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Existence of Optimal Controls for Control Systems Governed by Nonlinear Partial Differential Equations (*).

MARSHALL SLEMROD (**)

Introduction.

In the theory of optimal control one of the main problems is to show the existence of optimal controllers for the system being considered. Consider for example the finite dimensional control system

$$\dot{x}(t) = f\big(x(t), h(t), t\big)\,, \qquad x(0) = x_0 \tag{E}$$

where $x \in R^n$, $h \in R^m$, and $f\colon R^n \times R^m \to R^n$, $0 \leqslant t \leqslant T$. We restrict h to lie in some admissible set of controls V. We may then pose two standard problems:

(1) *Time optimal problem.* Given a closed set $K \subset R^n$, does there exist a control $h^* \in V$ such that the solution $x^*(t)$ to (E) with $h = h^*$ is such that $x^*(\tau_0) \in K$ and τ_0 is the minimal time that can be achieved with a control $h \in V^m$?.

(2) *Cost optimal problem.* Let $K \subset R^n$ be a closed set and F be a cost functional, $F\colon C\big([0, T]; R^n\big) \to R$. Does there exist a control $h^* \in V$ such that the solution $x^*(t)$ to (E) with $h = h^*$ is such that $x^*(\tau) \in K$ for some $\tau > 0$ and F is minimized by this choice of h in V?

Since we are in general dealing with nonlinear problems where results on controllability are sparse we usually make the assumption in both the time optimal and cost optimal problems that there exists some controller $h \in V$ so that the solution $x(t)$ of (E) for this h is such that $x(\tau) \in K$ for some $\tau > 0$. Given this controllability assumption it is then possible to proceed

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forward and obtain results on existence of optimal controls. See for example [12].

In this paper we should like to consider both the time optimal and cost optimal control problems under a similar controllability hypothesis for control systems governed by nonlinear partial differential equation. Such problems have already been considered by Lions in [2, 3, 4] for special classes of equations. We attempt here to provide a unified theory based on the theory of nonlinear evolution equations in Banach space. Our main technique is to imbed the control problems into theory of nonlinear evolution equations as given in [1] and make use of certain approximation results presented there. While our results are in some sense more unified than those of Lions they do not at present include all the examples presented in [2, 3, 4]. They do, however, include certain results for control systems in nonreflexive state spaces which seem inaccessible via the techniques of [2, 3, 4].

The paper contains seven sections. Section 1 contains preliminaries on evolution operators in Banach spaces and the relationship between nonlinear evolution equations and control systems governed by nonlinear partial differential equations. Section 2 and Section 3 give results on existence of optimal controls for the time optimal problem and cost optimal problem, respectively. Section 4 discusses the problem of existence of solutions to nonlinear evolution equations and Section 5 presents results which show when the hypothesis of Sections 2 and 3 are fulfilled. Section 6 is devoted to a nonlinear wave equation with control in the damping coefficient. Section 7 applies our theory to control of a nonlinear conservation law used as a model for control of traffic on an infinite highway.

1. – Preliminaries.

Let X be a real Banach space with norm $\|\cdot\|$. An operator $A(t)$, possibly multivalued, $0\leqslant t\leqslant T$ with t-independent domain $D\subset X$ will be said to be a *generator* of an *evolution operator* $U(t, s)$ if we can associate with $A(t)$ an operator $U(t,s)\colon X\to X$ satisfying the relations

(i) $U(s, s) = I$ (the identity),

(ii) $U(t, s)\,U(s, r) = U(t, r)$ for $0\leqslant r\leqslant s\leqslant t\leqslant T$,

(iii) $U(t, s)x$ is continuous in the pair (t, s) on the triangle $0\leqslant s\leqslant t\leqslant T$,

(iv) $\|U(t, s)x - U(t, s)y\|\leqslant \exp[\omega(t-s)]\|x-y\|$, ω real.

If $U(t, s)$ is generated by $A(t)$ we would like to think of $u(t) = U(t, s)x$ as being in some sense a generalized solution to the Cauchy problem

$$\dot u(t) + A(t)u(t)\ni 0\,, \qquad u(s) = x,\ s\leqslant t\leqslant T\,. \tag{1.1}$$

An important question is what classes of operators $A(t)$ actually do generate evolution operators and in what sense is $U(t, s)x$ a solution to the initial value problem (1.1). We will return to this point later and for now we only use the initial value problem as motivation for our definitions of generator and evolution operator.

Let $\mathcal{C}$ be a set of possibly unbounded single valued nonlinear operators on X with common domain D. We shall say $\mathcal{C}$ is *sequentially compact* if for any sequence $\{A^\beta\} \subset \mathcal{C}$, $0 \leqslant \beta < 1$, there exists a subsequence $\{A^{\beta'}\}$ so that $A^{\beta'}x \to Ax$ for some $A \in \mathcal{C}$, all $x \in D$, as $\beta' \downarrow 0$.

