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In this paper we introduce the notion of a Fredholm vector field on Banach manifolds (modelled on a real Banach space) with S-structures (see below). We define an equivalence relation between vector fields with a finite number of isolated zeros. For a special class of Fredholm vector fields X we define the Euler-characteristic $\chi(X)$ and show that it depends only on the equivalence class of X. For Hilbert manifolds $\mathcal{M}$ of finite codimension and of finite type $\chi(X) = \chi(\mathcal{M})$ for all X. This theorem gives a new proof of the Euler-Hopf theorem in finite dimensions.

In section VI we mention some direct generalizations of the theory of vector fields presented in the first five sections. The reader wishing to see the most general results should read the first two short sections and then proceed directly to the principal results in VI. In § VII we show how the theory could apply to the problem of geodesics on closed finite dimensional Riemannian manifolds.

I. – Fredholm structures, layer structures and S-structures.

Let $GL(E)$ be the general linear group of a real Banach space E and $GL_C(E)$ the Lie subgroup of invertible linear transformations of the form $I + K$, where K is completely continuous. It is shown in [1] that $GL_C(E)$ has the homotopy type of $GL_\infty(E)$; in particular $\pi_0(GL_C(E)) \approx Z_2$.

Let $\mathcal{M}$ be a paracompact C^r ($r \geqslant 3$) Banach manifold (possibly with boundary) admitting C^r partitions of unity. A *Fredholm structure* $\mathcal{F}$ for $\mathcal{M}$ is a collection of C^r charts $(\varphi_i, U_i)_{i \in I}$ covering $\mathcal{M}$ with the property that $D(\varphi_i \circ \varphi_j^{-1})_x \in GL_C(E)$ for all $x \in U_i \cap U_j$, where $D(\varphi_i \circ \varphi_j^{-1})_x$ denotes the Frechet derivative of $\varphi_i \circ \varphi_j^{-1}$ at x. We assume $\mathcal{F}$ to be maximal with respect to this property. Such structures were first defined in [1] and [2].

A layer structure $\mathfrak{L}$ for $\mathcal{M}$ is a maximal atlas of charts $(\varphi_i, \ U_i)_{i \in I}$ with $(\varphi_i \circ \varphi_j^{-1})(x) = x + K(x)$ where K is finite dimensional; i.e., $K(\varphi_j(U_i \cap U_j))$ is contained in a finite dimensional subspace of E. From [1] we have the following

PROPOSITION 1. Let $\mathcal{M}$ be as above. Then given any C^r Fredholm structure $\mathcal{F}$ for $\mathcal{M}$ there is a unique layer structure $\mathfrak{L} \subset \mathcal{F}$.

It is also shown in [1] that if $T\mathcal{M}$ admits a reduction to $GL_C(E)$ {in particular if $GL(E)$ is contractible} then $\mathcal{M}$ admits an $\mathcal{F}$ structure, and consequently a layer structure.

It turns out that Fredholm structures are too large a class of coordinate charts for vector fields, and layer structures are somewhat more restrictive. We shall introduce the notion of an S-structure for a Banach manifold which is somewhere in between these two structures.

An S-structure for $\mathcal{M}$ is a maximal atlas of charts $(\varphi_i, \ U_i)_{i \in I}$ covering $\mathcal{M}$ with $(\varphi_i \circ \varphi_j^{-1})(x) = x + K(x)$ where K is compact. Layer structures are S-structures and S-structures are Fredholm structures; moreover given an S-structure S there is a unique layer structure $\mathfrak{L}$ and a unique Fredholm structure $\mathcal{F}$ with $\mathfrak{L} \subset S \subset \mathcal{F}$. Consequently if $T\mathcal{M}$ admits a reduction to $GL_C(E)$, $\mathcal{M}$ admits an S-structure. In order to define Fredholm section of the tangent bundle we shall need to develop a few facts about S-structures.

DEFINITION. Let $K \colon \Omega \to E$ be a C^r compact mapping of an open region $\Omega \subset E$ into E. We say that K is S^1-compact if $DK \colon \Omega \times E \to E$ is also compact; i.e., the map $(p, h) \to DK_p(h)$ takes $\Omega \times$ (bounded sets) into a compact set.

Similarly K is S^r-compact if $(p, h_1, \ldots, h_r) \to D^r K_p(h_1, \ldots, h_0)$ is compact, where the h_i are all in some fixed bounded set in $E \times E \times \ldots \times E$. K is locally S^r-compact if each $p_0 \in \Omega$ has a neighborhood V with respect to which $D^r K \colon V \times E \times E \times \ldots \times E \to E$ is compact.

PROPOSITION 2. If K is C^{r+1} and compact then K is locally S^r-compact. If $\max_r \sup_{p \in \Omega} \|D^{r+1} K_p\| < \infty$ then K is S^r-compact.

PROOF. We do only the local case when $r = 1$. Let V_s be a ball of radius s on which $\sup_{p \in V} \|D^2 K_p\| \leqslant M$, and so that $V_{2s} \subset \Omega$. Then on V_s, K is uniformly Frechet differentiable. Assume that $DK \colon V_s \times E \to E$ is not compact. Therefore for some ball of radius l, say $B_l \subset E$ there does not exist a finite ε-net for $DK(V_s \times B_l)$ for every ε. Thus for some $\varepsilon > 0$ there exist sequences $(p_i, v_i) \in V \times E$ with

$$\|DK_{p_i}(v_i) - DK_{p_j}(v_j)\| \geqslant 3\varepsilon$$

for all pairs $(p_i, v_i) \neq (p_j, v_j)$.

Since K is uniformly C^1 on V_s, $K(p+h) - K(p) = DK_p(h) + \omega(p, h)$ where $\|\omega(p, h)\| < (\varepsilon/l)\|h\|$ if $\|h\| < \delta$ for some $\delta > 0$. Let $\tilde{\delta} = (1/l) \min(\delta, s)$. Then

$$\|K(p_i + \tilde{\delta}v_i) - K(p_j + \tilde{\delta}v_j)\| =$$
$$= \|DK_{p_i}(\tilde{\delta}v_i) - DK_{p_j}(\tilde{\delta}v_j) + \omega(p_i, \tilde{\delta}v_i) - \omega(p_j, \tilde{\delta}v_j)\|$$
$$\geqslant 3\varepsilon\tilde{\delta} - \|\omega(p_i, \tilde{\delta}v_i)\| - \|\omega(p_j, \tilde{\delta}v_j)\| \geqslant 3\varepsilon\tilde{\delta} - 2\varepsilon\tilde{\delta} = \varepsilon\tilde{\delta}.$$

This contradicts the compactness of K.

COROLLARY. If (φ, U) and (ψ, V) belong to some S-structure S of $\mathcal{M}$ then for all $x \in \psi(U \cap V)$, $D^2(\varphi \circ \psi^{-1})_x(h, k)$ has the property that for each fixed $k \in E$, $h \to D^2(\varphi \circ \psi^{-1})_x(h, k)$ is linear completely continuous.

II. – Fredholm vector fields.

One would like to define a C^r Fredholm vector field as a C^r section X of $T\mathcal{M}$ so that, locally, the principal part $\tilde{X}_\varphi$ of a local representation X_φ of X with respect to a coordinate map φ is Fredholm of index zero. It is not difficult to see that in general this definition depends on the coordinate chart φ.

