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**Flat vector bundles and the fundamental group in  
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## Flat Vector Bundles and the Fundamental Group in Non-Zero Characteristics (\*).

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**§ 0.** – Let  $k$  be an algebraically closed field, and let  $V$  be a smooth variety over  $k$ , and  $\mathcal{D}_m$  be the sheaf  $\text{Diff}_{V/k}^m(\mathcal{O}_V, \mathcal{O}_V)$  [EGA IV, 16.8.1]. The sections of  $\mathcal{D}_m$  will be called differential operators of order  $\leq m$  on  $V$ . Recall that the differential operators may be regarded as  $k$ -linear maps of  $\mathcal{O}_V$  to  $\mathcal{O}_V$ . Further, we have the following characterization which may be regarded as a definition:

**PROPOSITION 0.1** [EGA IV, 16.8.8]. Let  $D$  be a  $k$ -linear map from  $\mathcal{O}_V$  to  $\mathcal{O}_V$ . Then the following conditions are equivalent:

- a)  $D$  is a differential operator of degree  $\leq n$ .
- b) For each section  $a$  of  $\mathcal{O}_V$  over an open  $U$  of  $V$ , the morphism  $D_a$  of  $\mathcal{O}_U$  to  $\mathcal{O}_U$  defined by

$$D_a(t) = D(at) - aD(t)$$

where  $t$  is a section of  $\mathcal{O}_W$  for some open  $W \subseteq U$ , is a differential operator of degree  $\leq n - 1$  over  $U$ .

For a  $k$ -tuple of integers  $(n_1, \dots, n_k) = \mathbf{n}$ , define  $|\mathbf{n}| = \sum_i n_i$  and  $\mathbf{n}! = \prod_i n_i!$ , and if  $\mathbf{m}$  is another  $k$ -tuple of integers, set

$$\binom{\mathbf{n}}{\mathbf{m}} = \begin{cases} \frac{\mathbf{n}!}{\mathbf{m}!(\mathbf{n}-\mathbf{m})!} & \text{if } \mathbf{m} \leq \mathbf{n} \\ 0 & \text{otherwise} \end{cases}.$$

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PROPOSITION 0.2 [EGA IV, 16.11.2]. Suppose  $z_1, \dots, z_k$  are functions on  $V$  so that  $dz_1, \dots, dz_k$  form a basis of  $\mathcal{O}_{V/k}^1$ . Then there are differential operators  $D_{\mathbf{n}}$  for each  $\mathbf{n} = (n_1, \dots, n_k) \geq 0$  so that

$$* \quad D_{\mathbf{n}}(z^{\mathbf{q}}) = \binom{\mathbf{q}}{\mathbf{n}} z^{\mathbf{q}-\mathbf{n}}$$

where  $z^{\mathbf{l}} = z_1^{l_1} \dots z_k^{l_k}$ ,  $\mathbf{l} = (l_1, \dots, l_k)$ . The family  $(D_{\mathbf{n}})$  is uniquely determined by  $*$ , and the set  $\{D_{\mathbf{n}} \mid |\mathbf{n}| \leq m\}$  forms a basis for the differential operators of degree  $\leq m$ . We denote  $D_{(l_1, \dots, l_k)}$  by

$$\frac{\partial^{l_1}/\partial x_1^{l_1}}{l_1!} \cdot \frac{\partial^{l_2}/\partial x_2^{l_2}}{l_2!} \cdots \frac{\partial^{l_k}/\partial x_k^{l_k}}{l_k!}.$$

DEFINITION 0.3. A stratification  $\nabla$  on a locally free  $\mathcal{O}_V$ -module  $E$  is a ring homomorphism from  $\mathcal{D}$ , the sheaf of all differential operators on  $V$ , to  $\text{End}_k(E) = \text{Hom}_k(E, E)$  so that for any  $U$  open in  $V$  and  $D$  a section of  $\mathcal{D}$  over  $U$ ,  $s$  a section of  $E$  over  $U$  and  $f$  a section of  $\mathcal{O}_U$ , then

$$* \quad \nabla(D)(fs) = f\nabla(D)(s) + \nabla(D_f)(s)$$

and so that  $\nabla(fD) = f\nabla(D)$ . A map between two stratified bundles is said to be horizontal if it commutes with the action of  $\nabla$ . We thus get a category  $S(V)$  of stratified bundles and horizontal maps.

NOTE.  $*$  is called Leibniz's rule.

EXAMPLE. Let  $E_r = \mathcal{O}_V^r$  and  $\nabla(D)(f_1, \dots, f_r) = (Df_1, \dots, Df_r)$ . A stratified bundle is said to be trivial if it is isomorphic in  $S(V)$  to  $E_r$  for some  $r$ .

Suppose the characteristic of  $k$  is zero.  $\mathcal{D}$  is then the sheaf of universal enveloping algebras on  $\text{Der}_{V/k}$ , the sheaf of derivations on  $V$ . Consequently, a stratification on  $E$  is nothing but an  $\mathcal{O}_V$  morphism  $\nabla$  from  $\text{Der}_V$  to  $\text{End}_k(E)$  which is a Lie algebra homomorphism and which satisfies

$$\nabla(D)(fs) = f\nabla(D)(s) + D(f)(s).$$

That is,  $\nabla$  is an integrable connection.

Now suppose  $k = \mathbb{C}$ , and that  $V$  is proper over  $\mathbb{C}$ . Then there is a natural correspondence between holomorphic vector bundles of rank  $r$  with integrable connections and representations of the topological fundamental group  $\pi_1(V)$ , as is given in [2, I, §1, 2]. We let  $\hat{\pi}_1$  be the pro-finite completion of  $\pi_1(V)$ ,

and a stratified bundle will be called irreducible if it contains no non-trivial stratified sub-bundles.

**THEOREM 0.4** ( $k = \mathbb{C}$ ). (i)  $\hat{\pi}_1$  is abelian if and only if every irreducible stratified bundle is one dimensional.

(ii)  $\hat{\pi}_1$  is trivial if and only if every stratified bundle is trivial.

**PROOF.** Suppose  $\hat{\pi}_1$  is abelian. If  $G$  is  $\pi_1$  modulo its commutator subgroup, and  $\varphi: \pi_1 \rightarrow G$  is the natural map, then  $\hat{\varphi}: \hat{\pi}_1 \rightarrow \hat{G}$  is an isomorphism. Consequently, every representation of  $\pi_1$  factors through  $G$ . [Theorem 1.2, 3]. Since the only irreducible representations of  $G$  on  $\mathbb{C}^n$  are one dimensional, we see the only irreducible stratified bundles on  $V$  are one dimensional.

Conversely, if the only irreducible stratified bundles are one dimensional, then every representation  $\rho$  of  $\pi_1$  on  $\mathbb{C}^n$  has an eigenvector. Thus  $\rho$  is conjugate to a representation on the upper triangular  $n \times n$  matrices. If  $G$  is a finite quotient of  $\pi_1$ , and  $\rho$  is a faithful representation of  $G$  on upper triangular matrices, then  $\rho$  of any commutator must be unipotent and so the identity, since every unipotent matrix of finite order is the identity. Thus  $G$  is abelian, and so  $\hat{\pi}_1$  is abelian.

*ii*) is proved similarly.

Note that Theorem 0.4 is an algebraic theorem, since both  $\hat{\pi}_1$  and stratified bundles are algebraic. The purpose of this paper is to present some evidence that 0.4 may still be valid if the characteristic of  $k$  is positive, where  $\hat{\pi}_1$  is replaced by the Grothendieck fundamental group. First, we study stratified bundles over some projective varieties. Next, we introduce the notion of a stratified bundle with regular singular points on certain open varieties, and then we classify all stratified bundles with regular singular points on linear algebraic groups, and on  $\mathbb{P}^n - D$ , where  $D$  is a divisor with normal crossings in  $\mathbb{P}^n$ .

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**§1.** – Throughout the rest of this paper, all schemes, formal schemes, etc., will be over  $\mathbb{Z}/(p)$ . Let  $E$  be a locally free sheaf on  $V$ , a formal scheme. We denote  $F^*(E)$  by  $E^{(p)}$ , where  $F$  is the Frobenius endomorphism of  $V$ .

**DEFINITION 1.1.** A flat bundle  $\{E_i, \sigma_i\}$  on  $V$  is sequence of bundles  $E_i$  on  $V$  and  $\mathcal{O}_V$ -isomorphisms  $\sigma_i$  of  $E_{i+1}^{(p)}$  with  $E_i$ .

There is a natural  $p$ -linear map from  $E_{i+1}$  to  $E_{i+1}^{(p)}$ , and composing this map with  $\sigma_i$ , we get a  $p$ -linear map  $\lambda_i$  from  $E_{i+1}$  to  $E_i$ . Let  $\psi_i$  be the composition  $\lambda_0 \circ \dots \circ \lambda_{i-1}$ . For each affine open  $U$  in  $V$ , define  $(E)_r(U)$  to be the intersection of  $\psi_n(E_n(U))$ , and let  $(E)_r$  be the associated sheaf.  $(E)_r$  is a sub-sheaf of  $E_0$ , and since  $\psi_n(E_n(U))$  is the section of the image sheaf of  $E_n$  in  $E_0$  over  $U$  if  $U$  is affine, we see that the sections of  $(E)_r$  over  $U$  are just  $(E_f)(U)$ . If  $E = \{E_i, \sigma_i\}$  and  $F = \{E'_i, \sigma'_i\}$  are flat bundles over  $V$ , we define  $E \otimes F$  by  $\{E_i \otimes_{\mathcal{O}_V} E'_i, \sigma_i \otimes \sigma'_i\}$  and  $E^V$  by  $\{E_i^V, (\sigma_i^V)^{-1}\}$ .

**DEFINITION 1.2.** An  $\mathcal{O}_V$  morphism  $\sigma$  from  $E_0$  to  $F_0$  is horizontal if  $\sigma$  is a section of  $(E^V \otimes F)_r$ . A section of  $E_0$  is horizontal if it is a section of  $(E)_r$ .

**THEOREM 1.3 (Katz).** If  $V$  is smooth over  $k$ , there is an equivalence of categories between flat bundles and horizontal maps and stratified bundles and horizontal maps: Explicitly, if  $\nabla$  is a stratification on a bundle  $E$ , we define  $E_i$  to be the sub-sheaf of sections of  $E$  annihilated by  $\nabla(D)$  where the order of  $D$  is  $< p^i$  and  $D(1) = 0$ . Then  $E_i$  can be regarded as an  $\mathcal{O}_V$  module via  $p^i$ -th powers and  $E_{i+1}^{(p)} \cong E_i$ . Conversely, if  $E = \{E_i, \sigma_i\}$  is flat and  $U$  is an open set so that  $s_1, \dots, s_r \in \Gamma(U, E_i)$  give a trivialization of  $E_i$  over  $U$ , then any section  $s$  of  $E$  over  $U$  may be written

$$s = \sum f_k \psi_i(s_k).$$

If  $D$  is a differential operator of degree less than  $p^i$ , we set

$$\nabla(D)(s) = \sum D(f_k) \psi_i(s_k)$$

to define a stratification on  $E$ .

**PROOF.** Suppose first that  $\nabla$  is given and that the  $E_i$  are as above. We show by induction that the  $E_i$  are locally free, and the natural map  $E_{i+1}$  to  $E_i$  induces an isomorphism  $E_{i+1}^{(p)} \xrightarrow{\sim} E_i$ . Notice  $E_i$  has an integrable connection  $\nabla'$  which can be defined as follows: Let  $U$  be an affine open set and suppose  $x_1, \dots, x_n$  are defined on  $U$  and give an étale map to  $\mathbb{A}^n$ . Let  $D = \sum f_k (\partial / \partial x_k)$  be a derivation over  $U$  and  $D'$  a differential operator of degree  $< p^i$  so that

$$* \quad D'(f^{p^i}) = (D(f))^{p^i}.$$

Such an operator is given by

$$** \quad \sum_k f_k^{p^i} \frac{\partial^{p^i} / \partial x_k^{p^i}}{p^i!}.$$

If  $D''$  is another operator of degree  $\leq p^i$  satisfying  $*$ , then  $D' - D''$  is in the ring generated by operators of degree  $< p^i$  as can be seen by expanding  $D' - D''$  in terms of monomials  $D_I$  and evaluating on  $p^i$ -th powers. Thus if we set

$$\nabla'(D)(s) = \nabla(D')(s),$$

we see  $\nabla'$  is well defined. The expression  $*$  for  $D'$  commutes with all monomials  $D_I$  with  $|I| < p^i$ , so  $\nabla(D)(s)$  is a section of  $E_i$ , and hence  $\nabla'(D) \in \text{End}_k(E_i)$ . Now if  $s$  and  $f$  are sections of  $E_i$  and  $\mathcal{O}_V$ , we define  $f_*s$  to be  $f^{p^i} \cdot s$ . Notice  $f_*s$  is a section of  $E_i$ , and  $E_i$  is by induction a locally free sheaf with this  $\mathcal{O}_V$  structure, as this is what is meant by regarding  $E_i$  as a module via  $p^i$ -th powers. Now

$$\begin{aligned} \nabla'(D)(f_*s) &= \nabla(D')(f^{p^i}s) \\ &= f^{p^i}\nabla(D')(s) + \nabla(D'_{p^i})(s) \\ &= f_*\nabla(D')(s) + (Df)_*s \end{aligned}$$

since

$$\begin{aligned} D'_{p^i}(g) &= D'(f^{p^i}g) - f^{p^i}D'(g) \\ &= D'_o(f^{p^i}) + g(D'(f^{p^i})) - f^{p^i}D'(g) \\ &= f^{p^i}D_o^i(1) + g(D'(f^{p^i})) - f^{p^i}D'(g) \\ &= (Df)^{p^n} \cdot g. \end{aligned}$$

Notice  $\nabla'$  is integrable and has  $p$ -curvature zero, i.e.,  $\nabla'(D^p) = (\nabla'(D))^p$ , and that the sheaf of all sections annihilated by  $\nabla'$  is exactly  $E_{i+1}$ . Since  $k$  is perfect, a theorem of Cartier [7, § 5] shows that  $E_{i+1}$  is a locally free  $\mathcal{O}_V$  module via  $p^{i+1}$  powers, and that the  $p$ -linear inclusion of  $E_{i+1}$  into  $E_i$  induces an isomorphism  $E_{i+1}^{(p)} \xrightarrow{\sim} E_i$ . Hence the  $E_i$  define the structure of a flat bundle.