We shall say a set $\mathcal{C}$ of generators is *strongly compact* if for every sequence $\{A^\beta(t)\}$ of generators in $\mathcal{C}$, $0 \leqslant \beta < 1$, there exists a subsequence $\{A^{\beta'}(t)\}$ and $A(t) \in \mathcal{C}$ such that the evolution operators $\{U^{\beta'}(t, s)\}$ associated with $\{A^{\beta'}(t)\}$ and $U(t, s)$ associated with $A(t)$ have the property that $U^{\beta'}(t, s)x \to U(t, s)x$ as $\beta' \downarrow 0$ for all $x \in X$, $0 \leqslant s \leqslant t \leqslant T$, and the limit is uniform in $t \in [s, T]$.

With above definitions we can immediately consider some problems dealing with existence of optimal controls. First let us make some preliminary remarks to make the above definitions more clear. In applications a set $\mathcal{C}$ of generators will be determined by a differential equation for the state of a system in X and a set of admissible controls. For example the state of a system $u(t)$ at time t may be determined by an equation of the form

$$\dot{u}(t) + A(t)u(t) \ni 0\,, \qquad u(0) = x \tag{1.2}$$

where $A(t)x = A_1x + B(t)x$ where A_1 will be some known, possibly unbounded, multivalued, nonlinear operator on $D(A_1) \subset X$ and $B(t)$ (the control) an operator $X \to X$ lying in a set of admissible controls V. This type of equation represents a system in which the control is the coefficient of the zero order term of the operator $A(t)$. Another possibility is that the state of the system is again determined by (1.2) where now $A(t)x = A_1x - h(t)$. Here again A_1 is some known, possibly unbounded, multivalued, nonlinear operator on $D(A_1) \subset X$ and $h(t)$ (the control) εX, $0 \leqslant t \leqslant T$, and $h(t)$ lies in some set of admissible controls V. In examples, this type of equation would represent a system with distributed forcing controls.

2. – Time optimal control problem.

Let K be a closed set in our real Banach Space X. Let $\mathcal{C}$ be a set of generators with common domain D, $0 \leqslant t \leqslant T$. Assume for some $\tau \in [0, T]$ and some $A(t) \in \mathcal{C}$ the evolution operator $U(t, s)$ associated with $A(t)$ is such that $U(\tau, 0)x \in K$ for $x \in X$. Set τ_0 (the optimal time) $= \inf \tau$. We would like to show the existence of an « optimal generator » $A^*(t) \in \mathcal{C}$ such that $A^*(t)$ generates an «optimal evolution operator» $U^*(t, s)$ so that $U^*(\tau_0, 0)x \in K$. In this case we will say that the *time optimal control problem has a solution.*

THEOREM 2.1. If $\mathcal{C}$ is strongly compact the time optimal control possesses a solution.

PROOF. Let $\{\tau_\beta\}$, $0 \leqslant \beta \leqslant 1$ be a minimizing sequence of times so that $\tau_\beta \to \tau_0$ as $\beta \downarrow 0$. Let $\{A^\beta(t)\}$ be the associated sequence of generators. Since $\mathcal{C}$ is strongly compact there exists a subsequence $\{A^{\beta'}(t)\}$ of $\{A^\beta(t)\}$ and $A^*(t)$, a generator in $\mathcal{C}$, generating evolution operators $\{U^{\beta'}(t, s)\}$ and $U^*(t, s)$, respectively, where we know that for the associated sequence of evolution operators $\lim\limits_{\beta' \downarrow 0} U^{\beta'}(t, 0)x = U^*(t, 0)x$ uniformly in $t \in [0, \tau]$. So we have

$$\| U^{\beta'}(\tau_\beta, 0)x - U^*(\tau_0, 0)x \| \leqslant \| U^{\beta'}(\tau_\beta, 0)x - U^*(\tau_\beta, 0)x \| + \\ + \| U^*(\tau_\beta, 0)x - U^*(\tau_0, 0)x \| .$$

Since $\lim\limits_{\beta' \downarrow 0} U^{\beta'}(t, 0)x = U^*(t, 0)x$ uniformly in $t \in [0, T]$, the first term in the right-hand side of the above inequality goes to zero as $\beta' \downarrow 0$. Also, since $U^*(t, 0)x$ is continuous in t, the second term in the inequality goes to zero as $\beta' \downarrow 0$ so that $U^*(\tau_0, 0)x \in K$.

3. – The cost optimization problem.

