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Homology theory for real analytic and semianalytic sets

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Homology Theory for Real Analytic and Semianalytic Sets. (*)

ROBERT M. HARDT (**)

CONTENTS

1. – Introduction.

Here the homology of real semianalytic sets is treated using real analytic chains. A subset A of an m dimensional (separable) real analytic, Riemannian manifold M is called (real) *analytic* [respectively, *semianalytic*] in M if M can be covered by open balls U for which there is a function f [respectively, a finite family $\mathcal{F}$ of functions] analytic in U such that $U \cap A = f^{-1}\{0\}$ [respectively, $U \cap A$ is a union of sets each of which is a connected component of $f^{-1}\{0\} \sim g^{-1}\{0\}$ for some $f, g \in \mathcal{F}$]. For any nonnegative integer j, a j dimensional (real) analytic chain in M ([3, 4.2.28], [6, § 4]) is a locally finite sum of integral multiples of chains given by integration over a regular, j dimensional, oriented semianalytic set. Suppose $M \supset A \supset B$. Using the group of *real analytic cycles*

$$\mathfrak{Z}_j(A, B) = \{T\colon\ T \text{ is a } j \text{ dimensional analytic chain, } \operatorname{spt} T \text{ is compact, } \operatorname{spt} T \subset A, \text{ and } \operatorname{spt} \partial T \subset B \text{ or } j = 0\},$$

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the subgroup of *real analytic boundaries*

$$\mathfrak{B}_j(A, B) = \{R + \partial S : R \in \mathfrak{Z}_j(B, B) \text{ and } S \in \mathfrak{Z}_j(A, A)\},$$

and the *real analytic homology groups*

$$\boldsymbol{H}_j(A, B) = \mathfrak{Z}_j(A, B)/\mathfrak{B}_j(A, B), \quad \boldsymbol{H}_j(A) = \boldsymbol{H}_j(A, \emptyset),$$

we prove in § 4 our main results.

THEOREM. *If $A \supset B$ are semianalytic sets, then there exists an arbitrarily small open neighborhood W of B such that $\boldsymbol{H}_j(A \cap W, B) \simeq 0$ for all j.*

COROLLARY. *There exist arbitrarily small open neighborhoods U of A in M and V of B in U such that the inclusion map of $\mathfrak{Z}_j(A, B)$ into $\mathfrak{Z}_j(U, V)$ induces an isomorphism, $\boldsymbol{H}_j(A, B) \simeq \boldsymbol{H}_j(U, V)$, for all j.*

The corollary has two consequences. First in § 5 we define, by approximation, the homomorphism

$$\boldsymbol{H}_j(f) : \boldsymbol{H}_j(C, D) \to \boldsymbol{H}_j(A, B)$$

for any continuous map $f: (C, D) \to (A, B)$ where $C \supset D$ are semianalytic subsets of an analytic manifold; the axioms of Eilenberg-Steenrod follow as in [3, 4.4.1]. Second, in § 6, a homology intersection product

$$\bigcap : \boldsymbol{H}_i(A, B) \times \boldsymbol{H}_j(A, B) \to \boldsymbol{H}_{i+j-m}(A, B),$$

where i is any nonnegative integer with $i + j \geqslant m$, results by use of the intersection theory for real analytic chains of [6, § 5].

In [1] A. Borel and A. Haefliger, employing the Borel-Moore homology for locally-compact spaces, proved the orientability modulo 2 of real analytic sets and established a formula equating the modulo 2 cycle of the real part of the intersection of two holomorphic varieties with the intersection of the modulo 2 cycles of the real parts of the varieties. These facts are reproven in § 6 and § 8, using analytic chains and Federer's theory of slicing ([3,4.3], [6, § 4]). We observe in Example 7.2 that analytic sets are not necessarily locally orientable over $\boldsymbol{Z}$ even though those of dimension or codimension one are (7.1). We also note in 5.7 that the homology of a relatively compact pair of semianalytic sets is finitely generated.

The proofs of our main results in § 4 involve, for bounded semianalytic subsets of $\boldsymbol{R}^n$, a certain stratification (2.8) and system of neighborhoods (2.9) built up from finitely many local stratifications; the required local stratification (2.6) is established by Lojasiewicz in [11, § 11-§ 15] or [13, § 13] using the Weierstrass Preparation Theorem and classical elimination theory. The main complication in § 4 is that the projection of a bounded semianalytic set may fail to be semianalytic ([13, p. 133]). Readers interested in other aspects of semianalytic sets and their projections are referred to [4], [6, § 2], [8], [11], [12], [13], [14] and [18].

Replacing, for any integer $\nu \geqslant 2$, « analytic chain and spt » by « analytic chain modulo ν and spt^ν » [7], we obtain the real analytic homology group $\boldsymbol{H}_j(A, B; \boldsymbol{Z}_\nu)$ with coefficients in $\boldsymbol{Z}_\nu = \boldsymbol{Z}/\nu\boldsymbol{Z}$. All of the proofs and results of § 2 through § 6 carry over to the modulo ν case. We also note that, by replacing everywhere « (real) analytic set, semianalytic set, and analytic mapping » by « (real) algebraic set, semialgebraic set, and algebraic (polynomial) mapping » we may define real algebraic chains and transfer the methods and results of this paper to the real algebraic case. In fact here the situation is simpler because, by [16, Theorem 1], the projection of a bounded semialgebraic set in $\boldsymbol{R}^n$ is semialgebraic. Thus section 4.4 would be unnecessary.

Real analytic chains are suitable for studying the homology of real analytic objects because of their geometric content, their applicability to arbitrary semianalytic sets, and their economy as the smallest group of singular chains containing the orienting cycles of orientable semianalytic sets. However, the fact that they are singular chains, i.e., that semianalytic sets are triangulable ([5], [12]), will not be used here.

Our notation, except for the symbols, $\mathfrak{Z}_j(A, B)$, $\mathfrak{B}_j(A, B)$, $\boldsymbol{H}_j(A, B)$, $\boldsymbol{H}_j(A)$, $\boldsymbol{H}_j(A, B; \boldsymbol{Z}_\nu)$, defined above, is consistent with [3] and [6] (See the glossaries on [6, pp. 669-671]). In addition we define, for any subset G of a topological space, the frontier of G, denoted $\mathrm{Fr}\, G$, as $(\mathrm{Clos}\, G) \sim G$. The author wishes to thank Herbert Federer for suggesting many of the problems treated here, showing him Example 7.2, and offering needed encouragement and criticism.

2. – Semianalytic Sets.

Observing that the product, the sum of squares, or the cartesian product of two analytic functions is analytic, we readily verify that the union, intersection, difference, or cartesian product of two semianalytic sets is semianalytic. Moreover, a connected component of or the inverse image under an analytic map of a semianalytic set is semianalytic. However, the direct image under an analytic map of even a compact analytic set may fail to be semianalytic ([13, p. 133]).

2.1. *Real analytic dimension.* The real analytic dimension of a subset of M, which is defined in [6, 2.2], may be described as follows:

If A is a semianalytic subset of M, then

$\dim A = \sup\{-1, k: A$ contains a k dimensional analytic submanifold of $M\}$ (hence, $\dim \emptyset = -1$).

If E is an arbitrary subset of M, then

$$\dim E = \inf\{\dim A: A \supset E \text{ and } A \text{ is semianalytic}\}\,.$$

2.2. *Semianalytic subsets of* $\boldsymbol{R}^n$. Let n be a fixed positive integer. We will use the following *notations*. With $\boldsymbol{R}^0 = \{0\}$, let $p_k: \boldsymbol{R}^n \to \boldsymbol{R}^k$ for $k \in \{0, \dots, n\}$ and $q_l: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^l$ for $l \in \{0, \dots, n-1\}$ be given by $p_0(x_1, \dots, x_n) = 0$, $q_0(x_1, \dots, x_n) = 0$, $p_k(x_1, \dots, x_n) = (x_1, \dots, x_k)$, and $q_l(x_1, \dots, x_{n-1}) = (x_1, \dots, x_l)$ for $k \in \{1, \dots, n\}$, $l \in \{1, \dots, n-1\}$, and $(x_1, \dots, x_n) \in \boldsymbol{R}^n$. We also abbreviate $p = p_{n-1}$, and let $q: \boldsymbol{R}^n \to \boldsymbol{R}$ be the complementary projection, $q(x_1, \dots, x_n) = x_n$ for $(x_1, \dots, x_n) \in \boldsymbol{R}^n$.

An affine line L in $\boldsymbol{R}^n$ is *nonsingular for a semianalytic subset* A of $\boldsymbol{R}^n$ if A may be described locally using U, $\mathcal{F}$ as in §1 so that $f|(U \cap L) \not\equiv 0$ whenever $f \in \mathcal{F}$ and $f \not\equiv 0$. If L is nonsingular for two semianalytic sets, then it is clearly nonsingular for their union, intersection, or difference. Moreover, *if* $A_0 \subset A$ *are semianalytic and* L *is nonsingular for* A, *then* L *is nonsingular for* A_0. In fact, if $C \subset A$ is a component of $f^{-1}\{0\} \sim g^{-1}\{0\}$ for some $f, g \in \mathcal{F}$, f_0 and g_0 are analytic in an open subset U_0 of $\boldsymbol{R}^n$, and $C_0 \subset A_0$ is a component of $f_0^{-1}\{0\} \sim g_0^{-1}\{0\}$, then $C_0 \cap C$ is a union of components of

$$U_0 \cap U \cap [(f_0^2 + f^2)^{-1}\{0\} \sim (g_0^2 + f^2)^{-1}\{0\}]\,,$$

$$(f_0^2 + f^2)|(U_0 \cap U \cap L) \not\equiv 0 \quad \text{and} \quad (g_0^2 + f^2)|(U_0 \cap U \cap L) \not\equiv 0\,.$$

LEMMA 2.3. *If* f *is analytic in a connected open subset* U *of* $\boldsymbol{R}^n$ *and* $f \not\equiv 0$, *then, for* $\mathcal{H}^{n-1}$ *almost all* $\xi \in \boldsymbol{S}^{n-1}$,

$$f|(U \cap \{x + t\xi: t \in \boldsymbol{R}\}) \not\equiv 0 \quad \textit{for all } x \in \boldsymbol{R}^n\,.$$

PROOF. The proof of [12, Lemma 3] (or even of [10, Theorem 51]) shows that

$$Z = \boldsymbol{S}^{n-1} \cap \{\xi: f|(U \cap \{x + t\xi: t \in \boldsymbol{R}\}) \equiv 0 \text{ for some } x \in \boldsymbol{R}^n\}$$

is contained in the countable union of sets $\varphi(\Lambda^*)$ where Λ^* is some connected analytic manifold, φ is an analytic map, and $\dim D\varphi(y)[\mathrm{Tan}(\Lambda^*, y)] \leq n-2$ for $y \in \Lambda^*$. By partitioning, as in [6, 2.9], Λ^* into submanifolds on which φ has constant rank and using [3, 3.1.18], we may obtain a countable cover $\mathcal{C}$ of Λ^*, consisting of submanifolds C, of various dimensions, such that $\mathcal{H}^{n-2}[\varphi(C)]$ is finite; hence

$$\mathcal{H}^{n-1}[\varphi(\Lambda^*)] = \mathcal{H}^{n-1}[\varphi(\cup\, \mathcal{C})] = 0 .$$

Thus $\mathcal{H}^{n-1}(Z) = 0$.

COROLLARY 2.4. *If $\mathcal{B}$ is a countable family of semianalytic sets in $\boldsymbol{R}^n$, then, for $\mathcal{H}^{n-1}$ almost all $\xi \in S^{n-1}$, $p^{-1}\{y\}$ is nonsingular for $b(B)$ whenever $y \in \boldsymbol{R}^{n-1}$, $b \in \boldsymbol{O}(n)$, $B \in \mathcal{B}$, and $b(\xi) = (0, \ldots, 0, 1)$.*

2.5. For any open subset Y of $\boldsymbol{R}^{n-1}$ a function H on $Y \times \boldsymbol{R}$ is called a *monic pseudo-polynomial* if there exists a positive integer d and real functions $c_1, \ldots, c_d$ analytic in a neighborhood of Clos Y such that

$$H(y, z) = z^d + c_1(y)^{d-1} + \ldots + c_d(y) \qquad \text{for } (y, z) \in Y \times \boldsymbol{R} .$$

If for every $y \in Y$, $D(y)$ is the discriminant of $H(y, \cdot)$ and $\varrho_1(y) \leq \varrho_2(y) \leq \ldots \leq \varrho_d(y)$ is a complete list, counting multiplicities of the real parts of the complex roots of $H(y, \cdot)$, then D is analytic on Y $\big([19, 5.7]\big)$ and $\varrho_1, \ldots, \varrho_d$ are continuous on Y $\big([3, \text{p. } 450]\big)$ and analytic on $Y \sim D^{-1}\{0\}$.

We will say that a family $\mathfrak{I}$ of sets is *compatible with a set A* if for every $\Gamma \in \mathfrak{I}$, either $\Gamma \cap A = \emptyset$ or $\Gamma \subset A$. In addition we will call a semianalytic set that is a connected analytic submanifold a *semianalytic stratum.*

THEOREM 2.6. (*Local stratification*) *If $\mathcal{B}$ is a finite family of semianalytic subsets of $\boldsymbol{R}^n$ and $p^{-1}\{0\}$ is nonsingular for every member of $\mathcal{B}$, then there exists an $h \in \boldsymbol{O}(n)$ with $h(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$, $Q_0 = \boldsymbol{R}^0$, $H_0 = 1$, and, for $i \in \{1, \ldots, n\}$, positive δ_i,*

$$Q_i = \boldsymbol{R}^i \cap \{(x_1, \ldots, x_i) : |x_1| < \delta_1, \ldots, |x_i| < \delta_i\} ,$$

and monic pseudo-polynomials H_i on $Q_{i-1} \times \boldsymbol{R}$ with discriminants D_i on Q_{i-1} such that:

(1) $H_n(y, 0) = 0$ *for* $y \in Q_{n-1}$,

(2) $D_i^{-1}\{0\} \subset Q_{i-1} \cap \{(x_1, \ldots, x_{i-1}) : H_{i-1}[(x_1, \ldots, x_{i-2}), x_{i-1}] = 0\}$,

(3) *the partition* $Q_n = \cup \mathfrak{J}$, *where* $\mathfrak{J}$ *is the family of connected components of*

$$Q_n \cap \{(x_1, \ldots, x_n): H_n[(x_1, \ldots, x_{n-1}), x_n] = \ldots = H_{i+1}[(x_1, \ldots, x_i), x_{i+1}] = 0\,,$$
$$H_i[(x_1, \ldots, x_{i-1}), x_i] \neq 0\}$$

for $i \in \{0, \ldots, n\}$, *is finite and compatible with* $h(B)$ *for every* $B \in \mathfrak{B}$,

(4) *each* $\Delta \in \mathfrak{J}$ *is a semianalytic stratum,* $0 \in \operatorname{Clos} \Delta$, $p_{\dim \Delta}|\Delta$ *is an analytic isomorphism, and*

$$Q_n \cap \operatorname{Fr} \Delta = \cup \mathfrak{J} \cap \{\Gamma: \dim \Gamma < \dim \Delta \text{ and } \Gamma \cap \operatorname{Fr} \Delta \neq \emptyset\}\,.$$

PROOF. Either [11, §11-§15] or [13, §13] using the family $\mathfrak{B} \cup \{q^{-1}\{0\}\}$.

2.7. From 2.6 we infer that *if* $\varrho: Q_{n-1} \to \boldsymbol{R}$ *is continuous and* $H_n[y, \varrho(y)] = 0$ for $y \in Q_{n-1}$, *then set*

$$E = p^{-1}(Q_{n-1}) \cap \{x: q(x) = \varrho[p(x)]\}$$

is semianalytic because it equals

$$\cup \mathfrak{J} \cap \{\Gamma: \Gamma \subset \operatorname{Clos} \Delta \text{ for some } \Delta \in \mathfrak{J} \text{ with } \Delta \subset E \text{ and } \dim \Delta = n-1\}\,.$$

We also deduce from 2.4 and 2.6 that each of the semianalytic sets $B \in \mathfrak{B}$ have the following properties.

(1) B *is locally finite.*

(2) $\operatorname{Clos} B$, $\operatorname{Fr} B$, *and* $\operatorname{Bdry} B$ *are semianalytic sets with* $\dim \operatorname{Clos} B = \dim B$, $\dim \operatorname{Fr} B < \dim B$, *and* $\dim \operatorname{Bdry} B < n$.

(3) *If* $\delta \leqslant \inf\{\delta_1, \ldots, \delta_{n-1}\}$, *then* $p[B \cap p^{-1}\boldsymbol{U}(0, \delta) \cap q^{-1}\boldsymbol{U}(0, \delta_n)]$ *is an at most* $\dim B$ *dimensional semianalytic set in* $\boldsymbol{R}^{n-1}$.

THEOREM 2.8. (*Global stratification*) *If* $\mathfrak{A}$ *is a finite family of bounded semianalytic subsets of* $\boldsymbol{R}^n$ *and* $p^{-1}\{y\}$ *is nonsingular for* A *whenever* $y \in \boldsymbol{R}^{n-1}$ *and* $A \in \mathfrak{A}$, *then there exist* $g \in \boldsymbol{O}(n)$ *with* $g(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$ *and finite partitions* $\mathfrak{S}$, $\overline{\mathfrak{S}}$ *of* $\boldsymbol{R}^n$ *into semianalytic strata with the following six properties*:

(1) $\mathfrak{T} = \mathfrak{S} \cap \{\Gamma: \dim \Gamma < n\} \subset \overline{\mathfrak{T}} = \overline{\mathfrak{S}} \cap \{\Gamma: \dim \Gamma < n\}$.

(2) *For each* $\Gamma \in \overline{\mathfrak{T}}$, $p_{\dim \Gamma}|\Gamma$ *is an analytic isomorphism, and there are an open neighborhood* Y_Γ *of* $\operatorname{Clos} p(\Gamma)$, *a monic pseudopolynomial* H_Γ *on*

$Y_\Gamma \times \boldsymbol{R}$ having zero as one root, and a continuous function $\varrho\colon Y_\Gamma \to \boldsymbol{R}$ such that $H_\Gamma[y, \varrho(y)] = 0$ for $y \in Y_\Gamma$ and

$$\Gamma \subset E_\Gamma = p^{-1}(Y_\Gamma) \cap \{x\colon q(x) = \varrho[p(x)]\}$$

(hence $p^{-1}\{y\}$ is nonsingular for Γ whenever $y \in \boldsymbol{R}^{n-1}$).

(3) *For each $\Gamma \in \mathcal{T}$, there is an open semianalytic neighborhood Z_Γ of* $\operatorname{Clos} p(\Gamma)$ *in Y: such that*

$$p^{-1}(Z_\Gamma) \cap \bigcup \overline{\mathcal{T}} = p^{-1}(Z_\Gamma) \cap \{x\colon H_\Gamma[p(x), q(x)] = 0\}\,.$$

(4) *For each $\Gamma \in \overline{\mathcal{T}}$ and $\Delta \in \mathcal{S} \cup \overline{\mathcal{S}}$, $\Gamma \subset \operatorname{Clos} \Delta$ whenever $\Gamma \cap \operatorname{Clos} \Delta \neq \emptyset$ and $p(\Gamma) \subset p(\Delta)$ whenever $p(\Gamma) \cap p(\Delta) \neq \emptyset$.*

(5) *If $\mathcal{S}^\#$ is the partition of $\boldsymbol{R}^{n-1}$ consisting of $\{p(\Gamma)\colon \Gamma \in \overline{\mathcal{T}}\}$ along with the family of connected components of $\boldsymbol{R}^{n-1} \sim p(\bigcup \overline{\mathcal{T}})$, then statements* (1) *through* (4) *hold with n, $\mathcal{S}$, $\overline{\mathcal{S}}$ replaced by $n-1$, $\overline{\mathcal{S}}^\#$, and $\mathcal{S}^\#$ for some partition $\overline{\mathcal{S}}^\#$ of $\boldsymbol{R}^{n-1}$.*

(6) *$\mathcal{S}$, and hence $\overline{\mathcal{S}}$, is compatible with $g(A)$ for every $A \in \mathcal{A}$.*

PROOF. We use induction on n. Since the case $n = 1$ readily follows from 2.6, we assume $n \geqslant 2$. For each positive integer k and $u \in \boldsymbol{R}^k$, let $\boldsymbol{\tau}_u\colon \boldsymbol{R}^k \to \boldsymbol{R}^k$, $\boldsymbol{\tau}_u(v) = u + v$ for $v \in \boldsymbol{R}^k$.