Conversely, suppose  $\{E_i, \sigma_i\}$  is a flat bundle, and  $\nabla$  is defined as in the theorem. First, since  $D(f^{p^i}g) = f^{p^i}D(g)$  if  $\deg D < p^i$ , we see  $\nabla$  is well defined. It is also a ring homomorphism, and satisfies Leibniz's rule, and so is a stratification. Equally obviously, the processes of passing from stratified bundles to flat bundles and vice versa are inverse, and define an equivalence of categories.

LEMMA 1.4. Let  $\mathfrak{J}$  be a defining sheaf of ideals for  $V$ , and we assume  $\bigcap_{n=0}^{\infty} \mathfrak{J}^n = (0)$ , and  $\{E_i, \sigma_i\}$  be a flat bundle over  $V$ .  $V_0$  will be the closed sub-

scheme defined by  $\mathfrak{J}$  and  $\{E'_i, \sigma'_i\}$  the restriction of  $\{E_i, \sigma_i\}$  to  $V_0$ . Then the map  $\varphi$  from  $E_f$  to  $E'_f$  is an isomorphism.

PROOF. We may assume  $V_0$  is affine and reduced. If  $s$  is a section of  $E_f$ , we may write  $s = \varphi_n(s_n)$  where  $s_n$  is a section of  $E_n$ . The map  $\lambda'_n$  from  $E'_n$  to  $E'_0$  is injective, since  $V_0$  is reduced. Suppose  $\varphi(s) = 0$ . Then we must have  $s_n \in \Gamma(\mathfrak{J} \cdot E_n)$ . Since  $\varphi_n$  is  $p^n$  linear,  $s \in \mathfrak{J}^{p^n} E_0$ , and since  $\bigcap_{n=1}^{\infty} \mathfrak{J}^{p^n} = 0$ , we see  $s = 0$ . Thus  $\varphi$  is injective.

Suppose  $s$  is a section of  $(E')_f$ . Then there are sections  $s_n$  of  $E_n$  so that  $\varphi_n(s_n)$  reduces to  $s$  on  $V_0$ . For any  $k$  and  $l$ , the image of  $s_n - \varphi_{n+1}(s_{n+1})$  in  $E_k$  is in  $\mathfrak{J}^{p^l} \cdot E_k$  for  $n > l + k$ . Thus the sequence of images of  $\{s_n\}$  in  $E_k$  converges to an element  $t_k$  and  $\varphi_k(t_k)$  is independent of  $k$ , and gives a lifting of  $s'$ .

PROPOSITION 1.5. If  $\mathfrak{J}$  is a defining sheaf of ideals for  $V$  and  $\bigcap_{n=1}^{\infty} \mathfrak{J}^n = 0$ . Then there is an equivalence of categories between flat bundles on  $V$  and on  $V_0$ .

PROOF. In view of Lemma 1.4, we need only show that every flat bundle  $\{E_i, \sigma_i\}$  on  $V_0$  lifts to a flat bundle on  $V$ . Let  $V_n$  be the sub-scheme of  $V$  defined by  $\mathfrak{J}^{p^n}$ . Notice  $\mathcal{O}_{V_n}$  is an  $\mathcal{O}_V$  module via  $p^n$  powers, so we define  $E_m^n = E_{m+n} \otimes_{\mathcal{O}_V} \mathcal{O}_{V_n}$ . For fixed  $n$ , the sequence  $E_m^n$  defines a flat bundle on  $V_n$  lifting  $E$ . Further, there are canonical isomorphisms of  $E_m^{n+1} \otimes_{\mathcal{O}_{V_{n+1}}} \mathcal{O}_{V_n}$  with  $E_m^n$ , so letting  $E_m^\infty = \lim_{n \rightarrow \infty} E_m^n$ , we see  $E_m^\infty$  is a lifting of  $E$ .

COROLLARY 1.6. Let  $E$  be a flat bundle over  $\mathrm{Spf} k[[x_1, \dots, x_n]]$ . Then  $E$  is trivial.

PROOF. The flat bundles over  $\mathrm{Spec} k$ ,  $k$  a perfect field, are trivial.

PROPOSITION 1.7. Let  $\{E_i, \sigma_i\}$  and  $\{F_i, \sigma'_i\}$  be flat bundles on a variety  $V$  proper over  $k$ , and suppose  $E_i$  and  $F_i$  are isomorphic as bundles for each  $i$ . Then there is a horizontal isomorphism of  $\{E_i, \sigma_i\}$  with  $\{F_i, \sigma'_i\}$ .

PROOF. There is an integer  $K \geq 0$  that if  $k \geq K$  then the  $p$ -linear maps from  $\mathrm{End}(E_{k+1})$  to  $\mathrm{End}(E_k)$  and from  $\mathrm{End}(F_{k+1})$  to  $\mathrm{End}(F_k)$  are bijective, since they are always injective, and  $\mathrm{End}(E)$  is finite dimensional over  $k$ . If  $k > K$ , there is an isomorphism  $\varphi$  of  $E_k$  with  $F_k$  which maps  $E_{k+1}$  to  $F_{k+1}$ , namely,  $\varphi = \varphi^p$ ,  $\varphi$  an isomorphism of  $E_{k+1}$  with  $F_{k+1}$ . (Since  $V$  is reduced, we may identify  $F_{k+1}$  as a sub-sheaf of  $F_k$ .) On the other hand, since any

automorphism of  $E_k$  is of the form  $\psi^p$  for some automorphism  $\psi$  of  $E_{k+1}$ , it follows that any isomorphism of  $E_k$  with  $F_k$  maps  $E_{k+1}$  to  $F_{k+1}$ , and by induction maps  $E_n$  to  $F_n$  for all  $n \geq k$ . Thus  $\varphi^{p^k}$  defines an isomorphism of  $\{E_i, \sigma_i\}$  with  $\{F_i, \sigma'_i\}$ .

**THEOREM 1.8 (Katz).** Let  $V$  be a smooth variety projective over  $k$ . A line bundle  $L$  on  $V$  has a stratification if and only if the order of  $L$  in the Neron-Severi group of  $V$  is finite of order prime to  $p$ . The group of stratifications on  $\mathcal{O}_V$  is the Tate group  $T_p$ .

**PROOF.** If the order of  $L$  in  $NS(V)$  is of order prime to  $p$ , then there is a bundle  $L_1$  so that  $L_1^{(p)}$  is  $L$  by the  $p$ -divisibility of  $\text{Pic}^0(V)$ . Since the order of  $L_1$  in  $NS$  is prime to  $p$ , we may find  $L_2$  so that  $L_2^{(p)}$  is  $L_1$ . Continuing, we obtain a stratification of  $L$ . The second statement of 1.8 follows from 1.7.

Let  $V$  be a smooth variety over  $k$ . We show continuous representations of  $\pi_1(V)$  on  $k^n$  give stratified bundles on  $V$ . Let  $f: V' \rightarrow V$  be an étale cover of  $V$ , i.e.,  $f$  is étale and finite. Then  $f_*\mathcal{O}_{V'}$  has a natural stratification, since the differential operators on  $V$  act naturally on  $\mathcal{O}_{V'}$  and hence on  $f_*\mathcal{O}_{V'}$ . Suppose that  $K(V')$  over  $K(V)$  is normal, where  $K(V)$  is the function field of  $V$ , and let  $G$  be the Galois group, and  $\varrho$  a representation of  $G$  on  $k^n$ . Then  $G$  acts on  $\mathcal{O}_{V'}^n$ , and thus  $G$  is represented on the horizontal endomorphisms of  $(f_*\mathcal{O}_{V'})^n$ . Let  $E_\varrho$  be the subsheaf consisting of sections  $(s_1, \dots, s_n)$  so that

$$\varrho(\sigma)(s_1, \dots, s_r) = (s_1^\sigma, \dots, s_r^\sigma).$$

Then  $E_\varrho$  is locally free of rank  $n$ , and  $f^*E_\varrho$  is isomorphic to  $\mathcal{O}_{V'}^n$ , as stratified modules.

**PROPOSITION 1.9.** The above construction defines a faithful map from the category of continuous finite dimensional representations of  $\pi_1(V)$  to stratified bundles on  $V$ .

**PROOF.** Suppose we are given  $E_\varrho$  and  $E_{\varrho'}$ , where  $\varrho$  and  $\varrho'$  are representations of  $G$  on  $k^n$  and  $k^m$  respectively. A horizontal map from  $E_\varrho$  to  $E_{\varrho'}$  induces a horizontal map  $\varphi'$  on  $\mathcal{O}_{V'}^n$  to  $\mathcal{O}_{V'}^m$ , by pull-back. Since the only horizontal sections of  $\mathcal{O}_{V'}$  are constant,  $\varphi'$  is given by a matrix of constants, and this matrix gives a map from  $\varrho$  to  $\varrho'$ .

**THEOREM 1.10.** Let  $V$  be non-singular over  $k$ , then

- i) If every stratified bundle is trivial, then  $\pi_1$  is trivial.



ii) If the only irreducible stratified bundles are one dimensional, then  $[\pi_1, \pi_1]$  is a  $p$ -group.

iii) If every stratified bundle is a direct sum of stratified line bundles, then  $\pi_1$  is abelian with no  $p$ -part.

PROOF. Using 1.9, we see i) follows from the fact that every non-trivial finite group has a non-trivial representation on  $k^n$  for some  $n$ . In case ii), let  $G$  be a finite quotient of  $\pi_1$ . The hypothesis shows that every representation  $\rho$  of  $G$  has an invariant vector, so  $G$  can be represented on upper triangular matrices. Thus any commutator in  $G$  goes to a unipotent matrix, and so if  $\rho$  is faithful, any commutator in  $G$  has order  $p^n$  for some  $n$ . If each stratified bundle is a direct sum, then  $\rho$  is diagonalizable, and so abelian.

**§ 2.** – Information on the category of stratified modules gives knowledge of the fundamental group. On the other hand, it seems possible that the converses of the statements of 1.10 might be true. At any rate, the purpose of this paper is to examine the categories of stratified modules on some varieties whose fundamental group is known to be abelian.

LEMMA 2.1. Let  $V$  be quasi-projective and smooth over  $k$  and let  $E$  be a bundle which admits a stratification. Then all the Chern classes  $C_i(E)$  are divisible by  $p^n$  for all  $n$ . (The  $C_i(E)$  are in the Chow ring of cycles modulo rational equivalence.)

PROOF.  $E$  is isomorphic to  $E_i^{(p^i)}$  and so

$$C_k(E) = p^{ik} C_k(E_i).$$

THEOREM 2.2. Every stratified bundle on  $\mathbf{P}_k^n$  is trivial.

PROOF. Let  $\{E_i\}$  be a flat bundle. Since the bundle  $E_i$  has a stratification defined by  $\{E_{i+j}\}$ , we see the Chern classes of  $E_i$  are same as those of  $\mathcal{O}_{\mathbf{P}^n}^r$ , where  $r$  is the rank of  $E_i$ . The Riemann-Roch theorem tells us that

$$\chi(E) = \chi(\mathcal{O}^r) = r.$$

We argue by induction on  $n$ . Let  $H$  be a hyperplane in  $\mathbf{P}^n$ , and consider the exact sequence

$$0 \rightarrow E_i(k) \rightarrow E_i(k+1) \rightarrow \mathcal{O}_H(k+1)^r \rightarrow 0,$$

where we have used the induction hypothesis to replace  $E_i \otimes \mathcal{O}_H$  by  $\mathcal{O}_H^r$ . Using descending induction on  $k$  and the fact that  $H^l(E_i(k)) = 0$  for  $l > 0$  and  $k \gg 0$ , we see that  $H^l(E_i) = 0$  for  $l > 1$ . Thus

$$h^0(E_i) \geq h^0(E_i) - h^1(E_i) = \chi(E_i) = r.$$

Since the  $p^i$  linear map from  $H^0(E_i)$  to  $H^0(E_0)$  is injective, the dimension of the flat sections of  $E$  is greater than  $r - 1$ . Let  $s_1, \dots, s_r$  be flat sections of  $E$  linearly independent over  $k$ . If  $Q$  is a closed point of  $\mathbb{P}^n$ , Corollary 1.6 shows that the residue classes of the  $\{s_j\}$  in  $E_0/E_0 \cdot m_Q$  are linearly independent, where  $m_Q$  is the ideal of  $Q$ . So  $s_1, \dots, s_r$  define a horizontal isomorphism of  $E_0$  with  $\mathcal{O}_{\mathbb{P}^n}^r$ .

