for $x>0$: the first term at the right member may be treated as in the proof of lemma 3.4 and we get an inequality like (3.44); by taking account of (4.14), the right member of this inequality may be estimated by const $k^{-3/2}\|f\|^2_{L^2(Y_+)}$; the second term is treated as in the proof of Th. 3.1 and we get an inequality like (3.3); combining the two inequalities so obtained gives (4.7).

4.2. *The homogeneous equation* (*Wiener Hopf procedure*).

THEOREM 4.3. *Let $h \in \mathcal{H}^1(R_+)$ be given; there exists a* (*unique*) *function u such that*

$$u \in W_+(Y_+) \tag{4.15}$$

$$u_{xx} - xu_y - ku = 0 \qquad (k > 0) \tag{4.16}$$

$$u(x, 0) = h(x) \qquad \text{a.e. in } 0 < x < \infty\,; \tag{4.17}$$

this function satisfies the inequality

$$\|u\|_{W_+(Y_+)} \leqslant \text{const}\,(k)\,\|h\|_{\mathcal{H}^1(R_+)}\,. \tag{4.18}$$

PROOF. To prove this theorem, we will use a technique previously used to discuss a piecewise constant coefficients equation of the same kind as (4.16) [10]. Let us briefly describe this technique. We will consider separately the equation in the two regions $x > 0$ and $x < 0$ and then we impose a « smooth » match at $x = 0$ for the solutions. Thus, in the first quadrant of R^2, we will determine a solution u to eq. (4.16) satisfying the conditions: $u(\cdot, 0+) = h$ and $u_x(0+, \cdot) = \varphi_+$ where φ_+ is an auxiliary function; in the second quadrant, where the equation is backward parabolic, we will determine a solution u such that: $u_x(0-, \cdot) = \varphi_-$, where φ_- is another auxiliary function. Then the condition that u and u_x match smoothly at $x = 0$ will determine the auxiliary functions φ_+ and φ_-; it will be $\varphi_+ = \varphi_-$ and the common value, φ say, must satisfy an integral equation in convolution on the half-line (a Wiener-Hopf equation).

By following the above sketched procedure, let us look for solutions u represented in this way:

$$u(x, y) = \begin{cases} \displaystyle\int_{-\infty}^{y} \varphi_+(t) B(x, y - t)\, dt + U(x, y) & \text{for } x > 0,\ y > 0 \\ \displaystyle -\int_{y}^{\infty} \varphi_-(t) B(-x, t - y)\, dt & \text{for } x < 0,\ y > 0\,. \end{cases} \tag{4.19}$$

Here φ_+ and φ_- are functions to be determined such that

$$\varphi_+ \in H^{1/6}(R)\,, \qquad \text{spt (the support of) } \varphi_+ \subseteq \overline{R}_+ \tag{4.20}$$

$$\varphi_- \in H^{1/6}(R) \tag{4.21}$$

and U is the (unique) solution of class $W_+(X_+\cap Y_+)$ to the following problem:

$$\left\{\begin{array}{l} U_{xx}-xU_y-kU=0 \\ U(x,0+)=h(x) \\ U_x(0+,y)=0\,. \end{array}\right. \tag{4.22}$$

Such a function actually exists, thanks to Th. 3.4. The function B appearing in (4.19) is the same as in Th. 3.3; thus we see that the function u given by (4.19) belongs to $W_+(Y_+\setminus$ the half-line $x=0)$, verifies the equation $u_{xx}-xu_y-ku=0$ and (by the second of (4.20)) $u(x,0+)=U(x,0+)$ if $x>0$; thus the condition $u(x,0+)=h(x)$ if $x>0$ holds true.