However suppose now that $\mathcal{M}$ is endowed with a fixed S-structure. We say that $X \in \Gamma(T\mathcal{M})$, $\{\Gamma(T\mathcal{M}) = \text{space of sections of } T\mathcal{M}\}$ is Fredholm with respect to S if for each $(\varphi_i, U_i) \in S$, $\tilde{X}_\varphi$ the principal part of X with respect to φ is Fredholm of index zero.

PROPOSITION 3. The notion of Fredholm section is well defined.

PROOF. Suppose $\tilde{X}_\varphi : \mathcal{O} \to E$ is Fredholm of index zero, $\mathcal{O} = \varphi(V)$. Let $\psi, U)$ be another chart in S with $U \cap V \neq \emptyset$. If $\alpha = \psi \circ \varphi^{-1}$, then

$$\tilde{X}_\psi(x) = D\alpha_y \tilde{X}_\varphi(\alpha^{-1}(x)).$$

∴ For $x \in \varphi(U \cap V)$, $y = \alpha^{-1}(x)$.

$$D\tilde{X}_\psi(x)h = D^2\alpha_y\big(\tilde{X}_\varphi(\alpha^{-1}(x)), h\big) + D\alpha_y D\tilde{X}_\varphi D\alpha_x^{-1}(h).$$

The second term on the right is clearly, for each x, a Fredholm map of index zero. The first term has the property (by the Corollary to Proposition 2) that for each x, the linear map, $h \to D^2\alpha_y\big(\tilde{X}_\varphi(\alpha^{-1}(x)), h\big)$ is completely con-

tinuous. But if T is linear Fredholm and K is completely continuous then $T + K$ is Fredholm of the same index. This proves Proposition 3.

Let $\mathcal{M}$ be a manifold with an S-structure. $\mathcal{N} \subset \mathcal{M}$ is an S-submanifold of M if for each $x \in N$ there is a chart $(\varphi, U) \in S$ and a closed subspace E_0 of E with $\varphi(U \cap N) \subset E_0$.

One can readily show that an S-submanifold $\mathcal{N}$ inherits an S-structure from $\mathcal{M}$. We have

PROPOSITION 4. Let $\mathcal{N} \subset E$ be a Banach submanifold of finite codimension in E. Then $\mathcal{N}$ is an S-submanifold of E (in fact a layer submanifold), where E is given the natural S-structure.

PROOF. This essentially is Corollary 2.18, page 69 of [1].

We say that a manifold $\mathcal{M}$ is of *finite type* with respect to a C^r S-structure S if $\mathcal{M}$ can be covered by only finitely many coordinate neighborhoods of S.

The following is a partial converse to Proposition 4.

PROPOSITION 5. Let $\mathcal{N}$ be a C^r (admitting C^r partitions of unity) complete Finsler (e.g. see [5]) layer manifold modelled on E and of finite type; i.e., $\mathcal{N}$ can be covered by finitely many charts (U_i, φ_i), $i = 1, ..., N$, with $\varphi_i \circ \varphi_j^{-1}(x) = x + K_{ij}(x)$ where $K_{ij}(\varphi_i(U_i \cap U_j))$ is contained in a finite dimensional subspace F_{ij} of E. Then $\mathcal{N}$ can be embedded as a finite codimensional submanifold of a Banach space F. If E is isomorphic to $E \times R$, F can be taken to be E.

PROOF. Let π be a projection of E onto the span G of the F_{ij}. Let $\{\mu_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then $f(x) = \sum \mu_i(x)\varphi_i(x)$ is a Fredholm map of index zero from $\mathcal{N}$ to E. One readily checks that the map $\Psi \colon \mathcal{N} \to \mathbf{R}^N \times G^N \times E$ defined by

$$\Psi(x) = \big(\mu_1(x), ..., \mu_N(x), \pi\mu_1(x)\varphi_1(x), ..., \pi\mu_N(x)\varphi_N(x), f(x)\big)$$

is an embedding.

III. – The Euler characteristic.

Let $GL_C^\lambda(E)$ denote the group of invertible linear operators of the form $\lambda I + C$, where C is completely continuous, I is the identity and $\lambda > 0$. It follows easily that $\pi_0\big(GL_C^\lambda(E)\big) \approx \pi_0\big(GL_C(E)\big) \approx Z_2$. If we write $E = F \times R$, the operators I and J, J given by $J(y, r) = (y, -r)$ are in the two distinct components of $GL_C^\lambda(E)$.

A Fredholm vector field on an S-manifold $\mathcal{N}$ is *admissible* if given any $x_0 \in \mathcal{M}$ there is a coordinate neighborhood $(\varphi, U) \in S$ about x_0 so that the principal part $\tilde{X}_\varphi$ of X with respect to φ is bounded on $\varphi(U)$ and of the *local normal form* $\tilde{X}_\varphi(x) = \lambda(x)x + K(x)$ where K is compact, $\lambda : \varphi(U) \to R$ is C^r and bounded with $\inf_x \lambda(x) \geqslant \delta > 0$.

The following proposition shows that this notion is well defined; i.e., independent of the chart φ.

PROPOSITION 6. *If* (φ, U) *and* $(\psi, U) \in S$ *are two coordinate neighborhoods of* x_0 *and* $\tilde{X}_\varphi$ *is of the form* $\tilde{X}_\varphi(x) = \lambda(x)x + K^1(x)$ *as described above, then if* U *is sufficiently small,* $\tilde{X}_\psi$ *is of the same form.*

PROOF. $\tilde{X}_\psi(x) = D\alpha_y \tilde{X}_\varphi(\alpha^{-1}(x))$, where $\alpha = \psi \circ \varphi^{-1}$ and $y = \alpha^{-1}(x)$. Suppose U is small enough so that $\alpha(y) = y + K^0(y)$ with K^0, S^1-compact. Then $\alpha^{-1}(x) = x + K^2(x)$, K^2 S^1-compact. Hence

$$
\begin{aligned}
D\alpha_y \tilde{X}_\varphi(\alpha^{-1}(x)) &= D\alpha_y \big[\lambda(\alpha^{-1}(x))\alpha^{-1}(x) + K^1(\alpha^{-1}(x))\big] \\
&= \lambda(\alpha^{-1}(x))\alpha^{-1}(x) + K^1(\alpha^{-1}(x)) + \\
&\quad + DK_y^0\big[\lambda(\alpha^{-1}(x))\alpha^{-1}(x) + K^1(\alpha^{-1}(x))\big] \\
&= \lambda(\alpha^{-1}(x))x + \ldots .
\end{aligned}
$$

Now everything after the first term is a compact map which concludes the proposition.

It is important to note that if $\tilde{X}_\varphi$) is in the normal form above then $D\tilde{X}_\varphi(x) \in GL_C^\lambda(E)$ for all $x \in \varphi(U)$.

In addition $\tilde{X}_\varphi$ is proper (by proper we mean the inverse image of a compact set is relatively compact; e.g., see Smale [7]).

DEFINITION. A Fredholm vector field on $\mathcal{M}$ is said to be *weakly admissible* if for each $x \in \mathcal{M}$ and some chart $(\varphi, U) \in S$ about x, $D\tilde{X}_\varphi(x) \in GL_C^\lambda(E)$.

One can quickly check that this does depend on φ. Moreover it is immediate that admissible implies weakly admissible.

From here through § IV all Fredholm vector fields on $\mathcal{M}$ will be weakly admissible. Let X and Y be two such vector fields on $\mathcal{M}$ with finitely many isolated zeros. We say that X is equivalent to Y ($X \sim Y$) if the path $F(x) = t X(x) + (1 - t) Y(x)$ between X and Y has the property that $F^{-1}(\mathfrak{Z}(T\mathcal{M}))$ is compact, where $\mathfrak{Z}(T\mathcal{M})$, denotes the zero section of $T\mathcal{M}$.