The following theorem applies to  $K3$  surfaces.

**THEOREM 2.3.** Let  $X$  be a non-singular projective surface and suppose  $H^1(X, \mathcal{O}_X) = 0$  and that the canonical bundle  $K$  of  $X$  is trivial. Then every stratified bundle on  $X$  is trivial. In particular,  $X$  is simply connected.

**PROOF.** If the rank of  $E$  is  $r$ , we see  $\chi(E_i) = 2r$ , since  $E_i$  has the same Chern classes as  $\mathcal{O}_X^r$  and  $\chi(\mathcal{O}_X) = 2$  by Serre duality. We must either have  $h^0(E_i) \geq r$  or  $h^0(E_i^v) = h^2(E_i) \geq r$ . Since it suffices to prove the theorem for  $E$  or  $E^v$ , we may assume  $h^0(E_i) \geq r$ . The triviality of  $\{E_i\}$  follows as in the previous theorem.

**PROPOSITION 2.4.** Let  $X$  and  $Y$  be varieties over  $k$ , with  $X$  proper over  $k$  and let  $\{E_i, \sigma_i\} = E$  be a flat bundle on  $X \times_k Y$ . For each closed point  $y \in Y$  the fiber over  $Y$  is canonically identified with  $X$ , and so for each  $y$  a closed point of  $Y$ , there is a flat bundle  $E_y$  on  $X$ . Then  $E_y$  and  $E_{y'}$  are isomorphic as flat bundles for  $y, y'$  closed points of  $Y$ .

**PROOF.** Let  $\mathfrak{X}$  be the completion of  $X \times_k Y$  along the inverse image of  $y$ , and let  $E'$  be the flat bundle obtained as the inverse image of  $E_y$  by the projection of  $X \times_k Y$  onto  $X$ . Then  $E$  and  $E'$  are isomorphic when restricted by  $\mathfrak{X}$  by 1.5. If  $p$  is the projection of  $X \times_k Y$  to  $Y$  and  $\pi$  is the restriction of  $p$  to  $\mathfrak{X}$ , then  $\pi_*(\text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(E_i, E'_i))$  is a free  $\hat{\mathcal{O}}_{Y,y}$  module, and  $\pi_*(\text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(E_i, E'_i)) \otimes k_y$  is  $\text{End}_{\mathcal{O}_{\mathfrak{X}}}(E_y)_i$ . Thus  $p_*(\text{Hom}_{\mathcal{O}_{X \times Y}}(E_i, E'_i))$  is locally free at  $y$ , and  $p_*(\text{Hom}_{\mathcal{O}_{X \times Y}}(E_i, E'_i)) \otimes k_y$  is  $\text{End}_{\mathcal{O}_X}(E_y)_i$ . [EGA III, 4.1.5]. In particular, the identity map of  $(E_y)_i$  extends to an isomorphism of  $E_i$  with  $E'_i$  over  $X \times_k U$  where  $U$  is some neighborhood of  $y$ . For each closed  $y' \in Y$ ,  $(E_i)_y$  is isomorphic to  $(E_{y'})_i$ , and so by 1.7,  $E_y$  and  $E_{y'}$  are isomorphic as flat bundles.

**DEFINITION 2.5.** Let  $E$  be a flat bundle on  $V$ , a variety over  $k$ .  $E$  is said to be abelian if there is a flat bundle on  $V \times_k V$  and a closed point  $x_0 \in V$  so that  $E'$  restricted to either  $p_1^{-1}(x_0)$  or  $p_2^{-1}(x_0)$  is isomorphic to  $E$ . (The  $p_i$  are the projections.)

Note that if  $V$  is a smooth variety proper over  $\mathbf{C}$ , and flat bundles are identified with bundles with integrable connections, then a flat bundle is abelian if and only if the corresponding representation of  $\pi_1(V)$  factors through  $H_1(V, \mathbf{Z})$ .

Returning to characteristic  $p$ , any flat bundle on an abelian variety is abelian. For let  $f$  be the addition map from  $A \times_k A$  to  $A$ , and let  $E' = f^*(E)$  and  $x_0$  be the identity.

**THEOREM 2.6.** Let  $E$  be an abelian flat bundle on a variety  $X$  proper over  $k$ , and suppose  $E$  is irreducible. Then  $E$  has rank one.

**PROOF.** If  $\{E_i\}$  is abelian, then the new bundle  $E'_i = E_{i+n}$  is also abelian. For  $n \gg 0$ , all the endomorphisms of  $E'_0$  are horizontal, so replacing  $E$  by  $E'$ , we may suppose all endomorphisms of  $E_0$  are horizontal. If  $\text{End}_{\mathcal{O}_x}(E)$  is larger than  $k$ , then there is a non-zero element  $\varphi$  of  $\text{End}_{\mathcal{O}_x}(E)$  which is not an automorphism. For each closed point  $y$ , Proposition 1.5 applied to the formal scheme  $\text{Spf } \hat{\mathcal{O}}_{X,y}$  and the sub-scheme  $\text{Spec } k$  of definition, shows that the rank of  $\varphi$  is constant, and so  $\ker \varphi$  is a sub-bundle. So  $\text{End}_{\mathcal{O}_x}(E) = k$ , since  $E$  is irreducible. Now let  $E'$  be the flat bundle on  $X \times X$  of 2.5, and let  $E''_n$  be the bundle  $p_1^*(E_n)$ . Then

$$p_{2,*}(\text{Hom}_{\mathcal{O}_{X \times X}}(E''_n, E'_n)) = L_n$$

is a line bundle, and we have the restriction map

$$L_n \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^r, E_n) = E_n^r$$

by restricting to  $\{x_0\} \times X = X$ . Noting that  $L_{n+1}^{(p)} = L_n$ , we have constructed a flat sub-line bundle  $L$  of  $E^r$ . The projection of  $L$  onto some factor  $E$  of  $E^r$  is non-zero, and so must be an isomorphism, and so  $E$  has rank one.

Thus every flat bundle  $E$  on an abelian variety has a filtration by sub-stratified bundles  $E_i$  of  $E$  so that the rank of  $E_i$  is  $i$ . This can be done very explicitly when  $X$  is an elliptic curve using the Atiyah-Oda classification [8]. Recall there is an indecomposable bundle  $E_{r,0}$  over an elliptic curve  $X$  of degree zero, and any other indecomposable bundle of degree zero may be written uniquely as  $L \otimes E_{r,0}$ , with  $L \in \text{Pic}^0(X)$ . Further, if  $X$  is not supersingular, then  $E_{r,0}^{(p)} \cong E_{r,0}$ , whereas if  $X$  is supersingular, then

$E_{r,0}^{(p)} \cong \mathcal{O}_X^r$ . Thus if  $X$  is supersingular, then  $E_{r,0}$  cannot be given a flat structure, since it is not a  $p$ -th power. Thus all flat bundles have rank one, and the flat structure on a line bundle is unique, since  $T_p(X) = 0$ . On the other hand, if  $X$  is not supersingular, then any indecomposable flat bundle  $\{E_i\}$  may be identified with a line bundle  $\{L_i\}$   $E_i \cong L_i \otimes E_{r,0}$ . The indecomposable flat bundle  $E$  corresponding to the trivial line bundle may also be described in the following way: Let  $M$  be the  $r \times r$  matrix with ones on the diagonal and superdiagonal and zeros elsewhere, and suppose  $p^n > r$ . Then there is a unique étale covering of  $X$  with Galois group  $G = \mathbb{Z}/(p^n)$ . Represent  $G$  on  $k^r$  by letting  $\rho(n)$  be  $M^n$ . Then  $E_\rho \cong E$ .

Our next theorem will show that all flat bundles on a complete, smooth unirational variety  $V$  come from representations of the fundamental group of  $V$ .

PROPOSITION 2.7. Let  $X$  be a non-singular variety,  $U$  an open sub-variety so that the codimension of  $X - U$  in  $X$  is  $\geq 2$ . Then the restriction map from flat bundles on  $X$  to flat bundles on  $U$  is an equivalence of categories.

PROOF. This will appear as Theorem 3.14.

LEMMA 2.8. Let  $X$  and  $Y$  be non-singular varieties and  $F$  a rational map from  $X$  to  $Y$  which is generically surjective and generically finite, and suppose  $\pi_1(Y)$  is trivial. Let  $E$  be a flat bundle on  $Y$  so that  $F^*(E)$  is trivial. Then  $E$  is trivial.

PROOF. Let  $X'$  be the normalization of the closure of the graph of  $F$ , and  $G$  the map from  $X'$  to  $Y$ . If  $W$  is an open sub-set of  $X'$ , and  $H$  is a stratified bundle on  $X'$  and  $s$  is a flat section of  $H$  over  $W$ , then  $s$  extends to a flat section of  $H$  over  $X'$ . Indeed, if  $P$  is any closed point of  $X' - W$ , and  $s_1, \dots, s_r$  are a horizontal basis of  $H \otimes \hat{\mathcal{O}}_P$ , then

$$s = \sum f_i s_i$$

where the  $f_i$  are in the quotient field of  $\hat{\mathcal{O}}_P$ . But then the  $f_i$  are  $p^n$ -th powers for all  $n$ , so the  $f_i$  are in  $k$ . Setting  $H = G^*(E)$ , we see that there are horizontal sections  $s_1, \dots, s_r$  in  $G^*(E)$  which trivialize  $G^*(E)$ . Let  $s'_1, \dots, s'_r$  be sections of  $E$  over the generic point of  $Y$ . Then

$$s_i = \sum f_{ij} s'_j$$

where the  $f_{ij}$  are rational functions on  $X$ . Let  $K$  be the field generated by the  $f_{ij}$  over  $K(Y)$ , the function field on  $Y$ .  $K$  clearly does not depend on

$s'_1, \dots, s'_r$ . We will show  $K = K(Y)$ , and then the argument at the beginning of this proof will show  $E$  is trivial. First,  $K$  is separable over  $K(Y)$ , for there is an  $n$  so that  $f^n$  is separable over  $K(Y)$ , if  $f \in K(X)$ . So we may choose  $s'_i = \psi_n(t'_i)$ , where the  $t'_i$  are sections of  $E_n$  over the generic point of  $Y$ . By definition, we have  $s_i = \psi_n(t_i)$ , and

$$t_i = \sum g_{ij} t'_j.$$

Thus  $f_{ij} = g_{ij}^n$ , so  $K$  is separable over  $K(Y)$ . We next show  $K$  is unramified at any closed point  $y$  of  $Y$ .

Let  $s'_1, \dots, s'_r$  be a basis of  $E$  localized at  $y$ . We may also choose a horizontal basis  $s''_1, \dots, s''_r$  for  $E \otimes \hat{\mathcal{O}}_{Y,y}$ , and we may assume that the images of  $s''_i$  and  $s_i$  are the same in  $E \otimes \hat{\mathcal{O}}_{X',x}$ , where  $x$  is a closed point of  $X'$  over  $y$ . Thus the  $f_{ij}$  are elements of  $\hat{\mathcal{O}}_{Y,y}$  which are separable over  $K(Y)$ . It follows that  $K$  is unramified over  $K(Y)$ , and since  $\pi_1(Y) = 0$ ,  $K = K(Y)$ .

**THEOREM 2.10.** Let  $X$  and  $Y$  be smooth complete varieties of the same dimension, and suppose the function field of  $X$  is a finite extension of that of  $Y$ . Then if every stratified bundle on  $X$  is trivial, every stratified bundle on  $Y$  comes from a representation of the fundamental group of  $Y$ .

**PROOF.** Let  $Y'$  be an étale cover of  $Y$ . There is an open subset  $U$  of  $X$  and a generically surjective map  $f$  from  $U$  to  $Y$  so that the codimension of  $X - U$  is  $\geq 2$ . Then  $X$  is simply connected by 1.10, so  $U$  is simply connected. Then  $Y' \times_Y U$  is an étale cover of  $U$ , and so is a disjoint union of copies of  $U$ . Thus  $K(Y') \subseteq K(X)$ . It follows that the universal cover  $Y'$  of  $Y$  exists, and  $K(Y') \subseteq K(X)$ . If  $E$  is a stratified bundle on  $Y$ ,  $f^*(E)$  is trivial by 2.7. If  $\pi$  is the projection from  $Y'$  to  $Y$ , 2.8 shows  $\pi^*(E)$  is trivial. But  $\pi_1(Y)$  operates by a representation  $\rho$  on  $H^0(Y', \pi^*(E))$ , a vector space of dimension  $r$ , and  $E_\rho$  is just  $E$ .