Again let K be a closed set in our real Banach space X. Let $\mathcal{C}$ be a set of generators with common domain D. Assume for some $\tau \in [0, T]$ and some $A(t) \in \mathcal{C}$ the evolution operator $U(t, s)$ associated with $A(t)$ satisfies $U(\tau, 0)x \in K$ for $x \in X$. Let F be a « cost functional » such that F: $C([0, \tau]; X) \to R$ so that if $w_n \to w$ as $n \to \infty$ in $C([0, \tau]; X)$ we have $F(w) \leqslant \liminf F(w_n)$. We would like to show the existence of an « optimal generator » $A^*(t) \in \mathcal{C}$ such that $A^*(t)$ generates an « optimal evolution operator » $U^*(t, s)$ so that

i) $U^*(\tau, 0)x \in K$;

ii) $F\big(U^*(\cdot, 0)x\big) \leqslant F\big(U(\cdot, 0)x\big)$ for all other evolution operators $U(t, s)$ generated by $A(t) \in \mathcal{C}$.

If we can find such $A^*(t)$ and $U^*(t, s)$ we shall say *the cost* optimization problem has a solution.

THEOREM 3.1. If $\mathcal{C}$ is strongly compact the cost optimization problem possesses a solution.

PROOF. Let $\{A^{\beta}(t)\}$ be a minimizing sequence of generators where the corresponding evolution operators satisfy $U^{\beta}(\tau, 0)x \in K$, $0 \leqslant \beta < 1$, and $F_0 = \inf_{\beta} F(U^{\beta}(,\cdot\, 0)x)$. For convenience we order the minimizing sequence so that

$$\lim_{\beta \downarrow 0} F(U^{\beta}(\cdot, 0)x) = F_0 \,.$$

Since $\mathcal{C}$ is strongly compact there exists a subsequence $\{A^{\beta'}(t)\}$ of $\{A^{\beta}(t)\}$ and $A^*(t)$, a generator in $\mathcal{C}$, generating evolution operators $\{U^{\beta'}(t, s)\}$ and $U^*(t, s)$, respectively, where we know

$$\lim_{\beta' \downarrow 0} U^{\beta'}(t, 0)x = U^*(t, 0) \qquad \text{uniformly}$$

in $\tau \in [0, T]$. Thus $\|U^{\beta'}(\tau, 0)x - U^*(\tau, 0)x\| \to 0$ as $\beta' \downarrow 0$ so $U^*(\tau, 0)x \in K$. Furthermore $F(U^*(\cdot, 0)x) \leqslant \liminf F(U^{\beta'}(\cdot, 0)x) = F_0$ by the assumption on the cost functional F. Thus $F(U^*(\cdot, 0)x) = F_0$ and the proof is complete.

4. – Evolution operators, generators, and generalized solutions.

In section 1 we motivated the concept of generator and evolution operator. In this section we propose to give a partial answer to the question when does an operator generate an evolution operator and in what sense does the evolution operator provide a solution to (1.1). Our results are taken from [1] and the reader should consult [1] for further explanation and references.

Let X be a real Banach space with norm $\|\cdot\|$. A subset A of $X \times X$ is in the class $\mathcal{A}(\omega)$ if for $\lambda > 0$ such that $\lambda\omega < 1$ and each pair $[x_i, y_i] \in A$, $i = 1, 2$ we have $\|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geqslant (1 - \lambda\omega)\|x_1 - x_2\|$. We define for subsets A and B of $X \times X$ and any real number α the following:

(i) $D(A) = \{x : [x, y] \in A \text{ for some } y\}$

(ii) $R(A) = \{y : [x, y] \in A \text{ for some } x\}$

(iii) $Ax = \{y : [x, y] \in A \text{ for } x \in D(A)\}$

(iv) $\alpha A = \{[x, \alpha y] : [x, y] \in A\}$

(v) $A^{-1} = \{[y, x] : [x, y] \in A\}$

(vi) $A+B = \{[x, y+z]: [x, y] \in A \text{ and } [x, z] \in B\}$

(vii) $AB \quad = \{[x, z]: \text{for some } y[x, y] \in B \text{ and } [y, z] \in A\}$.

A is called *accretive* if $A \in \mathcal{A}(0)$. If λ is real, J_λ will denote the set $(I+\lambda A)^{-1}$ and $D_\lambda = D(J_\lambda) = R(I+\lambda A)$. It is not hard to see that for $A \in \mathcal{A}(\omega)$, $\lambda \geqslant 0$, $\lambda\omega < 1$, J_λ is actually a function (even though A viewed as an operator may be multivalued) and for $x, y \in D_\lambda$

$$\|J_\lambda x - J_\lambda y\| \leqslant (1-\lambda\omega)^{-1}\|x-y\| .$$

Now let T, ω denote real numbers, $T > 0$, and assume $A(t)$ satisfies:

(A.1) $A(t) \in \mathcal{A}(\omega)$ for $0 \leqslant t \leqslant T$,

(A.2) $\overline{D(A(t))} = X$ is independent of t,

(A.3) $R(I+\lambda A(t)) = X$ for $0 \leqslant t \leqslant T$ and $0 < \lambda < \lambda_0$, where $\lambda_0 > 0$, $\lambda_0\omega < 1$.