Let z be a zero of X, $z \notin \partial\mathcal{M}$, and let $(\varphi, U) \in S$ be a coordinate neighborhood about z. Let $W \subset \overline{W} \subset \varphi(U)$, $\varphi^{-1}(\overline{W}) \cap \partial M = 0$ be a bounded open neighborhood of $\varphi(z)$ with $\overline{W}$ containing no other zero of X and so that $\tilde{X}_\varphi : \overline{W} \to E$ is proper.

Define the index of X at z to be $\deg[\tilde{X}_\varphi, \overline{W}, 0]$, the Leray-Schauder degree of the principal part of X with respect to φ, $\overline{W}$, and 0 (e.g., see [1]).

The fact that the index of X at z is independent of W is a standard result in degree theory (again see [1]). The fact that it is also independent of φ is a consequence of the following.

LEMMA. Let α be a C^2 origin preserving diffeomorphism of a bounded convex neighborhood of $0 \in E$ to a neighborhood of 0 with α of the form $\alpha(x) = x + K(x)$, KS^2-compact. Then α is isotopic to either I or J where J is the operator introduced in the beginning of § III. Moreover the isotopy α^t can be chosen so that for each t, $\alpha^t(x) = x + K^t(x)$, where K^t is S^1-compact.

PROOF. Write $\alpha(x) = x + K(x)$ where K is strongly compact. Define

$$H(t, x) = \begin{cases} \dfrac{\alpha(tx)}{t}, & 0 < t \leqslant 1, \\ D\alpha_0(x), & t = 0. \end{cases}$$

H is smooth; moreover $H(t, x) = x + K(tx)/t$. But $K(tx)/t = \int_0^1 DK_{stx}\text{-}(x)\, ds$. Consequently, $x \to K(tx)/t$ is S^1-compact. Now $D\alpha_0$ is either in the component of I or J. This concludes the proof of the lemma.

PROPOSITION 7. The index of X at z does not depend on the selection of coordinate chart φ.

PROOF. Let ψ be another coordinate chart in S about z. We can assume, without loss of generality that $\varphi(z) = \psi(z) = 0$. Then $\tilde{X}_\psi(x) = D\alpha_y \tilde{X}_\varphi(\alpha^{-1}(x))$, $y = \alpha^{-1}(x)$, where $\alpha = \psi \circ \varphi^{-1}$ is an origin preserving diffeomorphism of a neighborhood $\overline{W}$ of zero. Restricting α to a suitable bounded convex neighborhood of zero, so that $\alpha(x) = x + K(x)$, where K is S^2-compact, we get an isotopy $\alpha^t(x)$ between α and either I or J. If $\alpha^0 = I$ then from the homotopy property of degree it follows that

$$\deg[\tilde{X}_\psi, \overline{W}, 0] = \deg[\tilde{X}_\varphi, \overline{W}, 0].$$

If $\alpha^0 = J$ we get

$$\deg[\tilde{X}_\varphi, \overline{W}, 0] = \deg[J\tilde{X}_\varphi J^{-1}, \overline{W}, 0] = \deg[\tilde{X}_\varphi, \overline{W}, 0].$$

Thus the proposition is proved.

Let X be a (weakly admissible) Fredholm vector field with finitely many zeros $\{z_i\}_{i=1}^k$. We define the Euler characteristic of X $\big(\chi(X)\big)$ by the formula

$$\chi(X) = \sum_{i=1}^k (\text{index } X \text{ at } z_i).$$

If X has no zeros its characteristic is zero.

THEOREM A. If $X \sim Y$, $\partial \mathcal{M} = \emptyset$, then $\chi(X) = \chi(Y)$. If $\partial \mathcal{M} \neq \emptyset$ and X and Y both point outward or inward on the same components of $\partial \mathcal{M}$ then we still have $\chi(X) = \chi(Y)$.

REMARK. If $X \sim Y$ then it is generally not true that $\chi(X) = \chi(Y)$. For compact manifolds all vector fields with isolated zeros are equivalent and so we obtain one number which is in fact the Euler characteristic of the manifold. For example consider the constant vector field $X(x) = (0, 1)$ on $\mathbf{R}^2$, and the vector field $Y = \nabla f$, where $f(x, y) = x^2 + y^2$. $\chi(X) = 0 \neq \chi(Y) = \chi(\mathbf{R}^2) = 1$. One easily checks that $X \sim Y$. In § V it will be clear why $\chi(Y) = \chi(\mathbf{R}^2)$, and why this theory extends the finite dimensional theory to non-compact finite dimensional manifolds.

PROOF OF THEOREM A. Assume that $X \sim Y$, and $\partial \mathcal{M} = \emptyset$. Around each zero of X, we may, using the Smale-Sard theorem perturb X locally (in a neighborhood in which the principal parts are proper) obtaining a new vector field $\hat{X} \sim X$ which has non-degenerate zeros (i.e., $\hat{X} \pitchfork \mathfrak{Z}(T\mathcal{M})$, $\hat{X}$ is transverse to the zero section), which agrees with X outside a neighborhood of its zeros and with $\chi(\hat{X}) = \chi(X)$.

Similarly we can do the same for Y and obtain a new vector field $\hat{Y}$ with $\hat{Y} \pitchfork \mathfrak{Z}(T\mathcal{M})$ and $\chi(\hat{Y}) = \chi(Y)$. Now consider $F_j(x) = t\hat{X}(x) + (1 - t)\hat{Y}(x)$. $F^{-1}(\mathfrak{Z}(T\mathcal{M}))$ is compact (recall $\hat{X}$ and $\hat{Y}$ agree with X and Y outside neighborhoods in which the respective principal parts are proper).

Notice that F_0, $F_1 \colon \to T\mathcal{M}$ are transverse to $\mathfrak{Z}(T\mathcal{M})$, since X and Y have non-degenerate zeros. By the Thom transversality theorem we can modify F to a map $\hat{F}$ which agrees with F on $\mathcal{M} \times \{0\}$ and $\mathcal{M} \times \{1\}$ and so that F_t is a weakly admissible vector field for each t, $\hat{F} \pitchfork \mathfrak{Z}(T\mathcal{M})$ and $\hat{F}^{-1}(\mathfrak{Z}(T\mathcal{M}))$ is compact and thus a 1-dimensional submanifold of $\mathcal{M} \times I$ with $\partial \hat{F}^{-1}(\mathfrak{Z}(T\mathcal{M}))$ equal to the union of the zeros of $\hat{X}$ and the zeros of $\hat{Y}$. $\hat{F}^{-1}$ then yields a cobordism between the zeros of $\hat{X}$ and the zeros of $\hat{Y}$. Let $\mathfrak{I}$ be a component of $\hat{F}^{-1}(\mathfrak{Z}(T\mathcal{M}))$ and let $\{z_i\}_{i=1}^k$ and $\{w_j\}_{j=1}^m$ denote the zeros of $\hat{X}$ and $\hat{Y}$

respectively. From standard and well used techniques in elementary differential topology it follows that if $\partial \mathcal{F} = \{z_i\} \cup \{z_j\}$ then (index $\hat{X}$ at z_i) $= -$ (index $\hat{X}$ at z_j) $= \pm 1$. If $\partial \mathcal{F} = \{z_i\} \cup \{w_j\}$ then (index $\hat{X}$ at z_i) $=$ (index $\hat{Y}$ at w_j). Thus $\chi(X) = \chi(\hat{X}) = \chi(\hat{Y}) = \chi(Y)$ and Theorem A is proved in the case $\partial \mathcal{M} = \emptyset$. If $\partial \mathcal{M} \neq \emptyset$, and X and Y both point outward (or inward) along the boundary, we observe that F_t has no zeros on ∂ for any t. Consequently the cobordism affected by $\hat{F}$ does not intersect the boundary and the same proof works.