**§ 3.** – The previous section dealt with varieties complete over  $k$ . In this section, we will be dealing with varieties  $U$  which are open sub-sets of smooth, complete varieties  $X$  and  $X - U$  is a divisor with normal crossings, *i.e.*,  $D = X - U$  is a union of smooth divisors, and for each point  $P \in D$ , there is a sequence of local parameters  $x_1, \dots, x_n$  at  $P$  so that  $D$  is defined by  $x_1, x_2, \dots, x_k = 0$  for some  $k$ . On such  $U$ , we define the concept of a stratified bundle on  $U$  with regular singular points on  $D$ . Continuous representations of the tame fundamental group of  $U$  will give stratified bundles with regular singular points. We begin with the «local» theory, which is due to Katz in the case  $n = 1$ .

DEFINITION 3.1. Let  $M$  be a module over  $R = k[[x_1, \dots, x_n]]$  and let  $\mathcal{D}'$  be the ring of all differential operators on  $\text{Spec } R$  generated by  $x_i^n(\partial^n/\partial x_i^n)/n!$ . A stratification on  $M$  with regular singular points is an  $R$ -linear ring homomorphism  $\nabla$  from  $\mathcal{D}'$  to  $\text{End}_k(M)$  satisfying Leibniz's rule (0.3\*).

NOTATION. We denote  $(x_i^n(\partial^n/\partial x_i^n)/n!)$  by  $\binom{x_i(\partial/\partial x_i)}{n}$ .

EXAMPLE. Let  $M = R$  and let  $\alpha_1, \dots, \alpha_n$  be  $p$ -adic integers. We define

$$* \quad \nabla \binom{x_i(\partial/\partial x_i)}{n} (1) = \binom{\alpha_i}{n}.$$

Note that if  $N$  is an integer and  $p^k > n$ , then

$$\binom{N + lp^k}{n} \equiv \binom{N}{n} \pmod{p}.$$

So the right hand side is well defined, and  $\nabla \binom{x_i(\partial/\partial x_i)}{n} (s)$  can be computed by Leibniz's rule. We denote  $(M, \nabla)$  by  $\mathcal{O}(\alpha_1, \dots, \alpha_n)$ . If  $\beta_i$  is a non-negative integer congruent to  $-\alpha_i \pmod{p^l}$ , then

$$\nabla \binom{x_i(\partial/\partial x_i)}{k} (x_1^{\beta_1} \dots x_n^{\beta_n}) = 0$$

if  $0 < k < p^l$ .

If  $(M, \nabla)$  and  $(M', \nabla')$  are stratified modules with regular singular points, a map  $\varphi$  from  $M$  to  $M'$  is called horizontal if it commutes with  $\nabla$  and  $\nabla'$ .

Before giving our next example we need a lemma.

NOTATION. Let  $C_n$  denote the set of all maps of  $\mathbb{Z}/(p^n)$  to  $\mathbb{Z}/(p)$ , and for each  $\alpha \in \mathbb{Z}/(p^n)$ , define  $h_\alpha \in C_n$  by

$$h_\alpha(\beta) = \binom{\beta}{\alpha}$$

where we have identified  $\mathbb{Z}/(p^n)$  with  $\{0, 1, \dots, p^n - 1\}$ .

LEMMA 3.2. The  $h_\alpha$  give a  $\mathbb{Z}/(p)$  basis for  $C_n$ .

PROOF. It suffices to show the  $h_\alpha$  are independent. Suppose there were  $k_\alpha \in \mathbb{Z}/(p)$  so that

$$\sum_{\alpha} k_{\alpha} \binom{\beta}{\alpha} = 0.$$

and suppose  $k_0 = k_1 = \dots = k_\gamma = 0$ . Setting  $\beta = \gamma + 1$ , we see  $k_{\gamma+1} = 0$ .

EXAMPLE. Let  $M = k[[y, x_2, \dots, x_n]]$ , where  $y^m = x_1$  and  $p$  does not divide  $m$ . Then  $M$  is naturally a stratified bundle with regular singular points. Let  $N$  be a positive integer and let  $m'$  be a positive integer so that

$$mm' \equiv 1 \pmod{p^N}$$

and suppose  $n < p^N$ . There are  $\alpha_i \in \mathbb{Z}/(p)$  so that

$$\binom{m'k}{n} \equiv \sum_i \alpha_i \binom{k}{l} \pmod{p}.$$

Then we have

$$\binom{k}{n} \equiv \sum_i \alpha_i \binom{mk}{l}.$$

Now

$$\binom{y(\partial/\partial y)}{l} (x_i^k) \equiv \binom{mk}{l} x_i^k$$

so

$$\binom{x_1(\partial/\partial x_1)}{n} = \sum \alpha_i \binom{y(\partial/\partial y)}{l}$$

so  $\binom{x_1(\partial/\partial x_1)}{n}$  acts on  $M$ .

If  $\varrho$  is any representation of  $\mathbb{Z}/(m)$ , the Galois group of  $k[[y, x_2, \dots, x_n]]$  over  $k[[x_1, \dots, x_n]]$ , on  $k^n$ , we may define a submodule  $M_\varrho$  of  $M^n$  consisting of all  $s = (s_1, \dots, s_n)$  so that  $\varrho(\sigma)(s) = \sigma(s)$  for  $\sigma \in \mathbb{Z}/(m)$ . Then  $M_\varrho$  is naturally a bundle with regular singular points.

**THEOREM 3.3.** Suppose  $M$  is a free  $R$ -module of finite type with regular singular points.

Then  $M$  is isomorphic as a bundle with regular singular points to

$$\bigoplus_{j=1}^n \mathcal{O}(\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{rj}).$$

PROOF. Let  $m$  be the maximal ideal of  $R$ , and let  $F_n = M/m^n M$ . For each  $i$ , define a  $\mathbb{Z}/(p)$  linear map from  $C_i$  to  $\text{End}_k(F_n)$  by setting

$$\varphi_i(h_\alpha) = \nabla \begin{pmatrix} x_i(\partial/\partial x_i) \\ \alpha \end{pmatrix}$$

for  $0 \leq \alpha < p^l$ .  $\varphi_i$  are well defined, since  $\nabla \begin{pmatrix} x_i(\partial/\partial x_i) \\ \alpha \end{pmatrix}$  maps  $m^n M$  to  $m^n M$ , and the  $\varphi_i$  are ring homomorphisms. Indeed, if  $\alpha$  and  $\beta$  are non-negative integers less than  $p^n$ , we have

$$\begin{pmatrix} \gamma \\ \alpha \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \equiv \sum_{\delta} l_{\delta} \begin{pmatrix} \gamma \\ \delta \end{pmatrix},$$

so

$$\begin{pmatrix} x_i(\partial/\partial x_i) \\ \alpha \end{pmatrix} \begin{pmatrix} x_i(\partial/\partial x_i) \\ \beta \end{pmatrix} = \sum_{\delta} l_{\delta} \begin{pmatrix} x_i(\partial/\partial x_i) \\ \delta \end{pmatrix},$$

as can be seen by evaluating both sides on monomials in  $x_1, \dots, x_n$ . For each  $a \in \mathbb{Z}/(p^l)$ , let  $\chi_a \in C_i$  be the characteristic function of  $a$ . Then the  $\varphi_i(\chi_a)$  are all commuting, orthogonal idempotents, so we may write  $F_n = \bigoplus V_{n,k}$ , where  $\varphi_i(h_\alpha)$  acts on  $V_{n,k}$  by scalar multiplication by an element of  $\mathbb{Z}/(p)$ , and any sub-space on which all the  $\varphi_i$  act by scalar multiplication is contained in some  $V_{n,k}$ . (For  $l \gg 0$ , the decomposition does not depend on  $l$ , since  $F_n$  is finite dimensional.) If  $m \geq n$ , then given a subspace  $V_{n,k}$ , we can find a  $k'$  so that  $V_{m,k'}$  maps surjectively to  $V_{n,k}$ . Indeed, each  $V_{m,k'}$  maps into some  $V_{n,k''}$  and the images of the  $V_{m,k'}$  generate  $F_n$ . So there are  $k_1, \dots, k_l$  so that the images of  $V_{m,k_i}$  are non-zero and contained in  $V_{n,k}$ , and generate  $V_{n,k}$ . The maximality of  $V_{m,k_i}$  shows that  $l = 1$ .

Thus given an element  $s_n$  in  $V_{n,k}$ , we may lift it to an element of  $V_{n+1,k'}$ . The sequence  $\{s_n\}$  then defines an element  $s$  in  $M = \lim F_n$ , and

$$\nabla \begin{pmatrix} x_i(\partial/\partial x_i) \\ l \end{pmatrix} (s) = \alpha_i(l)(s)$$

where  $\alpha_i$  is a function from  $\mathbb{Z}$  to  $\mathbb{Z}/(p)$ . There is a  $p$ -adic integer  $\alpha_i$  so that

$$\alpha_i(l) = \begin{pmatrix} \alpha_i \\ l \end{pmatrix}.$$

For in  $C_i$ , we may write

$$h_k = \sum \begin{pmatrix} \alpha \\ k \end{pmatrix} \chi_\alpha.$$



But  $\varphi_i(\chi_\alpha)(s) = 0$  except for one value of  $\alpha$ , which we denote  $\alpha(l)$ , and  $\varphi_i(\chi_{\alpha(l)})(s) = s$ . Thus

$$\varphi(\hbar_k)(s) = \binom{\alpha(l)}{k}(s)$$

and clearly the  $\alpha(l)$  converge to a  $p$ -adic integer  $\alpha_i$ .

Thus we may lift a basis of  $F_0$  on which  $\varphi_i$  acts by scalar multiplication, and this basis gives an isomorphism of  $E$  with  $\bigoplus_{j=1}^r \mathcal{O}(\alpha_{ij}, \dots, \alpha_{nj})$ .

**DEFINITION 3.4.**  $\alpha_{k1}, \dots, \alpha_{kr}$  are called the exponents of  $E$  along  $x_k = 0$ . If  $E$  is a torsion free module with regular singular points, the exponents of  $E$  along  $x_j = 0$  are defined to be the exponents of  $E \otimes K[[x_j]]$  where  $K = k((x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n))$ .

**THEOREM 3.5.** Let  $E$  be an  $R$ -module with regular singular points and suppose for each point  $p \in \text{Spec } R$  of codimension 2, we have  $\text{depth}_p(E) \geq 2$ . Further, assume one of the following conditions holds:

i) If  $T_j$  is the set of exponents of  $E$  along  $x_j = 0$ , then the map from  $T_j$  to  $\mathbb{Z}_p/\mathbb{Z}$  is injective.

ii)  $\nabla \binom{x_n(\partial/\partial x_n)}{k}$  maps  $E$  to  $x_n^k E$  and if  $U$  is the open set over which  $E$  is locally free, then  $H^0(U, E/(x_n E))$  is a free  $k[[x_1, \dots, x_{n-1}]] = R'$  module.

Then  $E$  is free.

**PROOF.** First assume i). The  $R'$  module  $E'_k = H^0(U, E/x_n^k E)$  is an  $R'$  module in regular singular points, and it satisfies the depth condition with respect to  $R'$ . (Theorem 3.8, 5). It also satisfies i) with respect to  $R'$ . Indeed, the exponents of  $E'_k$  along  $x_j = 0$  are just  $T_j$ , as  $E'_k$  is the successive extension of the bundles  $F_i = x_n^i E/x_n^{i-1} E$ , and  $F_i$  is isomorphic to  $E'_1$  as an  $R'$  module with regular singular points.

By induction,  $E'_k$  may be written as a direct sum of line bundles  $\mathcal{O}(\alpha_1, \dots, \alpha_{n-1})$  as an  $R'$  module with singular points, and the maps  $E'_k$  to  $E'_{k-1}$  are horizontal. Our assumption on  $T_j$  insures that a horizontal map from  $\mathcal{O}(\alpha_1, \dots, \alpha_{n-1})$  to  $\mathcal{O}(\beta_1, \dots, \beta_{n-1})$  is either identically zero, or an isomorphism. Thus the map from  $E'_k$  to  $E'_{k-1}$  is surjective, and so  $H^0(U, E/(x_n^k E)) = E'_k$  is a free  $R/(x_n^k E)$  module. On the other hand, by SGA 2, Prop. 1.4, we have  $H^0(U, E) = \lim H^0(U, E/(x_n^k E))$ , so  $H^0(U, E)$  is free. Thus  $E$  is free, since  $H^0(U, E) = E$  by the depth condition.

Now suppose ii) is satisfied. We claim the map from  $E/(x_n^k E)$  to  $E/(x_n E)$  has a section over  $U$ . For  $\nabla \left( \frac{x_n(\partial/\partial x_n)}{k-1} \right)$  maps  $E/(x_n^k E)$  to  $x_n^{k-1} E/x_n^k E$  and is easily seen to be the identity on  $x_n^{k-1} E/x_n^k E$ . Again  $H^0(U, E/(x_n^k))$  is a free  $R/(x_n^k)$  module, and so  $E$  is free.

NOTATION. Let  $X$  be a smooth variety and the  $E_i$  smooth divisors on  $X$  with normal crossings,  $E = \sum E_i$ . Let  $\mathcal{D}^E$  be the sheaf of differential operators  $D$  which locally map  $\mathfrak{J}_{E_i}^k$  to  $\mathfrak{J}_{E_i}^k$ , where  $\mathfrak{J}_{E_i}$  is the ideal sheaf of  $E_i$ .