Moreover we have

$$u_x(0+,y)=\varphi_+(y)\,,\qquad u_x(0-,y)=\varphi_-(y)\qquad \text{if } y>0 \tag{4.23}$$

and

$$\left\{\begin{array}{l} u(0+,y)=\displaystyle\int_{-\infty}^{y}\varphi_+(t)B(0,y-t)\,dt+U(0,y) \\ u(0-,y)=-\displaystyle\int_{y}^{\infty}\varphi_-(t)B(0,t-y)\,dt\,. \end{array}\right. \tag{4.24}$$

Now, the function u given by (4.19) will belong actually to $W_+(Y_+)$ if and only if $u(0-,\cdot)=u(0+,\cdot)$ and $u_x(0-,\cdot)=u_x(0+,\cdot)$. This means that the functions φ_+ and φ_- must satisfy the following equations, for $y>0$

$$\left\{\begin{array}{l} \varphi_+(y)=\varphi_-(y) \\ \displaystyle\int_{-\infty}^{y}\varphi_+(t)B(0,y-t)\,dt+\int_{y}^{\infty}\varphi_-(t)B(0,t-y)\,dt=-U(0,y)\,. \end{array}\right. \tag{4.25}$$

A solution of (4.25), verifying (4.20) and (4.21), can be obtained in this way: $\varphi(=\varphi_+=\varphi_-)$ is a solution, such that

$$\varphi\in H^{1/6}(R)\,,\qquad \operatorname{spt}\varphi\subseteq\bar{R}_+ \tag{4.26}$$

of the integral equation

$$\int_0^\infty K(y-t)\varphi(t)\,dt=l(y)\qquad(0<y<\infty) \tag{4.27}$$

where the kernel K is given by

$$K(y) = -B(0, |y|) \tag{4.28}$$

and l is any function such that

$$l \in H^{1/2}(R)\,; \quad l(y) = U(0, y) \quad \text{for } 0 < y < \infty \tag{4.29}$$

Let us remark that we can find functions l verifying (4.29), since the function $0 < y \to U(0, y)$ belongs to $H^{1/2}(R_+)$; in fact it is the trace on the positive y-axis of $U(x, y)$ which belongs to $W_+(X_+ \cap Y_+)$ (remember Th. 2.1).

Thus we have to solve a Wiener-Hopf problem. For such problems a formal technique of solution has been developed long ago and only recently it has been arranged in a satisfactory way, at least for a large class of these equations, in the context of Sobolev spaces; for every detail and bibliography see [12].

In the particular case we are dealing with, we can apply Th. 4.1.1 of [12]; this theorem states that a (unique) solution φ of (4.27), verifying (4.26), exists, provided the kernel K satisfies certain hypotheses; such hypotheses are verified in the following lemma:

LEMMA 4.1. *Consider the function* $y \to \hat{K}(y)$ *given by* (4.28); *let* $\hat{K}$ *be its Fourier transform. There exists two functions,* K_+ *and* K_-, *such that*

i) $\zeta = \eta + i\chi \to K_+(\zeta)$ *is holomorphic in the half-plane* $\operatorname{Im}\zeta > 0$ *and continuous in the closure* $\operatorname{Im}\zeta \geqslant 0$; $\zeta \to K_-(\zeta)$ *is holomorphic in the half-plane* $\operatorname{Im}\zeta < 0$ *and continuous in the closure* $\operatorname{Im}\zeta \leqslant 0$.

ii) $0 < \text{const} \leqslant (k^3 + |\zeta|^2)^{1/12}|K_+(\zeta)| \leqslant \text{const}$ if $\operatorname{Im}\zeta \geqslant 0$
$0 < \text{const} \leqslant (k^3 + |\zeta|^2)^{1/12}|K_-(\zeta)| \leqslant \text{const}$ if $\operatorname{Im}\zeta \leqslant 0$

iii) $\hat{K}(\eta) = K_+(\eta)K_-(\eta)$ *for every real* η (the couple (K_+, K_-) is called a factorization of K).