IV. – The vector field characteristic of a manifold (the intrinsic theory).

The question now naturally arises as to when $\chi(X)$ does not depend on X and can be described by topological data alone. We answer the first part of the question below and the second part in § V.

DEFINITION. Suppose $\mathcal{M}$ is a manifold with an S-structure of finite type (cf. § II). An *admissible* vector field X is of finite type if $\mathcal{M}$ is the finite union of closed coordinate neighborhoods (U_i, φ_i) with respect to which the principal parts of X, $\tilde{X}_{\varphi_i}$ are in normal form; i.e., of the form $x \to \lambda_i(x)x + K_i(x)$, $\lambda_i \geqslant \alpha > 0$ and K compact.

PROPOSITION 8. If $\mathcal{M}$ is a complete Finsler manifold (e.g., see [5]) of finite type and X and Y are also of finite type and have isolated zeros then $X \sim Y$.

PROOF. Cover $\mathcal{M}$ by finitely many neighborhoods (U_i, φ_i) with respect to which $\tilde{X}_{\varphi_i}(x) = \lambda_i(x)x + K_i(x)$ and $\tilde{Y}_{\varphi_i}(x) = \nu_i(x)x + C_i(x)$. The map $(t, x) \to t\tilde{X}_{\varphi_i}(x) + (1 - t)\tilde{Y}_{\varphi_i}(x)$ is proper on $U_i \times I$. Thus the closure of the set of zeros of F, $F(x, t) = tX(x) + (1 - t)Y(x)$, in $U_i \times I$ is compact. But $\mathcal{M} \times I = \bigcup \overline{U}_i \times I$ and therefore $F^{-1}(\mathfrak{z}(T\mathcal{M}))$ is compact and $X \sim Y$.

If $\mathcal{M}$ is of finite type, then by the last proposition all vector fields of finite type are equivalent. Thus by Theorem A, $\chi = \chi(X)$ does not depend on X, if we restrict X to be of finite type and to point outward along the boundary. Therefore χ should be describable in terms of topological data. In the next section we define the notion of compact vector field on manifolds of finite codimension in some Banach space. All compact vector fields are equivalent; thus $\chi(X)$ depends on $\mathcal{M}$. We show that $\chi(X) = \chi(\mathcal{M})$ for any compact X.

V. – Relation between $\chi(M)$ and χ (the extrinsic theory - compact vector fields).

Let $\mathcal{M} \subset E$ be a submanifold of the Banach space E of finite codimension. Then $\mathcal{M}$ inherits a natural S-structure S from E. A C^r vector field $X: \mathcal{M} \to T\mathcal{M}$ can then be identified as a map $X: \mathcal{M} \to E$. X is *compact* if $X(x) = \lambda(x)x + K(x)$, where $K(\mathcal{M})$ is compact and $\inf_{x \in \mathcal{M}} \lambda(x) \geqslant \alpha > 0$. It is easy to see that the set of zeros of X is a compact set and moreover that compact vector fields are admissible with respect to S and are all equivalent. In the remainder of this section we shall study only the extrinsic theory of vector fields, namely, we shall consider only finite codimensional manifolds $\mathcal{M}$ embedded in an ambient space E and compact vector fields X on $\mathcal{M}$.

In particular we shall take $\mathcal{M}$ to be a C^r closed and *bounded* Hilbert submanifold of a real Hilbert space H with inner product $\langle \, , \, \rangle$, with $\mathcal{M}^0$ ($\mathcal{M}$ interior) is an open submanifold of H. We shall show that if $\mathcal{M}$ admits a bounded C^r outward normal vector field $V: \partial \mathcal{M} \to H$ [i.e., V admits a C^r extension to a neighborhood of every point $p \in \partial \mathcal{M}$] of the form $V(p) = p + K(p)$, K compact, then $\chi(\mathcal{M})$ is defined $(H^i(\mathcal{M}) = 0, i \geqslant N_0$, for some N_0, where H^i denotes the i-th singular cohomology group with real coefficients and is a finite dimensional vector space for all i and moreover, $\chi(X) = \chi(\mathcal{M})$ for any admissible vector field X which points outward on $\partial \mathcal{M}$.

Since by Propositions 7 and 8, $\chi(X)$ is independent of X all we need to do is construct one admissible vector field X with $\chi(X) = \chi(\mathcal{M})$. We shall construct a smooth real valued function $f: \mathcal{M} \to R$ with non-degenerate critical points and with ∇f compact, and therefore having finitely many zeros. The from standard results in Morse theory (see e.g., Palais [4]), it follows that $\chi(\mathcal{M})$ is defined and $\chi(\nabla f) = \chi(\mathcal{M})$.

PROPOSITION 9 (Collar Neighborhood Theorem). Let δ be any positive real number and let $V: \partial \mathcal{M} \to H$ be a C^r vector field which is transverse to $\partial \mathcal{M}$ of the form $V(p) = p + K(p)$ with K compact. Then $\partial \mathcal{M}$ has a neighborhood in H, C^r diffeomorphic to $\partial \mathcal{M} \times (-\delta, \delta)$.

PROOF. Define the C^r endpoint map $E: \partial \mathcal{M} \times R \to H$ by $E(p, \lambda) = p + \lambda(p + K(p))$. Then for $(h, v) \in T(\partial \mathcal{M})_p \times R, DE_{(p,0)}(h, v) = h + v(p + K(p))$. Since $V \pitchfork \partial \mathcal{M}$, $DE_{(p,0)}$ is isomorphism, and therefore by the inverse function theorem, E is a local diffeomorphism. Since K is compact and $V \pitchfork \partial \mathcal{M}$ we have that $\inf_{p \in \partial \mathcal{M}} \|V(p)\| \geqslant \alpha > 0$ which guarantees that E is uniformly a local diffeomorphism in the sense that there exists an $\eta > 0$ so that given any $p \in \partial \mathcal{M}$, $E|(\partial \mathcal{M} \cap B_\eta(p)) \times (-\eta, \eta)$ is a diffeomorphism, where $B_\eta(p)$ de-

notes the ball of radius η about p. If we can show that for some $\varepsilon > 0$, $E|\partial\mathcal{M} \times (-\varepsilon, \varepsilon)$ is injective the proof of Proposition 8 will be complete [$\partial\mathcal{M} \times (-\varepsilon, \varepsilon)$ is clearly C^r diffeomorphic to $\partial\mathcal{M} \times (-\delta, \delta)$].