LEMMA 3.6. Let  $U$  be an open sub-set of  $X$  and suppose there are functions  $x_1, \dots, x_n$  on  $U$  so that  $dx_1, \dots, dx_n$  are a basis for  $\Omega_{U/k}^1$ , and suppose  $E$  is defined by  $x_1 \dots x_l = 0$ . Then  $\mathcal{D}_E$  is generated as a sheaf of  $\mathcal{O}_U$  rings by

$$\left( \frac{x_i(\partial/\partial x_i)}{k} \right) (1 \leq i \leq l) \quad \text{and} \quad \frac{\partial^k/\partial x_i^k}{k!} \quad \text{for } l < i \leq n.$$

PROOF. Every  $D$  may be expanded in terms of the  $D_I$ . For fixed  $n$ , let  $D$  be a differential operator of degree  $\leq n$ , not in  $\mathcal{D}^E$ , so that the number of non-zero terms in the expansion of  $D$  in terms of the  $D_I$  is minimal.

$$D = \sum c_I D_I.$$

Choose  $I = (i_1, \dots, i_n)$  so that  $c_J = 0$  if  $J = (j_1, \dots, j_n)$  and  $|J| < |I|$ . Then

$$D(x_1^{i_1} \dots x_n^{i_n}) = c_I.$$

So  $c_I$  is divisible by  $x_1^{i_1} \dots x_n^{i_n}$ , and so  $c_I D_I \in \mathcal{D}^E$ . This contradicts the minimality of  $D$ .

DEFINITION 3.7. Let  $X$  be smooth, and  $E = \sum E_i$  be a divisor with normal crossings. Let  $F$  be a coherent torsion free sheaf on  $X$  which is locally free over  $U = X - E$ . A stratified module on  $U$  with regular singular points on  $E$  is a map  $\nabla$  from  $\mathcal{D}^E$  to  $\text{End}_k(F)$  so that the restriction of  $\nabla$  to  $U$  defines a stratification of  $F$  restricted to  $U$ . A stratified bundle  $F$  on  $U$  is said to have regular singular points at infinity if  $U$  is an open sub-set of some smooth complete  $X$  so that  $X - U$  is a divisor with normal crossings, and  $F$  extends to a module on  $X$  with regular singular points on  $X - U$ .

Note that if  $E$  is smooth there is a map from  $\mathcal{D}^E$  to the differential operators on  $E$ . If  $D$  is a section of  $\mathcal{D}^E$ , we denote by  $\bar{D}$  its image as a differential operator on  $E$ .

LEMMA 3.8. Let  $E$  be a smooth divisor on  $X$  and  $F$  a bundle on  $X$ , stratified on  $X - E$  and having regular singular points on  $E$ . Then we may write

$$F \otimes \mathcal{O}_E = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha,$$

where if  $s$  is any section of  $F_\alpha$  over an open set  $U \subseteq X$  and  $D$  is a differential operator on  $U$  with  $\bar{D} = 0$ , then

$$\nabla(D)(s) = \alpha_0 s$$

where  $\alpha_0$  has the property that if  $f$  is any local defining equation for  $E$ , then

$$D(f^\beta) \equiv \alpha_0 f^\beta \pmod{(f^{\beta+1})}$$

where  $\beta$  is some positive integer congruent to  $\alpha \pmod{p^N}$ ,  $p^N > \text{order } D$ .

PROOF. Let  $x_1, \dots, x_n$  be local co-ordinates at a point  $P$  of  $E$  so that  $E$  is defined by  $x_1 = 0$ . Any differential operator  $D$  in  $\mathcal{D}^E$  so that  $\bar{D} = 0$  is a linear combination of  $\binom{x_1(\partial/\partial x_1)}{k} \pmod{x_1 \mathcal{D}^E}$ . As in the proof of 3.3, we get a ring homomorphism  $\varphi$  of  $\mathcal{O}_i$  to  $\text{End}_{\mathcal{O}_E}(F \otimes \mathcal{O}_E)$  by sending  $h_\alpha$  to  $\nabla \binom{x_1(\partial/\partial x_1)}{\alpha}$ . Thus the  $\varphi(h_\alpha)$  are sums of orthogonal idempotents in  $\text{End}(F_{\mathcal{O}_E} \otimes \mathcal{O}_E)$ , and so we may write

$$F = \bigoplus_{\alpha \in \mathbb{Z}_p} F_\alpha$$

so that if  $s$  is a section of  $F_\alpha$ , then

$$\nabla \binom{x_1(\partial/\partial x_1)}{k} (s) = \binom{\alpha}{k} s.$$

Thus we get the decomposition  $F \otimes \mathcal{O}_E = \bigoplus F_\alpha$  in a neighborhood of  $P$ . The decomposition does not depend on  $x_1, \dots, x_n$  and so we get a global decomposition.

DEFINITION 3.9. If  $E, F$  and  $X$  are as in Lemma 3.8, we say  $\alpha$  is an exponent of  $F$  along  $E$  if  $F_\alpha \neq 0$ . Its multiplicity is the rank of  $F_\alpha$ .

LEMMA 3.10. Let  $x_1, \dots, x_n$  be functions on  $X$ , which we assume affine, so that  $dx_1, \dots, dx_n$  to give a basis of  $\Omega_{X/k}^1$ , and suppose  $E$  is defined by

$x_1 = 0$  and  $F$  is a bundle with regular singular points on  $E$ . Then there is a bundle  $F'$  with regular singular points so that the set of exponents of  $F'$  maps injectively to  $\mathbb{Z}_p/\mathbb{Z}$ , and  $F$  and  $F'$  are isomorphic as stratified bundles on  $X - E$ .

PROOF. Let  $T$  be the set of exponents of  $F$ , and for  $\alpha$  a  $p$ -adic integer, define  $\lambda(\alpha)$  to be  $\alpha$  if  $\alpha \in \mathbb{Z}^+$  and  $\lambda(\alpha)$  to be zero otherwise, and finally define

$$\mu(F) = \sum_{\alpha, \beta \in T} \lambda(\alpha - \beta).$$

The lemma will be established by induction if we construct a bundle  $F'$  so that  $F'$  is isomorphic to  $F$  on  $X - E$  and  $\mu(F') < \mu(F)$ . Let  $\alpha \in T$  and suppose  $\alpha - n \notin T$  for  $n \in \mathbb{Z}^+$ , but  $\alpha + l \in T$  for some  $l \in \mathbb{Z}^+$ , and let  $F'$  be the sub-module of  $E$  generated by  $x_1 F$  and the images of  $F$  under the endomorphisms  $\nabla \binom{x_1(\partial/\partial x_1)}{k} - \binom{\alpha}{k} I$ , for all  $k \in \mathbb{Z}^+$ , where  $I$  is the identity endomorphism. Clearly  $F'$  is stable under the action of  $\nabla$ .  $F'$  is locally free. For let  $P$  be a point of  $E$  and  $s_1, \dots, s_r$  be a basis of  $F$  in a neighborhood so that

$$\nabla \binom{x_1(\partial/\partial x_1)}{k} (s_i) = \binom{\alpha_i}{k} s_i \quad \text{mod } x_1 F.$$

Suppose  $\alpha_1 = \alpha_2 = \dots = \alpha_l = \alpha$  and  $\alpha_i \neq \alpha$  for  $i > l$ . Then  $\{x_1 s_1, \dots, x_1 s_l, s_{l+1}, \dots, s_r\} = \{s'_1, \dots, s'_r\}$  is a basis for  $F'$  at  $P$ . The exponents of  $F'$  are

$$\{\alpha + 1, \alpha + 1, \dots, \alpha_{l+1}, \dots, \alpha_r\},$$

since

$$\binom{x_1(\partial/\partial x_1)}{k} (s'_i) \equiv \binom{\alpha + 1}{k} s'_i \quad \text{mod } x_1 F' \quad i \leq l$$

and

$$\nabla \binom{x_1(\partial/\partial x_1)}{k} (s'_i) \equiv \binom{\alpha_i}{k} s'_i \quad \text{mod } x_1 F' \quad i > l.$$

Clearly  $\mu(F') < \mu(F)$ .

LEMMA 3.11. Let  $X, E, F, U$  and  $\nabla$  be as in Definition 3.7. Then there is a bundle  $F'$  stratified over  $U$  with regular singular points on  $E$  so that  $F$  and  $F'$  are isomorphic as stratified bundles over  $U$ . Further, set  $T_i$  of exponents along  $E_i$  maps injectively to  $\mathbb{Z}_p/\mathbb{Z}$ .

PROOF. Let  $E = \sum E_i$  and let  $\{W_i\}$  be affine open sets such that  $E \cap W_i = E_i \cap W_i \neq \emptyset$  and so that each  $W_i$  has a local system of parameters  $x_{i1}, \dots, x_{in}$  so that  $E$  is defined by  $x_{i1} = 0$ , and  $F$  is locally free over the  $W_i$ . Then Lemma 3.10 shows there are bundles  $F'_i$  over  $W_i$  so that the set exponents  $T_i$  of  $F'_i$  along  $E_i$  maps injectively to  $\mathbb{Z}_p/\mathbb{Z}$ . The  $F'_i$  glue to give a bundle  $F''$  on  $U \cup W_1 \cup \dots \cup W_k = U'$ . If  $j$  is the inclusion of  $U'$  into  $X$ , then  $j_* F'' = F'$  is a module stratified over  $U$  with regular singular points on  $E$ . Let  $P$  be a closed point of  $E$ , and consider  $F' \otimes_{\mathcal{O}_{X,P}} = G$ . Clearly  $G$  satisfies the depth condition of Theorem 3.5, and the exponents of  $G$  along  $E_j$  are  $T_j$ . Thus  $G$  is free, and so  $F'$  is locally free.

LEMMA 3.12. Let  $R = k[[x_1, \dots, x_n]]$ , and  $R'$  be  $R$  localized at the ideal generated by element  $x_1 x_2 \dots x_n$ , and suppose  $\nabla$  is an  $R$ -linear ring homomorphism from  $\mathcal{D}'$  to  $\text{End}_k(R')$ , where  $\mathcal{D}'$  is the ring generated by  $\begin{pmatrix} x_i(\partial/\partial x_i) \\ k \end{pmatrix}$ , and suppose  $\nabla$  satisfies Leibniz's rule. Then  $\nabla(D)$  maps  $R$  to  $R$  for any  $D \in \mathcal{D}'$ . More briefly, the singular points of a line bundle stratified over  $\text{Spec } R - \text{Spec } R/(x_1 \dots x_n)$  are automatically regular.

PROOF. Let  $n$  be an integer. There is always a non-zero section  $s$  of  $R$  such that  $\nabla(D)(s) = 0$  if order  $D < p^n$  and  $D(1) = 0$ . For let

$$D' = \sum_{\mathbf{w}} (-1)^{|\mathbf{w}|} \begin{pmatrix} x_1(\partial/\partial x_1) \\ w_1 \end{pmatrix} \dots \begin{pmatrix} x_n(\partial/\partial x_n) \\ w_n \end{pmatrix}$$

where  $\mathbf{w} = (w_1, \dots, w_n)$  runs over all  $n$ -tuples such that  $0 \leq w_i < p^n$ . One verifies that  $D'f$  is always a  $p^n$ -th power, and so  $DD' = 0$  if order  $D < p^n$  and  $D(1) = 0$ . Now  $\nabla(D')(s')$  is non-zero for some  $s'$ , and take  $s = \nabla(D')(s')$ . By multiplying  $s$  by a  $p^n$ -th power, we may assume that  $s \in R$ . Thus we may write

$$\nabla(D)(1) = \nabla(D)(s^{-1} \cdot s) = D(s^{-1}) \cdot s$$

if  $D(1) = 0$ . But the order of the pole of  $D(s^{-1})$  along  $x_i = 0$  is less than or equal to the order of zero of  $s$  along  $x_i = 0$ , we see  $D(s^{-1}) \cdot s \in R$ .

THEOREM 3.13. Let  $U$  be a variety and  $F$  a stratified bundle on  $U$  with regular singular points at infinity, and let  $X$  be a complete smooth variety which contains  $U$  as an open sub-set so that  $X - U$  has normal crossings. Then  $F$  may be extended to a bundle  $F'$  over  $X$  so that  $F'$  has regular singular points on  $X - U$ .

PROOF. The hypothesis and Lemma 3.11 show that  $U$  is an open sub-set of an  $X'$  and  $F$  extends to a bundle  $F''$  on  $X'$  with regular singular points on  $X' - U$ . Now the rational map  $G$  from  $X$  to  $X'$  is defined on an open sub-set  $W$  of  $X$  so that  $W \supset U$  and the codimension of  $X - W$  is  $\geq 2$ . The bundle  $G^*(F'')$  has regular singular points on  $W - U$ . For let  $P$  be a closed point of  $W - U$ , and  $Q$  be its image in  $X'$ . Then  $G^*(\widehat{F''}) = \widehat{F''} \otimes_{\widehat{\mathcal{O}}_{X',Q}} \widehat{\mathcal{O}}_{X',P}$ . But  $\widehat{F''}$  is a direct sum of line bundles. It follows that  $G^*(\widehat{F''})$  is a direct sum of line bundles invariant under the action of  $\nabla$ . Finally, Lemma 3.12 shows the singular points of  $G^*(\widehat{F''})$  are regular, and so  $F''$  has regular singular points. Finally, letting  $j$  be the inclusion map of  $W$  into  $X$ , and letting  $F''' = j_* F''$ , we get a stratified module with regular singular points on  $X - U$ , and Lemma 3.11 completes the proof.