Now we can apply Th. 4.1.1 of [12] which, in our case, gives us the following result:

LEMMA 4.2. *Consider eq.* (4.27), *where the kernel K is given by* (4.28) *and the function l by* (4.29). *Then a unique solution φ verifying* (4.26) *exists and satisfies the inequality*

$$\|\varphi\|_{H^{1/6}(R)} \leqslant \text{const} \inf_{\alpha \in C_0^\infty(R_-)} \|l - \alpha\|_{H^{1/2}(R)}\,. \tag{4.30}$$

REMARK. The inf at the right of (4.30) is the norm, in the space $H^{1/2}(R_+)$ of the restriction of l to R^+; thus, since this restriction is given by $U(0,\cdot)$, we have, recalling (2.16) and (3.46),

$$\inf_{\alpha\in C_0^\infty(R_-)} \|l-\alpha\|_{H^{1/2}(R)} \leqslant \mathrm{const}\,(k)\,\|h\|_{\mathcal{H}^1(R_+)}\,. \tag{4.31}$$

Thus Th. 4.3 is proved.

PROOF OF LEMMA 4.1. From the definition of $\hat{B}$ in (3.39) we obtain

$$\hat{K}(\eta)=2\,\mathrm{Re}\,\hat{B}(0,\eta)=\frac{2}{\sqrt{k}}\,\mathrm{Im}\left[\frac{H^{(1)}_{1/3}(2k^{3/2}/3|\eta|)}{H^{(1)}_{-2/3}\,(2k^{3/2}/3|\eta|)}\right].$$

By recalling now some elementary properties of Hankel functions, namely: $\bar{H}^{(1)}_\nu(z)=H^{(2)}_\nu(\bar{z})$, $\bar{H}^{(2)}_\nu(z)=H^{(1)}_\nu(\bar{z})$ and the wronskian $W(H^{(1)}_\nu, H^{(2)}_\nu)=-4i/\pi z$, we obtain

$$\hat{K}(\eta)=\frac{-6|\eta|}{\pi k^2}\left[H^{(1)}_{-2/3}\left(\frac{2}{3}\,\frac{k^{3/2}}{|\eta|}\right)H^{(2)}_{-2/3}\left(\frac{2}{3}\,\frac{k^{3/2}}{|\eta|}\right)\right]^{-1} \tag{4.32}$$

Now, let us consider the function

$$\mathfrak{K}(t)=\frac{|t|(1+t^2)^{1/6}}{H^{(1)}_{-2/3}(|t|^{-1})\,H^{(2)}_{-2/3}(|t|^{-1})}\,. \tag{4.33}$$

An inspection of $\mathfrak{K}$ shows that the following properties hold: i) $R\ni t\to\mathfrak{K}(t)$ is a continuous real-valued positive function (at the origin it is defined as $\lim_{t\to 0}\mathfrak{K}(t)=\pi/2$); ii) $\mathfrak{K}(-\infty)=\mathfrak{K}(+\infty)=3\cdot 2^{-10/3}\Gamma^2(\frac{1}{3})$; ind $\mathfrak{K}=0$ (*). Then (see Th. 2.1.1 in [12]) we can assert that a factorization of $\mathfrak{K}$ exists with the following properties

$$\begin{aligned} &0<\mathrm{const}\leqslant|\mathfrak{K}_+(\zeta)|\leqslant\mathrm{const} && \chi\geqslant 0\\ &&\zeta=t+i\chi\\ &0<\mathrm{const}\leqslant|\mathfrak{K}_-(\zeta)|\leqslant\mathrm{const} && \chi\leqslant 0\end{aligned}$$

Then, the assertions of lemma follow, since

$$\hat{K}(\eta)=-\frac{4}{\pi}\left(k^3+\frac{9}{4}\eta^2\right)^{-1/6}\mathfrak{K}\left(\frac{3}{2}\,\frac{\eta}{k^{3/2}}\right).$$

(*) ind $\mathfrak{K}$, the index of $\mathfrak{K}$, is the index, with respect to the origin, of the curve $[-\pi,+\pi]\ni\theta\to\mathfrak{K}(\mathrm{tg}\,\theta/2)$. Roughly speaking, ind $\mathfrak{K}$ is the number of times the curve $t\to\mathfrak{K}(t)$ turns around the origin, when t varies from $-\infty$ to $+\infty$.

5. – The time-dependent problem.