For each fixed $a \in \boldsymbol{R}^n$ we use the family $\mathcal{B}_a = \{\boldsymbol{\tau}_{-a}(A)\colon A \in \mathcal{A}\}$ to select h, δ_i, Q_i, H_i, and $\mathfrak{J}$ as in 2.6 and $h^\# \in \boldsymbol{O}(n-1)$ so that $p \circ h = h^\# \circ p$. Fixing positive numbers $\delta < \bar{\delta} < \inf\{\delta_1, \ldots, \delta_{n-1}\}$, we define

$$Q_a = \boldsymbol{\tau}_a[h^{-1}(Q_n)]\,, \qquad Y_a = p(Q_a)\,, \qquad Z_a = \boldsymbol{U}[p(a), \bar{\delta}]\,,$$

$$H_a(y, z) = z \cdot H_n\big(h^\#[y - p(a)], z - q(a)\big) \quad \text{for } (y, z) \in Y_a \times \boldsymbol{R}\,,$$

$$\mathfrak{J}_a = \{(\boldsymbol{\tau}_a \circ h^{-1})[\Gamma \cap p^{-1}\boldsymbol{U}(0, \delta)]\colon \Gamma \in \mathfrak{J}\} \cup \{(\boldsymbol{\tau}_a \circ h^{-1})[\Gamma \cap p^{-1}\operatorname{Fr}\boldsymbol{U}(0, \delta)]\colon \Gamma \in \mathfrak{J}\}\,,$$

$$\bar{\mathfrak{J}}_a = \{(\boldsymbol{\tau}_a \circ h^{-1})[\Gamma \cap p^{-1}\boldsymbol{U}(0, \bar{\delta})]\colon \Gamma \in \mathfrak{J}\} \cup \{(\boldsymbol{\tau}_a \circ h^{-1})[\Gamma \cap p^{-1}\operatorname{Fr}\boldsymbol{U}(0, \bar{\delta})]\colon \Gamma \in \mathfrak{J}\}\,.$$

There is a finite subset F of $\boldsymbol{R}^n$ with $\operatorname{Clos} \bigcup \mathcal{A} \subset \bigcup_{a \in F} Q_a \cap p^{-1}\boldsymbol{U}[p(a), \delta]$. Then, by 2.2 and 2.7(3),

$$\mathcal{B} = \big\{p(G_{\lambda(1)} \cap \ldots \cap G_{\lambda(m)})\colon G_a \in \mathfrak{J}_a \cup \bar{\mathfrak{J}}_a,\ m \in \{1, \ldots, \operatorname{card} F\}\,, \quad \text{and } \lambda\colon \{1, \ldots, m\} \to F\big\}$$

is a finite family of bounded semianalytic subsets of $\boldsymbol{R}^{n-1}$. Choosing, by 2.4, $b \in \boldsymbol{O}(n-1)$ so that $q_{n-2}^{-1}\{w\}$ is nonsingular for $b(B)$ whenever $w \in \boldsymbol{R}^{n-2}$ and $B \in \mathfrak{B}$, we find, by induction, an orthogonal transformation $f \in \boldsymbol{O}(n-1)$ and partitions $\mathcal{S}^{\#}$, $\overline{\mathcal{S}}^{\#}$ of $\boldsymbol{R}^{n-1}$ which satisfy the theorem with n, $\mathfrak{A}$, g, $\mathcal{S}$, $\overline{\mathcal{S}}$ replaced by $n-1$, $\{b(B): B \in \mathfrak{B}\}$, f, $\mathcal{S}^{\#}$, $\overline{\mathcal{S}}^{\#}$. Letting $g^{\#} = f \circ b$, $g \in \boldsymbol{O}(n)$ satisfy $p \circ g = g^{\#} \circ p$ and $g(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$, and $\mathcal{S}$ [respectively, $\overline{\mathcal{S}}$] be the partition of $\boldsymbol{R}^n$ consisting of

$$\mathcal{T}[\text{respectively, } \overline{\mathcal{T}}] = \{g(G) \cap p^{-1}(\Gamma^{\#}) : G \in \bigcup_{a \in F} \mathfrak{J}_a[\text{respectively, } \bar{\mathfrak{J}}_a],\ \dim G < n, \quad \text{and } \Gamma^{\#} \in \mathcal{S}^{\#}\}$$

along with the family of connected components of $\boldsymbol{R}^n \sim \bigcup \mathcal{T}$ [resp., $\overline{\mathcal{T}}$], and use 2.6 to verify (1), (4), (5), and (6).

From 2.6 we also infer that, for each $\Gamma \in \overline{\mathcal{T}}$, $p|\Gamma$ is an analytic isomorphism and obtain (2) by letting

$$F_\Gamma = F \cap \{a : p(\Gamma) \subset p[g(\cup \bar{\mathfrak{J}}_a)]\},$$

$$Y_\Gamma = g^{\#}\Big(\bigcap_{a \in F_\Gamma} Y_a\Big), \qquad H_\Gamma = \prod_{a \in F_\Gamma} (H_a \circ [(g^{\#})^{-1} \boxdot \mathbf{1}_{\boldsymbol{R}}])|(Y_\Gamma \times \boldsymbol{R}).$$

Finally for $\Gamma \in \mathcal{T}$, (3) follows with $Z_\Gamma = g^{\#}\Big(\bigcap_{a \in F_\Gamma} Z_a\Big)$.

THEOREM 2.9 (*System of Neighborhoods*). *If* $\mathcal{S}$, $\overline{\mathcal{S}}$, $\mathcal{T}$, $\overline{\mathcal{T}}$, Z_Γ, $\mathcal{S}^{\#}$, $\overline{\mathcal{S}}^{\#}$ *are as in* 2.8, *and, for each* $\Gamma \in \mathcal{S} \cup \overline{\mathcal{S}}$, V_Γ *is a neighborhood of* Γ, *then there exists a family* $\{U_\Gamma : \Gamma \in \mathcal{S} \cup \overline{\mathcal{S}}\}$ *of open sets with the following four properties*:

(1) *For each* $\Gamma \in \mathcal{S} \cup \overline{\mathcal{S}}$, $\Gamma \subset U_\Gamma \subset V_\Gamma$.

(2) *For each* $\Gamma \in \mathcal{T}$, $p(U_\Gamma) \subset Z_\Gamma$.

(3) *For each* Γ *and* Δ *both belonging to either* $\mathcal{S}$ *or* $\overline{\mathcal{S}}$, $U_\Gamma \cap U_\Delta = \emptyset$ *whenever* $\Gamma \cap \operatorname{Clos} \Delta = \emptyset = \Delta \cap \operatorname{Clos} \Gamma$, *and* $U_\Gamma \cap U_\Delta = p_{\dim \Delta}^{-1}[p_{\dim \Delta}(U_\Gamma)] \cap U_\Delta$ *whenever* $\Gamma \cap \operatorname{Clos} \Delta \neq \emptyset$.

(4) *There exists a family* $\{U^{\#}_{\Gamma^{\#}} : \Gamma^{\#} \in \mathcal{S}^{\#} \cup \overline{\mathcal{S}}^{\#}\}$ *satisfying* (1) *through* (3) *with* n, $\mathcal{S}$, $\overline{\mathcal{S}}$ *replaced by* $n-1$, $\mathcal{S}$, $\overline{\mathcal{S}}^{\#}$ *such that* $U^{\#}_{p(\Gamma)} = p(U_\Gamma)$ *whenever* $\Gamma \in \overline{\mathcal{T}}$.

PROOF. Letting $U_\Delta = \Delta$ for every n dimensional member of $\mathcal{S} \cup \overline{\mathcal{S}}$, we will define U_Γ for $\Gamma \in \overline{\mathcal{T}}$ and establish 2.9 by induction on n. The case $n = 1$ is easily treated.

To handle the inductive step, choose, by 2.8(2), for each $k \in \{0, \ldots, n-1\}$ and each k dimensional $\Gamma \in \overline{\mathcal{T}}$ a continuous function α_Γ on $\boldsymbol{R}^k$ such that

$$\Gamma = p^{-1}[p(\Gamma)] \cap \{x : q(x) = \alpha_\Gamma[p_k(x)]\},$$

let $\delta_\Gamma(u)$, for $u \in p_k(\Gamma)$, be the infemum of the four numbers,

$$1, \operatorname{dist}[u, \operatorname{Fr} p_k(\Gamma)],\ \tfrac{1}{2}\operatorname{dist}[\Gamma \cap p_k^{-1}\{u\}, \operatorname{Fr} V_\Gamma]\,,$$
$$\tfrac{1}{2}\operatorname{dist}[\Gamma \cap p_k^{-1}\{u\},\ \operatorname{Clos} \cup \overline{\mathcal{S}} \cap \{C\colon \Gamma \cap \operatorname{Clos} C = \emptyset\}]\,,$$

(here $\operatorname{dist}(u, \emptyset) = +\infty$), and let ε_Γ be the extension to $\boldsymbol{R}^k$ of δ_Γ and $0|[\boldsymbol{R}^k \sim p_k(\Gamma)]$. Then ε_Γ is continuous, and

$$X_\Gamma = \boldsymbol{R}^{n-1} \cap \{y\colon \operatorname{dist}(p^{-1}\{y\} \cap q^{-1}\{\alpha_\Gamma[q_k(y)]\}, \operatorname{Fr} V_\Gamma) < \varepsilon_\Gamma[q_k(y)]\}$$

is an open neighborhood of $p(\Gamma)$. Moreover for each $l \in \{k+1, \ldots, n-1\}$ and l dimensional $\Delta \in \overline{\mathcal{C}}$ with $\Gamma \subset \operatorname{Fr} \Delta$,

$$X_\Gamma^\Delta = \boldsymbol{R}^{n-1} \cap \{y\colon \pm(\alpha_\Gamma \pm \varepsilon_\Gamma)[q_k(y)] \mp (\alpha_\Delta \pm \varepsilon_\Delta)[q_l(y)] > 0\}$$

is also an open neighborhood of $p(\Gamma)$ because $\varepsilon_\Gamma[q_k(y)] > 0$, $\varepsilon_\Delta[q_l(y)] = 0$, and $\alpha_\Gamma[q_k(y)] = \alpha_\Delta[q_l(y)]$ whenever $y \in p(\Gamma)$. With

$$X_\Gamma^\Delta = \boldsymbol{R}^{n-1} \quad \text{for } \Gamma \in \overline{\mathcal{C}} \text{ and } \Delta \in \overline{\mathcal{S}} \sim \overline{\mathcal{C}}, \qquad Z_\Gamma = \boldsymbol{R}^{n-1} \quad \text{for } \Gamma \in \overline{\mathcal{C}} \sim \mathcal{C},$$
$$V_{\Gamma^\#}^\# = \{Z_\Gamma \cap X_\Gamma \cap X_\Gamma^\Delta\colon \Gamma \in \overline{\mathcal{C}},\ p(\Gamma) = \Gamma^\#,\ \Delta \in \overline{\mathcal{S}},\ \Gamma \subset \operatorname{Fr} \Delta\} \quad \text{for } \Gamma^\# \in \mathcal{S}^\#,$$
$$V_{\Gamma^\#}^\# = \boldsymbol{R}^{n-1} \quad \text{for } \Gamma^\# \in \overline{\mathcal{S}}^\# \sim \mathcal{S}^\#,$$

we inductively choose a family $\{U_{\Gamma^\#}^\#\colon \Gamma^\# \in \mathcal{S}^\# \cup \overline{\mathcal{S}}^\#\}$ as in (4) such that $\Gamma^\# \subset U_{\Gamma^\#}^\# \subset V_{\Gamma^\#}^\#$ for $\Gamma^\# \in \mathcal{S}^\# \cup \overline{\mathcal{S}}^\#$, define

$$U_\Gamma = p^{-1}[U_{p(\Gamma)}^\#] \cap \{x\colon |q(x) - \alpha_\Gamma[p_k(x)]| < \varepsilon_\Gamma[p_k(x)]\}\,,$$

and verify the theorem by using 2.8(4)-(5).

3. – Real Analytic Chains.

H. Federer has proven in [3, 3.4.8 (13)] that, for any nonnegative integer j, the restriction of j dimensional Hausdorff measure, $\mathcal{H}^j$, to any j dimensional semianalytic set in M is locally finite.

By [3, 4.2.28] a current T is a j dimensional analytic chain in M if and only if it satisfies one of the two equivalent conditions:

(1) *There exist a locally finite disjointed family $\mathcal{B}$ of j dimensional orientable semianalytic strata, orienting j vectorfields ξ_B and integers m_B for $B \in \mathcal{B}$, such that $T = \sum_{B \in \mathcal{B}} m_B(\mathcal{H}^j \llcorner B) \wedge \xi_B$; that is,*

$$T(\varphi) = \sum_{B \in \mathcal{B}} m_B \int_B \langle \varphi(x), \xi_B(x) \rangle \, d\mathcal{H}_j x \qquad \textit{for } \varphi \in \mathcal{D}^j(M)\,.$$

(2) $T \in \mathcal{F}_j^{\mathrm{loc}}(M)$, $\dim(\operatorname{spt} T) \leqslant j$, *and* $\dim(\operatorname{spt} \partial T) \leqslant j-1$ ([3, 4.1.24]).

From (2) it follows for positive j that the current ∂T [where $(\partial T)(\psi) = T(d\psi)$ for $\psi \in \mathfrak{D}^{j-1}(M)$] is a $j-1$ dimensional analytic chain in M.

From (1), 2.4, 2.7 (2), and 2.1 we infer that if $T \neq 0$, then

$$\operatorname{spt} T = \cup \{\operatorname{Clos} B : B \in \mathfrak{B} \text{ and } m_B \neq 0\}$$

is a j dimensional semianalytic subset of M.

From (1) we also see that if A is semianalytic subset of M, then the current $T \llcorner A$ [3, p. 356] is also an analytic chain in M. In fact, for each $B \in \mathfrak{B}$, we may, by 2.4 and 2.6, choose a locally finite disjointed family $\mathfrak{C}_B$ of j dimensional semianalytic strata $C \subset A \cap B$ such that $\dim[(A \cap B) \sim \cup \mathfrak{C}_B] < j$; hence

$$T \llcorner A = \sum_{B \in \mathfrak{B}} m_B(\mathcal{H}^j \llcorner A \cap B) \wedge \xi_B = \sum_{B \in \mathfrak{B}} \sum_{C \in \mathfrak{C}} m_B(\mathcal{H}^j \llcorner C) \wedge \xi_B .$$

From either (1) or (2) we infer that if N is an analytic submanifold of M with $\operatorname{spt} T \subset N$, then the above equation defining $T(\varphi)$ gives us, for $\varphi \in \mathfrak{D}^j(N)$, an analytic chain $T|N$ in N, called the *restriction of* T *to* N.

LEMMA 3.1. *Suppose* $f: M \to N$ *is an analytic map of analytic manifolds and* $C \subset M$ *and* $D \subset N$ *are semianalytic. If* $\dim(C \cap f^{-1}\{y\}) \leqslant 0$ *for all* $y \in D$, *then* $\dim[C \cap f^{-1}(D)] \leqslant \dim D$.

PROOF. If x is a regular point of $E = C \cap f^{-1}(D)$ such that $\dim Df(x) \cdot [\operatorname{Tan}(E, x)]$ is maximal, then by [6, 2.2 (4)] and [3, 3.1.18, 3.4.11],

$$\dim E = \dim \operatorname{Tan}(E, x) = \dim \operatorname{Tan}(E \cap f^{-1}\{f(x)\}, x) + \dim Df(x)[\operatorname{Tan}(E, x)] \leqslant$$
$$\leqslant 0 + \dim \operatorname{Tan}(D, x) \leqslant \dim D .$$

COROLLARY 3.2. *If* M *and* N *are orientable,* f *maps* C *homeomorphically onto an open subset of* N, $\dim[\operatorname{im} Df(x)] = \dim N$ *for* $x \in M$, *and* j *is a non-negative integer, then there exists a unique homomorphism*

$$\Upsilon_j : \mathfrak{Z}_j[f(C), f(C)] \to \mathfrak{Z}_j(C, C)$$

such that $f_\# \circ \Upsilon_j = \mathbf{1}_{\mathfrak{Z}_j[f(C), f(C)]}$.

PROOF. For any semianalytic subset A of C we infer from the proof of [6, 2.9] and [3, 3.1.18] that $\dim[A \sim G(A)] < \dim A$ where

$$G(A) = A \cap \{x : x \text{ is a regular point of } A \text{ with } \dim Df(x)[\operatorname{Tan}(A, x)] = \dim A\} .$$

Inasmuch as $\operatorname{spt} f_\# T \supset f[G(\operatorname{spt} T)]$ for $T \in \mathfrak{Z}_j(C, C)$ by [3, 4.1.30], the homomorphism $f_\# | \mathfrak{Z}_j(C, C)$ is injective; thus Υ_j is unique.

To prove existence, let $k = \dim N$, ω and η be dual ([3, 1.7.5]) orienting k form and k vectorfield for N, and $\mathcal{N} = \mathcal{H}^k \wedge \eta$ the corresponding orienting cycle for N. The submanifold $G(C)$ is then oriented by the vectorfield Υ which is dual to the k form $[f|G(C)]^\# \omega / |[f|G(C)]^\# \omega|$. By [3, 4.1.28] and the estimates

$$\dim G(C) \leqslant k\,, \qquad \dim \operatorname{Fr} G(C) \leqslant \dim([C \sim \operatorname{Fr} G(C)] \cup \operatorname{Fr} C) \leqslant k-1\,,$$

$\mathfrak{I} = [\mathcal{H}^k \llcorner G(C)] \wedge \Upsilon$ is an analytic chain in M; moreover, $f_\# \mathfrak{I} = \mathcal{N} \llcorner f(C)$ because $\operatorname{spt}[f_\# \mathfrak{I} - \mathcal{N} \llcorner f(C)]$ is contained in the $\mathcal{H}^k$ null subset $f([C \sim G(C)] \cup \cup \operatorname{Fr} C)$ of N. It follows that $f^{-1}[f(C)] \cap \operatorname{spt} \partial \mathfrak{I} = \emptyset$ because

$$f(C) \cap \operatorname{spt} f_\# \partial \mathfrak{I} = f(C) \cap \operatorname{spt} \partial[\mathcal{N} \llcorner f(C)] = \emptyset$$

and $f_\# | \mathfrak{Z}_{k-1}(C, C)$ is injective. For $Q \in \mathfrak{Z}_j[f(C), f(C)]$ we infer from [3, 3.1.18], 2.1, and 3.1 that

$$\dim f^{-1}(\operatorname{spt} Q) \leqslant j + \dim M - k\,, \qquad \dim f^{-1}(\operatorname{spt} \partial Q) \leqslant j + \dim M - k - 1\,,$$
$$\dim [f^{-1}(\operatorname{spt} Q) \cap \operatorname{spt} \mathfrak{I}] \leqslant j\,, \qquad \dim [f^{-1}(\operatorname{spt} \partial Q) \cap \mathfrak{I}] \leqslant j - 1\,,$$
$$f^{-1}(\operatorname{spt} Q) \cap \operatorname{spt} \partial \mathfrak{I} \subset f^{-1}[f(C)] \cap \operatorname{spt} \partial \mathfrak{I} = \emptyset\,,$$

and use [6, 5.8(11)] to define $\Upsilon_j(Q) = (f^\# Q) \cap \mathfrak{I} \in \mathfrak{Z}_j(C, C)$ and verify that $f_\# \Upsilon_j(Q) = Q \cap f_\# \mathfrak{I} = Q \cap [\mathcal{N} \llcorner f(C)] = Q \cap \mathcal{N} = Q$.

4. – Homology Neighborhood Theorem.