Assume that the injectivity assertion on E is false. Then there exist sequences $(p_n, \lambda_n) \neq (p'_n, \lambda'_n)$, with $\lambda_n \to 0$, $\lambda'_n \to 0$ and with $E(p_n, \lambda_n) = E(p'_n, \lambda'_n)$. Thus

$$(p_n - p'_n) = -\lambda_n p_n + \lambda'_n p'_n + \lambda'_n K(p'_n) - \lambda_n K(p_n) .$$

Since $\mathcal{M}$ is bounded and K compact we may conclude that $p_n - p'_n$ converges to zero. Therefore for n sufficiently large $(p'_n, \lambda'_n) \in \mathcal{O} \times \widetilde{\mathcal{O}}$ where $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are open sets in $\partial\mathcal{M}$ and R about p_n and 0 respectively with $\mathcal{O} \times \widetilde{\mathcal{O}} \subset B_\eta(p_n) \times (-\eta, \eta)$, whence $E|\mathcal{O} \times \widetilde{\mathcal{O}}$ is a diffeomorphism.

This implies that $(p_n, \lambda_n) = (p'_n, \lambda'_n)$ a contradiction, and the proof of Proposition 8 is complete.

The next proposition is the crucial step in the construction of the map $f \colon \mathcal{M} \to R$.

PROPOSITION 10. Let $\mathcal{M}$ and V be as before with $V = I + K$, outward and normal along $\partial\mathcal{M}$. Moreover assume that the map $DK^* \colon \partial\mathcal{M} \times H \to H$ defined by $(p, h) \to DK_p^*(h)$ is compact (restricted to bounded sets) where DK_p^* denotes the Hilbert space adjoint of DK_p. Then there exists a closed collar neighborhood $\overline{W} = E(\partial\mathcal{M} \times [-\varepsilon, \varepsilon])$, $|\varepsilon| \leqslant \frac{1}{2}$, of $\partial\mathcal{M}$ in H and a C^r map, $g \colon \overline{W} \to R$ with $g^{-1}(2) = \partial\mathcal{M}$, ∇g compact and with $\nabla g \pitchfork \partial\mathcal{M}$ (in fact ∇g will point outward along $\partial\mathcal{M}$).

REMARK. If K is S^1-compact with DK_p self adjoint for each p then the compactness condition on DK^* above automatically follows.

PROOF. Let $E \colon \partial\mathcal{M} \times (-\delta, \delta) \to U$ be a diffeomorphism where U is a collar neighborhood of $\partial\mathcal{M}$ given by Proposition 9. Define $g \colon U \to R$ by $g(E(p, \lambda)) = \lambda + 2$. Clearly $g^{-1}(2) = \partial\mathcal{M}$. By the chain rule we have that

$$Dg_{E(p,\lambda)}(dE_{(p,\lambda)}(h, v)) = \langle \nabla g(E(p, \lambda)), DE_{(p,\lambda)}(h, v) \rangle = v .$$

Denote $\nabla g(E(p, \lambda))$ by $L(p, \lambda)$.

Therefore for all $(h, v) \in T(\partial\mathcal{M})_p \times R$

$$\langle L(p, \lambda), (1 + \lambda) h + \lambda DK_p(h) + v(p + K(p)) \rangle = v$$

or

$$\langle (1 + \lambda) L(p, \lambda) + \lambda DK_p^*(L(p, \lambda)), h \rangle + v\langle L(p, \lambda), p + K(p) \rangle = v$$

This implies that

$$\langle L(p, \lambda), p + K(p)\rangle = 1 \quad \text{and} \quad (1 + \lambda) L(p, \lambda) + \lambda DK_p^*(L(p, \lambda))$$

is normal to $\partial \mathcal{M}$ at p. (DK_p^* denotes the adjoint of DKp.)
Thus

$$(1 + \lambda) L(p, \lambda) + \lambda DK_p^*(L(p, \lambda)) = \xi(p + K(p)),$$

$\xi = \xi(p, \lambda)$ a real number.

Before proceeding we shall need the next lemma.

LEMMA 1. There exists positive numbers m_1, M_2, and an $\varepsilon > 0$ depending only on K so that for $|\lambda| \leqslant \varepsilon$, $2M_1 \geqslant \xi(p, \lambda) \geqslant m_1/2 > 0$ for all $p \in \partial \mathcal{M}$.

PROOF. First note that $\langle L(p, \lambda), p + K(p)\rangle = 1$ so

(∗) $\langle (1 + \lambda) L(p, \lambda), L(p, \lambda)\rangle + \langle \lambda DK_p(L(p, \lambda), L(p, \lambda))\rangle = \xi(p, \lambda)$

$$= \|L(p, \lambda)\|^2 + \lambda \|L(p, \lambda)\|^2 + \lambda \langle DK_p(L(p, \lambda), L(p, \lambda))\rangle.$$

From the fact that $\inf_p \|V(p)\| \geqslant \alpha > 0$ it follows that there is a $\delta_1 > 0$ and positive numbers m_0, M_0 with

$$0 < m_0 \leqslant \|DE_{(p, \lambda)}\| \leqslant M_0 \quad \text{for all } (p, \lambda) \in \partial \mathcal{M} \times [-\delta_1, \delta_1].$$

This together with the relation $\langle L(p, \lambda), DE_{(p, \lambda)}(h, \nu)\rangle = \nu$ implies that there exist positive m_1, M_1 with $0 < m_1 \leqslant \|L(p, \lambda)\|^2 \leqslant M_1$ for all $(p, \lambda) \in \partial \mathcal{M} \times [-\delta_1, \delta_1]$. Using (∗) we see that we can find an $\varepsilon \leqslant \delta_1$ with

$$2M_1 \geqslant \xi(p, \lambda) \geqslant m_1/2 \quad \text{for all } (p, \lambda) \in \partial \mathcal{M} \times [-\varepsilon, \varepsilon],$$

which concludes the lemma.

Returning to the proof of Proposition 10 we see that since

$$(1 + \lambda) L(p, \lambda) + \lambda DK_p^*(L(p, \lambda)) = \xi(p + K(p)),$$

$$L(p, \lambda) = \frac{-\lambda}{1 + \lambda} DK_p^*(L(p, \lambda)) + \frac{\xi}{1 + \lambda}(p + K(p)),$$

$$= \frac{\xi}{(1 + \lambda)^2} E(p, \lambda) + \frac{\xi}{(1 + \lambda)^2} K(p) - \frac{\lambda}{1 + \lambda} DK_p^*(L(p, \lambda)).$$

By the compactness assumptions on K it follows that

$$(p, \lambda) \to \frac{\xi}{(1+\lambda)^2} K(p) - \frac{\lambda}{1+\lambda} DK_p^*(L(p, \lambda))$$

is also a compact map, say $C(p, \lambda)$. Consequently

$$\nabla g\big(E(p, \lambda)\big) = = L(p, \lambda) = \frac{\xi}{1+\lambda} E(p, \lambda) + C(p, \lambda)$$

or

$$\nabla g(q) = \left(\frac{\xi(q)}{(1+\lambda)^2}\right) \cdot q + C(q) .$$

Therefore ∇g is compact on $\overline{W} = E(\partial \mathcal{M} \times [-\varepsilon, \varepsilon])$. Moreover for $p \in \partial \mathcal{M}$,

$$\nabla g(p) = \frac{p + K(p)}{\|p + K(p)\|^2} .$$

Thus $\nabla g \pitchfork \partial \mathcal{M}$ and points outward. This finishes the proof of Proposition 10.

In the preceding proposition we constructed a C^r mapping $g \colon \overline{W} \to [2-\varepsilon, 2+\varepsilon]$. We would like to extend g to a C^r map $f \colon \mathcal{M} \to R$ with ∇f compact and having non-degenerate zeros. We shall do this in a couple of steps.