Let $J_\lambda(t) = (I+\lambda A(t))^{-1}$. The t-dependence of $A(t)$ will be restricted by the condition:

(C.1) There is a continuous function $f: [0, T] \to Y$ and a monotone increasing function $L: [0, \infty] \to [0, \infty]$ such that $\|J_\lambda(t)x - J_\lambda(\tau)x\| \leqslant \|f(t) - f(\tau)\| \cdot L(\|x\|)$ for $0 < \lambda < \lambda_0$, $0 \leqslant t$, $\tau \leqslant T$, and $x \in X$, Y any Banach space (1).

It follows from conditions (A.1) to (A.3) that for each fixed τ, $0 \leqslant \tau \leqslant T$, there is a semigroup $S_\tau(t)$ on X associated with $A(\tau)$. See [5]. The condition (C.1) assumes only the continuity of f and corresponds roughly to the case when $A(t)$ has the form $A(0) + B(t)$, where $B(t)$ is well—behaved in x.

As mentioned in Section 1 one would hope that the evolution operator $U(t, s)x$ will provide in some sense a generalized solution to the Cauchy problem (1.1). This is indeed the case when $A(t)$ satisfies (A.1) to (A.3) and (C.1). Since $A(t)$ may in general be multivalued, we will again write our Cauchy problem in the form

$$\dot{u}(t) + A(t)u(t) \ni 0\,, \qquad u(s) = x\,. \tag{4.1}$$

This representation of the equation while seemingly at first a bit unusual has become an inherent concept in the theory of nonlinear semigroups and evolution equations. (See for example [5] and the references given there.) The main justification for needing to consider multivalued operators $A(t)$

(1) In [1] condition (C.1) was given with $Y = X$. It has been noted in [13] that an arbitrary Banach space Y will also work.

is that condition (A.3) may not be satisfied for $A(t)$ single valued but only for its multivalued extension.

We now make the following definition. A function $u\colon [s, T] \to X$ is called a *strong solution* of (4.1) if:

(i) u is continuous on $[s, T]$ and $u(s) = x$,

(ii) u is absolutely continuous on compact subsets of (s, T),

(iii) u is differentiable a.e. on (s, T) and satisfies (1.1) a.e.

We can now state the following fundamental theorem.

THEOREM. 4.1. [1] Let $A(t)$ satisfy (A.1), (A.2), (A.3), (C.1). Then there exists an evolution operator $U(t, s)$ on X. Furthermore if there exists a strong solution $u(t)$ of (4.1) then $u(t) = U(t, s)x$ for $0 \leqslant s \leqslant t \leqslant T$.

Theorem 4.1 provides us with an answer to our questions on existence of generators and evolution operators and yields explicitly in what sense we take $U(t, s)x$ to be a generalized solution of (4.1).

There is a class of equations to which Theorem 4.1 may easily be applied. This is the class of equations where $A(t) = A - h(t)$. More specifically we call an initial value problem of the form

$$\begin{cases} \dot{u}(t) + Au(t) \ni h(t) & s \leqslant t \leqslant T\,, \\ u(s) = x\,, \end{cases} \tag{4.2}$$

quasi-autonomous if A is t-independent and $h\colon [0, T] \to X$ is single-valued. For quasi-autonomous equations, Theorem 4.1 has been applied in [1] to yield the following result which we state here in a more restricted and simplified form.

THEOREM. 4.2. Let $A \in \mathcal{A}(\omega)$, $R(I + \lambda A) = X$ for $0 < \lambda < \lambda_0$, $\overline{D(A)} = X$, and $h\colon [0, T] \to X$ be continuous. Then there exists an evolution operator $U(t, s)$ on X. Furthermore if u is a strong solution of (4.2), then $u(t) = U(t, s)x$.

5. – Strong compactness of sets of generators.

As was shown in Section 2 and 3 strong compactness of a set of generators is a sufficient condition for existence of solutions to the time optimal and cost optimal control problems. It is apparent that we must now pose the question: When is a set of generators strongly compact? Let us make the following definition. A set $\mathcal{C}$ of, possibly unbounded, nonlinear operators on X with common domain D is said to be a *uniform set of generators* if the hypothesis of Theorem 4.1 are satisfied uniformly for $A(t)\in\mathcal{C}$, $0\leqslant t\leqslant T$. (By this we mean the same f, L, λ_0, ω work for each $A(t)\in\mathcal{C}$.)

We can now state our basic result on strong compactness of sets of generators.

THM. 5.1. Let $\mathcal{C}$ be a uniform set of generators. Assume that for $A(t)\in\mathcal{C}$, $0\leqslant t\leqslant T$, we have

(i) $A(t)$ are single valued and closed,

(ii) $D(A(t))=D(A(0))$.