4.1. If T is a j dimensional analytic chain in $\boldsymbol{R}^n$ satisfying *condition* (#) $\dim p(\operatorname{spt} T) \leqslant j$ *and* $\dim p(\operatorname{spt} \partial T) \leqslant j-1$, then $p_\# T$ is, by § 3 (2), a j dimensional analytic chain in $\boldsymbol{R}^{n-1}$. In particular, if E is an at most $n-1$ dimensional semianalytic set in $\boldsymbol{R}^n$ and $p^{-1}\{y\}$ is nonsingular for E whenever $y \in \boldsymbol{R}^{n-1}$, then, by 2.2 and 2.7 (3), any analytic chain with support in E satisfies condition (#). For $\boldsymbol{R}^n \supset A \supset B$, let

$$\mathfrak{Z}_j^\#(A, B) = \mathfrak{Z}_j(A, B) \cap \{T\colon\ T \text{ satisfies condition } (\#)\}\,,$$
$$\mathcal{B}_j^\#(A, B) = \{R + \partial S\colon R \in \mathfrak{Z}_j^\#(B, B) \quad \text{and} \quad S \in \mathfrak{Z}_{j+1}^\#(A, A)\}\,,$$
$$\boldsymbol{H}_j^\#(A, B) = \mathfrak{Z}_j^\#(A, B) / \mathcal{B}_j^\#(A, B)\,.$$

We will prove by induction that the following two propositions are true for every positive integer n.

PROPOSITION B_n [respectively, $B_n^\#$]. *Suppose* $\mathcal{S}$ *is a partition of* R^n *and* $\{U_\Gamma : \Gamma \in \mathcal{S}\}$ *is a system of neighborhoods as in* 2.8 *and* 2.9. *If* $\mathcal{C} \subset \mathcal{S}$, $\mathcal{D} \subset \mathcal{S}$, *and* V *is an open subset of* R^k *where* $k = \inf\{\dim \Gamma : \Gamma \in \mathcal{C}\}$, *then*

$$H_j\Big[\bigcup_{\Gamma \in \mathcal{C}} \bigcup_{\Delta \in \mathcal{D}} U_\Gamma \cap p_k^{-1}(V) \cap (\Gamma \cup \Delta)\,, \quad \bigcup_{\Gamma \in \mathcal{C}} p_k^{-1}(V) \cap \Gamma\Big] \simeq 0$$

[*respectively,* $H_j^\#[\ ,\ ] \simeq 0$] *for all* j.

Using the homotopy formula for currents (3, 4.1.9]) we readily verify Propositions B_1 and $B_1^\#$. Assuming now that $n \geqslant 2$ and

PROPOSITION A_n [*respectively* $A_n^\#$] *is Proposition* B_n [*respectively* $B_n^\#$] *in case* $\mathcal{C}$ *has only one member* Γ,

we establish the induction in the following four sections:

4.2. *Proposition* A_{n-1} *implies proposition* $A_n^\#$.

PROOF. We assume $\Gamma \in \mathcal{D}$ and $k = \dim \Gamma < n$ and abbreviate $W = U_\Gamma \cap p_k^{-1}(V)$, $\Gamma^\# = p(\Gamma)$, $W^\# = p(W) = p(U_\Gamma) \cap q_k^{-1}(V)$.

First to treat the case $\dim \cup \mathcal{D} < n$, we will prove the stronger assertion,

PROPOSITION $A_n^\#$ *is true if* $\mathcal{D}$ *is replaced by any subfamily* $\overline{\mathcal{D}}$ *of* $\overline{\mathcal{S}}$ *with* $\dim \cup \overline{\mathcal{D}} < n$,

by induction on $\dim \cup \overline{\mathcal{D}}$. If $\dim \cup \overline{\mathcal{D}} \leqslant k$, then $W \cap \cup \overline{\mathcal{D}} \subset \Gamma$, and the assertion is trivial. We now assume

$$\dim \cup \overline{\mathcal{D}} \in \{k+1, \ldots, n-1\}\,, \qquad T \in \mathcal{Z}_j^\#(W \cap \cup \overline{\mathcal{D}},\, W \cap \Gamma)\,,$$

and

$$\mathcal{F} = \overline{\mathcal{D}} \cap \{\Delta : \dim \Delta = \dim \cup \overline{\mathcal{D}}\}\,, \qquad \mathcal{I} = \overline{\mathcal{D}} \sim \mathcal{F}\,, \quad \text{and for each } \Delta \in \mathcal{F}$$

$$\mathcal{D}_\Delta = \overline{\mathcal{D}} \cap \{D : D \subset \operatorname{Clos} \Delta\}\,, \qquad \mathcal{I}_\Delta = \mathcal{D}_\Delta \cap \mathcal{I} = \mathcal{D}_\Delta \sim \{\Delta\}\,,$$

$$\mathcal{D}_\Delta^\# = \mathcal{S}^\# \cap \{p(D) : D \in \mathcal{D}_\Delta\}\,, \qquad \mathcal{I}_\Delta^\# = \mathcal{D}_\Delta^\# \sim \{p(\Delta)\}\,.$$

For each $\Delta \in \mathcal{F}$, $p|\operatorname{Clos} \Delta$ is a homeomorphism, by 2.8 (2), and the analytic chain $T_\Delta = T \llcorner \Delta$ satisfies

$$\operatorname{spt}\ T_\Delta \subset (\operatorname{spt} T) \cap \operatorname{Clos} \Delta \subset W \cap \cup \mathcal{D}_\Delta\,,$$
$$\operatorname{spt} \partial T_\Delta \subset (\operatorname{spt} T) \cap \operatorname{Fr} \Delta \ \ \subset W \cap \cup \mathcal{I}_\Delta\,,$$

and condition (#) by 2.8 (2) and 4.1. We apply Proposition A_{n-1} twice — with Γ, $\mathcal{D}$ replaced:

first, by $\Gamma^\#$, $\mathcal{I}_\Delta^\#$, to choose an analytic chain $P_\Delta^\#$ in R^{n-1} with

$$\operatorname{spt} P_\Delta^\# \subset W^\# \cap \cup \mathcal{I}_\Delta^\#\,, \qquad \operatorname{spt} \partial(p_\# T_\Delta - P_\Delta^\#) \subset W^\# \cap \Gamma^\#\,,$$

and second, by $\Gamma^{\#}$, $\mathfrak{D}_\Delta^{\#}$, to choose an analytic chain $S_\Delta^{\#}$ in $\boldsymbol{R}^{n-1}$ with

$$\operatorname{spt} S_\Delta^{\#} \subset W^{\#} \cap \cup \mathfrak{D}_\Delta^{\#} = p(W \cap \cup \mathfrak{D}_\Delta),$$
$$\operatorname{spt}(p_{\#} T_\Delta - \partial S_\Delta^{\#}) \subset (\operatorname{spt} P_\Delta^{\#}) \cup \operatorname{spt}(p_{\#} T_\Delta - P_\Delta^{\#} - \partial S_\Delta^{\#}) \subset p(W \cap \bigcup \mathfrak{J}_\Delta).$$

By 2.8 (2), 4.1, and 3.2, $S_\Delta^{\#}$ lifts to an analytic chain S_Δ in $\boldsymbol{R}^n$ satisfying condition (#) and

$$\operatorname{spt} S_\Delta \subset W \cap \cup \mathfrak{D}_\Delta, \qquad \operatorname{spt}(T_\Delta - \partial S_\Delta) \subset W \cap \cup \mathfrak{J}_\Delta.$$

Inasmuch as $\operatorname{spt}\left(\sum_{\Delta \in \mathcal{F}} S_\Delta\right) \subset W \cap \cup \mathfrak{D}$ and

$$\operatorname{spt}\left(T - \partial \sum_{\Delta \in \mathcal{F}} S_\Delta\right) \subset \operatorname{spt}\left(T - \sum_{\Delta \in \mathcal{F}} T_\Delta\right) \cup \bigcup_{\Delta \in \mathcal{F}} \operatorname{spt}(T_\Delta - \partial S_\Delta) \subset W \cap \bigcup \mathfrak{J},$$

there is, by induction an analytic chain S satisfying condition (#) and

$$\operatorname{spt} S \subset W \cap \cup \mathfrak{J}, \qquad \operatorname{spt}\left[T - \partial\left(\sum_{\Delta \in \mathcal{F}} S_\Delta\right) - \partial S\right] \subset W \cap \Gamma;$$

thus $T \in \mathfrak{B}_j^{\#}(W \cap \cup \overline{\mathfrak{D}}, W \cap \Gamma)$.

Having verified the assertion, we now assume that $\dim \cup \mathfrak{D} = n$ and $T \in \mathfrak{Z}_j^{\#}(W \cap \cup \mathfrak{D}, W \cap \Gamma)$, and define $\mathcal{F}$, $\mathfrak{J}$, $\mathfrak{D}_\Delta$, $\mathfrak{J}_\Delta$, and $\mathfrak{D}_\Delta^{\#}$ as above with $\overline{\mathfrak{D}} = \overline{\mathfrak{S}} \cap \{\Delta : \Delta \subset \cup \mathfrak{D}\}$; hence, $\cup \overline{\mathfrak{D}} = \cup \mathfrak{D}$. Thus for each $\Delta \in \mathcal{F}$ the analytic chain $T_\Delta = T \llcorner \Delta$ satisfies

$$\operatorname{spt} T_\Delta \subset W \cap \cup \mathfrak{D}_\Delta, \qquad \operatorname{spt} \partial T_\Delta \subset W \cap \cup \mathfrak{J}_\Delta,$$

and condition (#) by 2.8 (2), 4.1, and the inclusion $p(\operatorname{spt} T_\Delta) \subset p(\operatorname{spt} T)$. We first apply the assertion with $\mathfrak{D}$ replaced by $\mathfrak{J}_\Delta$, to obtain an analytic chain P_Δ satisfying condition (#) and

$$\operatorname{spt} P_\Delta \subset W \cap \cup \mathfrak{J}_\Delta, \qquad \operatorname{spt} \partial(T_\Delta - P_\Delta) \subset W \cap \Gamma,$$

second use Proposition A_{n-1}, with Γ, $\mathfrak{D}$ replaced by $\Gamma^{\#}$, $\mathfrak{D}_\Delta^{\#}$ to obtain an analytic chain $S_\Delta^{\#}$ with

$$\operatorname{spt} S_\Delta^{\#} \subset W^{\#} \cap \cup \mathfrak{D}_\Delta^{\#} = p(W \cap \cup \mathfrak{D}_\Delta),$$
$$\operatorname{spt}[p_{\#}(T_\Delta - P_\Delta) - \partial S_\Delta^{\#}] \subset W^{\#} \cap \Gamma^{\#} = p(W \cap \Gamma),$$

and third recall again 2.8 (2), 4.1, and 3.2 to select an analytic chain Q_Δ

in $\boldsymbol{R}^n$ satisfying condition (#), $\operatorname{spt} Q_\Delta \subset W \cap \Gamma$, and $p_\# Q_\Delta = p_\#(T_\Delta - P_\Delta) - \partial S^\#_\Delta$; hence $\partial(T_\Delta - P_\Delta - Q_\Delta) = 0$.

There exists a semianalytic set C_Δ in $\boldsymbol{R}^n$ such that $\operatorname{spt} S^\#_\Delta \subset p(C_\Delta \cap W \cap \cup \mathfrak{D}_\Delta)$ and p maps C_Δ homeomorphically onto an open subset of $\boldsymbol{R}^{n-1}$. In fact, with Y_Γ, H_Γ as in 2.8 (2),

$$\delta = \inf\{1, \operatorname{dist}[\Gamma \cap p^{-1}(\operatorname{spt} S^\#_\Delta), \operatorname{Fr} U_\Gamma]\}$$

and

$$\varepsilon = \sup\{|z| : H_\Gamma(y, z) = 0 \text{ for some } y \in Y_\Gamma\}$$

are finite positive numbers. If $r_1, r_2, \ldots, r_d$ is a complete list, counting multiplicities of the complex roots of H_Γ such that $\varrho_1 = \mathcal{R}e\, r_1 \leqslant \varrho_2 = \mathcal{R}e\, r_2 \leqslant \ldots \leqslant \varrho_d = \mathcal{R}e\, r_d$, $\varrho_0 = -\infty$, and $\varrho_{d+1} = +\infty$, then

$$\Gamma \subset E_\Gamma = p^{-1}(Y_\Gamma) \cap \{x : q(x) = \varrho_l[p(x)]\}\,, \tag{1}$$

$p^{-1}(Y_\Gamma) \cap \Delta = p^{-1}(Y_\Gamma) \cap \{x : q(x) \text{ is strictly between } \varrho_l(x) \text{ and } \varrho_m(x)\}$ for some $l \in \{1, \ldots, d\}$ and $m \in \{l-1, l+1\}$. With $\sigma_0 = \varrho_1 - 1$, $\sigma_1 = \varrho_1, \ldots$, $\sigma_d = \varrho_d$, $\sigma_{d+1} = \varrho_d + 1$, the set

$$C_\Delta = p^{-1}(Y_\Gamma) \cap \{x : q(x) = [\sigma_l + (2\varepsilon)^{-1}\delta(\sigma_m - \sigma_l)][p(x)]\}$$

satisfies the above inclusion. The function I_Γ on $Y_\Gamma \times \boldsymbol{R}$ whose value at $(y, z) \in Y_\Gamma \times \boldsymbol{R}$ equals

$$\prod_{\lambda=1}^{d} \prod_{\mu=1}^{d} [z - [r_\lambda + (2\varepsilon)^{-1}\delta(r_\mu - r_\lambda)](y)] \cdot \\ \cdot [z - [r_\lambda(y) + (2\varepsilon)^{-1}\delta]]\, [z - [r_\lambda(y) - (2\varepsilon)^{-1}\delta]]$$

is a monic pseudo-polynomial because its coefficients, being symmetric polynomial functions of $r_1, \ldots, r_d$ are polynomial function of the coefficients of H_Γ ([19, 5.7]), hence analytic in Y_Γ. Thus C_Δ, being the graph of a continuous root of I_Γ, is semianalytic by 2.6.

In the following, our construction (and our reason for using the substratification $\overline{\mathcal{S}}$ of $\mathcal{S}$) is based on the observation (2.8 (3), 2.9 (2)(3)):

(2) *$tx + (1-t)y \in W \cap \cup \mathfrak{D}_\Delta$ whenever $0 \leqslant t \leqslant 1$, $p(x) = p(y)$, and $x, y \in W \cap \cup \mathfrak{D}_\Delta$.*

Choosing Υ_j as in 3.2 with f, M, N, C replaced by p, $\boldsymbol{R}^n$, $\boldsymbol{R}^{n-1}$, C_Δ, we let $S_\Delta = \Upsilon_j(S^\#_\Delta)$,

$$\tilde{C} = (\boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n) \cap \{(t, x, y, z) : p(x) = p(y),\ y \in C_\Delta,\ \ z = tx + (1-t)y\},$$

$$f\colon \boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n \to \boldsymbol{R}\times\boldsymbol{R}^n, \qquad h\colon \boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n \to \boldsymbol{R}^n,$$

$$f(t, x, y, z) = (t, x) \text{ and } h(t, x, y, z) = z \text{ for } (t, x, y, z) \in \boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n.$$

Then S_Δ satisfies condition (#) by 2.2 and 2.7 (3), $\tilde{C}$ is semianalytic, f maps $\tilde{C}$ homeomorphically onto an open subset of $\boldsymbol{R}\times\boldsymbol{R}^n$, $h[(f|\tilde{C})^{-1}(0, x)] \in C_\Delta$ and $h[(f|\tilde{C})^{-1}(1, x)] = x$ whenever $x \in p^{-1}[p(C_\Delta)]$ and $p[h(t, x, y, z)] = p(x)$ whenever $(t, x, y, z) \in \tilde{C}$. Applying 3.2 again, this time with f, M, N, C replaced by f, $\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$, $\boldsymbol{R}\times\boldsymbol{R}^n$, $\tilde{C}$ to obtain a lifting $\tilde{\Upsilon}_j$, we let

$$J_\Delta = h_\# \tilde{\Upsilon}_{j+1}[[0, 1]\times(T_\Delta - P_\Delta - Q_\Delta)]$$

and compute, using [3, 4.1.15],

$$\begin{aligned} T_\Delta - P_\Delta - Q_\Delta - \partial J_\Delta &= h_\# \tilde{\Upsilon}_j[\boldsymbol{\delta}_0\times(T_\Delta - P_\Delta - Q_\Delta)] = \\ &= (\Upsilon_j \circ p_\# \circ h_\# \circ \tilde{\Upsilon}_j)[\boldsymbol{\delta}_0\times(T_\Delta - P_\Delta - Q_\Delta)] = \\ &= \Upsilon_j[p_\#(T_\Delta - P_\Delta - Q_\Delta)] = \Upsilon_j(\partial S_\Delta^\#) = \partial\Upsilon_{j+1}(S_\Delta^\#) = \partial S_\Delta. \end{aligned}$$

From the inequalities

$$\dim(\operatorname{spt} J_\Delta) \leqslant \dim p(\operatorname{spt} J_\Delta) + 1 \leqslant \dim p[\operatorname{spt}(T_\Delta - P_\Delta - Q_\Delta)] + 1 \leqslant j + 1$$

we see that J_Δ is an analytic chain satisfying condition (#). Moreover,

$$\begin{aligned} \operatorname{spt} J_\Delta &= \{tx + (1-t)y\colon 0 \leqslant t \leqslant 1,\ x \in \operatorname{spt}(T_\Delta - P_\Delta - Q_\Delta), \\ &\qquad y = (p|C_\Delta)^{-1}[p(x)]\} \subset W \cap \cup\, \mathfrak{D}_\Delta, \\ \operatorname{spt} S_\Delta &= C_\Delta \cap p^{-1}(\operatorname{spt} S_\Delta^\#) \subset W \cap \cup\, \mathfrak{D}_\Delta, \end{aligned}$$

Finally since $\operatorname{spt} \sum_{\Delta\in\mathfrak{F}} (J_\Delta + S_\Delta) \subset W \cap \cup\, \mathfrak{D}$ and

$$\operatorname{spt}\Big[T - \partial\sum_{\Delta\in\mathfrak{F}}(J_\Delta + S_\Delta)\Big] \subset \operatorname{spt}\Big(T - \sum_{\Delta\in\mathfrak{F}} T_\Delta\Big) \cup \bigcup_{\Delta\in\mathfrak{F}} \operatorname{spt}(T_\Delta - \partial J_\Delta - \partial S_\Delta) \subset W \cap \cup\, \mathfrak{J},$$

another application of the assertion, with $\mathfrak{D}$ replaced by $\mathfrak{J}$, provides an analytic chain S satisfying condition (#) and

$$\operatorname{spt} S \subset W \cap \cup\, \mathfrak{J}, \qquad \operatorname{spt}\Big[T - \partial\sum_{\Delta\in\mathfrak{F}}(J_\Delta + S_\Delta) - \partial S\Big] \subset W \cap \Gamma;$$

whence $T \in \mathfrak{B}_j^\#(W \cap \cup\, \mathfrak{D}, W \cap \Gamma)$.

4.3. *Proposition $A_n^\#$ implies proposition $B_n^\#$.*

PROOF. We use induction on $i = \dim \cup \mathcal{C}$. For each $\Gamma \in \mathcal{C}$ we abbreviate $W_\Gamma = U_\Gamma \cap p_k^{-1}(V)$, hence $W_\Gamma \cap \Gamma = p_k^{-1}(V) \cap \Gamma$, and assume $0 \neq T \in$ $\in Z_j^\#\Big[\bigcup_{\Gamma\in\mathcal{C}} \bigcup_{\Delta\in\mathfrak{D}} W_\Gamma \cap (\Gamma \cup \Delta),\ \bigcup_{\Gamma\in\mathcal{C}} W_\Gamma \cap \Gamma\Big]$.