Let $\varphi \colon R \to R$ be a C^∞-map which is 1 on $[-\varepsilon/2, \varepsilon/2]$ and zero outside $[-\varepsilon, \varepsilon]$ with $\varphi'(t) \geqslant 0$ for all $t \leqslant 0$. Define C^r maps φ_1, $\varphi_0 \colon \overline{W} \to R$ by

$$\varphi_0\big(E(p, \lambda)\big) = \varphi(\lambda) , \qquad \varphi_1 = 1 - \varphi_0$$

φ_0, φ_1 clearly extend to C^r maps on all of H.

LEMMA 2. $\nabla \varphi_0(q) = \eta(q) \cdot q + C_1(q)$, where $\eta(q) \geqslant 0$ for $q \in \mathcal{M}$, C_1 compact, and $\sup\limits_{q \in \overline{W}} \eta(q) = \Omega < \infty$.

PROOF. The proof is essentially a minor modification of Proposition 9.

Recall that the real valued function ξ of Proposition 9 had the bounds (by Lemma 1 of this section), $2M_1 \geqslant \xi \geqslant m_1/2$. Let $\Gamma = \sup\limits_{p \in \mathcal{M}} \|p\|^2$, $w = \varepsilon/2\Gamma$, and g be the map constructed in 9. Define $g_1 \colon H \to R$ by $g_1(q) = w\|q\|^2$ and $\tilde{f} \colon M \to R$ by $\tilde{f}(q) = \varphi_0(q) g(q) + \varphi_1(q) g_1(q)$. We shall show that $\nabla \tilde{f}$ is compact with $\nabla \tilde{f} \pitchfork \partial \mathcal{M}$ and pointing outward.

Surely $\nabla \tilde{f}$ points outward along $\partial \mathcal{M}$ (and is in fact normal to $\partial \mathcal{M}$) since $\nabla \tilde{f}(p) = \nabla g(p)$ for $p \in \partial \mathcal{M}$. Recall further that $\nabla g(q) = \tilde{\xi}(q) \cdot q + C(q)$,

$q = E(p, \lambda)$, $\tilde{\xi}(q) = \xi(q)/(1 + \lambda)^2$. The restriction $|\lambda| \leqslant \frac{1}{2}$ implies that $4M_1 \geqslant \geqslant \tilde{\xi}(q) \geqslant m_1/8$ for all $q \in \overline{W}$. Now $2 - \varepsilon \leqslant g(q) \leqslant 2 + \varepsilon$, and $0 \leqslant g_1(q) \leqslant \varepsilon/2$. Computing $\nabla \tilde{f}$ we get

$$\nabla \tilde{f} = \varphi_0 \nabla g + \varphi_1 \nabla g_1 + g \nabla \varphi_0 + g_1 \nabla \varphi_1 = \varphi_0 \nabla g + \varphi_1 \nabla g_1 + (g - g_1) \nabla \varphi_0 \,.$$

The sum of the first two terms is of the form

$$\big(\varphi_0(q) \tilde{\xi}(q) + \varphi_1(q) w\big) \cdot q + C(q) \,,$$

whereas the sum of the last terms is of the form

$$\big(g(q) - g_1(q)\big) \cdot \eta(q) \cdot q + C_1(q) \,.$$

Therefore

$$\nabla \tilde{f}(q) = [\varphi_0(q) \tilde{\xi}(q) + \varphi_1(q) w + \big(g(q) - g_1(q)\big) \eta(q)] \cdot q + C(q) + C_1(q)$$

$$= \gamma(q) \cdot q + C_2(q) \,.$$

Let $\beta = \min(m_1/8, w)$. Then for $q = E(p, \lambda)$, $g(q) = 2 + \lambda$, and $|g_1(q)| \leqslant \leqslant \varepsilon/2 (0 < \varepsilon < \frac{1}{2})$, $g(q) - g_1(q) \geqslant 1$, $\big(g(q) - g_1(q)\big) \eta(q) \geqslant 0$, and $\varphi_0(q) \tilde{\xi}(q) + \varphi_1(q) w \geqslant \beta$. Whence $\gamma(q) \geqslant \beta > 0$, and C_2 compact. $\nabla \tilde{f}$ is therefore compact.

The last remaining road block to applying the results of Morse theory to $\tilde{f}$ is that $\tilde{f}$ may have degenerate critical points. However $\nabla \tilde{f}$ is Fredholm of index 0 and consequently by the Smale-Sard theorem [7] $\nabla \tilde{f}$ has a regular value v where $\|v\|$ may be chosen arbitrarily small. Pick v so close to $0 \in H$ so that $(\nabla \tilde{f} + v) \pitchfork \partial \mathcal{M}$ and $\nabla \tilde{f} + v$ points outward. $\nabla \tilde{f} + v$ has zero as a regular value. Define $f: \mathcal{M} \to R$ by $f(q) = \tilde{f}(q) + \langle q, v \rangle$. Then $\nabla f = \nabla \tilde{f} + v$ and so ∇f has only non-degenerate zeros, is transverse to $\partial \mathcal{M}$ and points outward.

In addition f is bounded on $\mathcal{M}$ and satisfies condition C. Therefore f has only finitely many critical points in $\mathcal{M}^0$. Since f is admissible each critical point is of finite index. This implies (using Morse theory) that $\mathcal{M}$ has a handle body decomposition where there are only finitely many handles. Thus $H^i(\mathcal{M}) = 0$ if i is sufficiently large, and $\dim H^i < \infty$ for all i. The Morse equality says that $\chi(\mathcal{M}) = \sum_{i=0}^{\infty} (-1)^i C_i$, where C_i = number of critical points of index i. If z is a critical point of index i of X, then index X at z

is equal to $(-1)'$. Thus the Morse equality says that $\chi(\nabla f) = \chi(\mathcal{M})$. By our earlier remarks it follows that $\chi(\mathcal{M}) = \chi$.

We finalize our results as a theorem.

THEOREM B. Let $\mathcal{M}$ be a closed (as a subset) bounded C^r submanifold of H of codimension 0. Suppose $\partial\mathcal{M}$ admits a bounded C^r, outward normal vector field $V: \partial\mathcal{M} \to H$ of the form $p \to p + K(p)$, K and DK^* compact. Then $H^i(\mathcal{M}) = 0$ (the i-th singular cohomology group with real coefficients) for i sufficiently large, and $\dim H^i < \infty$ for all i. Thus $\chi(\mathcal{M})$ is defined. Finally for any compact vector field X which points outward on $\partial\mathcal{M}$, $\chi(X) = \chi(\mathcal{M})$.

REMARK. Theorem B above immediately generalizes to the case where $\mathcal{M}$ is a closed (as a subset) C^r finite codimensional submanifold of H with the property that the map $x \to \tilde{P}_x(x)$ from $\mathcal{M}$ to H, where $\tilde{P}_x$ is the orthogonal proejction of H onto $T\mathcal{M}_x^\perp$, is compact.

VI. – Generalizations of the theory.

In Section V we introduced the notion of admissible vector field X. We shall now considerably extend this notion.

Let E be a Banach space and C a convex set of invertible linear operators containing the identity (e.g., if $E = H$, C could be taken to be the linear space of strongly positive operators). A vector field X on an S-manifold $\mathcal{M}$ is admissible if locally the principal part $\tilde{X}_\varphi$ of X with respect to φ is bounded and in the normal form $\tilde{X}_\varphi(x) = T_x(x) + K(x)$, where $x \to T_x$ is a compact map of $\varphi(U)$ into $\mathcal{L}(E)$, the linear operators on E, K compact, $T_x \in C$ and $\|T_x h\| \geqslant \varrho(x)\|h\|$ for all h where $\inf_{x \in \varphi(U)} \varrho(x) \geqslant \alpha > 0$.