Then if $\mathcal{C}$ is compact it is strongly compact.

PROOF. The proof is a direct combination of Theorem 4.1 of [1] and Theorem 4.1 of [6] where we have made some additional simplifying assumptions to reduce the complexity of presentation.

In the quasi-autonomous case a much simpler condition for strong compactness may be given.

THEOREM 5.2. Let $\mathcal{C}$ be a set of generators of the form $A(t)=A-h(t)$ where A is fixed, all h belong to a compact subset of $C([0,T];X)$, and A and all h satisfy the conditions of Theorem 4.2. Then $\mathcal{C}$ is strongly compact.

PROOF. The proof follows directly from Lemma 5.2 of [1].

Having stated sufficient conditions for strong compactness of a set $\mathcal{C}$ of generators we may now proceed to some examples.

6. – A nonlinear wave equation with control in the damping coefficient.

Let Ω be a bounded domain in R^n with smooth boundary. Let $H_2(\Omega)$, $H_1^0(\Omega)$, $H_0(\Omega)$ denote the usual Sobolev spaces [2]. We will consider the nonlinear hyperbolic control system by

$$(6.1)\qquad \begin{cases} \ddot y-\Delta y+\gamma(\dot y)+b(t)\dot y=0\,, & 0\leqslant t\leqslant T\,,\\ \qquad\qquad\qquad y=0 & \text{on } \partial\Omega\,,\end{cases}$$

where Δ is the Laplacian, $\gamma\colon R\to R$ is monotone non-decreasing, $\gamma(0)=0$, and γ viewed as a map on $H_0(\Omega)$ into $H_0(\Omega)$ is continuous. For example $\gamma(\varphi)=\|\varphi\|^{p-1}_{H_0(\Omega)}\varphi$, $p>1$, is a possible choice; see [2, p. 344].

The controls $b(t)$ lie in the *admissible control set*
$V=\{$a set of nonnegative equi-bounded and equi-lipschitzian functions on $[0,T]\to R$, i.e. $|b(t)-b(\tau)|\leqslant M|t-\tau|$ (M independent of b), $N\geqslant b(t)\geqslant 0\}$.

By the Ascoli-Arzela Theorem V is compact. We write (6.1) as a system

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 - \Delta y_1 + \gamma(y_2) + b(t) y_2 = 0$$
and let $u = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Let X (the state space) be given by $H_1^0(\Omega) \oplus H_0(\Omega)$ which is endowed with the « energy » inner product

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$(v, w)_X = \int_\Omega \{\nabla v_1 \cdot \nabla w_1 + v_2 w_2\}\, dx .$$

Let us introduce the operators

$$A_1 = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \qquad A_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & \gamma + b(t) \end{pmatrix}$$

where $D(A_1) = H_2(\Omega) \cap H_1^0(\Omega) \oplus H_1(\Omega)$ and $D(A_2(t)) = X$. In this case it is well known that A_1 is the infinitesimal generator of a C_0 contraction semigroup on X (the generalized solution to the wave equation with homogeneous boundary conditions). A_1 is closed on $D(A_1)$ and satisfies (A.1), (A.2), (A.3) where $\omega = 0$ (say by the Hille-Yosida-Phillips Theorem).

Define $A(t) = -A_1 + A_2(t)$. $\mathcal{C}$ will denote the set of operators $A(t)$ of this form where $b \in V$.

We now see that our original control system (6.1) can be written as

$$\dot{u}(t) + A(t)\, u(t) = 0 \qquad \text{where } D(A(t)) = D(A_1) .$$

We now wish to show $\mathcal{C}$ is a uniform set of generators. Clearly for $A(t) \in \mathcal{C}$, $D(A(t)) = D(A_1)$ so all members of $\mathcal{C}$ possess common domain $D(A_1)$. For $A(t) \in \mathcal{C}$ integration by parts shows $(A(t)x - A(t)y, x - y)_X \geq 0$ so $A(t)$ is monotone and it follows that $A(t) \in \mathcal{A}(0)$. Thus $\omega = 0$ works for all $A(t) \in \mathcal{C}$ and (A.1) is satisfied. $\overline{D(A(t))} = \overline{D(A_1)} = H_1^0(\Omega) \oplus H_0(\Omega) = X$ so condition (A.2) is satisfied. To check the range condition (A.3) we will employ the following theorem of Webb [7].

THM. 6.1. [7] Let A be a closed, densely defined, linear accretive operator from a Banach space X to itself, $R(I + \lambda A) = X$ for some $\lambda > 0$.

Let B be a continuous everywhere defined nonlinear accretive operator from X to itself. Then $A+B$ is accretive and $R(I+\lambda A(t))=X$ for some $\lambda>0$.