Recalling 2.9 (3), we let

$$\mathfrak{B} = \mathcal{C} \cap \{B : \dim B < i\}, \qquad \mathcal{E} = \mathcal{C} \cap \{\Gamma : \dim \Gamma = i\},$$

$$U_{\mathfrak{B}} = \bigcup_{B\in\mathfrak{B}} U_B, \qquad U_{\mathcal{E}} = \bigcup_{\Gamma\in\mathcal{E}} U_\Gamma, \qquad V' = p_\iota[U_{\mathfrak{B}} \cap p_k^{-1}(V)],$$

and for each $\Gamma \in \mathcal{E}$,

$$\mathfrak{D}_\Gamma = \mathfrak{D} \cap \{\Delta : \Gamma \subset \operatorname{Clos} \Delta\}, \qquad W'_\Gamma = U_{\mathfrak{B}} \cap W_\Gamma = U_\Gamma \cap p_k^{-1}(V'),$$

and observe that $\{U_\Gamma : \Gamma \in \mathcal{E}\}$ is disjointed by 2.8 (3) and 2.9 (3). Thus if $\operatorname{spt} T \subset U_{\mathcal{E}}$, for example, if $i = 0$, then we may apply Proposition $A_n^\#$ to each of the analytic chains $T \llcorner U_\Gamma$ for $\Gamma \in \mathcal{E}$.

We now assume $U_{\mathfrak{B}} \cap \operatorname{spt} T \neq \emptyset$. Since $p^{-1}[p(U_{\mathfrak{B}})] \cap U_{\mathcal{E}} = U_{\mathfrak{B}} \cap U_{\mathcal{E}}$ by 2.9 (3),

$$\varepsilon = \operatorname{dist}[\operatorname{Fr} p(U_{\mathfrak{B}}), p(\operatorname{spt} T \sim U_{\mathcal{E}})]$$

is positive. We may choose, first by the Stone-Weierstrass theorem ([9, p. 244]), a polynomial α on $\mathbf{R}^{n-1}$ such that

$$|\alpha(y) - \operatorname{dist}[y, p(\operatorname{spt} T \sim U_{\mathcal{E}})]| < \varepsilon/3 \quad \text{for } y \in p(\operatorname{spt} T),$$

and then, by [6, 2.2 (7)], a number r between $\varepsilon/3$ and $2\varepsilon/3$ such that

$$\dim[(\alpha\circ p)^{-1}\{r\} \cap \operatorname{spt} T[\leqslant j-1,\ \dim[(\alpha\circ p)^{-1}\{r\} \cap \operatorname{spt} \partial T] \leqslant j-2,$$
$$\dim[\alpha^{-1}\{r\} \cap p(\operatorname{spt} T)] \leqslant j-1,\ \dim[\alpha^{-1}\{r\} \cap p(\operatorname{spt} \partial T)] \leqslant j-2.$$

We infer by [3, 4.2.1, 4.3.4] and [6, 4.3] that

$$\begin{aligned}\langle T, \alpha\circ p, r\rangle &= (\partial T) \llcorner \{x : (\alpha\circ p)(x) \geqslant r\} - \partial\big(T \llcorner \{x : (\alpha\circ p)(x) \geqslant r\}\big) \\ &= \partial\big(T \llcorner \{x : (\alpha\circ p)(x) < r\}\big) - (\partial T) \llcorner \{x : (\alpha\circ p)(x) < r\}\end{aligned}$$

are all analytic chains. We obtain the decompositions

$$\langle T, \alpha\circ p, r\rangle = \sum_{\Gamma\in\mathcal{E}} R_\Gamma, \qquad T \llcorner \{x : (\alpha\circ p)(x) \geqslant r\} = \sum_{\Gamma\in\mathcal{E}} T_\Gamma$$

where, for each $\Gamma \in \mathcal{E}$, R_Γ and T_Γ are analytic chains which have supports contained in U_Γ and which therefore satisfy

$$\operatorname{spt} R_\Gamma \subset U_{\mathfrak{B}} \cap W_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}) \subset W'_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}_\Gamma), \qquad \operatorname{spt} \partial R_\Gamma \subset W'_\Gamma \cap \Gamma,$$
$$\operatorname{spt} T_\Gamma \subset W_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}_\Gamma), \qquad \partial T_\Gamma + R_\Gamma = (\partial T) \llcorner U_\Gamma \cap \{x: (\alpha \circ p)(x) \geq r\},$$

and condition (#) because

$$p(\operatorname{spt} R_\Gamma) \subset p[(\alpha \circ p)^{-1}\{r\} \cap \operatorname{spt} T] \subset \alpha^{-1}\{r\} \cap p(\operatorname{spt} T),$$
$$p(\operatorname{spt} \partial R_\Gamma) \subset \alpha^{-1}\{r\} \cap p(\operatorname{spt} \partial T),$$
$$p(\operatorname{spt} T_\Gamma) \subset p(\operatorname{spt} T), \qquad p(\operatorname{spt} \partial T_\Gamma) \subset p(\operatorname{spt} R_\Gamma) \cup p(\operatorname{spt} \partial T).$$

First, for each $\Gamma \in \mathcal{E}$, we use Proposition $A_n^\#$, with k, V replaced by i, V' to choose analytic chains P_Γ, Q_Γ satisfying condition (#) and

$$P_\Gamma + \partial Q_\Gamma = R_\Gamma, \qquad \operatorname{spt} P_\Gamma \subset W'_\Gamma \cap \Gamma, \qquad \operatorname{spt} Q_\Gamma \subset W'_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}_\Gamma).$$

Second, since $\operatorname{spt}(T_\Gamma + Q_\Gamma) \subset W_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}_\Gamma)$ and $\operatorname{spt} \partial(T_\Gamma + Q_\Gamma) = \operatorname{spt}(\partial T_\Gamma + R_\Gamma - P_\Gamma) \subset W_\Gamma \cap \Gamma$, we may again apply Proposition $A_n^\#$, this time to select an analytic chain S_Γ satisfying condition (#) and

$$\operatorname{spt} S_\Gamma \subset W_\Gamma \cap (\Gamma \cup \cup \mathfrak{D}_\Gamma), \qquad \operatorname{spt}(T_\Gamma + Q_\Gamma - \partial S_\Gamma) \subset W_\Gamma \cap \Gamma.$$

Third we observe that

$$\operatorname{spt} \partial\big(T \llcorner \{x: (\alpha \circ p)(x) < r\} - \sum_{\Gamma \in \mathcal{E}} Q_\Gamma\big) = \operatorname{spt}\big[(\partial T) \llcorner \{x: (\alpha \circ p)(x) < r\} + \sum_{\Gamma \in \mathcal{E}} P_\Gamma\big] \subset$$
$$\subset U_{\mathfrak{B}} \cap \bigcup_{\Gamma \in \mathcal{E}} p_k^{-1}(V) \cap \Gamma \subset \bigcup_{B \in \mathfrak{B}} \bigcup_{\Gamma \in \mathcal{E}} W_B \cap (B \cup \Gamma),$$

and that $T \llcorner \{x: (\alpha \circ p)(x) < r\} - \sum_{\Gamma \in \mathcal{E}} Q_\Gamma$ satisfies condition (#). Since $\dim \cup \mathfrak{B} < i$, there exists, by induction, an analytic chain Q satisfying condition (#) and

$$\operatorname{spt} Q \subset \bigcup_{\Gamma \in \mathcal{E}} \bigcup_{B \in \mathfrak{B}} W_B \cap (B \cup \Gamma),$$
$$\operatorname{spt} \partial\big(T \llcorner \{x: (\alpha \circ p)(x) < r\} - \sum_{\Gamma \in \mathcal{E}} Q_\Gamma - Q\big) \subset \bigcup_{B \in \mathfrak{B}} W_B \cap B.$$

Fourth we note that

$$\operatorname{spt}\big(T \llcorner \{x: (\alpha \circ p)(x) < r\} - \sum_{\Gamma \in \mathcal{E}} Q_\Gamma - Q\big) \subset \bigcup_{B \in \mathfrak{B}} \bigcup_{\Delta \in \mathfrak{D} \cup \mathcal{E}} W_B \cap (B \cup \Delta),$$

apply induction to choose an analytic chain S satisfying condition (#) and

$$\operatorname{spt} S \subset \bigcup_{B\in\mathfrak{B}} \bigcup_{\Delta\in\mathfrak{D}\cup\mathcal{E}} W_B \cap (B \cup \Delta),$$
$$\operatorname{spt}\Big(T \llcorner \{x\colon (\alpha\circ p)(x) < r\} - \sum_{\Gamma\in\mathcal{E}} Q_\Gamma - Q - \partial S\Big) \subset \bigcup_{B\in\mathfrak{B}} W_B \cap B,$$

and conclude that

$$\begin{aligned} T &= T \llcorner \{x\colon (\alpha\circ p)(x) \geqslant r\} + T \llcorner \{x\colon (\alpha\circ p)(x) < r\} \\ &= \sum_{\Gamma\in\mathcal{E}} (T_\Gamma + Q_\Gamma - \partial S_\Gamma) + \Big(T \llcorner \{x\colon (\alpha\circ p)(x) < r\} - \sum_{\Gamma\in\mathcal{E}} Q_\Gamma - Q - \partial S\Big) + \\ &\qquad + Q + \partial\Big(S + \sum_{\Gamma\in\mathcal{E}} S_\Gamma\Big) \end{aligned}$$

belongs to $\mathfrak{B}_j^\#\Big[\bigcup_{\Gamma\in\mathfrak{C}} \bigcup_{\Delta\in\mathfrak{D}} W_\Gamma \cap (\Gamma \cup \Delta),\ \bigcup_{\Gamma\in\mathfrak{C}} W_\Gamma \cap \Gamma\Big]$.

4.4. *Proposition $\boldsymbol{B}_n^\#$ implies proposition $\boldsymbol{A}_n$.*

Proof. We assume $\Gamma \in \mathfrak{D}$, $k = \dim \Gamma < n$, $W = U_\Gamma \cap p_k^{-1}(V)$, and $T \in \mathfrak{Z}_j(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma)$. Since, for $\dim \cup\, \mathfrak{D} < n$,

$$\begin{aligned} \mathfrak{Z}_j(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma) &= \mathfrak{Z}_j^\#(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma) = \\ &= \mathfrak{B}_j^\#(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma) \subset \mathfrak{B}_j(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma), \end{aligned}$$

by 2.8 (2), 4.1, and Proposition $\boldsymbol{B}_n^\#$, we also assume $\dim \cup\, \mathfrak{D} = n$. From 4.2 we recall the following notations

$$\overline{\mathfrak{D}},\ \mathcal{F},\ \mathfrak{I},\ \mathfrak{D}_\Delta,\ T_\Delta,\ E_\Gamma, \quad \text{and} \quad C_\Delta \text{ (which depends on } \delta).$$

It will be sufficient to find, for each $\Delta \in \mathcal{F}$, an analytic chain S_Δ such that $\operatorname{spt} S_\Delta \subset W \cap \cup\, \mathfrak{D}$ and $T_\Delta - \partial S_\Delta$ satisfies condition (#) because then

$$T - \partial \sum_{\Delta\in\mathcal{F}} S_\Delta = \Big(T - \sum_{\Delta\in\mathcal{F}} T_\Delta\Big) + \sum_{\Delta\in\mathcal{F}} (T_\Delta - \partial S_\Delta)$$

would, by 2.8 (2) and 4.1, satisfy condition (#) and belong to $\mathfrak{B}_j^\#(W \cap \cup\, \mathfrak{I},\ W \cap \Gamma) \subset \mathfrak{B}_j(W \cap \cup\, \mathfrak{D},\ W \cap \Gamma)$ by Proposition $\boldsymbol{B}_n^\#$.

Fixing $\Delta \in \mathcal{F}$, we note that $\operatorname{spt} \partial T_\Delta \subset W \cap \cup\, \mathfrak{I}_\Delta$ and that ∂T_Δ satisfies, by 2.8 (2) and 4.1, condition (#) (even though T_Δ may not). Recalling the assertion in 4.2 (or repeating the proof of the assertion) with $\mathfrak{D}$ replaced by $\mathfrak{I}_\Delta$, we choose an analytic chain P_Δ with

$$\operatorname{spt} P_\Delta \subset W \cap \cup\, \mathfrak{I}_\Delta \quad \text{and} \quad \operatorname{spt} \partial(T_\Delta - P_\Delta) \subset W \cap \Gamma.$$

By 2.9 (1)(3) and 4.1 (1)(2) we may select an open semianalytic set X which has compact closure in W and contains

$$\{tx+(1-t)y: 0\leqslant t\leqslant 1,\ x\in \operatorname{spt}(T_\Delta-P_\Delta),\ y=(p|E_\Gamma)^{-1}[p(x)]\}\,.$$

Since any semianalytic set is, by 2.6 (4), a countable union of compact sets, there is a countable family $\mathcal{N}$ of open neighborhoods of $\cup\,\mathfrak{J}$ such that any neighborhood of $\cup\,\mathfrak{J}$ contains some member of $\mathcal{N}$.

As a first approximation to S_Δ we will select for every $N\in\mathcal{N}$ analytic chains R^N_Δ and S^N_Δ such that R^N_Δ satisfies condition (#),

$$(\operatorname{spt} R^N_\Delta)\cup \operatorname{spt} S^N_\Delta\subset X\cap\cup\,\mathfrak{D}\,,\qquad \operatorname{spt}\partial R^N_\Delta\subset X\cap\cup\,\mathfrak{J}\,,$$
$$\operatorname{spt}(T_\Delta-R^N_\Delta-\partial S^N_\Delta)\subset X\cap N\,.$$

Fixing $N\in\mathcal{N}$ and choosing, by 2.8 (2), 2.8, and 2.9, an orthogonal transformation $g'\in\boldsymbol{O}(n)$, a partition $\mathcal{S}'$ of $\boldsymbol{R}^n$, and a system of neighborhoods $\{U'_{\Gamma'}:\Gamma'\in\mathcal{S}'\}$ so that $\mathcal{S}'$ is compatible with $A'=g'(X\cap\cup\,\mathfrak{D})$ and $B'=g'(X\cap\cup\,\mathfrak{J})$, $g'(0,\dots,0,1)=(0,\dots,0,1)$, and

$$N'=(g')^{-1}(\cup\{U'_{\Gamma'}:\Gamma'\in\mathcal{S}' \text{ and } \Gamma'\subset B'\})\subset X\cap N\,,$$

we infer from Proposition $\boldsymbol{B}^\#_n$—with $\mathcal{S}$, $\mathcal{C}$, $\mathfrak{D}$, V replaced by $\mathcal{S}'$, $\mathcal{S}'\cap\{\Gamma':\Gamma'\subset B'\}$, $\mathcal{S}'\cap\{\Gamma':\Gamma'\subset A'\}$, $\boldsymbol{R}^k$—that

$$\begin{aligned}\mathfrak{Z}^\#_{j-1}[(X\cap\cup\,\mathfrak{D})\cap N', X\cap\cup\,\mathfrak{J}]&=(g')^{-1}_\#(\mathfrak{Z}^\#_{j-1}[A'\cap g'(N'),B'])\\&=(g')^{-1}_\#(\mathfrak{B}^\#_{j-1}[A'\cap g'(N'),B'])=\mathfrak{B}^\#_{j-1}[(X\cap\cup\,\mathfrak{D})\cap N',X\cap\cup\,\mathfrak{J}]\,.\end{aligned}$$

Recalling from 4.2 the construction of C_Δ, we may replace δ by a smaller positive number in order that C_Δ be close enough to E_Γ so that

$$\{tx+(1-t)y: 0\leqslant t\leqslant 1, x\in\operatorname{spt}\partial(T_\Delta-P_\Delta),\ y=(p|C_\Delta)^{-1}[p(x)]\}\subset N'\,,$$
$$\{tx+(1-t)y: 0\leqslant t\leqslant 1,\ x\in\operatorname{spt}(T_\Delta-P_\Delta),\ y=(p|C_\Delta)^{-1}[p(x)]\}\subset X\cap\cup\,\mathfrak{D}\,.$$

By 2.4, 2.7 (3), and 4.2 (1), we may choose an orthogonal transformation $\gamma\in\boldsymbol{O}(n)$ near $\mathbf{1}_{\boldsymbol{R}^n}$ so that $\gamma^{-1}(C_\Delta)$ is nonsingular for $p^{-1}\{y\}$ whenever $y\in\boldsymbol{R}^{n-1}$,

$$\dim(p\circ\gamma)[\operatorname{spt}(T_\Delta-P_\Delta)]\leqslant k\,,\qquad \dim(p\circ\gamma)[\operatorname{spt}\partial(T_\Delta-P_\Delta)]\leqslant k-1\,,$$
$$\{tx+(1-t)y: 0\leqslant t\leqslant 1,\ x\in\operatorname{spt}\partial(T_\Delta-P_\Delta),\ y=[p\circ\gamma|\gamma^{-1}(C_\Delta)]^{-1}[(p\circ\gamma)(x)]\}\subset N'\,,$$
$$\{tx+(1-t)y: 0\leqslant t\leqslant 1,\ x\in\operatorname{spt}(T_\Delta-P_\Delta),\ y=[p\circ\gamma|\gamma^{-1}(C_\Delta)]^{-1}](p\circ\gamma)(x)]\}$$
$$\subset(X\cap\Delta)\cup N'\,.$$

With f and h as in 4.2 and

$$\tilde{C} = (\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n) \cap$$
$$\cap \{(t, x, y, z): (p\circ\gamma)(x) = (p\circ\gamma)(y),\ y \in \gamma^{-1}(C_\Delta),\ z = tx + (1-t)y\}$$

we infer that $\tilde{C}$ is semianalytic, that f maps $\tilde{C}$ homeomorphically onto an open subset of $\boldsymbol{R}\times\boldsymbol{R}^n$, that $h[(f|\tilde{C})^{-1}(0, x)] \in \gamma^{-1}(C_\Delta)$ and $x = h[(f|\tilde{C})^{-1}(1, x)]$ whenever $x \in (p\circ\gamma)^{-1}[p(C_\Delta)]$, and that $(p\circ\gamma)[h(t, x, y, z)] = (p\circ\gamma)(x)$ whenever $(t, x, y, z) \in \tilde{C}$. Applying 3.2, twice, with f, M, N, C replaced by $p\circ\gamma$, $\boldsymbol{R}^n$, $\boldsymbol{R}^{n-1}$, $\gamma^{-1}(C_\Delta)$ and f, $\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$, $\boldsymbol{R}\times\boldsymbol{R}^n$, $\tilde{C}$ to obtain liftings Υ_j and $\tilde{\Upsilon}_j$ respectively, we let

$$R = \Upsilon_j[(p\circ\gamma)_\#(T_\Delta - P_\Delta)]\,,$$
$$I = h_\#\,\tilde{\Upsilon}_{j+1}[[\![0, 1]\!]\times\partial(T_\Delta - P_\Delta)]\,,$$
$$J = h_\#\,\tilde{\Upsilon}_j[[\![0, 1]\!]\times(T_\Delta - P_\Delta)]\,,$$

observe that

$$\operatorname{spt} R \subset \gamma^{-1}(C_\Delta) \cap [(X \cap \Delta) \cup N']\,, \qquad \operatorname{spt} H \subset (X \cap \Delta) \cup N'\,, \qquad \operatorname{spt} I \subset N'\,,$$

and compute, using [3, 4.1.15], that

$$T_\Delta - P_\Delta + I - \partial J = h_\#\,\tilde{\Upsilon}_j[\boldsymbol{\delta}_0\times(T_\Delta - P_\Delta)] =$$
$$= (\Upsilon_j\circ(p\circ\gamma)_\#\circ h_\#\circ\tilde{\Upsilon}_j)[\boldsymbol{\delta}_0\times(T_\Delta - P_\Delta)] = R\,.$$

From the inequalities

$$\dim(\operatorname{spt} I) \leqslant \dim(p\circ\gamma)(\operatorname{spt} I) + 1 \leqslant \dim(p\circ\gamma)[\operatorname{spt}\partial(T_\Delta - P_\Delta)] + 1 \leqslant j\,,$$
$$\dim(\operatorname{spt} J) \leqslant \dim(p\circ\gamma)[\operatorname{spt}(T_\Delta - P_\Delta)] + 1 \leqslant j + 1\,,$$

we see that I and J are analytic chains in $\boldsymbol{R}^n$. Choosing an open semianalytic set D with

$$\operatorname{Clos} D \subset (X \cap \Delta) \quad \text{and} \quad (\operatorname{spt} R \cup \operatorname{spt} J) \sim D \subset N'\,,$$

we infer from 4.1, with $E = \gamma^{-1}(C_\Delta)$, that $R \llcorner D$ satisfies condition (#). Then since $\partial(R \llcorner D)$ belongs to $\mathfrak{Z}^\#_{j-1}[(X \cap \cup\,\mathfrak{D}) \cap N', X \cap \cup\,\mathfrak{J}]$, we may select an analytic chain Q satisfying condition (#) and

$$\operatorname{spt} Q \subset (X \cap \cup\,\mathfrak{D}) \cap N'\,, \qquad \operatorname{spt}\partial[(R \llcorner D) - Q] \subset X \cap \cup\,\mathfrak{J}\,.$$