The notion can again be seen to be independent of φ. If we denote by C_K the set of linear invertible operators of the form $\{T + K, T \in C, K$ linear completely continuous$\}$. Then $\pi_0(C_K) = 2$ (C_K is not necessarily a group so $\pi_0(C_K)$ is just the number of path components). Clearly $D\tilde{X}_\varphi(x) \in C_K$ for each $x \in \varphi(U)$. We can then define the notion of the index of X at an isolated zero z. If z is such a zero with $(\varphi, U) \in S$ a chart about z with $\varphi(z) \in W \subset \overline{W} \subset \varphi(U)$, we define (index X at z) $= \deg[\tilde{X}_\varphi, \overline{W}, 0]$ where the degree is the Brouwer degree introduced by Elworthy and the author [1].

If X has only finitely many zeros we define $\chi(X) = \sum\limits_{\text{zeros } z}$ (index X at z).

If both X and Y have finitely many zeros we say X is equivalent to Y, $X \sim Y$, if $F: \mathcal{M} \times I \to T\mathcal{M}$ defined by $F(x, t) = tX(x) + (1 - t) Y(x)$ has the property that $F^{-1}(\mathfrak{Z}(T\mathcal{M}))$ is compact, $\mathfrak{Z}(T\mathcal{M})$ the zero section of $T\mathcal{M}$. So we have

THEOREM A$'$. If $X \sim Y$, $\partial \mathcal{M} = \emptyset$, then $\chi(X) = \chi(Y)$. If $\partial \mathcal{M} \neq \emptyset$ and X and Y both point outward or inward along the same components of $\partial \mathcal{M}$ then we still have $\chi(X) = \chi(Y)$.

If $\mathcal{M}$ is of finite type and X and Y are of finite type and either both point inward or outward on $\partial \mathcal{M}$ then $X \sim Y$. Thus we get a vector field characteristic χ of a manifold $\mathcal{M}$ of finite type by defining $\chi = \chi(X)$ for any admissible X of finite type which points out ward along $\partial \mathcal{M}$.

The proof of the following theorem is essentially a minor modification of that of Theorem B.

THEOREM B$'$. Let $\mathcal{M}$ be a closed bounded C^r submanifold of a Hilbert space H of codimension 0. Suppose $\partial \mathcal{M}$ admits a bounded C^r outward non zero normal vector field $V : \partial \mathcal{M} \to H$ of the form $p \to T_p(p) + K(p)$, K, DK^*, $p \to T_p$, and $(p, h, k) \to DT_p(h)(k)$ compact where D means derivative with respect to the subscript variable. In addition suppose that for each p, $T_p \in \mathcal{C}$, $T_p^* = T_p$, $\sup_p \|T_p\| < \infty$, and $\|T_p h\| \geqslant \varrho(p) \|h\|$ where $\inf_{p \in \partial \mathcal{M}} \varrho(p) \geqslant \beta > 0$.

Then $H^i(\mathcal{M})$, the i-th singular cohomology group with integer or real coefficients is zero for i sufficiently large, and $\dim H^i < \infty$ for all i. Thus $\chi(\mathcal{M})$ is defined. Finally for any compact vector field X which points outward on $\partial \mathcal{M}$, $\chi(X) = \chi(\mathcal{M})$.

REMARK 1. By altering the Riemannian structure of H in a neighborhood of $\partial \mathcal{M}$ the transversality assumption on V in B$'$ can probably be weakened to V being uniformly transverse to $\partial \mathcal{M}$. By uniformly transverse to $\partial \mathcal{M}$ we mean that there is a δ, $1 > \delta > 0$ with

$$\sup_{\substack{\|h\| \leqslant 1 \\ h \in T(\partial \mathcal{M})_x}} \frac{\langle V(x), h \rangle}{\|V(x)\|} \leqslant 1 - \delta \qquad \text{for all } x \in \partial \mathcal{M} \,.$$

REMARK 2. Theorem B$'$ can be generalized to bounded closed submanifolds of finite codimension in H for which the map $x \mapsto \tilde{P}_x(x)$ from $\mathcal{M}$ to H is compact where $\tilde{P}_x$ is the orthogonal projection of H onto $T\mathcal{M}_p^\perp$.

REMARK 3. Let $\mathcal{M}$ be a submanifold of H as in Theorem B$'$ and let $V(p) = T_p(p) + K(p)$ be a normal vector field with T_p positive definite for each p. Then if $f : \mathcal{M} \to R$ is a function constructed as in Section V, ∇f is compact with critical points of finite index. Then $-f$ is a smooth map satisfying condition C in the sense of Palais and Smale and has only critical points of infinite index. Applying the Morse deformation theorem to $-f$ we can conclude that $\partial \mathcal{M}$ has the same homotopy type of $\mathcal{M}$.

VII. – A conjecture on the problem of geodesics on closed finite dimensional Riemannian manifolds.

In the following paragraphs we see how the problem of existence of geodesics can be put in the framework of Section V. Unfortunately there is a certain gap which we shall leave in the form of a conjecture.

Let M^n be a complete n-dimensional C^{k+4} Riemannian manifold $(k \geqslant 1)$. By a celebrated theorem of John Nash, M^n can be isometrically embedded in some Euclidean N-space, for N sufficiently large. Let $H_1(I, \mathbf{R}^N)$ denote the Hilbert space of functions from the unit interval I to $\mathbf{R}^N$ which are absolutely continuous with square integrable derivatives. The inner product $\langle\,,\,\rangle$ on this space is given by

$$\langle \sigma, \varrho \rangle = \int\limits_0^1 \langle \sigma(t), \varrho(t) \rangle_{\mathbf{R}^N} dt + \int\limits_0^1 \langle \sigma'(t), \varrho'(t) \rangle_{\mathbf{R}^n} dt$$

where $\langle\,,\,\rangle_{\mathbf{R}^N}$ denotes the canonical inner product on $\mathbf{R}^N$.

Let $H_1(I, M^n)$ denote the smooth (C^k) Hilbert submanifold of $H_1(I, \mathbf{R}^N)$ consisting of those maps which take I into M^n. For a more detailed discussion of this Hilbert manifold see Palais [5]. Let P, Q be points in M^n and let $\Omega(I, P) \subset H_1(I, M^n)$ be the Hilbert submanifold consisting of maps σ in $H_1(I, M^n)$ with $\sigma(0) = P$ and $\Omega(I, P, Q) \subset \Omega(I, P)$ the Hilbert submanifold consisting of those maps σ with $\sigma(0) = P$ and $\sigma(1) = Q$. We shall give $\Omega(I, P)$ and $\Omega(I, P, Q)$ the Riemannian structure induced from $H_1(I, \mathbf{R}^N)$.

The tangent space $\Omega(I, P, Q)_\varrho$ to $\Omega(I, P, Q)$ at σ can be characterized as the subspace of $H_1(I, \mathbf{R}^N)$ of maps $h: I \to \mathbf{R}^N$ with $h(t) \in TM_{\sigma(t)}^n$ and $h(0) = h(1) = 0$. These are maps which « lift » over σ.