Using the above theorem with $A=-A_1$ and $B=A_2(t)$ we see $R(I+\lambda A(t))=X$ for some $\lambda>0$. (Any $\lambda>0$ is sufficient for our purposes since $\omega>0$.) Thus (A.3) is uniformly satisfied.

We must now check to see if (C.1) is uniformly satisfied for $A(t)\in \mathfrak{C}$. Let us define, as usual, $J_\lambda(t)=(I+\lambda A(t))^{-1}$. Then for $x\in X$, $\lambda>0$, and $z(t)=J_\lambda(t)x$ we have

$$z(t)-z(\tau)=-\lambda\big(A(t)z(t)-A(\tau)z(\tau)\big)\,.$$

Taking the inner product in X with $z(t)-z(\tau)$ we get

$$\|z(t)-z(\tau)\|_X^2=-\lambda\big(A_2(t)(t)-A_2(t)\,z(\tau),\,z(t)-z(\tau)\big)_X$$

since A_1 is skew self-adjoint on $D(A_1)\subset X$. From the monotonicity of γ on $H_0(\Omega)$ we get

$$\|z(t)-z(\tau)\|_X^2\leqslant-\lambda\left(\begin{pmatrix}0&0\\0&b(t)\end{pmatrix}z(t)-\begin{pmatrix}0&0\\0&b(t)\end{pmatrix}z(\tau)+\right.$$
$$\left.+\begin{pmatrix}0&0\\0&b(t)\end{pmatrix}z(\tau)-\begin{pmatrix}0&0\\0&b(\tau)\end{pmatrix}z(\tau)\,,\;z(t)-z(\tau)\right)_X.$$

Since $b(t)\geqslant 0$ we have

$$\|z(t)-z(\tau)\|_X^2\leqslant-\lambda\left(\begin{pmatrix}0&0\\0&b(t)\end{pmatrix}z(\tau)-\begin{pmatrix}0&0\\0&b(\tau)\end{pmatrix}z(\tau),\;z(t)-z(\tau)\right)_X.$$

Since $J_\lambda(t)0=0$ we have $\|z(t)\|_X\leqslant\|x\|_X$ and because $|b(t)-b(\tau)|\leqslant M|t-\tau|$ for all $b\in V$ we have from the Schwarz inequality that $\|J_\lambda(t)x-J_\lambda(\tau)x\|_X\leqslant$ $\leqslant\lambda M|t-\tau|\,\|x\|_X$ for all $x\in X$, $\lambda>0$, and for all $A(t)\in\mathfrak{C}$. Thus condition (C.1) is satisfied and we have proven

LEMMA 6.1. $\mathfrak{C}$ is a uniform set of generators.

Now let

$$\{A^\beta(t)\}=\left\{-A_1+\begin{pmatrix}0&0\\0&\gamma+b_\beta(t)\end{pmatrix}\right\}$$

be a sequence in $\mathfrak{C}$, $0\leqslant\beta<1$. Since $\{b_\beta\}\subset V$ is compact there exists a subsequence $\{b'_\beta\}$ of $\{b_\beta\}$ so that $b'_\beta(t)\to b^*(t)\in V$ uniformly for $0\leqslant t\leqslant T$ as $\beta'\downarrow 0$.

Denoting

$$A^*(t) = -A_1 + \begin{pmatrix} 0 & 0 \\ 0 & \gamma + b^*(t) \end{pmatrix}$$

we see $A^*(t) \in \mathcal{C}$ and it is obvious that $A^{\beta'}(t)x \to A^*(t)x$ for all $x \in D(A_1)$. Thus we have proven

LEMMA 6.2. $\mathcal{C}$ is a compact set of generators.

Finally we must check the hypothesis of Theorem 5.1. Clearly, since A_1 is closed and single valued and $A_2(t)$ is continuous and single-valued, $A(t)$ is closed and single-valued. Also $D(A(t)) = D(A_1)$. Thus the hypothesis of Theorem 5.1 are satisfied and we have proven

LEMMA 6.3. $\mathcal{C}$ is a strongly compact set of generators.

Applying Lemma 6.3 to Theorem 2.1 and 3.1 we obtain

THM. 6.2. For system (6.1) with the set of admissible controls V the time optimal and cost optimal control problems possess solutions.

REMARK. We may also consider other admissible sets of controls. Following a suggestion of J. L. Lions we take

$$V = \{b;\ b \text{ in a bounded set of } H_1[0, T],\ b(t) \geqslant 0,\ 0 \leqslant t \leqslant T\}.$$

As before conditions (A.1), (A.2) are satisfied and condition (A.3) is satisfied uniformly. We will check condition (C.1) later. To see that $\mathcal{C}$ is a uniform set of generators we note that for a sequence $\{b_\beta\} \subset V$ there exists a subsequence $\{b_{\beta'}\} \subset V$ so that $b_{\beta'} \to b^* \in V$ weakly in $H_1[0, T]$ and strongly in $C[0, T]$. Thus as before $A^{\beta'}(t)x \to A^*(t)x$ for all $x \in D(A_1)$ and $\mathcal{C}$ is a uniform set of generators. Lemmas 6.1 and 6.2 follow as before and Thm. 6.2 will be proven for our new V if we can verify (C.1).