With $R^N_\Delta = R \llcorner D - Q$ and $S^N_\Delta = J \llcorner D$ we obtain the desired inclusions:

$$(\operatorname{spt} R^N_\Delta) \cup \operatorname{spt} S^N_\Delta \subset X \cap \cup \mathfrak{D}, \qquad \operatorname{spt} \partial R^N_\Delta \subset X \cap \cup \mathfrak{J},$$

$$\operatorname{spt}(T_\Delta - R^N_\Delta - \partial S^N_\Delta) =$$
$$= \operatorname{spt}[Q + (R - R \llcorner D) + \partial(J - J \llcorner D) + P_\Delta + I] \subset N' \subset X \cap N.$$

Next we fix a bounded semianalytic set Y with $\operatorname{Clos} X \subset Y \subset W$ and use 2.4, 2.7 (3), 2.8, and 2.9 to select an orthogonal transformation $g^* \in \boldsymbol{O}(n)$, a partition $\mathcal{S}^*$ of $\boldsymbol{R}^n$, and a system of neighborhoods $\{U^*_{\Gamma^*}: \Gamma^* \in \mathcal{C}^*\}$ such that $\mathcal{S}^*$ is compatible with $A^* = g^*(Y \cap \cup \mathfrak{D})$ and $B^* = g^*(Y \cap \cup \mathfrak{J})$, $g^*_\#(T_\Delta - R^N_\Delta - \partial S^N_\Delta)$ satisfies condition (#) for all $N \in \mathcal{N}$, and

$$U^* = \cup \{U^*_{\Gamma^*}: \Gamma^* \in \mathcal{S}^* \text{ and } \Gamma^* \subset B^*\} \subset g^*(W).$$

Finally choosing $N \in \mathcal{N}$ so that

$$N \subset (g^*)^{-1}(U^*) \cup (\boldsymbol{R}^n \sim \operatorname{Clos} X); \qquad \text{hence } g^*(X \cap N) \subset U^*,$$

we observe that $g^*_\#(T_\Delta - R^N_\Delta - \partial S^N_\Delta) \in \mathfrak{Z}^\#_j(A^* \cap U^*, B^*)$ and apply Proposition $\boldsymbol{B}^\#_n$—with $\mathcal{S}$, $\mathcal{C}$, $\mathfrak{D}$, V replaced by $\mathcal{S}^*$, $\mathcal{S}^* \cap \{\Gamma^*: \Gamma^* \subset B^*\}$, $\mathcal{S}^* \cap \{\Gamma^*: \Gamma^* \subset A^*\}$, $\boldsymbol{R}^k$—to obtain an analytic chain S^*_Δ such that

$$\operatorname{spt} S^*_\Delta \subset A^* \cap U^* \subset g^*(W \cap \cup \mathfrak{D}),$$
$$\operatorname{spt}[g^*_\#(T_\Delta - R^N_\Delta - \partial S^N_\Delta) - \partial S^*_\Delta] \subset B^* \subset g^*(W \cap \cup \mathfrak{J}).$$

With $S_\Delta = S^N_\Delta + (g^*)^{-1}_\# S^*_\Delta$, we conclude that $\operatorname{spt} S_\Delta \subset W \cap \cup \mathfrak{D}$ and, by 2.8 (2) and 4.1, that $T_\Delta - \partial S_\Delta$ satisfies condition (#) because R^N_Δ does and

$$\operatorname{spt}(T_\Delta - R^N_\Delta - \partial S_\Delta) \subset (g^*)^{-1}[g^*(W \cap \cup \mathfrak{J})] \subset \cup \mathfrak{J},$$

which completes the proof.

4.5. *Proposition* $\boldsymbol{A}_n$ *implies proposition* $\boldsymbol{B}_n$.

PROOF. Here we may repeat the argument of 4.3. Specifically we should, from that proof, drop all superscripts # and omit any statements concerning condition (#) and any dimensional estimates involving the projection p.

COROLLARY 4.6. *If $A \supset B$ are semianalytic subsets of a real analytic manifold M, then there exists an arbitrarily small open neighborhood W of B such that $\boldsymbol{H}_j(A \cap W, B) = 0$ for all j.*

PROOF. Since there exists ([5, Theorem 3]) a proper analytic embedding of M into some Euclidean space, we assume that M equals $\boldsymbol{R}^m$ for some positive integer m.

Suppose X is an open neighborhood of B. With $B_i = \boldsymbol{B}(0, i)$ for $i \in \{1, 2, \ldots\}$ we use 2.4, 2.8, and 2.9 to choose inductively orthogonal transformations $g_1, g_2, \ldots \in \boldsymbol{O}(m)$ and partitions $\mathcal{S}_1, \mathcal{S}_2, \ldots$ of $\boldsymbol{R}^m$ with corresponding families $\{U^i_\Gamma : \Gamma \in \mathcal{S}_i\}$ of neighborhoods such that $\mathcal{S}_1$ is compatible with B_1, $g_1(A \cap B_1)$, and $g_1(B \cap B_1)$ and, for $i = 1, 2, \ldots$, $\mathcal{S}_{i+1}$ is compatible with

$$B_{i+1}, \quad g_{i+1}(A \cap B_{i+1}), \quad g_{i+1}(B \cap B_{i+1}), \quad \text{and} \quad g_{i+1}[g_i^{-1}(\Gamma')] \qquad \text{for } \Gamma' \in \mathcal{S}_i,$$

$U^{i+1}_\Gamma \subset g_{i+1}(X)$ whenever $\Gamma \in \mathcal{S}_{i+1}$ and $\Gamma \subset g_{i+1}(B \cap B_{i+1})$, and $W^{i+1}_\Gamma = g^{-1}_{i+1}(U^{i+1}_\Gamma) \subset \subset \boldsymbol{U}(0, i+2) \cap g_i^{-1}(U^i_{\Gamma'})$ whenever $\Gamma \in \mathcal{S}_{i+1}$, $\Gamma' \in \mathcal{S}_i$, and $\Gamma \subset g_{i+1}[g_i^{-1}(\Gamma')]$; letting, for $i \in \{1, 2, \ldots\}$,

$$\mathcal{C}_i = \mathcal{S}_i \cap \{\Gamma : \Gamma \subset g_i(B \cap B_i)\}, \qquad \mathcal{D}_i = \mathcal{S}_i \cap \{\Delta : \Delta \subset g_i(A \cap B_i)\},$$
$$W_i = \cup \{W^i_\Gamma : \Gamma \in \mathcal{C}_i\},$$

we conclude from Proposition $\boldsymbol{B}_n$ that

$$\boldsymbol{H}_j(A \cap W_i, B \cap W_i) \simeq \boldsymbol{H}_j[g_i(A \cap W_i), g_i(R \cap W_i)] =$$
$$= \boldsymbol{H}_j\Big[\bigcup_{\Gamma \in \mathcal{C}_i} \bigcup_{\Delta \in \mathcal{D}_i} U^i_\Gamma \cap (\Gamma \cup \Delta), \bigcup_{\Gamma \in \mathcal{C}_i} \Gamma\Big] \simeq 0 \qquad \text{for all } j.$$

The set $W = \bigcup_{i=1}^{\infty} W_i$ is an open neighborhood B in X. Suppose $T \in \mathfrak{Z}_j(A \cap W, B)$. To see that T belongs to $\mathfrak{B}_j(A \cap W, B)$, we will first let $W_0 = \emptyset$ and choose inductively, for $i = 0, 1, \ldots$, an analytic chain S_i such that

$$\operatorname{spt} S_i \subset A \cap W_i \quad \text{and} \quad \operatorname{spt}\Big(T - \partial \sum_{l=0}^{i} S_l\Big) \subset \bigcup_{l=i}^{\infty} W_l.$$

Suppose $S_0 = 0$ and $S_1, S_2, \ldots, S_{i-1}$ have been chosen. Selecting an analytic chain R such that

$$\operatorname{spt} R \subset A \cap W_{i-1}, \quad \operatorname{spt}\Big[\Big(T - \partial \sum_{l=0}^{i} S_l\Big) - R\Big] \subset \bigcup_{l=i}^{\infty} W_l;$$

hence $\operatorname{spt} \partial R \subset A \cap B_i \cap \bigcup_{l=i}^{\infty} W_l \subset A \cap W_i$, we choose an analytic chain Q with

$$\operatorname{spt} Q \subset A \cap W_i, \quad \operatorname{spt} \partial(R - Q) \subset B \cap W_i,$$

and then an analytic chain S_i such that

$$\operatorname{spt} S_i \subset A \cap W_i\,, \qquad \operatorname{spt}(R - Q - \partial S_i) \subset B \cap W_i :$$

thus $\operatorname{spt}\Bigl(T - \partial \sum_{l=0}^{i} S_l\Bigr) \subset \bigcup_{l=i}^{\infty} W_l$.

Finally we take i large enough so that $\operatorname{spt} T \subset B_i$, infer that

$$\operatorname{spt}\Bigl(T - \partial \sum_{l=0}^{i} S_l\Bigr) \subset A \cap \Bigl(\bigcup_{l=i}^{\infty} W_l\Bigr) \cap \Bigl(B_i \cup \bigcup_{l=0}^{i} W_l\Bigr) \subset A \cap W_i\,,$$

and select an analytic chain S with

$$\operatorname{spt} S \subset A \cap W_i \quad \text{and} \quad \operatorname{spt}\Bigl(T - \partial\Bigl(\sum_{l=0}^{i} S_l\Bigr) - \partial S\Bigr) \subset B \cap W_i\,.$$

COROLLARY 4.7. *There exist arbitrarily small open neighborhoods U of A in M and V of B in U such that the inclusion map of $\mathfrak{Z}_j(A, B)$ into $\mathfrak{Z}_j(U, V)$ induces an isomorphism Φ_j mapping $\boldsymbol{H}_j(A, B)$ onto $\boldsymbol{H}_j(U, V)$ for all j.*

PROOF. Apply 4.6 twice—with A, B replaced:

first, by M, A to obtain an open neighborhood U of A in M with $\boldsymbol{H}_j(U, A) = 0$ and

second, by M, B to obtain an open neighborhood V of B in U with $\boldsymbol{H}_j(V, B) = 0$.

5. – Real Analytic Homology Theory.

Suppose M and N are m and n dimensional real analytic manifolds, $M \supset A \supset B$ and $N \supset C \supset D$ are semianalytic sets, and f maps the pair (C, D) continuously into (A, B). Using 4.7 we will, by approximation, define the group homomorphism

$$\boldsymbol{H}_j(f) : \boldsymbol{H}_j(C, D) \to \boldsymbol{H}_j(A, B) \qquad \text{for } j \in \{0, 1, \ldots\}\,.$$

LEMMA 5.1. *If $U \supset V$ are open subsets of M, $j \in \{0, 1, \ldots\}$, $Q \subset \boldsymbol{I}_j(M)$, $\operatorname{spt} Q \subset U$, and either $j = 0$ or $\operatorname{spt} \partial Q \subset V$, then there exist an analytic chain $R \in \mathfrak{Z}_j(U, V)$ and an integral current $S \in \boldsymbol{I}_{j+1}(M)$ such that $\operatorname{spt} S \subset U$ and $\operatorname{spt}(Q - R - \partial S) \subset V$.*

PROOF. Let $\alpha\colon M \to \boldsymbol{R}^a$ be a proper, real analytic embedding ([5]), and $A' \supset B'$ be relatively open semianalytic subsets of $\alpha(M)$ with $\alpha(\operatorname{spt} Q) \subset A' \subset \alpha(U)$ and $\alpha(\operatorname{spt} \partial Q) \subset B' \subset \alpha(V)$. Also let ϱ be a class ∞ retraction mapping an open subset W of $\boldsymbol{R}^a$ onto A' ([3, 3.1.20]). We select open neighborhoods U' of A' in W and V' of B' in $U' \cap \varrho^{-1}(B')$ such that $\boldsymbol{H}_j(A', B') \simeq \boldsymbol{H}_j(U', V')$ as in 4.7.

By the polyhedral approximation of [3, 4.2.9 (1)(4)(6)], there is a real analytic chain $R' \in \mathfrak{Z}_j(U', V')$ and an integral current $S' \in \boldsymbol{I}_{j+1}(M)$ with $\operatorname{spt} S' \subset U'$ and $\operatorname{spt}(\alpha_\# Q - R' - \partial S') \subset V'$. With analytic chains $R'' \in \mathfrak{Z}_j(A', B')$ and $S'' \in \mathfrak{Z}_{j+1}(U', U')$ chosen so that $\operatorname{spt}(R' - R'' - \partial S'') \subset V'$, the lemma is satisfied by the two currents R and S in M which are characterized by the conditions,

$$\alpha_\# R = R'' \quad \text{and} \quad \alpha_\# S = \varrho_\#(S' + S'') .$$

COROLLARY 5.2. *The inclusion map of* $\mathfrak{Z}_j(U, V)$ *into* $\boldsymbol{I}_j(M) \cap \{Q\colon \operatorname{spt} Q \subset U,$ $\operatorname{spt} \partial Q \subset V\}$ *induces an isomorphism* Ψ *mapping* $\boldsymbol{H}_j(U, V)$ *onto the integral current homology group* ([3, 4.4.5])

$$\boldsymbol{I}_j(M) \cap \{Q\colon \operatorname{spt} Q \subset U,\ \operatorname{spt} \partial Q \subset V\}/\{R + \partial S\colon R \in \boldsymbol{I}_j(M),\ S \in \boldsymbol{I}_{j+1}(M),\ \operatorname{spt} R \subset V,\ \operatorname{spt} S \subset U\} .$$

LEMMA 5.3. *If* K *is a compact subset of* C *and* $\varepsilon > 0$, *then there exists a class* ∞ *function* g *mapping* N *into* M *such that* $\operatorname{dist}[f(x), g(x)] < \varepsilon$ *for all* $x \in K$.

PROOF. We consider the commutative diagram

$$\begin{array}{ccc} N \subset K & \xrightarrow{f|K} & M \\ \downarrow{\scriptstyle \beta|K} & & \downarrow{\scriptstyle \alpha} \\ \boldsymbol{R}^b & \xrightarrow{F} & \boldsymbol{R}^a \end{array}$$

where $\alpha\colon M \to \boldsymbol{R}^a$ and $\beta\colon N \to \boldsymbol{R}^b$ are class ∞ proper imbeddings ([20, p. 113]), and F is a continuous extension to $\boldsymbol{R}^b$ of the map $(\alpha \circ f \circ \beta^{-1})|\beta(K)$. We also choose a class ∞ retraction ϱ of an open neighborhood W of $\alpha(M)$ onto $\alpha(M)$ and a compact subset L of W with $\alpha[f(K)] \subset \operatorname{Int} L$. With

$$\lambda = \sup \{\operatorname{Lip}(\varrho|L), \operatorname{Lip}[\alpha^{-1}|\varrho(L)]\} ,$$

we may, by regularization ([3, 4.1.2]), choose a class ∞ mapping $G\colon \boldsymbol{R}^b \to \boldsymbol{R}^a$ such that $G[\beta(K)] \subset \operatorname{Int} L$ and $\operatorname{dist}[F(y), G(y)] < \varepsilon/\lambda^2$ for $y \in \beta(K)$; the lemma follows with $g = \alpha^{-1} \circ \varrho \circ G \circ \beta$.

LEMMA 5.4. *For any* $T \in \mathbf{I}_j(N)$ *with* $\operatorname{spt} T \subset C$ *and open neighborhoods* U *of* $f(\operatorname{spt} T)$ *and* V *of* $f(\operatorname{spt} \partial T)$ *in* M, *there is an* $\varepsilon > 0$ *such that if* g *and* h *are class* ∞ *mappings of* N *into* M *with* $\operatorname{dist}[f(x), g(x)] < \varepsilon$ *and* $\operatorname{dist}[f(x), h(x)] < \varepsilon$ *for* $x \in \operatorname{spt} T$, *then*

$$(\operatorname{spt} g_\# T) \cup (\operatorname{spt} h_\# T) \subset U\,, \qquad (\operatorname{spt} g_\# \partial T) \cup (\operatorname{spt} h_\# \partial T) \subset V\,,$$

and there exists an integral current $S \in \mathbf{I}_{j+1}(M)$ *with* $\operatorname{spt} S \subset U$ *and*

$$\operatorname{spt}(g_\# T - h_\# T - \partial S) \subset V\,.$$

PROOF. With α, $\mathbf{R}^a$, ϱ, W as in 5.3, we choose $\varepsilon < 0$ so that

$$\begin{aligned}
&\{y\colon \operatorname{dist}[y, (\alpha\circ f)(\operatorname{spt} T)] < \varepsilon\} \subset W\,,\\
&\varrho\big(\{y\colon \operatorname{dist}[y, (\alpha\circ f)(\operatorname{spt} T)] < \varepsilon\}\big) \subset \alpha(U)\,,\\
&\varrho\big(\{y\colon \operatorname{dist}[y, (\alpha\circ f)(\operatorname{spt} \partial T)] < \varepsilon\}\big) \subset \alpha(V)\,.
\end{aligned}$$

If g and h satisfy the hypothesis and

$$\sigma\colon \mathbf{R}\times N \to \mathbf{R}^b\,, \qquad \sigma(t, x) = (1-t)g(x) + \operatorname{th}(x) \qquad \text{for } (t, x) \in \mathbf{R}\times N\,,$$

then $S = \alpha_\#^{-1}\varrho_\#\sigma_\#([\![0, 1]\!]\times T) \in \mathbf{I}_{j+1}(M)$ satisfies, by [3, 4.1.9],

$$\begin{aligned}
&\alpha(\operatorname{spt} S) \subset \varrho\big[\sigma\big(\{t\colon 0 \leqslant t \leqslant 1\}\times \operatorname{spt} T\big)\big] \subset \alpha(U)\,,\\
&\alpha\big[\operatorname{spt}(g_\# T - h_\# T - \partial S) \subset \varrho\big[\sigma\big(\{t\colon 0 \leqslant t \leqslant 1\}\times \operatorname{spt} \partial T\big)\big] \subset \alpha(V)\,.
\end{aligned}$$

5.5. Let $T \in \mathfrak{Z}_j(C, D)$ and $K = \operatorname{spt} T$. With U, V as in 4.7, ε as in 5.4, and g as in 5.3, let ω be the integral current homology class (5.2) of the integral current $g_\# T$; the function which associates ω with T is, by 5.4, a well-defined group homomorphism with kernel containing $\mathfrak{B}_j(C, D)$. Letting Ω denote the induced homomorphism on $\mathbf{H}_j(C, D)$, we recall 4.7 and 5.2 and define the homomorphism

$$\mathbf{H}_j(f)\colon \mathbf{H}_j(C, D) \to \mathbf{H}_j(A, B)\,, \qquad \mathbf{H}_j(f) = \Phi_i^{-1}\circ\psi^{-1}\circ\Omega\,.$$

The axioms of Eilenberg and Steenrod ([2, p. 10]), which for integral current homology in the local Lipschitz category readily follow by elementary properties of integral currents as in [3, 4.4.1, 4.4.5], are also easily verified, by approximation, for our real analytic homology theory on the category of semianalytic sets and continuous maps.

5.6. The homology groups $\boldsymbol{H}_j(A, B)$ for $j \in \{0, 1, \ldots\}$ are isomorphic to the homology groups of the *chain complex* ([2, p. 124]) with chain groups $C_j = \mathfrak{Z}_j(A, A)/\mathfrak{Z}_j(B, B)$ for $j \geq 0$, $C_j = \{0\}$ for $j < 0$, and with boundary homomorphisms $\partial_j\colon C_j \to C_{j-1}$ induced by ∂ for $j > 0$.