THEOREM 3.14. Let  $U$  be a smooth variety, and  $W$  a closed sub-set of codimension  $\geq 2$ . There is an equivalence of categories induced by restriction between stratified bundles on  $U$  and stratified bundles on  $U - W$ .

PROOF. If  $E$  is a stratified bundle on  $U - W$  and  $j$  is the inclusion of  $U - W$  into  $U$ , let  $E' = j_*(E)$ . If  $P$  is a point of  $W$ , then  $E' \otimes \widehat{\mathcal{O}}_{U,P}$  is a free  $\widehat{\mathcal{O}}_{U,P}$  module by Theorem 3.5, so  $E'$  is locally free.

EXAMPLE. Let  $X = \mathbb{P}^2$  and  $E = L_0 + L_1 + L_2$ , where  $L_i$  is defined by  $X_i = 0$ . Let  $M_i$  be the ideal sheaf of  $L_i$ .  $M_i$  is a naturally stratified bundle on  $\mathbb{P}^2 - E$  with regular singular points on  $E$ , and there is a horizontal map of  $M_i$  to  $\mathcal{O}_{\mathbb{P}^1}$ . Let  $F$  be the kernel of the map of  $M_0 \oplus M_1 \oplus M_2$  to  $\mathcal{O}_{\mathbb{P}^1}$ .  $F$  is just  $\Omega_{\mathbb{P}^1/k}^1$ , and is stratified with regular singular points. Its exponents are 0 and 1 along any line  $L_i$ . If  $\mathbb{P}^2$  is embedded as a plane in  $\mathbb{P}^3$  and  $Q$  is some closed point not on  $\mathbb{P}^2$ , and  $\pi$  is the projection of  $\mathbb{P}^3 - Q$  to  $\mathbb{P}^2$ , then  $\pi^*(F)$  is stratified with regular singular points on  $\pi^{-1}(E)$ , but does not extend to a bundle on  $\mathbb{P}^3$ .

Notice two facts. Since we may only have horizontal maps between  $\mathcal{O}(\alpha_1, \dots, \alpha_n)$  and  $\mathcal{O}(\beta_1, \dots, \beta_n)$  if  $\alpha_i \equiv \beta_i \pmod{\mathbb{Z}}$ , we have that if  $F$  and  $F'$  are two bundles with regular singular points on  $X - U$ , which are isomorphic on  $U$  as stratified bundles, then their exponents on any component of  $X = U$  are the same mod  $\mathbb{Z}$ . Second, the representation of  $\pi_1(U)$  which factor through the tame fundamental group of  $U$  gives bundles with regular singular points, as the example on page 22 shows.

§ 4. - We wish to examine stratified bundles with regular singular points on some algebraic groups. We begin with  $G_m = \mathbb{P}^1 - \{0, \infty\}$ .

LEMMA 4.1. Let  $\mathcal{O} = k[x]$  and  $\hat{\mathcal{O}} = k[[x]]$ , and assume  $s_1, \dots, s_r \in \hat{\mathcal{O}}^n$  are a free basis for a sub-module  $E$  of  $\hat{\mathcal{O}}^n$ . Then there are  $t_i \in E \cap \mathcal{O}$  so that the  $t_i$  are  $\hat{\mathcal{O}}$  basis of  $E$ .

PROOF. Let  $\{e_j\}$  be the standard basis of  $\hat{\mathcal{O}}^n$ . We may write

$$s_i = \sum p_{ij} e_j$$

where  $p_{ij} = \sum_{n=0}^{\infty} a_n^{ij} x^n$ . Let  $p_{ij}^N = \sum_{n=0}^N a_n^{ij} x^n$ , and finally set

$$t_i = \sum p_{ij}^N e_j.$$

We denote the matrices  $(p_{ij})$  and  $(p_{ij}^N)$  by  $P$  and  $P^N$ . For  $N \gg 0$ , we wish to know  $P(P^N)^{-1} \in GL_n(\hat{\mathcal{O}})$ . This will show the  $\hat{\mathcal{O}}^n$  modules generated by the  $\{s_i\}$  and  $\{t_i\}$  are identical. But  $\det P^N$  approaches  $\det P$  as  $N \rightarrow \infty$ , so  $\det P^N / \det P$  is a unit for  $N \gg 0$ . But

$$P \operatorname{adj} P^N \rightarrow P \operatorname{adj} P = \det P \cdot I$$

where  $\operatorname{adj}$  is the adjoint. So

$$\frac{P \operatorname{adj} P^N}{\det P^N} = P(P^N)^{-1}$$

is holomorphic for  $N \gg 0$ .

LEMMA 4.2. Let  $E$  be a bundle on  $\mathbf{P}^1$  stratified over  $\mathbf{P}^1 - \{P_1, \dots, P_l\}$  with regular singular points at the  $P_i$ . Suppose  $E \cong \bigoplus_{i=1}^r \mathcal{O}(n_i)$ , where the  $n_i$  are non-negative integers, not all zero. Then there is a sub-sheaf  $E'$  contained in  $E$  so that  $E'$  is stratified on  $\mathbf{P}^1 - \{P_1, \dots, P_l\}$  and has regular singular points, and  $E' \cong \bigoplus_{i=1}^r \mathcal{O}(n'_i)$ ,  $n'_i \geq 0$  and  $\sum n'_i < \sum n_i$ .

PROOF. We may assume  $P_1 = 0$ . Let  $s_i$  be a section of  $\mathcal{O}(n_i)$  which vanishes to order  $n_i$  at zero. Then  $E$  is generated by  $s_1, \dots, s_r$  except at  $x = 0$ , and there it is generated by  $x^{-n_1} s_1, \dots, x^{-n_r} s_r$ . We may assume that  $n_1 = \dots = n_k = 0$  and  $n_l > 0$  for  $l > k$ , for if all  $n_i > 0$ , we may take  $E' = xE$ .  $\hat{E} = E \otimes \hat{\mathcal{O}}_{x,0}$  is isomorphic to  $\bigoplus_{i=1}^r \mathcal{O}(\alpha_i)$ ,  $\alpha_i \in \mathbf{Z}_p$ , and let  $t_i$  be generators for  $\hat{E}$  which are eigenvectors for  $\nabla \left( \begin{smallmatrix} x(d/dx) \\ k \end{smallmatrix} \right)$ . Then

$$x^{-n_l} s_l = \sum a_{ij} t_j.$$

By reordering the  $t_j$ , we may assume  $a_{r1}(0) \neq 0$ . For  $1 \leq i \leq k$ , set

$$s'_i = s_i - \varphi_i x^{-nr} s_r,$$

where the  $\varphi_i$  are chosen in  $k$  so that  $(a_{i1} - \varphi_i a_{r1})(0) = 0$ , and for  $i > k$ , set  $\varphi'_i = \varphi_i$ . Consider the locally free sheaf  $E'$  on  $\mathbb{P}^1$  which is  $E$  except at zero, and at zero,  $E'$  is generated by  $xt_1, t_2, \dots, t_n$ . This makes sense by Lemma 4.1. On the other hand, the  $s'_i$  are sections of  $E'$ , and so if  $E \cong \bigoplus \mathcal{O}(n'_i)$ , then  $n'_i \geq 0$ . The map from  $E'$  to  $E$  is not an isomorphism, so

$$\sum n'_i = \deg E' < \deg E = \sum n_i.$$

PROPOSITION 4.3. Let  $E$  be a bundle on  $\mathbb{P}^1$ , stratified over  $\mathbb{G}_m$  with regular singular points at 0 and  $\infty$ , and suppose  $E$  is generated by its global sections as a sheaf. If  $\alpha_1, \dots, \alpha_r$  are the exponents of  $E$  at zero, then there sections  $s_1, \dots, s_r$  so that  $s_1, \dots, s_r$  give a trivialization of  $E$  over  $\mathbb{P}^1 - \{\infty\}$ , and

$$\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix} (s_i) = \begin{pmatrix} \alpha_i \\ k \end{pmatrix} s_i.$$

PROOF. By Lemma 4.2, we may find  $E'$ , a sub-sheaf of  $E$ , so that  $E' \cong \bigoplus_r \mathcal{O}_{\mathbb{P}^1}$ , so that the inclusion of  $E'$  into  $E$  is an isomorphism over  $\mathbb{P}^1 - \{\infty\}$ , and so that the stratification  $\nabla$  on  $E$  induces a stratification on  $E'$  over  $\mathbb{G}^m$  with regular singular points. Let  $s_1, \dots, s_r$  be a basis of  $E'$  so that in the fiber of  $E'$  over 0,  $\bar{s}_1, \dots, \bar{s}_r$  are eigenvectors for  $\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix}$ , i.e.,

$$\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix} (s_i) \equiv \begin{pmatrix} \alpha_i \\ k \end{pmatrix} s_i \pmod{x E'}.$$

However,  $\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix} (s_i)$  is a section of  $E'$ , since if  $M$  is the divisor  $0 + \infty$ , then  $\begin{pmatrix} x(d/dx) \\ k \end{pmatrix} \in \mathcal{D}^M$ . Thus

$$\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix} (s_i) = \sum a_i s_i$$

where  $a_i \in k$ . Evaluating at zero, we see

$$\nabla \begin{pmatrix} x(d/dx) \\ k \end{pmatrix} (s_i) = \begin{pmatrix} \alpha_i \\ k \end{pmatrix} s_i.$$



We next examine stratified bundles over principal  $\mathbf{G}_m$  bundles. Let  $L$  be a line bundle over a smooth quasi-projective variety  $Y$  and let  $X = \mathbf{P}(L^{\vee} \oplus \mathcal{O}_Y)$  and  $X_0$  and  $X_{\infty}$  the sections of the map  $\pi$  from  $X$  to  $Y$  corresponding to  $L^{\vee}$  and  $\mathcal{O}_Y$ . We say a stratified bundle is indecomposable if it is not the direct sum of two non-trivial sub-stratified bundles.

**PROPOSITION 4.4.** Let  $E$  be an indecomposable stratified bundle on  $U = X - \{X_0, X_{\infty}\}$ , with regular singular points on  $X_0$  and  $X_{\infty}$ . Then the exponents of  $E$  along  $X_0$  are all equal in  $\mathbf{Z}_p/\mathbf{Z}$ . If  $L$  has a stratification, then for any  $\alpha \in \mathbf{Z}_p$ , there is a stratified line bundle on  $U$  with exponent  $\alpha$  on  $X_0$ .

**PROOF.** We may assume  $E$  is a bundle on  $X$ , and that the set of exponents of  $E$  along  $X_0$  maps injectively to  $\mathbf{Z}_p/\mathbf{Z}$ . By replacing  $E$  with the bundle  $E \otimes \mathcal{O}(NX_{\infty})$  for  $N \gg 0$ , we may assume that  $\pi_*(E)$  is locally free and  $\pi^*\pi_*(E)$  maps surjectively to  $E$ .

Now let  $P$  be a closed point of  $Y$  and  $V$  a neighborhood of  $P$  so that  $L$  is trivial over  $V$ , i.e., there is a nowhere vanishing section  $x$  of  $L$  over  $V$ . Let  $\pi$  be the projection of  $X - X_{\infty}$  to  $Y$ . Then

$$\Gamma(\pi^{-1}(V), \mathcal{O}_X) = \Gamma(V, \mathcal{O}_Y)[x].$$

Thus  $\begin{pmatrix} x(d/dx) \\ k \end{pmatrix}$  is a differential operator on  $X$ . Further, if  $x' = ux$ , where  $u$  is a nowhere vanishing function on  $V$ , then

$$\begin{pmatrix} x'(d/dx') \\ k \end{pmatrix} = \begin{pmatrix} x(d/dx) \\ k \end{pmatrix}.$$

For each  $\alpha \in \mathbf{Z}_p$  and  $k \in \mathbf{Z}^+$ , we get a well-defined  $\mathcal{O}_Y$  linear endomorphism  $T_{\alpha,k}$  of  $\pi_*(E)$

$$T_{\alpha,m} = \nabla \begin{pmatrix} x(d/dx) \\ m \end{pmatrix} - \begin{pmatrix} \alpha \\ m \end{pmatrix} I.$$

For a closed point  $P$  of  $Y$ , denote by  $\bar{T}_{\alpha,m}$  the induced endomorphism of  $H^0(\mathbf{P}^1, E_P) = \pi_*(E) \otimes k_P$ , where  $E_P = E \otimes k_P$ ,  $k_P$  the residue field of  $P$ . If  $\alpha_1, \dots, \alpha_l$  are the exponents of  $E$  with multiplicities  $r_i$ , then for  $M \gg 0$ , we have

$$\bar{E}_{\alpha_i} = \bigcap_{m=1}^M \text{Ker } \bar{T}_{\alpha_i,m}$$

is a vector space of dimension  $r_i$ , and  $\bar{E}_{\alpha_i} \cap \bar{E}_{\alpha_j} = 0$  if  $\alpha_i \neq \alpha_j$ , and the map from  $\bigoplus \pi^*(\bar{E}_{\alpha_i}) \rightarrow E_P$  is an isomorphism over  $\mathbb{P}^1 - \{\infty\}$ . It follows that

$$E_{\alpha_i} = \bigcap^{\infty} \text{Ker } T_{\alpha_i, m}$$

is locally free of rank  $r_i$ , and the map from  $\bigoplus \pi^*(E_{\alpha_i})$  to  $E$  is an isomorphism over  $X - X_{\infty}$ .