In an attempt to check (C.1) we note that for $b \in V$,

$$|b(t) - b(\tau)| = \left| \int_t^t \dot{b}(s)\, ds \right| \leqslant \text{const } |t - \tau|^{\frac{1}{2}}.$$

Hence following the argument for condition (C.1) as given before we will obtain the inequality

$$\|J_\lambda(t)x - J_\lambda(\tau)x\|_X \leqslant \text{const } |t - \tau|^{\frac{1}{2}} \|x\|_X.$$

At this point one might attempt to find a continuous function $f\colon [0, T] \to R$

so that

$$|f(t)-f(\tau)|\geqslant|t-\tau|^{\frac{1}{2}}.$$

Such a function f would fulfill the hypotheses of (C.1) where $Y=R$. It has been noted in [13] that a search for such a function f will prove fruitless for it is corollary of the Denjoy-Young-Saks Theorem that no such function exists. However following a trick given in [13] we note that for $Y=C[0,T]$ and

$$f(t)(s)=\begin{cases}0 & s\leqslant t\\ |t-s|^{\frac{1}{2}} & s>t\end{cases}$$

we obtain the inequality

$$\|f(t)-f(\tau)\|_Y\geqslant|t-\tau|^{\frac{1}{2}}$$

and f is continuous. Thus for this f we conclude

$$\|J_\lambda(t)x-J_\lambda(\tau)x\|_X\leqslant\text{const}\ \|f(t)-f(\tau)\|_Y\|x\|_X$$

and condition (C.1) is verified. Thus Thm. 6.2 follows for our new set of admissible controls as well.

7. – A continuum approach to highway traffic control.

Lighthill and Whitham [8] have formulated a continuum model of traffic flow which is presented here in a form given by Dafermos [9]. Consider an infinitely long highway $-\infty<x<\infty$, time t, $0\leqslant t\leqslant T$, and adopt the following notation:

$v(x,t)=$ vehicle speed at point x at time t (e.g. yds/sec.)

$k(x,t)=$ vehicle concentration at point x at time t (e.g. vehicles/yd)

$q(x,t)=$ rate of traffic flow at point x at time t (e.g. vehicles/sec.)

$h(x,t)=$ traffic influx rate from side roads at point x at time t (e.g. vehicles/yd sec.).

We note that $h(x,t)$ maybe greater or less than zero depending on whether there is vehicle influx or outflux, respectively.

We now note the compatibility condition that the rate of traffic flow equals the vehicle speed times vehicle concentration or

$$q=vk. \tag{7.1}$$

Also we make the hypothesis that the vehicle speed depends on vehicle concentration so $v = v(k)$ and hence

$$q = v(k)\,k = q(k) \tag{7.2}$$

i.e. the rate of traffic flow depends on vehicle concentration.

We assume that when there is a maximum concentration of vehicles no vehicle is moving, i.e. $v(k_{\max}) = 0$. Similarly when there is a minimum concentration of vehicles (say $k = 0$ approximately) the vehicles (of which there will be almost none) will move at the maximum velocity $v_{\max}$ allowed by law.

Also we assume that the vehicles can move only forward ($v \geqslant 0$) and vehicles take up a positive amount of space ($k \geqslant 0$).

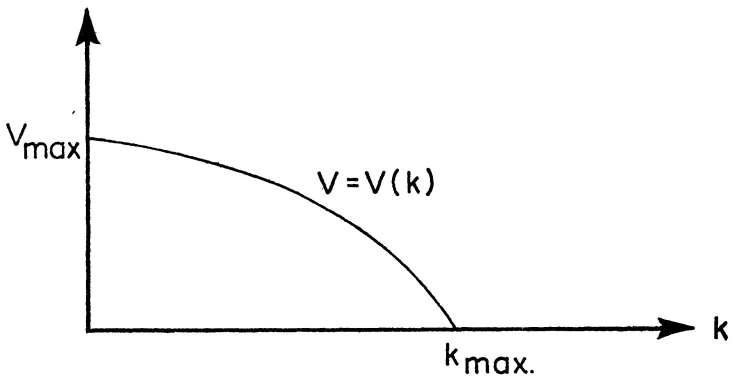

Figure 1

Since $q(k) = v(k)\,k$ we see $q(0) = 0$, $q(k_{\max}) = 0$ and therefore we have Fig. 2.