THEOREM 5.7. *If $A \supset B$ are relatively compact semianalytic subsets of M, then $\boldsymbol{H}_j(A, B)$ is finitely generated for all j.*

PROOF. By the fourth axiom (exactness) of Eilenberg-Steenrod, we assume that $B = \emptyset$. We also note that if E and F are semianalytic sets with $F \cap \operatorname{Clos} E \subset E$, then the inclusion $\mathfrak{Z}_j(E, E \cap F) \subset \mathfrak{Z}_j(E \cup F, F)$ and the map sending $T \in \mathfrak{Z}_j(E \cup F, F)$ to $T \llcorner E \in \mathfrak{Z}_j(E, E \cap F)$ induce inverse isomorphisms between $\boldsymbol{H}_j(E, E \cap F)$ and $\boldsymbol{H}_j(E \cup F, F)$ for all j; thus if $F \cap \operatorname{Clos} E \subset E$ and $E \cap \operatorname{Clos} F \subset F$, then there is, by [2, 1.4.1, 15.3], an exact Mayer-Vietoris sequence

$$0 \leftarrow \boldsymbol{H}_0(E \cup F) \leftarrow \boldsymbol{H}_0(E) \oplus \boldsymbol{H}_0(F) \leftarrow \boldsymbol{H}_0(E \cap F) \leftarrow \boldsymbol{H}_1(E \cup F) \leftarrow \ldots$$
$$\ldots \leftarrow \boldsymbol{H}_{j-1}(E \cap F) \leftarrow \boldsymbol{H}_j(E \cup F) \leftarrow \boldsymbol{H}_j(E) \oplus \boldsymbol{H}_j(F) \leftarrow \ldots$$

From this we observe, by induction, that if $\mathcal{E}$ is a finite family of semianalytic sets such that $E \cap \operatorname{Clos} F \subset F$, $F \cap \operatorname{Clos} E \subset E$, and $\boldsymbol{H}_j(E \cap \cup \mathcal{F})$ is finitely generated whenever $E \in \mathcal{E}$, $F \in \mathcal{E}$, and $\mathcal{F} \subset \mathcal{E}$, then $\boldsymbol{H}_j(\cup \mathcal{E})$ is finitely generated for all j. In particular, by covering $\operatorname{Clos} A$ by finitely many closed balls contained in coordinate neighborhoods, we may assume M is an open subset of $\boldsymbol{R}^n$.

We now use induction on n. For any interval or singleton set I in $\boldsymbol{R}^1$ and $a \in I$, there is a strong deformation retraction ([17, p. 30]) of I onto $\{a\}$; thus $\boldsymbol{H}_0(I) \simeq H_0(\{a\}) \simeq \boldsymbol{Z}$ and $\boldsymbol{H}_j(I) \simeq \boldsymbol{H}_j(\{a\}) \simeq 0$ for $j > 0$ by the first, fifth and seventh axioms of Eilenberg-Steenrod. The case $n = 1$ follows because any bounded semianalytic subset of $\boldsymbol{R}^1$ is a finite disjoint union of intervals and singleton sets.

To handle the inductive step, we assume, after an orthogonal transformation of $\boldsymbol{R}^n$, that $\mathcal{A}$, g, $\mathcal{S}$, $\mathcal{T}$, $\overline{\mathcal{S}}$, $\overline{\mathcal{T}}$, H_Γ, Z_Γ are as in 2.8 with $\mathcal{A} = \{A\}$ and $g = \mathbf{1}_{\boldsymbol{R}^n}$, and let

$$\mathcal{C} = \{A \cap \operatorname{Clos} \Gamma\colon \Gamma \in \overline{\mathcal{T}} \text{ and } \Gamma \subset A\}\,,$$

$$\mathcal{D} = \{A \cap \operatorname{Clos}(\Delta \cap p^{-1}[p(\Gamma)])\colon \Gamma \in \mathcal{T},\ \Delta \in \overline{\mathcal{S}} \sim \overline{\mathcal{T}},\text{ and } \Delta \subset A\}\,,$$

and $\mathcal{E} = \mathcal{C} \cup \mathcal{D}$. Then, being bounded, $A = \cup\, \mathcal{E}$. Moreover $F \cap \operatorname{Clos} E \subset E$ and $E \cap \operatorname{Clos} F \subset F$ whenever $E, F \in \mathcal{E}$.

If $E \subset \mathcal{C}$ and $\mathcal{F} \subset \mathcal{E}$, then, by 2.8 (2)(4)(6), p maps $E \cap \cup \mathcal{F}$ homeomorphically onto the semianalytic subset $p(E \cap \cup \mathcal{F})$ of $\boldsymbol{R}^{n-1}$; hence, $\boldsymbol{H}_j(E \cap \cup \mathcal{F}) \simeq \boldsymbol{H}_j[p(E \cap \cup \mathcal{F})]$ is finitely generated for all j. It follows, in particular, that for any $\mathcal{B} \subset \mathcal{C}$, $\boldsymbol{H}_j(\cup \mathcal{B})$ is finitely generated for all j.

Next if $E \in \mathcal{D}$ and $\mathcal{F} \subset \mathcal{E}$, then there are two possibilities. If $E \notin \mathcal{F}$, then $E \cap \cup \mathcal{F}$ is, by 2.8 (4), the union of a subfamily of $\mathcal{C}$; hence $\boldsymbol{H}_j(E \cap \cup \mathcal{F})$ is finitely generated for all j. If however $E \in \mathcal{F}$, then $E \cap \cup \mathcal{F} = E$. By 2.8 (4)(5)(6), $p(E)$ is a semianalytic subset of $\boldsymbol{R}^{n-1}$. Suppose $E = A \cap \operatorname{Clos}(\Delta \cap p^{-1}[p(\Gamma)]$ where $\Gamma \in \mathcal{T}$, $\Delta \in \overline{\mathcal{S}} \sim \overline{\mathcal{T}}$ and $\Delta \subset A$. There are, by 2.8 (3), continuous functions σ and τ on Z_Γ such that $H_\Gamma[y, \sigma(y)] = H_\Gamma[y, \tau(y)] = 0$ for $y \in Z_\Gamma$ and

$$\Delta \cap p^{-1}(Z_\Gamma) = p^{-1}(Z_\Gamma) \cap \{x\colon \sigma[p(x)] < q(x) < \tau[p(x)]\}\,.$$

Arguing as in 4.2 we see that

$$C = p^{-1}(Z_\Gamma) \cap \{x\colon q(x) = \tfrac{1}{2}(\sigma + \tau)[p(x)]\}$$

is a semianalytic set for which $p^{-1}\{y\}$ is nonsingular whenever $y \in \boldsymbol{R}^{n-1}$; thus, by 2.2 and 2.7 (3), p maps any semianalytic set in C homeomorphically onto a semianalytic subset of $\boldsymbol{R}^{n-1}$. Since by 2.8 (3)(4)(6)

$$h\colon \{t\colon 0 \leqslant t \leqslant 1\} \times E \to p^{-1}[p(E)]$$
$$h(t, x) = (1 - t)x + t(p|C)^{-1}[p(x)] \quad \text{for } 0 \leqslant t \leqslant 1 \text{ and } x \in E\,,$$

is a strong deformation retract of E onto $C \cap p^{-1}[p(E)]$, $\boldsymbol{H}_j(E) \simeq \simeq \boldsymbol{H}_j(C \cap p^{-1}[p(E)]) \simeq \boldsymbol{H}_j[p(E)]$ are finitely generated for all j.

It now follows from our previous observation that $\boldsymbol{H}_j(A) = \boldsymbol{H}_j(\cup \mathcal{E})$ is finitely generated for all j.

6. – Intersection Theory for $H^*(A, B)$.

Suppose M is an m dimensional orientable real analytic manifold, $M \supset A \supset B$ are semianalytic, and i and j are nonnegative integers with $i + j \geqslant m$. Using 4.7 and [6, § 5] we will define, in 6.4, for any two homology classes $\varrho \in \boldsymbol{H}_i(A, B)$ and $\tau \in \boldsymbol{H}_j(A, B)$ the *intersection class* $\varrho \cap \tau \in \boldsymbol{H}_{i+j-m}(A, B)$. Recall that for any i dimensional analytic chain R in M and j dimensional analytic chain T in M which *intersect suitably*, that is,

$$\dim(\operatorname{spt} R \cap \operatorname{spt} T) \leqslant i + j - m\,,$$
$$\dim[(\operatorname{spt} \partial R \cap \operatorname{spt} T) \cup (\operatorname{spt} R \cap \operatorname{spt} \partial T)] \leqslant i + j - m - 1\,,$$

an $i+j-m$ dimensional analytic chain $R\cap T$ has been defined and that real analytic intersection theory in M « at the chain level » has been treated in [6, § 5].

To define $\varrho\cap\tau$ we first observe that *if E and F are subsets of $\boldsymbol{R}^m$ with* $\dim E+\dim F\geqslant m-1$, *then*

$$\dim[\boldsymbol{\tau}_z(E)\cap F]\leqslant\dim E+\dim F-m \qquad \text{for } \mathfrak{L}^m \text{ almost all } z\in\boldsymbol{R}^m .$$

In fact by using the maps $f\colon \boldsymbol{R}^m\times\boldsymbol{R}^m\to\boldsymbol{R}^m$, $g_z\colon\boldsymbol{R}^m\to\boldsymbol{R}^m\times\boldsymbol{R}^m$,

$$f(x,y)=x-y\,,\qquad g_z(x)=(x+z,x)\qquad \text{for } x,y,z\in\boldsymbol{R}^m ,$$

we may infer that g_z is an analytic isomorphism mapping $\boldsymbol{\tau}_z(E)\cap F$ onto $(E\times F)\cap f^{-1}\{z\}$ and then apply [6, 2.2 (7)].

LEMMA 6.1. *If E and F are semianalytic subsets of $\boldsymbol{R}^m$ with* $\dim E+\dim F\geqslant m-1$, *then for $\mathcal{H}^{m-1}$ almost all $\xi\in\boldsymbol{S}^{m-1}$*

$$\dim[\{x+t\xi\colon x\in E\}\cap F]\leqslant\dim E+\dim F-m$$

for $\mathfrak{L}^1$ almost all $t\in\boldsymbol{R}$ and

$$\dim[\{x+t\xi\colon x\in E,\ t\in\boldsymbol{R}\}\cap F]\leqslant 1+\dim E+\dim F-m\ .$$

PROOF. We abbreviate $l=1+\dim E+\dim F-m$ and for $(x,\xi)\in\boldsymbol{R}^m\times\boldsymbol{S}^{m-1}$, $L_{x,\xi}=\boldsymbol{R}^m\cap\{x+t\xi\colon t\in\boldsymbol{R}\}$. From the above observation, 2.4, and Fubini's theorem, we infer that, for $\mathcal{H}^{m-1}$ almost all $\xi\in\boldsymbol{S}^{m-1}$, the line $L_{x,\xi}$ is nonsingular for E and

$$\dim[\boldsymbol{\tau}_{t\xi}(E)\cap F]\leqslant\dim E+\dim F-m\,.$$

for all $x\in\boldsymbol{R}^m$ and $\mathfrak{L}^1$ almost most all $t\in\boldsymbol{R}$.

Fix such a $\xi\in\boldsymbol{S}^{m-1}$, let $h\colon\boldsymbol{R}\times\boldsymbol{R}^m\to\boldsymbol{R}^m$, $h(t,x)=x+t\xi$ for $(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^m$, and choose $b\in\boldsymbol{O}(m)$ with $b(\xi)=(0,\ldots,0,1)$. Then, by 2.7 (3), $(p\circ b)(E)$ and

$$h(\boldsymbol{R}\times E)\cap F=(p\circ b)^{-1}[(p\circ b)(E)]\cap F$$

are semianalytic sets. Assuming, for contradiction, that $\dim[h(\boldsymbol{R}\times E)\cap F]>l$, we choose, by [6, 2.2 (4)] a bounded interval $I\subset\boldsymbol{R}$ such that $\infty\geqslant\mathcal{H}^{l+1}[h(I\times E)\cap F]>0$. Moreover by 2.6 (4) there is a semianalytic stratum Γ in E such that $\mathcal{H}^{l+1}[h(I\times\Gamma)\cap F]$ is positive and $p|b(\Gamma)$ and hence, $h|(I\times\Gamma)$ are analytic isomorphisms. Letting

$$\varphi\colon I\times\Gamma\to I\,,\qquad \varphi(t,x)=t\qquad \text{for } (t,x)\in I\times\Gamma\,,$$
$$\psi=\varphi\circ[h|(I\times\Gamma)]^{-1}|h(I\times\Gamma)\cap F\,,$$

we infer that the approximate Jacobian $\operatorname{ap} J_1\psi(x)$ ([3, 3.2.20]) is positive for every regular point x of $h(I\times\Gamma)\cap F$. From [6, 2.2(3)(6)] and the coarea formula [3, 3.2.22 (3)] we deduce the contradiction

$$0=\int_R \mathcal{H}^l[\tau_{t\xi}(E)\cap F]\,d\mathcal{L}^1t\geqslant\int_I \mathcal{H}^l[\tau_{t\xi}(\Gamma)\cap F]\,d\mathcal{L}^1t$$
$$=\int_I\int_{\psi^{-1}\{t\}} d\mathcal{H}^l\,d\mathcal{L}^1t=\int_{h(I\times\Gamma)\cup F}\operatorname{ap} J_1\psi\,d\mathcal{H}^{l+1}>0\,.$$

LEMMA 6.2. *Suppose Q is an i dimensional analytic chain in M, $\mathcal{F}$ is a countable collection of semianalytic sets in M, U, V_0, W_0, V, W are open subsets of M, $\operatorname{spt} Q\subset U$, $\operatorname{Clos} V_0\subset W_0$, $\operatorname{Clos} V\subset W$, $W\cap\operatorname{spt}\partial Q=\emptyset$, $\operatorname{Clos} W$ is compact, and there exists an analytic isomorphism mapping W into $\mathbf{R}^m$. If R_0 and S_0 are analytic chains in M, $(\operatorname{spt} R_0)\cup\operatorname{spt} S_0\subset U$, $W_0\cap\operatorname{spt}(Q-R_0-\partial S_0)=\emptyset$, and*

$$\dim(F\cap\operatorname{spt} R_0)\leqslant i+(\dim F)-m\,,\qquad \dim(F\cap\operatorname{spt}\partial R_0)\leqslant i+(\dim F)-m-1$$

for all $F\in\mathcal{F}$, then there exist analytic chains R and S in M such that $(\operatorname{spt} R)\cup\operatorname{spt} S\subset U$,

$$(V_0\cup V)\cap\operatorname{spt}(Q-R-\partial S)=\emptyset\,,$$
$$V_0\cap[\operatorname{spt}(R-R_0)\cup\operatorname{spt}(S-S_0)]=\emptyset\,,$$

$$\dim(F\cap\operatorname{spt} R)\leqslant j+(\dim F)-m\,,\qquad \dim(F\cap\operatorname{spt}\partial R)\leqslant i+(\dim F)-m-1$$
for all $F\in\mathcal{F}$.

PROOF. Choosing, by [6, 2.2 (7)], an open semianalytic set D with $\operatorname{Clos} V\subset D\subset\operatorname{Clos} D\subset W$ and

$$\dim[(F\cap\operatorname{spt}\ R_0)\cap\operatorname{Fr} D]\leqslant i+\dim F-m-1\,,$$
$$\dim[(F\cap\operatorname{spt}\partial R_0)\cap\operatorname{Fr} D]\leqslant i+\dim F-m-2$$

for all $F\in\mathcal{F}$, it suffices to prove the lemma with Q, $\mathcal{F}$, U, V_0, W_0, W, R_0, S_0 replaced by $Q\llcorner D-(\partial S_0)\llcorner D+\partial(S_0\llcorner D)$,

$$\{F\cap\operatorname{Clos} D, F\cap\operatorname{Fr} D\colon F\in\mathcal{F}\},\quad U\cap W,\quad V_0\cap W,\quad W_0\cap W,\quad R_0\llcorner D,\quad S_0\llcorner D$$

to obtain suitable analytic chains R_1, S_1 and then let

$$R = R_0 \llcorner (M \sim D) + R_1 \llcorner D\,, \qquad S = S_0 \llcorner (M \sim D) + S_1 \llcorner D\,.$$

Thus we may assume M equals $\mathbf{R}^m$ and $K = \operatorname{spt} Q \cup \operatorname{spt} R_0 \cup \operatorname{spt} S_0$ is compact.

Let α be a polynomial on $\mathbf{R}^m$ with $\alpha(x) < 0$ for $x \in V_0 \cap K$ and $\alpha(x) > 1$ for $x \in K \sim W_0$ and choose, by [6, 2.2 (7)] r so that $0 < r < 1$ and

$$\dim\big(\alpha^{-1}\{r\} \cap \operatorname{spt} R_0\big) \leqslant i - 1\,, \qquad \dim\big(\alpha^{-1}\{r\} \cap \operatorname{spt} \partial R_0\big) \leqslant i - 2\,,$$

$$\dim\big(\alpha^{-1}\{r\} \cap F \cap \operatorname{spt} R_0\big) \leqslant i + (\dim F) - m - 1\,,$$

$$\dim\big(\alpha^{-1}\{r\} \cap F \cap \operatorname{spt} \partial R_0\big) \leqslant i + (\dim F) - m - 2$$

for all $F \in \mathcal{F}$. Thus $(Q - R_0 - \partial S_0) \llcorner \{x : \alpha(x) \leqslant r\} = 0$. With

$$\begin{aligned} \mathcal{E} = \big\{ &\operatorname{spt}[(Q - \partial S_0) \llcorner \{x : \alpha(x) \geqslant r\}],\ \operatorname{spt}\langle R_0, \alpha, r\rangle\,, \\ &\operatorname{spt} \partial[(Q - \partial S_0) \llcorner \{x : \alpha(x) \geqslant r\}],\ \operatorname{spt} \partial\langle R_0, \alpha, r\rangle \big\}\,, \end{aligned}$$

we use 6.1 to select $\xi \in \mathbf{S}^{m-1}$, $\varepsilon > 0$, and $h\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ so that

$$\begin{aligned} &h(t, x) = x + t\xi \qquad \text{for } (t, x) \in \mathbf{R} \times \mathbf{R}^m\,, \\ &h\big(\{t : 0 \leqslant t \leqslant \varepsilon\} \times K\big) \subset U\,, \\ &V \cap h\big(\{t : 0 \leqslant t \leqslant \varepsilon\} \times \operatorname{spt} \partial Q\big) = \emptyset\,, \\ &V_0 \cap h\big(\{t : 0 \leqslant t \leqslant \varepsilon\} \times K \cap \{x : \alpha(x) \geqslant r\}\big) = \emptyset\,, \\ &\dim[\boldsymbol{\tau}_{\varepsilon\xi}(E) \cap F] \leqslant \dim E + \dim F - m\,, \\ &\dim[h(\mathbf{R} \times E) \cap F] \leqslant 1 + \dim E + \dim F - m \end{aligned}$$

for all $E \in \mathcal{E}$ and $F \in \{\mathbf{R}^m\} \cup \mathcal{F}$. From these last two estimates we infer that

$$\begin{aligned} R &= R_0 \llcorner \{x : \alpha(x) < r\} + h_\#\big([\![0, \varepsilon]\!] \times \langle R_0, \alpha, r\rangle\big) + \boldsymbol{\delta}_{\varepsilon\xi\#}[(Q - \partial S_0) \llcorner \{x : \alpha(x) \geqslant r\}]\,, \\ S &= S_0 + h_\#\big([\![0, \varepsilon]\!] \times (Q - \partial S_0) \llcorner \{x : \alpha(x) \geqslant r\}\big) \end{aligned}$$

are analytic chains in $\mathbf{R}^m$ and

$$\dim(F \cap \operatorname{spt} R) \leqslant i + (\dim F) - m\,, \qquad \dim(F \cap \operatorname{spt} \partial R) \leqslant i + (\dim F) - m - 1$$

for all $F \in \mathcal{F}$. Using [3, 4.1.9, 4.2.1, 4.3.4] we conclude

$$(\operatorname{spt} R) \cup \operatorname{spt} S \subset U, \qquad V_0 \cap [\operatorname{spt}(R - R_0) \cup \operatorname{spt}(S - S_0)] = \emptyset,$$
$$(V_0 \cup V) \cap \operatorname{spt}(Q - R - \partial S) = \emptyset.$$

THEOREM 6.3. *If $U \supset V$ are open subsets of M, $\varrho \in \boldsymbol{H}_i(U, V)$, and $\mathcal{C}$ is a countable collection of analytic chains in M, then there exists an analytic chain $R \in \varrho$ such that R and T intersect suitably for all $T \in \mathcal{C}$.*

PROOF. Let $Q \in \varrho$ and choose first finite open covers $\{U_1, U_2, \dots, U_L\}$ and $\{V_1, V_2, \dots, V_L\}$ of $\operatorname{spt} Q$ such that $\operatorname{Clos}(U_1 \cup \dots \cup U_L)$ is a compact subset of U, $V_L \subset V$, $(\operatorname{spt} \partial Q) \cap (U_1 \cup \dots \cup U_{L-1}) = \emptyset$ and $\operatorname{Clos} V_l \subset U_l$ for $l \in \{1, \dots, L\}$, and then open sets $W_0, W_1, \dots, W_L$ so that $W_0 = \emptyset$,

$$\operatorname{Clos}(V_1 \cup \dots \cup V_l) \subset W_l \subset \operatorname{Clos} W_l \subset W_{l-1} \cup U_l$$

for $l \in \{1, 2, \dots, L\}$. With $U_0 = \emptyset = W_{-1}$, $R_0 = 0$, $S_0 = 0$, we inductively apply 6.2, for each $l \in \{1, 2, \dots, L-1\}$ with $\mathcal{F}$, U, V_0, W_0, V, W, R_0, S_0 replaced by $\{\operatorname{spt} T, \operatorname{spt} \partial T \colon T \in \mathcal{C}\}$, $V_1 \cup \dots \cup V_L$, W_{l-1}, $W_{l-2} \cup U_{l-1}$, V_l, U_l, R_{l-1}, S_{l-1} to obtain analytic chains R_l and S_l such that

$$(\operatorname{spt} R_l) \cup \operatorname{spt} S_l \subset V_1 \cup V_2 \cup \dots \cup V_L,$$
$$\operatorname{spt}(Q - R_l - \partial S_l) \subset M \sim (W_{l-1} \cup V_l) \subset V_{l+1} \cup \dots \cup V_L,$$

and R_l and T intersect suitably for all $T \in \mathcal{C}$, and then take $R = R_{L-1}$.