The tangent space to $\Omega(I, P)$ at a point σ can be defined similarly. We shall define the « energy » integral $J: \Omega(I, P) \to R$, by

$$J(\sigma) = \frac{1}{2} \int\limits_0^1 \| \sigma'(t) \|^2 dt$$

and denote by $\bar{J}$ the restriciton of J to $\Omega(I, P, Q)$. It is well known [5] that the critical points of $\bar{J}$ are geodesics of M^n joining P and Q parametrized by arc length.

We will now show how $\Omega(I, P, Q)$ can be embedded as a finite codimensional submanifold of some Hilbert space. First we shall show that the Hilbert manifold $\Omega(I, P)$ is diffeomorphic to the Hilbert space $H_1(I, TM_p^n) = \{\varrho \in H_1(I, \mathbf{R}^N) \,|\, \varrho(t) \in TM_p, \ \forall t, \ \varrho(0 = 0\}$. Let $\sigma \in \Omega(I, P)$ and $v \in TM_{\sigma(t)}^n$.

Denote by $\tau_t v$ the parallel translation of v [here we are giving M^n the symmetric affine connection compatible with its metric and are defining parallel translation with respect to this connection] along σ to $TM^n_{\sigma(0)} = TM^n_p$. Thus $\tau_t v \in TM^n_p$. Now $\sigma'(t) \in TM^n_{\sigma(t)}$, and so $t \to \tau_t \sigma'(t)$ defines a path in TM^n_p. The equation $\varrho'(t) = \tau_t \sigma'(t)$ can be solved uniquely and the map $\sigma \mapsto \varrho$ determines a diffeomorphism between $\Omega(I, P)$ and $H_1(I, TM^n_p)$ (e.g., see Kobayashi and Nomizu [8], p. 130). This diffeomorphism is called the development diffeomorphism. Denote this diffeomorphism by $\gamma \colon H_1(I, TM^n_p) \to \Omega(I, P)$. Thus $\Omega(I, P)$ is diffeomorphic to the Hilbert space $H_1(I, TM)^n_p$. Define a map $\varphi \colon \Omega(I, P) \to M^n$ by $\varphi(\sigma) = \sigma(1)$. The map φ is a submersion and $\varphi^{-1}(Q) = \Omega(I, P, Q)$. Consequently it follows that $\Omega(I, P, Q)$ is an n-codimensional submanifold of $\Omega(I, P)$ and therefore can also be embedded as an n-codimensional submanifold of $H_1(I, TM^n_p)$. Denote by $\Gamma(P, Q)$ the inverse image $\gamma^{-1}\big(\Omega(I, P, Q)\big)$.

$\Gamma(P, Q)$ has a Riemannian structure which it inherits as a submanifold of $H^1(I, TM^n_p)$ with its natural inner product; i.e., if ϱ_1, $\varrho_2 \in H^1(I, TM^n_p)$ then $\langle \varrho_1, \varrho_2 \rangle = \int_0^1 \langle \varrho_1'(t), \varrho_2'(t) \rangle_p \, dt$ where $\langle \, , \, \rangle_p$ denotes the inner product on TM^n_p. Define

$$E(\varrho) = J\big(\gamma(\varrho)\big) = \int_0^1 \left\| \frac{d}{dt} \gamma(\varrho) \right\|^2_{\gamma(p)} dt$$

$$= (\text{by parallel translation}) \int_0^1 \| \varrho'(t) \|^2 \, dt$$

and

$$\bar{E} = \bar{J} \circ \gamma .$$

Therefore $\nabla E(\varrho) = \varrho$, $\big(\nabla E(\varrho)$ is the gradient of E at $\varrho\big)$. Also $dE_\varrho(h)$ (the differential of E at ϱ operating on $h \in H_1(I, TM^n_p)$) is equal to $dJ_{\gamma(\varrho)}\big(d\gamma_\varrho(h)\big)$. Thus

$$\langle \nabla E(\varrho), h \rangle_{H^1(I, TM^n_p)} = \langle \nabla J\big(\gamma(\varrho)\big), d\gamma_\varrho(h) \rangle_{H^1(I, \mathbf{R}^N)}$$

and hence

$$\nabla E(\varrho) = d\gamma_\varrho^* \nabla J\big(\gamma(\varrho)\big)$$

where $d\gamma_\varrho^*$ denotes the adjoint of $d\gamma_\varrho$. In fact

$$\nabla \bar{E}(\varrho) = d\gamma_\varrho^* \nabla \bar{J}\big(\gamma(\varrho)\big) .$$

Whence γ determines a bijection between the critical points of $\bar{E}$ (zeros of $\nabla \bar{E}$) and the critical points of $\bar{J}$ (zeros of $\nabla \bar{J}$ which are geodesics parametrized by arc length). Let $\varrho \in \Gamma(P, Q)$. Then $\nabla \bar{E}(\varrho) = P_\varrho \nabla E(\varrho) = P_\varrho(\varrho)$, where P_ϱ is the orthogonal $\{$in $H_1(I, TM_p^n)\}$ projection of $H_1(I, TM_p^n)$ onto the tangent space of $\Gamma(P, Q)$ at ϱ. Therefore

$$\nabla \bar{E}(\varrho) = P_\sigma(\varrho) = \varrho + \tilde{P}_\varrho(\varrho)$$

where $\tilde{P}_\varrho = id - P_\varrho$ is the orthogonal projection onto the complement of the tangent space to $\Gamma(P, Q)$ at ϱ. Thus the range of $\tilde{P}_\varrho$ is n-dimensional.

Let $a > 0$ be a real number which is a regular value of $\bar{E}$ $\{$i.e., no critical points in $\bar{E}^{-1}(a)\}$. Denote by $\Gamma^a(P, Q)$ the manifold $\bar{E}^{-1}(-\infty, a)$. This is a bounded closed (as a subset) submanifold of $H_1(I, TM_p^n)$ with smooth boundary $\partial \Gamma^a = \bar{E}^{-1}(a)$. If one can show that $\nabla \bar{E}$ is a compact vector field (or equivalently if $\varrho \to \tilde{P}_\varrho(\varrho)$ is compact) one would then be able to put the theory of geodesics into the framework of § V (see remark following Theorem B). We have no proof of this. The main difficulty seems to be in the fact that the development diffeomorphism does not extend to a continuous map in the C^0 topology.

In a later paper we hope to surmount this difficulty by using a somewhat different approach to the theory of vector fields on Banach manifolds.

BIBLIOGRAPHY

[1] K. D. ELWORTHY - A. J. TROMBA, *Differential structures and Fredholm maps on Banach manifolds*, Proc. Symp. on Global Analysis, Berkeley, 1968.

[2] K. D. ELWORTHY - A. J. TROMBA, *Degree theory on Banach manifolds*, Proc. Symp. on Pure Math., **18**; Amer. Math. Soc., Providence, R. I., 1970.

[3] J. MILNOR, *Topology from the Differentiable Viewpoint*, Univ. of Virginia Press, Charlottesville, Va., 1965.

[4] R. PALAIS, *Morse theory on Hilbert manifolds*, Topology, **2** (1963), pp. 299-340.

[5] R. PALAIS, *Lectures on the differential topology of infinite dimensional manifolds*, Notes, Brandeis University, Waltham, Mass., 1964-65.

[6] R. PALAIS, *On the homotopy of certain groups of operators*, Topology, **3** (1965), pp. 271-279.

[7] S. SMALE, *An infinite dimensional version of Sard's theorem*, Amer. J. Math., **87** (1965), pp. 861-866.

[8] S. KOBAYASHI - K. NOMIZU, *Foundations of Differential Geometry*, Interscience. New York, 1963.