We have to show that if  $L$  admits a stratification  $\nabla$ , then there is a line bundle on  $X$  with exponent  $\alpha$  on  $X_0$ . For each  $n \in \mathbb{Z}^+$  and each closed point  $P$  in  $Y$ , there is a neighborhood  $U$  of  $P$  and a nowhere zero section  $x$  of  $L$  so that  $\nabla(D)(x) = 0$  if  $D$  is a differential operator of order  $\leq n$  so that  $D(1) = 0$ . If  $y_1, \dots, y_n$  are a local system of parameters at  $P$ , then  $x, y_1, \dots, y_n$  are a local system of parameters at any point  $Q$  in  $X - X_{\infty}$  above  $P$ . We define a stratification on  $\mathcal{O}_{X - X_{\infty}}$  by

$$\nabla \binom{x(d/dx)}{k} (\sum f_i x^i) = \sum \binom{\alpha + i}{k} f_i x^i$$

and

$$\nabla(D)(\sum f_i x^i) = \sum D(f_i) x^i,$$

where  $D$  is differential operator on  $Y$  of degree  $\leq n$  and  $k \leq n$ . These definitions do not depend on  $x$ , and so define the structure of a stratified module on  $\mathcal{O}_U$ , which has exponent  $\alpha$  on  $X_0$ .

Let  $X$  be a smooth variety and  $Y \xrightarrow{\pi} X$  a principal  $\mathbb{G}_a$  bundle over  $X$ ,  $\mathbb{G}_a = \mathbb{A}_k^1$ .  $Y$  may be embedded in a  $\mathbb{P}^1$  bundle  $Y'$  by the canonical inclusion of  $\mathbb{G}_a$  in  $\mathbb{P}^1$ .

**PROPOSITION 4.5.** There is an equivalence of categories between stratified bundles on  $X$  and stratified bundles on  $Y$  which may be extended to have regular singular points on  $Y' - Y$  given by  $\pi^*$ .

**PROOF.** Let  $E$  be a stratified bundle on  $Y$  with regular singular points on  $Y' - Y$ . We claim  $E$  extends to a stratified bundle on  $Y'$ . Now the exponents of  $E$  are all integers. Indeed, to compute the exponents of  $E$ , it suffices to restrict  $E$  to a fiber. However, on  $\mathbb{G}_a$  all stratified bundles with regular singular points at  $\infty$  are trivial, and so the exponents of  $E$  must all be zero mod  $\mathbb{Z}$ . Lemma 3.10 shows that  $E$  may be extended a bundle  $E'$  on an open sub-set  $W$  of  $Y'$  which properly contains  $Y$  and so that all the exponents of  $E'$  on  $W - Y$  are equal to some integer  $M$ . If  $\mathfrak{J}$  is the ideal sheaf of  $Y' - Y$ , then  $\mathfrak{J}$  is a line bundle stratified over  $Y$  with regular singular points on  $Y' - Y$  and exponent 1 on  $Y' - Y$ . Thus

$E' \otimes \mathcal{J}^{-M} = E''$  has exponents zero, i.e.,  $E''$  is stratified on  $W$ . Thus  $E$  extends to a stratified bundle on  $Y'$ .

We have to show there is an equivalence of categories between stratified bundles on  $Y'$  and stratified bundles on  $X$ . Let  $E$  be a stratified bundle on  $Y'$ ,  $p$  the projection of  $Y'$  to  $X$ , and consider  $p_*(E)$ . Since  $E$  is trivial on each fiber of  $Y'$  over  $X$ ,  $p_*(E)$  is locally free of the same rank as  $E$ , and the map  $\varphi$  from  $p^*p_*E$  to  $E$  is an isomorphism. Note  $p_*(E)$  is stratified. For if  $\mathcal{D}_X$  and  $\mathcal{D}_{Y'}$  are the sheaves of differential operators on  $X$  and  $Y'$ , there is a natural map  $\psi: \mathcal{D}_{Y'} \rightarrow p^*(\mathcal{D}_X)$ . Any section of  $p_*(E)$  is annihilated by any differential operator in the kernel of  $\psi$ , as can be seen by passing to local co-ordinates on a sufficiently small open sub-set  $U$  of  $X$  so that  $p^{-1}(U) \cong \mathbb{P}^1 \times U$ . Thus  $\mathcal{D}_X$  acts on  $p_*(E)$ , and  $\varphi$  is horizontal.

Let  $G$  be an algebraic group, and  $B$  a Borel sub-group, i.e.,  $B$  is a maximal, linear solvable sub-group of  $G$ . We do not know if in general  $G$  may be embedded in a smooth complete variety  $X$  so that  $G - X$  is a divisor with normal crossings, and so our definition of stratified bundle with regular points at infinity does not apply in this case. However, we may use the following *ad hoc* condition. Notice that  $B$  as an algebraic variety is a product of  $G_a$ 's and  $G_m$ 's, as can be seen by applying Lemma 2 of Chapter VII of [9] inductively. Mapping each  $G_a$  or  $G_m$  to  $\mathbb{P}^1$ , we get a map of  $B$  to  $(\mathbb{P}^1)^d = \bar{B}$ ,  $d$  the dimension of  $B$ , so that  $\bar{B} - B$  is a divisor with normal crossings. Let  $\pi$  be the map of  $G$  to  $G/B$ .

**THEOREM 4.6.** Let  $E$  be a stratified bundle on  $G$ , and suppose for some closed point  $P \in G/B$ , there is a neighborhood  $V$  of  $P$  so that the B-bundle  $\pi^{-1}(V)$  is trivial over  $V$ , i.e.,  $\pi^{-1}(V) = V \times B$  and furthermore  $E$  extends to a bundle  $E'$  on  $V \times \bar{B}$  with regular singular points on  $V \times (\bar{B} - B)$ . If  $G$  is linear, then  $E$  is a direct sum of stratified line bundles. If  $G$  is commutative, there are sub-stratified bundles  $0 = E_0 \subset E_1 \subset \dots \subset E_r = E$  so that  $E_i/E_{i-1}$  is a line bundle.

**PROOF.** Let  $\{1\} = U_0 \subset U_1 \subset \dots \subset U_n = U$  be a composition series for the unipotent part of  $B$ . We claim there is a stratified bundle  $E_i$  on  $G/U_i$  so that  $\pi_i^*(E_i)$  is  $E$ , where  $\pi_i$  is the map from  $G$  to  $G/U_i$ . We work by induction on  $i$ .  $G/U_i$  is a principal  $G_a$  bundle over  $G/U_{i+1}$ . We claim that if  $\overline{G/U_i}$  is the  $\mathbb{P}^1$  bundle over  $G/U_{i+1}$  associated to  $G/U_i$ , then  $E_i$  has regular singular points on  $\overline{G/U_i} - G/U_i$ . Over the inverse image of  $V$  in  $G/U_i$ , there is a section  $s$  mapping  $G/U_i$  to  $G$ . For some sub-set  $W$  of codimension two in  $\overline{G/U_i}$ , the map  $s$  extends to a map  $s'$  of  $\overline{G/U_i} - W$  to  $V \times \bar{B}$ . As in the proof of Theorem 3.13,  $(s')^*(E')$  has regular singular points on  $(\overline{G/U_i} - W) - (G/U_i)$ , and so  $E_i$  has regular singular points on  $\overline{G/U_i} - (G/U_i)$ , since  $s^*(E) = E_i$ . Proposition 4.5 shows that  $E_{i+1}$  exists.

Now  $G/U$  is a principal  $T$  bundle over  $H = G/B$ , where  $T = B/U$  is a torus,  $\mathbf{G}_m \times \dots \times \mathbf{G}_m$ . Thus  $G/U$  may be written  $G_1 \times_H G_2 \times_H \dots \times_H G_n$  where the  $G_i$  are principal  $\mathbf{G}_m$  bundles. Thus considering the natural inclusion of  $\mathbf{G}_m$  into  $\mathbf{P}^1$ , we get  $\mathbf{P}^1$  bundles  $H_i$  over  $H$  and open immersions from  $G_i$  to  $H_i$ . There are line bundles  $L_i$  so that  $H_i = \mathbf{P}(L_i \oplus \mathcal{O}_H)$ . If  $K = H_1 \times_H \dots \times_H H_n$ , then  $G/U$  is an open sub-set of  $H$ , and  $K - G/U$  is a divisor with normal crossings. We claim  $E_n$  has regular singular points on  $K - G/U$ . There is a map of the inverse image of  $V$  in  $G/U$  to  $G$ . In particular, we get a rational map of the inverse image of  $V$  in  $K$  to  $V \times \bar{B}$  and this map is defined except on a sub-set of codimension two. As before,  $E_n$  has regular singular points.

Suppose  $E$  is indecomposable. Then  $E_n$  is also indecomposable. We claim that the exponents of  $E_n$  on each component of  $K - G/U$  are all equal. This follows from Proposition 4.4 applied to the principal  $\mathbf{G}_m$  bundle  $G/U$  over

$$G_1 \times_H \dots \times_H G_{i-1} \times_H G_{i+1} \times_H \dots \times_H G_n.$$

Now suppose  $G$  is linear. Then the stratified bundle  $Hom(E_n, E_n)$  has zero exponents on every component of  $K - G/U$ , i.e.,  $Hom(E_n, E_n)$  can be extended to a stratified bundle on  $K$ . But  $K$  is a complete, rational variety, so  $Hom(E_n, E_n)$  is trivial. (Theorem 2.2, 2.10 and [1] 14.13.) The horizontal endomorphisms of  $E_n$  may be identified with  $M_r(k)$ , the ring of  $r \times r$  matrices,  $r = \text{rank } E$ . Let  $p_1, \dots, p_r$  be the canonical projections of  $k^n$  to  $k$ , and let  $P_1, \dots, P_r$  be the corresponding horizontal endomorphisms of  $E_n$ . Then  $E_n = \bigoplus \text{Ker } P_i$ . Thus  $r = 1$ .

Now suppose  $G$  is commutative. Let  $P$  be a closed point of  $G/B$ , and  $L$  a stratified line bundle on the fiber  $Y$  of  $G/U$  over  $P$ . We claim  $L$  extends to a stratified line bundle on all  $G/U$ . The bundles  $L_i$  above are all translation invariant, and therefore algebraically equivalent to zero, since  $G/B$  is an abelian variety. Thus  $L_i$  is stratifiable, and so for any  $\alpha_i \in \mathbf{Z}_p$ , there is a stratified line bundle  $L_i$  on  $G_i$  with exponent  $\alpha_i$  on one component of  $H_i - G_i$ , and  $-\alpha_i$  on the other component. [Proposition 4.4 and Theorem 1.8]. By choosing the exponents  $\alpha_i$  correctly and letting  $M_i$  be the pullback of  $L_i$  to  $G/U$ , we will have that  $L$  is  $M_1 \otimes M_2 \otimes \dots \otimes M_n$  restricted to  $Y$ , since any stratified line bundle on  $\mathbf{G}_m \times \mathbf{G}_m \times \dots \times \mathbf{G}_m$  is determined by its exponents.

Now let  $E$  be an indecomposable bundle on  $G$ . Then  $E$  restricted to  $Y$  is a direct sum of stratified line bundles by the first part of this theorem. Further, these bundles are all isomorphic to the same stratified bundle  $L$ , since they all have the same exponents. Extending  $L$  to a stratified bundle  $L'$  on  $G/U$ , the stratified bundle  $E \otimes (L')^\vee = E'$  extends to a stratified bundle

on  $K$ . Let  $\pi$  be the projection from  $K$  to  $G/B$ . Then  $\pi_*(E')$  is stratified on  $G/B$ , and  $\pi^*\pi_*(E') = E'$ , as in the second part of the proof of 4.5. Theorem 2.6 shows that  $\pi_*(E')$  has a filtration  $0 = E_0 \subset E_1 \subset \dots \subset E_n = \pi_*(E')$  such that  $E_i/E_{i-1}$  is a line bundle. Thus  $E$  also has such a filtration.

**COROLLARY 4.7.** Assume  $G$  is linear, and that there is a complete variety  $X$  so that  $G$  is an open sub-set of  $X$  and  $X - G$  is a divisor with normal crossings. Then every stratified bundle  $E$  on  $G$  with regular singular points on  $X - G$  is direct sum of stratified line bundles. In particular, the tame fundamental group of  $G$  is abelian and has no  $p$ -part.

**PROOF.** The map of  $V \times \bar{B}$  into  $X$  is defined except at a sub-set of co-dimension 2, and so  $E$  restricted to  $V \times \bar{B}$  has regular singular points.