6.4. $\varrho \cap \tau$. Let U and V be as in 4.7. For any homology classes $\varrho \in \boldsymbol{H}_i(A, B)$ and $\tau \in \boldsymbol{H}_j(A, B)$ we use 6.3 to choose analytic chains $R \in \Phi_i(\varrho)$ and $T \in \Phi_j(\tau)$ which intersect suitably and define the *intersection class* $\varrho \cap \tau \in \boldsymbol{H}_{i+j-m}(A, B)$ as *the Φ_{i+j-m} inverse image of the homology class in* $\boldsymbol{H}_{i+j-m}(U, V)$ of $R \cap T$.

The homology intersection class $\varrho \cap \tau$ is then well-defined. In fact suppose $R' \in \Phi_i(\varrho)$ and $T' \in \Phi_j(\tau)$ also intersect suitably. Then there are analytic chains Q and S such that $(\operatorname{spt} Q) \cup \operatorname{spt} S$ is a compact subset of U and

$$\operatorname{spt}(R - R' - \partial Q) \cup \operatorname{spt}(T - T' - \partial S') \subset V.$$

Using 6.3 to change, if necessary, first S and then Q, we may assume $\{S, R\}$, $\{S, R'\}$, $\{Q, T\}$, $\{Q, T'\}$, $\{Q, \partial S\}$, and hence $\{R - R' - \partial Q, T\}$ and $\{T - T' - \partial S, R\}$ intersect suitably. Thus, by [6, 5.8 (9)],

$$\operatorname{spt}(R \cap T - R' \cap T' - \partial[(-1)^{(m-i)(j+1)} R' \cap S + T' \cap Q + Q \cap \partial S]) \subset V.$$

Moreover we also infer that $\varrho \cap \tau$ is independent of the initial choice of U, V; if U', V' is a different pair of open sets satisfying 4.7, then we may by the previous paragraph, choose

$$R' \in \mathfrak{Z}_i(U \cap U', V \cap V') = \mathfrak{Z}_i(U, V) \cap \mathfrak{Z}_i(U', V'),$$
$$T' \in \mathfrak{Z}_j(U \cap U', V \cap V') = \mathfrak{Z}_j(U, V) \cap \mathfrak{Z}_j(U', V')$$

to compute $\varrho \cap \tau$.

Properties of the homology intersection product corresponding to [6, 5.11 (1)(2)(3)(4)(5)(6)(7)(8)] readily follow.

To prove associativity suppose $\varrho \in \boldsymbol{H}_i(A, B)$, $\sigma \in \boldsymbol{H}_j(A, B)$, and $\tau \in \boldsymbol{H}_k(A, B)$ where $i + j \geqslant m$, $j + k \geqslant m$, and $i + j + k \geqslant 2m$. Choose analytic chains $S \in \Phi_j(\sigma)$ and $T \in \Phi_k(\tau)$ which intersect suitably, apply 2.6 and 3.0 to construct a $j + k - m$ dimensional analytic chain P and a $j + k - m - 1$ dimensional analytic chain Q such that

$$(\operatorname{spt} S) \cap \operatorname{spt} T \subset \operatorname{spt} P, \qquad (\operatorname{spt} \partial S \cap \operatorname{spt} T) \cup (\operatorname{spt} S \cap \operatorname{spt} \partial T) \subset \operatorname{spt} Q,$$

and then select an analytic chain $R \in \Phi_i(\varrho)$ which intersects suitably with S, P, and Q. Thus $\{R, S\}$, $\{S, T\}$, and $\{R, S, T\}$ (See [6, 5.10]) intersect suitably; hence $(R \cap S) \cap T$ equals, by [6, 5.11 (6)], $R \cap (S \cap T)$.

7. – Real Analytic Sets.

In this section we assume that E is a k dimensional real analytic set in M and let

$$\operatorname{Reg} E = E \cap \{x\colon x \text{ has a neighborhood } U \text{ such that } U \cap E \text{ is a } k \text{ dimensional analytic submanifold of } M\}.$$

Thus E is closed and $\dim (E \sim \operatorname{Reg} E) \leqslant k$ by 2.6. We first study the extent to which E is locally orientable.

THEOREM 7.1. *If k equals either $m - 1$ or 1 and $y \in \operatorname{Clos} \operatorname{Reg} E$, then there exist an analytic chain T in M and an open ball U about x such that*

$$U \cap \operatorname{spt} T = U \cap \operatorname{Clos} \operatorname{Reg} E, \qquad U \cap \operatorname{spt} \partial T = \emptyset$$

and for every $x \in U \cap \operatorname{Reg} E$, $\Theta^k(\|T\|, x) = 1$; hence ([3, 4.1.31 (2)]) the homology class of T generates $\boldsymbol{H}_k(E, E \sim \{x\})$.

PROOF. We assume that $k \geqslant 1$, that M is an open subset of $\boldsymbol{R}^m$, that $y = 0$, and that the germ of E at 0, $\gamma_0(E)$, is irreducible ([3, 3.4.5]).

Case 1, $k = m - 1$. Recalling 2.6 (or [3, 3.4.8 (13)(10)]), we choose a connected neighborhood V of 0 in M, a function f analytic in V and a subset F of V such that $\dim F \leqslant k-1$, $V \cap E \subset f^{-1}\{0\}$, $0 \in \operatorname{Clos}(f^{-1}\{0\} \sim F)$, and $Df(a) \neq 0$ for all $a \in f^{-1}\{0\} \sim F$; by [3, 3.4.5, 3.4.7] we may also assume that $\gamma_0(f^{-1}\{0\})$ is irreducible. Then the inclusions

$$\gamma_0(E) \subset \gamma_0(f^{-1}\{0\}) \subsetneqq \gamma_0(\boldsymbol{R}^m)$$

and [3, 3.4.8 (15)] imply that $\gamma_0(E) = \gamma_0(f^{-1}\{0\})$. Choosing an open ball U about 0 such that $\operatorname{Clos} U \subset V$ and $U \cap E = U \cap f^{-1}\{0\}$, we define T to be the extension ([6, 3.3]) of $\langle \boldsymbol{E}^m | V, f, 0 \rangle$ U to M; therefore $U \cap \operatorname{spt} \partial T \subset$ $\subset \operatorname{spt} \partial \boldsymbol{E}^m = \emptyset$. Noting that for all points $a \in U \cap (f^{-1}\{0\} \sim F)$, $\operatorname{im} Df(a) = \boldsymbol{R}$, hence $\Theta^k(\|T\|, a) = 1$ by [3, 4.3.11], and that $U \cap \operatorname{Reg} E \subset \operatorname{Clos}(f^{-1}\{0\} \sim F)$, we conclude first that

$$U \cap \operatorname{spt} T = U \cap \operatorname{Clos}(f^{-1}\{0\} \sim F) = U \cap \operatorname{Clos} \operatorname{Reg} E$$

and second, by [3, 4.1.31 (2)], that $\Theta(\|T\|, x) = 1$ for $x \in \operatorname{Reg} E$.

Case 2, $k = 1$. Here we use [3, 3.4.8 (10)] to choose $r > 0$, orthogonal projections $\mu\colon \boldsymbol{R}^m \to \boldsymbol{R}^2$, $\nu\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ with $\nu(s, t) = s$ for $(s, t) \in \boldsymbol{R}^2$, and a finite family $\mathfrak{J}$ of one dimensional semianalytic strata in $W = (\nu \circ \mu)^{-1} \boldsymbol{U}(0, r)$ such that $\gamma_0(\{0\} \cup \cup \mathfrak{J}) = \gamma_0(E)$, $E' = \mu(\{0\} \cup \cup \mathfrak{J})$ is an analytic subset of $M' = \nu^{-1} \boldsymbol{U}(0, r)$, μ maps $\cup \mathfrak{J}$ isomorphically onto an analytic submanifold of M', and $\nu \circ \mu$ maps each $G \in \mathfrak{J}$ isomorphically onto either $\boldsymbol{R} \cap \{t\colon -r < t < 0\}$ or $\boldsymbol{R} \cap \{t\colon 0 < t < r\}$. We use Case 1 with M, E, y replaced by M', E', 0 to choose a suitable analytic chain T' and open ball U' in M'. Applying, for each $G \in \mathfrak{J}$, [3, 4.1.31 (2)] to the component C_G of $U' \cap \mu(G)$ whose closure contains 0, we obtain an orienting 1 vectorfield ξ_G of G such that

$$\operatorname{spt}(T' - \mu_{\#}[(\mathcal{H}^1 \llcorner G) \wedge \xi_G] | M') \subset M' \sim C_G .$$

Letting $T = \sum_{G \in \mathfrak{J}} (\mathcal{H}^1 \llcorner G) \wedge \xi_G$ and U be an open ball about 0 in $M \cap W$ such that $U \cap (\{0\} \cup \cup \mathfrak{J})$ equals $U \cap E$, we see that $U \cap \operatorname{spt} T = U \cap$ $\cap \operatorname{Clos} \operatorname{Reg} E$ and $\Theta^1(\|T\|, a) = 1$ for $a \in \cup \mathfrak{J}$. Noting that $\partial T | U = i \boldsymbol{\delta}_0 | U$ for some integer i, we compute

$$i \boldsymbol{\delta}_{\mu(0)} | U' = \mu_{\#}(\partial T) | U' = \partial T' | U' = 0$$

by Case 1, hence $i=0$. Finally if $0 \in \operatorname{Reg} E$, then $\Theta^1(\|T\|, 0) = 1$ by [3, 4.1.31 (2)] as before.

EXAMPLE 7.2. Letting $g: \boldsymbol{R}^6 \to \boldsymbol{R}^6$ be given by

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = \big(x_4^2 - 2x_1x_2,\, x_5^2 - 2x_2x_3,\, x_6^2 - 2x_1,\, x_3,\\ \sqrt{2}x_1x_5 - x_4x_6,\, \sqrt{2}x_2x_6 - x_4x_5,\, \sqrt{2}x_3x_4 - x_5x_6\big)$$

for $(x_1, x_2, x_3, x_4, x_5, x_6) \in \boldsymbol{R}^6$, we compute that $g(0) = 0$ and that $\dim Dg(x)(\boldsymbol{R}^6) = 3$ whenever $0 \neq x \in \boldsymbol{R}^6$. Thus $C = g^{-1}\{0\}$ is, by [3, 3.1.18] and 2.1, a three dimensional real analytic subset of $\boldsymbol{R}^6$. Being the double cone over a real projective plane, C is not locally orientable at 0. Specifically we show that

if $U \subset \boldsymbol{R}^6$ is open, T is a 3 dimensional analytic chain in $\boldsymbol{R}^6$ and $0 \in U \cap \cap \operatorname{spt} T \subset U \cap C$, then $0 \in \operatorname{spt} \partial T$ [hence $\boldsymbol{H}_3(C, C \sim \{0\}) \simeq \mathfrak{Z}_3(C, C \sim \{0\}) \simeq 0$].

In fact suppose $0 \notin \operatorname{spt} \partial T$. Letting $u: \boldsymbol{R}^6 \to \boldsymbol{R}$, $u(x) = |x|$ for $x \in \boldsymbol{R}^6$, we note that $(\operatorname{grad} u)(x) \in \operatorname{Tan}(C, x)$ whenever $0 \neq x \in C$ because C is a cone. Since the 3 vector $\vec{T}(x)$ is associated with $\operatorname{Tan}(C, x)$ for every nonzero regular point x of $(\operatorname{spt} T) \sim \operatorname{spt} \partial T$, $0 \in \operatorname{spt}(\vec{T} \llcorner Du)$, and we may, by [3, 4.3.2 (1)] and [6, 2.2 (7), 4.3] choose a positive $r < \operatorname{dist}[0, (\operatorname{Fr} U) \cup \operatorname{spt} \partial T]$ so that $\langle T, u, r\rangle$ is a nonzero two dimensional analytic cycle.

Using the map $f: \boldsymbol{S}^2 \to C$ given by

$$f(w_1, w_2, w_3) = r\big(w_1^2, w_2^2, w_2^2, \sqrt{2}w_1w_2, \sqrt{2}w_2w_3, \sqrt{2}w_1w_3\big) \text{ for } (w_1, w_2, w_3) \in \boldsymbol{S}^2,$$

we compute that $\dim Df(w)[\operatorname{Tan}(\boldsymbol{S}^2, w)] = 2$ for $w \in \boldsymbol{S}^2$ and $f(v) = f(w)$ if and only if $v = \pm w$ for $v, w \in \boldsymbol{S}^2$; thus, by [3, 3.1.18, 3.1.24], $f(\boldsymbol{S}^2)$ is a compact, connected analytic submanifold of $\boldsymbol{R}^6$. Moreover $V \cap u^{-1}\{r\}$ is the disjoint union of $f(\boldsymbol{S}^2)$ and $(-f)(\boldsymbol{S}^2)$. In fact, if $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \in V \cap u^{-1}\{r\}$, then $(x_1, x_2, x_3) \neq 0$. Let $\varrho = 1/\sqrt{r}$. In case $x_1 > 0$, $f(w) = x$ where $w = \varrho\big(\sqrt{x_1}, x_4/\sqrt{2x_1}, x_6/\sqrt{2x_1}\big)$ belongs to $\boldsymbol{S}^2$ because

$$|w|^4 = \varrho^4(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)/r^2 = 1\,.$$

In case $x_1 < 0$, $(-f)(v) = x$ where $v = \varrho\big(\sqrt{-x_1}, x_4/\sqrt{-2x_1}, x_6/\sqrt{-2x_1}\big) \in \boldsymbol{S}^2$. The remaining cases—$x_2 > 0$, $x_2 < 0$, $x_3 > 0$, $x_3 < 0$—are similarly treated.

Since $\emptyset \neq \operatorname{spt}\langle T, u, r\rangle \subset f(\boldsymbol{S}^2) \cup (-f)(\boldsymbol{S}^2)$ and $\partial\langle T, u, r\rangle = 0$, we infer from [3, 4.1.31 (2)] that one, and hence both of the components $f(\boldsymbol{S}^2)$ and $(-f)(\boldsymbol{S}^2)$ are orientable. If φ is an orienting 2 form for $f(\boldsymbol{S}^2)$ then $\psi = f^{\#}\varphi/|f^{\#}\varphi|$ is one of the two standard unit orienting 2 forms for S^2; hence for $v \in S^2$, $\psi(v) = -\psi(-v) = -\psi(v)$ or $\psi(v) = 0$, a contradiction.

THEOREM 7.3. *There exists a unique analytic chain modulo two Z_E in M with* $\operatorname{spt}^2 Z_E = \operatorname{Clos}\operatorname{Reg} E$; *moreover* $\partial Z_E = 0$ *for* $k \geq 1$ *and hence* ([7, 3.1]) *the homology class of* Z_E *generates* $H_k(E, E \sim \{x\}; Z_2)$ *for every* $x \in \operatorname{Reg} E$.

PROOF. If Q is any analytic chain modulo two in M with $\operatorname{spt}^2 Q = \operatorname{Clos}\operatorname{Reg} E$ and W is a relatively open semianalytic subset of $\operatorname{Reg} E$ with orienting k vectorfield ξ, then [7, 3.1], applied to each component of $W \sim \operatorname{spt}^2 \partial Q$, shows that $W \cap \operatorname{spt}^2(Q - [(\mathcal{H}^k \llcorner W) \wedge \xi]^2) \subset \operatorname{spt}^2 \partial Q$; since $\mathcal{H}^k(\operatorname{spt}^2 \partial Q) = 0$, it follows from [3, 4.2.26 (4.1.14)$''$] that $(Q - [(\mathcal{H}^k \llcorner W) \wedge \xi]^2) | W = 0$. Thus $\operatorname{spt}^2 \partial Q \subset \operatorname{Fr}\operatorname{Reg} E$, and uniqueness follows from [7, 3.1] and [3, 4.2.26 (4.1.14)$''$].

To prove existence let $\{V_1, V_2, \ldots\}$ be a cover of $\operatorname{Reg} E$ consisting of relatively open semianalytic subsets of $\operatorname{Reg} E$ with orienting k vectorfields $\xi_1, \xi_2, \ldots$. Also let $W_1 = V_1$, $W_i = V_i \sim \operatorname{Clos}(V_1 \cup \ldots \cup V_{i-1})$ for $i \in \{2, 3, \ldots\}$, $F = (\operatorname{Reg} E) \sim (W_1 \cup W_2 \cup \ldots)$, and

$$Z_E = \left(\sum_{i=1}^{\infty} (\mathcal{H}^k \llcorner W_i) \wedge \xi_i\right)^2 \in \mathcal{R}_k^{\mathrm{loc},2}(M) ;$$

hence $\mathcal{H}^k(F) = 0$ and $\operatorname{spt}^2 Z_E = \operatorname{Clos}\operatorname{Reg} E$. By the argument of the previous paragraph $(\operatorname{Reg} E) \cap \operatorname{spt}^2 \partial Z_E$, being contained in F by [7, 3.1], is empty by [3, 4.2.26 (4.1.14)$''$]. Since $\dim \operatorname{spt}^2 \partial Z_E \leq \dim \operatorname{Fr}\operatorname{Reg} E \leq k-1$, Z_E is an analytic chain modulo two in M.

Assuming now $k \geq 1$ we verify that $\partial Z_E = 0$ in two cases.

Case 1, $k = 1$. Here we assume $y \in \operatorname{spt}^2 \partial Z_E$. Choosing T, U as in 7.1, we infer from uniqueness, with M and E replaced by U and $U \cap E$, that $(T)^2 | U = Z_E$ hence,

$$U \cap \operatorname{spt}^2 \partial Z_E = U \cap \operatorname{spt}^2 \partial T \subset U \cap \operatorname{spt} \partial T = \emptyset ,$$

a contradiction.