Let us consider the problem outlined in the introduction; we can now pose this problem in a precise way. Let us consider the linear operator

$$L \equiv \frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} \tag{5.1}$$

acting in $L^2(Y_+)$ with domain $D(L)$ so defined

$$D(L) \equiv \{u;\, u \in W_+(Y_+);\, u = 0 \text{ for } y = 0,\ x > 0\}\,. \tag{5.2}$$

Then we have the problem to find a vector-valued function $]0, T[\ni t \to w(t)$, with the following properties

$$\begin{cases} w \in C^1(0, T; L^2(Y_+)) \cap C^0(0, T;\ W_+(Y_+)) \\ w' = Lw \\ w(0) = f \end{cases} \tag{5.3}$$

where f is a given function belonging to the domain of L (as one can see, here we took homogeneous conditions at the boundary: but it is clear how we can reduce to this case, starting from a different situation).

THEOREM 5.1. *Let f be a given function belonging to $D(L)$. Then, there exists only one function w with the properties* (5.3).

PROOF. *Uniqueness*: it is a consequence of the following inequality

$$\frac{d}{dt}\|w(t)\|^2_{L^2(Y_+)} \leqslant 0 \tag{5.4}$$

(5.4) follows from the property that, for every $w(t) \in D(L)$ we have

$$(Lw(t), w(t))_{L^2(Y_+)} \leqslant -\|w_x(t)\|^2_{L^2(Y_+)} \tag{5.5}$$

(5.5) is proved by integration by parts, as we did in the proof of uniqueness for Th. 4.1.

Existence: the existence of a solution is a consequence of Th. 4.1. For, we can say:

i) $D(L)$ is dense in $L^2(Y_+)$;

ii) L is a closed map in $L^2(Y_+)$ with domain $D(L)$;

iii) every $k>0$ belongs to the resolvent set of L, i.e., $(L-k)D(L)$ is the whole $L^2(Y_+)$ and $(L-k)$ has a bounded inverse;

iv) for every $k>0$ and every $f \in L^2(Y_+)$, we have

$$k\|(L-k)^{-1}f\|_{L^2(Y_+)} \leqslant \|f\|_{L^2(Y_+)}. \tag{5.6}$$

Assertions ii) to iv) follow from Th. 4.1 (in particular inequality (5.6) follows from (4.4)) while i) is obvious. These assertions imply (by Hille-Yosida theorem [14]) that L *is the infinitesimal generator of a strongly continuous semigroup of contraction operators*

$$\mathcal{C}(t): L^2(Y_+) \to L^2(Y_+), \qquad i.e., \qquad \|\mathcal{C}(t)f\|_{L^2(Y_+)} \leqslant \|f\|_{L^2(Y_+)}. \tag{5.7}$$

Thus, if $f \in D(L)$, then $w(t) = \mathcal{C}(t)f$ is the unique solution to pb. (5.3).

Remark. From theorems given in sect. 3, we can solve some other time-dependent problems for eq. (1.1), like the Cauchy problem in the region $R^2 \times]0, T[$ or the mixed problem in the region $X_+ \times]0, T[$. For instance, the following results could be easily proved:

i) Let $f \in Z(R^2)$ be given; let L be the operator defined by (5.1) acting in $L^2(R^2)$, with domain $Z(R^2)$; then there exists only one function w such that

$$w \in C^1(0, T; L^2(R^2)) \cap C^0(0, T; Z(R^2))$$

$$w' = Lw$$

$$w(0) = f.$$

ii) Let $f \in Z(X_+)$ be given and $f(0+, y) = 0$ for $-\infty < y < +\infty$; let L be the operator defined by (5.1) acting in $L^2(X_+)$ with domain $\mathfrak{D}(L)$ so defined:

$$\mathfrak{D}(L) = \{u;\ u \in Z(X_+);\ u = 0 \text{ for } x = 0\};$$

then there exists only one function w such that:

$$w \in C^1(0, T; L^2(X_+)) \cap C^0(0, T; Z(X_+))$$

$$w' = Lw$$

$$w(0) = f.$$

The problem considered in ii) has been studied in [11].

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