**§ 5.** – This section will examine stratified bundles with regular singular points on  $\mathbb{P}^n - D$ , where  $D$  is a divisor with normal crossings. Our main tool will be a theorem comparing stratifications on a variety to stratifications on a hyperplane section in the spirit of Lefschetz. We begin by studying the extension of stratified bundles with regular singular points from sub-schemes of definition to formal schemes.

**LEMMA 5.1.** Let  $U = \text{Spec } A$ , and let  $x_1, \dots, x_n$  be in  $A$  so that  $dx_1, \dots, dx_n$  give a basis of  $\Omega_{A/k}^1$ . Let  $U' = \text{Spec } A/(x_n)$ , and  $B'$  be the divisor on  $U'$  defined by  $x_1 \dots x_{n-1} = 0$ , and suppose  $E$  is a free  $A/(x_n)$  module, stratified on  $U' - B'$  with regular singular points on  $B'$ . If  $\hat{A}$  denotes completion of  $A$  with respect to the  $x_n$ -adic topology, and  $B$  the divisor on  $U$  defined by  $x_1 \dots x_{n-1} = 0$ , then there is a free  $\hat{A}$  module  $\hat{E}$ , an  $\hat{A}$  linear homomorphism  $\nabla$  from  $\hat{A} \otimes \mathcal{D}^B$  to  $\text{End}_k(\hat{E})$  satisfying Leibniz's rule, and an isomorphism  $\varphi$  from  $\hat{E}/x_n \hat{E}$  to  $E'$  so that the action of  $\mathcal{D}^B$  induced by  $\nabla$  on  $E'$  is the given one. If  $\hat{E}'$ ,  $\nabla'$  and  $\varphi'$  also satisfy the above conditions, there is a unique isomorphism of  $\hat{E}$  with  $\hat{E}'$  so that  $\nabla$  and  $\nabla'$ , and  $\varphi$  and  $\varphi'$  correspond.

**PROOF.** Let  $X_k = \text{Spec } A/(x_n^k)$ .  $E$  extends to a flat bundle  $\{E_i^k, \varphi_i\}$  on  $X_k - B'$  (Proposition 1.5). If  $l \leq k$ , and  $s$  is a section of  $E_1^k$ , we may define a section  $D_l(s)$  of  $E_1^{k-l}$  in the following way: write  $s = \sum_m f_m \psi_l(s_m)$  where the  $s_m$  are sections of  $E_l^k$ , and define

$$(*) \quad D_l(s) = \sum \left( \binom{\partial^l / \partial x_n^l}{l!} f_m \right) \psi_l(s_m).$$

Notice  $D_i(s)$  is well defined as a section of  $E_1^{k-1}$ . Now define  $A/(x_n^k)$  modules  $F_k$  contained in  $\Gamma(X_k - B, E_1^k) = G_k$

$$F_k = \{s | \bar{s} \in F_{k-1} \text{ and } D_{k-1}(s) \in F_1\}$$

where  $F_1 = E$ , and where  $\bar{s}$  is the reduction of  $s$  in  $G_{k-1}$ .

We claim the  $F_k$  are free  $A/(x_n^k)$  modules and that  $F_k$  maps surjectively to  $F_{k-1}$ . Let  $\bar{s}$  be a section of  $F_{k-1}$ . Then  $\bar{s}$  gives a section of  $E_1^{k-1}$ , and so there is a section  $s'$  of  $E_1^k$  which reduces to  $\bar{s}$ . Now consider

$$s = s' - x_n^{k-1} D_k(s').$$

Notice that  $x_n^{k-1} D_k(s')$  is a well-defined section of  $E_1^k$ , and that  $D_k(s) = 0$ . Therefore  $s \in F_k$ , and  $s$  reduces to  $\bar{s}$ .

Thus by induction we may find  $s_1, \dots, s_r$  in  $F_k$  so that  $\bar{s}_1, \dots, \bar{s}_r$  in  $F_{k-1}$  generate  $F_{k-1}$  freely. We claim they generate  $F_k$  freely. Indeed, there can be no linear dependence, since  $s_1, \dots, s_r$  give a basis of  $E_1^k$  by restriction to  $X_k - B'$ . We must show any  $s$  in  $F_k$  is a linear combination of  $s_1, \dots, s_r$ . At any rate,  $\bar{s} = \sum \bar{g}_i \bar{s}_i$ , and lifting  $\bar{g}_i$  to  $g_i$ , a section of  $\mathcal{O}_{X_k}$ , and considering  $\bar{s} - \sum g_i s_i$ , we may suppose  $\bar{s} = 0$ . Over  $X_k - B'$ , we have

$$s = \sum x_n^{k-1} h_i s_i$$

where the  $h_i$  are sections of  $\mathcal{O}_{X_1} - B$ . But

$$D_{k-1}(s) = \sum h_i \bar{s}_i$$

is a section of  $E$ , where  $\bar{s}_i$  are the images of the  $s_i$  in  $E$ . Thus since the  $\bar{s}_i$  are a basis for  $E$ , the  $h_i$  may be extended to sections of  $\mathcal{O}_{X_1}$ , and so  $s$  is a linear combination of the  $s_i$ 's.

Let  $\hat{E} = \lim F_k$ .  $\hat{E}$  is free over  $\hat{A}$ .  $D_i$  gives a well-defined endomorphism of  $\hat{E}$ , and we define this endomorphism to be  $\nabla \left( \frac{(\partial^l / \partial x_n^l)}{l!} \right)$ . Further, the operators  $\binom{x_j(\partial / \partial x_j)}{l}$  operate on  $F_k$  by a similar formula to  $*$  for  $j < n$ , and so we may define  $\nabla \left( \frac{x_j(\partial / \partial x_j)}{l} \right)$  to be limit of this action.

Next let  $\hat{E}'$ ,  $\nabla'$ , and  $\varphi'$  be another such extension, and assume we have constructed  $\varphi_k$ , an isomorphism of  $\hat{E}/x_n^k \hat{E}$  with  $\hat{E}'/x_n^k \hat{E}'$  compatible with  $\nabla$  and  $\nabla'$ . Let  $\bar{s}$  be a section of  $\hat{E}'/x_n^k \hat{E}'$ . We let  $s$  be a section of  $\hat{E}'/x_n^{k+1} \hat{E}'$

reducing to  $\bar{s}$ , and define

$$\psi(\bar{s}) = s - \nabla' \left( \begin{matrix} x_n(\partial/\partial x_n) \\ k \end{matrix} \right) s.$$

$\psi$  is well defined, and we define

$$\varphi_{k+1}(s) = \psi(\varphi_k(\bar{s})) - \nabla \left( \begin{matrix} x_n(\partial/\partial x_n) \\ k \end{matrix} \right) (s),$$

where we have identified  $x_n^k \hat{E}/x_n^{k+1} \hat{E}$  with  $x_n^k \hat{E}'/x_n^{k+1} \hat{E}'$ . Let  $\hat{\phi}$  be the limit of the  $\varphi_k$ . That  $\hat{\phi}$  makes  $\nabla$  and  $\nabla'$  correspond may be seen by closing any closed point of  $U'$ , and completing with respect to this point any using Theorem 3.3. The equivalence of categories follows similarly.

**THEOREM 5.2.** Let  $Y$  be a smooth ample divisor in  $X$ , which we assume smooth, and quasi-projective and of dimension  $\geq 3$ , and suppose  $Y$  is complete. Let the  $E_i$  be smooth divisors with normal crossings which intersect  $Y$  transversally and so that the divisors  $E_i \cap Y$  have normal crossing on  $Y$ . Let  $U = X - \sum E_i$  and  $U' = Y - \sum E_i \cap Y$ . Then restriction gives an equivalence of categories between stratified bundles on  $U$  with regular singular points at infinity and stratified bundles on  $U$  with regular singular points at infinity and stratified bundles on  $U'$  with regular singular points at infinity.

**PROOF.** Let  $F$  be a bundle on  $Y$ , stratified over  $U$  with regular singular points on  $\sum E_i \cap Y$ , and let  $U_i$  be affine open sub-sets of  $X$  so that there are functions  $x_1, \dots, x_n$  on  $U_i$ , so that  $Y \cap U_i$  is defined by  $x_n = 0$  and the  $E_i \cap Y$  are contained in the divisor defined by  $x_1 \cdot x_2 \dots x_{n-1} = 0$ , and that  $F$  is free over  $U_i$ , and the  $U_i$  cover  $Y$ . If  $U_i = \text{Spec } A_i$ , and  $\hat{A}_i$  is the completion of  $A_i$  with respect to the  $(x_n)$ -adic topology, we get a free  $\hat{A}_i$  module  $\hat{F}_i$  as in Lemma 5.1, with  $\nabla_i$ , and  $\varphi_i$ . The unicity statement of Lemma 5.1 shows that there is a canonical isomorphism  $\varphi_{ij}$  of  $\hat{F}_i$  with  $\hat{F}_j$ , on  $U_i \cap U_j \cap Y$ , and the  $\{\varphi_{ij}\}$  satisfy the cocycle condition. Thus there is a bundle  $\hat{F}$  on formal completion  $Z$  of  $X$  along  $Y$ , and  $\mathcal{D}^E \otimes \mathcal{O}_Z$  acts on  $\hat{F}$  by  $\nabla$  where  $E = \sum E_i$ . By the effective Lefschetz condition  $\text{Leff}(X, Y)$  [4, X.2], there is a bundle  $F'$  on some neighborhood of  $Y$  so that  $F' \otimes \mathcal{O}_Z = \hat{F}$ . We wish to make  $\mathcal{D}^E$  act on  $F'$  over  $U$ . Let  $n$  be a positive integer and  $\mathcal{D}_n^E$  the sheaf of operators in  $\mathcal{D}^E$  of order  $\leq n$ . Let  $H$  be a very ample divisor on  $X$  so that  $F' \otimes \mathcal{O}(H)$  and  $\mathcal{D}_n^E \otimes \mathcal{O}(H)$  are generated by their global sections. If  $s$  is a section of  $F' \otimes \mathcal{O}(H)$ , and  $D$  is a section of  $\mathcal{D}_n^E \otimes \mathcal{O}(H)$ , both define sections of  $\hat{F} \otimes \mathcal{O}(H)$  and  $\mathcal{D}_n^E \otimes \mathcal{O}(H) \otimes \mathcal{O}_Z$ . Thus we may form  $\nabla(D)(s)$  as a section of  $\hat{F} \otimes \mathcal{O}((n+1)H)$ . By  $\text{Lef}(X, Y)$ ,  $\nabla(D)(s)$  extends to a section of  $F'$ . Thus  $\mathcal{D}_E$  acts on  $F'$  over  $U$ , and so  $F'$  is stratified over  $U - E$  with regular singular points on  $E$ . If  $i$  is the inclusion of  $U$  into  $X$ ,

then  $i_*F'$  is a coherent module, locally free and stratified over  $X - E$ . So  $F'$  extends to a stratified bundle on  $U$  with regular singular points.

Note that the same proof works if we assume  $\text{Leff}(X, Y)$  instead of assuming  $Y$  is an ample divisor.

**THEOREM 5.3.** Let the  $E_i$  be smooth divisors with normal crossings in  $\mathbb{P}^n$ . Then every stratified bundle on  $U = \mathbb{P}^n - (\sum E_i)$  with regular singular points at infinity is a direct sum of stratified line bundles on  $U$ .

**PROOF.** First, suppose  $E_i$  is defined by  $x_i = 0$  for  $i = 0, \dots, n - 1$ . Then  $U = \mathbb{G}_m \times_k \mathbb{G}_m \times \dots \times_k \mathbb{G}_m$ , and so the theorem follows. Next, if we pick a generic  $\mathbb{P}^2$  in  $\mathbb{P}^n$ , Theorem 5.2 inductively gives us the theorem for  $U = \mathbb{P}^2 - (\sum L_i)$ , where the  $L_i$  are  $n$  lines in general position. Suppose the  $E_i$  are smooth divisors on  $\mathbb{P}^2$  having normal crossings, and suppose  $E_i$  is defined by the homogeneous equations  $\varphi_i(X_0, X_1, X_2) = 0$  of degree  $d_i$ . For each  $i$ , pick lines  $L_{i,1}, \dots, L_{i,d_i}$  in the collection of lines  $\{L_i\}$  above, and let  $\psi_i$  be the homogeneous equation for the divisor  $L_{i,1} + L_{i,2} + \dots + L_{i,d_i}$ . In  $\mathbb{P}^2$ , let  $E'_i$  be the divisor defined by

$$f_i(X_0, \dots, X_5) = \varphi_i(X_0, X_1, X_2) + \psi_i(X_3, X_4, X_5)$$

Let  $W$  be the set where  $E'_i$  do not have normal crossings,  $M_1$  (resp.  $M_2$ ) be the sub-variety defined by  $X_0 = X_1 = X_2 = 0$  (resp.  $X_3 = X_4 = X_5 = 0$ ), the theorem is valid for  $M_2 - (M_2 \cap \sum E'_i)$ , and therefore for  $\mathbb{P}^5 - W$  and the divisor  $\sum E'_i$ . It therefore holds for  $M_1 - \sum (E_i \cap M_1)$ , and so in general when  $n = 2$ . The theorem for general  $n$  follows from Theorem 5.2 by induction.

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