Case 2, $k > 1$. We assume M is an open subset of R^m, x is a regular point for $B = \operatorname{spt}^2 \partial Z_E$, and $\dim[\operatorname{Tan}(B, x)] = k-1$. The remainder of the proof will consist of choosing an orthogonal projection $\mu: R^m \to R^{k-1}$, a neighborhood U of x, and a point $b \in U \cap B$ satisfying the conditions:

$$\dim[E \cap \mu^{-1}\{\mu(b)\}] = 1 , \qquad U \cap B \cap \mu^{-1}\{\mu(b)\} = \{b\} ,$$
$$(\boldsymbol{\delta}_b)^2 = \langle \partial Z_E, \mu, \mu(b)\rangle^2 | U = \partial[\langle Z_E, \mu, \mu(b)\rangle^2 | U] = \partial[Z_{E \cap \mu^{-1}\{\mu(b)\}} | U] ,$$

which contradict Case 1 with M and E replaced by U and $U \cap E \cap \mu^{-1}\{\mu(b)\}$.

Let D be a countable dense subset of $\operatorname{Reg} E$, $\mu\colon \boldsymbol{R}^m \to \boldsymbol{R}^{k-1}$ be an orthogonal projection with

$$\dim \mu[\operatorname{Tan}(B, x)] = k-1 = \dim \mu[\operatorname{Tan}(E, d)] \quad \text{for all } d \in D$$

and U be an open neighborhood of x such that $G = U \cap B$ is a nonempty connected analytic submanifold of M and $\mu|G$ is an analytic isomorphism. Recalling the proof of 3.2, we see that the real analytic dimension of

$$F = (\operatorname{Reg} E) \cap \{a\colon \dim \mu[\operatorname{Tan}(E, a)] < k-1\}$$

is at most $k-1$ because every component of $\operatorname{Reg} E$ meets D. Fixing, by [7, 4.1], an analytic chain T in M with $(T)^2 = Z_E$ and $\operatorname{spt} T = \operatorname{spt}^2 T = \operatorname{Clos} \operatorname{Reg} E$ and choosing, by [6, 2.2 (7)], $b \in G$ so that

$$\dim\big(E \cap \mu^{-1}\{\mu(b)\}\big) \leqslant 1\,, \qquad \dim\big[(F \cup \operatorname{spt} \partial T) \cap \mu^{-1}\{\mu(b)\}\big] \leqslant 0\,,$$

and the slice $\langle T, \mu, \mu(b)\rangle$ is determined by integration along the fiber as in [3, 4.3.8 (2)], we conclude first by [7, 3.2 (6)(2)] that $\langle \partial Z_E, \mu, \mu(b)\rangle^2 | U = (\boldsymbol{\delta}_b)^2$, hence $\dim\big(E \cap \mu^{-1}\{\mu(b)\}\big) = 1$, and second, by [3, 4.3.8 (2)], that

$$\begin{aligned} U \cap \operatorname{spt}^2\langle Z_E, \mu, \mu(b)\rangle^2 &= U \cap \operatorname{spt}^2\langle T, \mu, \mu(b)\rangle = \\ &= U \cap \mu^{-1}\{\mu(b)\} \cap \operatorname{Clos}[(\operatorname{Reg} E) \sim F] = U \cap \operatorname{Clos} \operatorname{Reg}\big(E \cap \mu^{-1}\{\mu(b)\}\big)\,; \end{aligned}$$

hence $\langle Z_E, \mu, \mu(b)\rangle^2 | U = Z_{E \cap \mu^{-1}\{\mu(b)\}} | U$ by uniqueness.

REMARK 7.4. If

$$E = \boldsymbol{R}^3 \cap \{(x, y, z)\colon z(x^2 + y^2) = x^3\}\,, \qquad F = \boldsymbol{R}^3 \cap \{(x, y, z)\colon x = 0\}\,,$$
$$Y = \boldsymbol{R}^3 \cap \{(x, y, z)\colon x = 0 = z\}\,, \qquad Z = \boldsymbol{R}^3 \cap \{(x, y, z)\colon x = 0 = y\}\,,$$

then the closed semianalytic set $\operatorname{spt}^2 Z_E = \operatorname{Clos}(E \sim Z)$ is not analytic ([15, p. 106]). Moreover

$$Z_{E \cap F} = Z_{Y \cup Z} = Z_Y + Z_Z \neq Z_Y = Z_E \cap^2 Z_F$$

([7, 4.4]) even though $\dim(E \cap F) = 1 = \dim E + \dim F - 3$.

8. – The Real Part of a Holomorphic Chain.

In this section we assume that $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ are embedded in $\boldsymbol{C}^m$ and $\boldsymbol{C}^n$ in the usual fashion and the $U \subset \boldsymbol{C}^m$ and $V \subset \boldsymbol{C}^n$ are open sets. For any map

$f\colon U \to V$ with $f(\boldsymbol{R}^m \cap U) \subset \boldsymbol{R}^n$, we let $\mathcal{R}f\colon \boldsymbol{R}^m \cap U \to \boldsymbol{R}^m \cap V$, $\mathcal{R}f(x) = f(x)$ $\mathcal{R}f(x) = f(x)$ for $x \in \boldsymbol{R}^m \cap U$.

We first observe that *if D is a complex j dimensional holomorphic submanifold of $\boldsymbol{C}^m$ and $\boldsymbol{R}^m \cap D \neq \emptyset$, then $\boldsymbol{R}^m \cap D$ is a j dimensional real analytic submanifold of $\boldsymbol{R}^m$.* In fact for $x \in \boldsymbol{R}^m \cap D$ there is an open neighborhood W of x in $\boldsymbol{C}^m$ and a multi-index $\lambda \in \Lambda(m, j)$ such that

$$\pi_\lambda\colon W \cap D \to \boldsymbol{C}^j\,, \qquad \pi_\lambda(z_1, \ldots, z_m) = (z_{\lambda(1)}, \ldots, z_{\lambda(j)}) \qquad \text{for } (z_1, \ldots, z_m) \in W \cap D\,,$$

is a holomorphic isomorphism; then $\mathcal{R}(\pi_\lambda | W \cap D)^{-1}$ is a real analytic isomorphism mapping $\boldsymbol{R}^j \cap \pi_\lambda(W \cap D)$ onto $\boldsymbol{R}^m \cap W \cap D$.

Next, recalling [3, 3.4.12] and [6, 2.2], we define, for $\emptyset \neq E \subset \boldsymbol{C}^m$, the *complex dimension of E*, denoted $\dim_{\boldsymbol{C}} E$, as

$$\sup_{x \in \boldsymbol{C}^m} \inf \{\dim_{\boldsymbol{C}} \beta\colon \beta \text{ is the germ of a holomorphic subvariety at } x \text{ and } \beta \text{ contains the germ of } E \text{ at } x\}\,;$$

in addition, we let $\dim_{\boldsymbol{C}} \emptyset = -1$. We obtain the inequality

$$\dim(\boldsymbol{R}^m \cap E) \leqslant \dim_{\boldsymbol{C}} E\,.$$

In fact, if $x \in \boldsymbol{R}^m \cap E$, β is the germ of a holomorphic subvariety at x, β contains the germ of E at x, and ε is the complexification of $\gamma_x(\boldsymbol{R}^m) \cap \beta$ ([15, p. 91]), then, by [15, p. 93],

$$\dim \gamma_x(\boldsymbol{R}^m \cap E) \leqslant \dim[\gamma_x(\boldsymbol{R}^m) \cap \beta] = \dim_{\boldsymbol{C}} \varepsilon \leqslant \dim_{\boldsymbol{C}} \beta\,.$$

8.1. *Complex holomorphic chains.* Let H be a complex j dimensional holomorphic chain in U ([3, 4.2.29], [6, § 6]). From [3, 4.2.29] and [15, pp. 67-68] we recall that H is a locally finite sum of integral multiples of chains corresponding to integration over the global irreducible components of the holomorphic set $\operatorname{spt} H$; hence, $\operatorname{spt}^2 H$, being the union of those irreducible components occuring with odd multiplicity, is a (pure) complex j dimensional holomorphic subset of U. We now define a j dimensional (real) analytic chain modulo two in $\boldsymbol{R}^m \cap U$, $\mathcal{R}H$, called the *real part of H*, by

$$\mathcal{R}H = Z_{\boldsymbol{R}^m \cap \operatorname{spt}^2 H} \quad \textit{in case } \dim(\boldsymbol{R}^m \cap \operatorname{spt}^2 H) = j\,,$$
$$\mathcal{R}H = 0 \qquad \textit{in case } \dim(\boldsymbol{R}^m \cap \operatorname{spt}^2 H) < j\,.$$

If I is a complex holomorphix chain in V, then $\mathcal{R}(H \times I) = (\mathcal{R}H) \times (\mathcal{R}I)$.

8.2. PROPER MAPPING FORMULA. *If f maps U holomorphically into V, $f(\boldsymbol{R}^m \cap U) \subset \boldsymbol{R}^n$, H is a complex j dimensional holomorphic chain in U, and $f|\operatorname{spt} H$ is proper, then $\mathfrak{R}(f_\# H) = (\mathfrak{R}f)_\#(\mathfrak{R}H)$.*

PROOF. By the proper mapping theorem ([15, p. 129]), $f(\operatorname{spt} H)$ is holomorphic in V with $\dim_{\boldsymbol{C}} f(\operatorname{spt} H) = j$; hence, by the argument of [3, 4.2.28], $f_\# H$ is a holomorphic chain in V. Let

$$\begin{aligned} A = \ & (\operatorname{spt} H) \sim \{x\colon x \text{ is a regular point of } \operatorname{spt} H \\ & \quad \text{and } \dim_{\boldsymbol{C}} Df(x)[\operatorname{Tan}(\operatorname{spt} H, x)] = j\}\,, \\ B = \ & f(\operatorname{spt} H) \sim \{y\colon \text{is a regular point of } f(\operatorname{spt} H) \\ & \quad \text{and } \dim_{\boldsymbol{C}} \operatorname{Tan}[f(\operatorname{spt} H), y] = j\}\,, \end{aligned}$$

and observe, by the real and complex rank theorems, that the restriction of f induces a holomorphic covering map of holomorphic submanifolds

$$(\operatorname{spt} H) \sim f^{-1}[f(A) \cup B] \to f(\operatorname{spt} H) \sim [f(A) \cup B]$$

and a real analytic covering map of real analytic submanifolds

$$\boldsymbol{R}^m \cap (\operatorname{spt} H) \sim f^{-1}[f(A) \cup B] \to \boldsymbol{R}^n \cap f(\operatorname{spt} H) \sim [f(A) \cup B]\,.$$

For any connected component C of $\boldsymbol{R}^n \cap f(\operatorname{spt} H) \sim [f(A) \cup B]$ and $y \in C$ we compute

$$\begin{aligned} \Theta^j(\|\mathfrak{R}(f_\# H)\|^2, y) &\equiv \Theta^{2j}(\|f_\# H\|^2, y) \equiv \sum_{x \in f^{-1}\{y\} \cap \operatorname{spt} H} \Theta^{2j}(\|H\|^2, x) \equiv \\ &\equiv \sum_{x \in f^{-1}\{y\} \cap \operatorname{spt} H} \Theta^j(\|\mathfrak{R}H\|^2, x) \equiv \Theta^j(\|(\mathfrak{R}f)_\#(\mathfrak{R}H)\|^2, x) \bmod 2\,, \end{aligned}$$

and observe that

$$\partial \mathfrak{R}(f_\# H) = 0 = (\mathfrak{R}f)_\# \partial(\mathfrak{R}H) = \partial(\mathfrak{R}f)_\#(\mathfrak{R}H)$$

by 7.3, and deduce from [7, 3.1] that

$$C \cap \operatorname{spt}^2[\mathfrak{R}(f_\# H) - (\mathfrak{R}f)_\#(\mathfrak{R}H)] = \emptyset\,.$$

Thus $\operatorname{spt}^2[\mathfrak{R}(f_\# H) - (\mathfrak{R}f)_\#(\mathfrak{R}H)] \subset \boldsymbol{R}^n \cap [f(A) \cup B]$. Since, by [15, p. 65], $\dim_{\boldsymbol{C}} B < j$, to complete the proof it suffices by [3, 4.2.26 (4.2.14)$^\nu$] to show that $\dim_{\boldsymbol{C}} A < j$; hence $\dim(\boldsymbol{R}^m \cap A) < j$ and $\dim(\boldsymbol{R}^n \cap B) < j$ by 8.0 and

$$\mathcal{H}^j(\boldsymbol{R}^n \cap [f(A) \cup B]) = \mathcal{H}^j[f(\boldsymbol{R}^m \cap A) \cup (\boldsymbol{R}^n \cap B)] = 0\,.$$

By the reasoning of [6, 2.9], A is holomorphic in U. If $\dim_{\mathbf{C}} A = j$, then

$$D = A \cap \{x\colon x \text{ is a regular point of } A \text{ and } \dim \operatorname{Tan}(A, x) = j\}$$

would be nonempty. Choosing a point $d \in D$ so that $\dim_{\mathbf{C}} Df(d)[\operatorname{Tan}(D, d)]$ is maximal, we infer from the complex rank theorem that $\dim_{\mathbf{C}}[f^{-1}\{f(d)\} \cap D] \geq 1$. But this is impossible because $f^{-1}\{f(d)\} \cap \operatorname{spt} H$, being a compact holomorphic subset of $\mathbf{C}^m$, is finite by [15, p. 52].

THEOREM 8.3. *If f maps U holomorphically into $\mathbf{C}^n$, $f(\mathbf{R}^m \cap U) \subset \mathbf{R}^n$, H is a complex j dimensional holomorphic chain in U, $j \geq n$, $y \in \mathbf{R}^n$, and $\dim_{\mathbf{C}}(f^{-1}\{y\} \cap \operatorname{spt} H) \leq j - n$, then $\mathfrak{R}\langle H, f, y\rangle = \langle \mathfrak{R}H, \mathfrak{R}f, y\rangle^2$.*

PROOF. Both sides are defined because $\partial H = 0$ and

$$\dim[(\mathfrak{R}f)^{-1}\{y\} \cap \operatorname{spt}^2 \mathfrak{R}H] \leq \dim(\mathbf{R}^m \cap f^{-1}\{y\} \cap \operatorname{spt} H) \leq j - n .$$

Case 1, $j = n$. Here we assume $x \in \mathbf{R}^m \cap f^{-1}\{y\} \cap H$ and select $0 < \varrho <$ $< \operatorname{dist}(x, \operatorname{Fr} U)$ and $a, b \in \{0, 1\}$ such that

$$\mathbf{B}(x, \varrho) \cap f^{-1}\{y\} \cap \operatorname{spt} H = \{y\} ,$$
$$[\mathfrak{R}\langle H, f, y\rangle - (a\boldsymbol{\delta}_x)^2] | \mathbf{U}(x, \varrho) = 0 = [\langle \mathfrak{R}H, \mathfrak{R}f, y\rangle - (b\boldsymbol{\delta}_x)^2] | \mathbf{U}(x, \varrho) .$$

Recalling [7, 3.2 (7)] we also choose

$$0 < \sigma < \tfrac{1}{2} \inf \{|y - f(z)|\colon z \in [\operatorname{Fr} \mathbf{U}(x, \varrho)] \cap \operatorname{spt} H\} ,$$
$$W = \mathbf{U}(x, \varrho) \cap f^{-1}\mathbf{U}(y, \sigma) ,$$

so that $(b\mathbf{E}^n | \mathbf{R}^n \cap \mathbf{U}(y, \sigma))^2 = [\mathfrak{R}(f|W)]_\#[(\mathfrak{R}H)|\mathbf{R}^m \cap W]$. Moreover $f|W \cap \operatorname{spt} H$ is proper and

$$(a\mathbf{E}^n | \mathbf{R}^n \cap \mathbf{U}(y, \sigma))^2 = \mathfrak{R}[a\mathbf{E}^{2n} | \mathbf{U}(y, \sigma)] =$$
$$= \mathfrak{R}[(f|W)_\#(H|W)] = [\mathfrak{R}(f|W)]_\#[(\mathfrak{R}H)|\mathbf{R}^m \cap W]$$

by [6, 3.6 (1)(6)(8)] and 8.2; hence $a = b$.

Case 2, $j > n$. Here we assume the theorem false. Noting that the complex dimension of

$$X = (f^{-1}\{y\} \cap \operatorname{spt} H) \sim \{x\colon x \text{ is a regular point of } f^{-1}\{y\} \cap \operatorname{spt} H$$
$$\text{and } \dim_{\mathbf{C}} \operatorname{Tan}(f^{-1}\{y\} \cap \operatorname{spt} H, x) = j - n\}$$

does not exceed $j-n-1$, we choose first by [3, 4.2.26 (4.1.14)$^\nu$], [6, 2.2 (4)], and 4.1 a point

$$x \in \operatorname{spt}^2(\mathcal{R}\langle H, f, y\rangle - \langle \mathcal{R}H, \mathcal{R}f, y\rangle^2) \sim X\,,$$

then a neighborhood W of x along with a projection $\mu\colon \mathbf{C}^m \to \mathbf{C}^{j-n}$ whose restriction to $W \cap f^{-1}\{y\} \cap \operatorname{spt} H$ is a holomorphic isomorphism. Letting $I = H|W$, $g = f|W$, $h = \mu|W$ and using [7, 3.1, 3.3, 4.3 (4.5)$^\nu$], [6, 4.5], and Case 1 twice we obtain the contradiction

$$\begin{aligned} 0 &\neq \langle \mathcal{R}\langle I, g, y\rangle - \langle \mathcal{R}I, \mathcal{R}g, y\rangle^2, \mathcal{R}h, h(x)\rangle^2 \\ &= \mathcal{R}\langle\langle I, g, y\rangle, h, h(x)\rangle - \langle \mathcal{R}I, (\mathcal{R}g) \boxdot (\mathcal{R}h), (y, h(x))\rangle^2 \\ &= \mathcal{R}\langle I, g \boxdot h, (y, h(x))\rangle - \mathcal{R}\langle I, g \boxdot h, (y, h(x))\rangle = 0\,. \end{aligned}$$

8.4. Borel-Haefliger formula. *If I and J are complex i and j dimensional holomorphic chains in U, $i+j \geqslant m$, and* $\dim_{\mathbf{C}}(\operatorname{spt} I \cap \operatorname{spt} J) \leqslant i+j-m$, *then*

$$\mathcal{R}(I \cap J) = (\mathcal{R}I) \cap^2 (\mathcal{R}J)\,.$$

Proof. Using the two maps $f\colon U \times U \to \mathbf{C}^m$ and $\mu\colon U \times U \to U$, $f(z, w) = z - w$ and $\mu(z, w) = z$ for $(z, w) \in U \times U$, we recall the definitions ([6, § 5], [7, 4.3 (§ 5)$^\nu$])

$$I \cap J = \mu_\#\langle I \times J, f, 0\rangle\,,$$

$$(\mathcal{R}I) \cap^2 (\mathcal{R}J) = (\mathcal{R}\mu)_\#\langle(\mathcal{R}I) \times (\mathcal{R}J), \mathcal{R}f, 0\rangle^2\,,$$

note that $\mu|f^{-1}\{0\}$ is proper, and then apply 8.2 and 8.3.

Added in proof.

Here we mention some recent results relevant to the present paper. Dennis Sullivan's theorem ([23]) on the oddness of the local Euler characteristic $\chi(A, A \sim \{a\})$ for a point a in a real analytic set A has been established in [24] and [25]. Many of the properties enjoyed by the class of semianalytic sets have now been obtained for the larger class of *subanalytic sets* consisting of all proper analytic images of semianalytic sets. That subanalytic sets admit stratifications into subanalytic, real analytic submanifolds was proven independently in [21] (first) and in [8] (where they are called semianalytic shadows). Using his desingularization theorems to represent locally a subanalytic set as the finite union of proper analytic images of quadrants in Euclidean spaces, H. Hironaka has, in [22], generalized to subanalytic sets many of the results of [13] including the Lojasiewicz inequalities. The stratification of subanalytic sets leads in [26] to subanalytic CW decomposition and in [27] to triangulation by homeomorphisms with subanalytic graphs. In [26], where the subanalytic analogues of 4.6, § 5 and § 6 of the present paper are obtained, the homology of subanalytic pairs is represented by subanalytic chains whereas, here, for semianalytic pairs we use the smaller group of analytic chains.

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