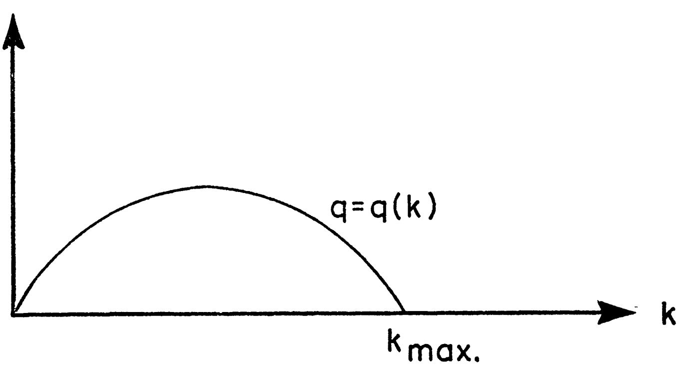

Figure 2

We assume $q(k)$ is continuous in k and set

$$\begin{cases} q(k) = 0 & k < 0 \\ q(k) = 0 & k > k_{\max} \end{cases}$$

so $q\colon R \to R$.

Now let us derive our state equation for the vehicle concentration $k(x, t)$. Let $x_2 > x_1$, $t_2 > t_1$.

The number of vehicles in (x_1, x_2) at time $t_1 = \int_{x_1}^{x_2} k(x, t_1)\, dx$.

The number of vehicles in (x_1, x_2) at time $t_2 = \int_{x_1}^{x_2} k(x, t_2)\, dx$.

The number of vehicles entering from side roads during (t_1, t_2)

$$= \int_{x_1}^{x_2} \int_{t_1}^{t_2} h(x, t)\, dt\, dx\,.$$

The number of vehicles entering through x_1 in (t_1, t_2)

$$= \int_{t_1}^{t_2} q(x_1, t)\, dt\,.$$

The number of vehicles exiting through x_2 in (t_1, t_2)

$$= \int_{t_1}^{t_2} q(x_2, t)\, dt\,.$$

The number of vehicles must then satisfy a conservation law, i.e. number of vehicles in (x_1, x_2) at time t_2 — number of vehicles in (x_1, x_2) at time t_1 = number of vehicles entering through x_1 during (t_1, t_2)-number of vehicles exiting through x_2 during (t_1, t_2) + number of vehicles entering through side roads during (t_1, t_2). This law may be written as

$$\int_{x_1}^{x_2} k(x, t_2)\, dx - \int_{x_1}^{x_2} k(x, t_1)\, dx = \int_{t_1}^{t_2} q(x_1, t)\, dt - \int_{t_1}^{t_2} q(x_2, t)\, dt + \int_{x_1}^{x_2} \int_{t_1}^{t_2} h(x, t)\, dt\, dx$$

or

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\left\{\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}-h\right\}dt\,dx=0\ .$$

Thus it makes sense to consider as our conservation law the differential equation

$$\left\{\begin{aligned}\frac{\partial k}{\partial t}(x,t)+\frac{\partial q}{\partial x}\big(k(x,t)\big)&=h(x,t)\\ k(x,0)&=k_0(x)\end{aligned}\right. \tag{7.3}$$

where our *state is the vehicle concentration k and our control is the traffic influx (or possibly outflux) rate h from side roads.*

It is not apparent that (7.3) possesses solutions in its present form so we will generalize (7.4) to make use of the results of Crandall [10] and Benilan [11].

DEFINITION 7.1. A_0 is the operator in $L^1(R)$ defined by: $y\in D(A_0)$ and $w\in A_0(y)$ if $y,\ w\in L^1(R)$, $q(y)\in L^1(R)$ and

$$\int_R \operatorname{sign}_0\big(y(x)-c\big)\{\big(q(y(x))-q(c)\big)f_x(x)+w(x)f(x)\}\,dx\geqslant 0$$

for every $f\in C_0^\infty(R)$ such that $f>0$ and every $c\in R$. Here

$$\operatorname{sign}_0 r=\left\{\begin{array}{rl}1 & \text{if } r>0\\ 0 & \text{if } r=0\\ -1 & \text{if } r<0\ .\end{array}\right.$$

To show the relation between A_0 and our original equation (7.3) we note

LEMMA 7.1. [10] Let $q\in C^1$ and A_0 be given by Definition 7.1. If $y\in C_0^1(R,R)$ then $y\in D(A_0)$ and $A_0y=\{q(y)_x\}$.

Thus we see that A_0y is in some sense a generalization of $\{q(y)_x\}$. We will need to generalize $\{q(y)_x\}$ one step further. Let A be *the closure of* A_0, i.e. $y\in D(A)$ and $w\in Av$ if there are sequences $\{y_k\}\subseteq D(A_0)$ and $\{w_k\}$ such that $w_k\in A_0y_k$ and $y_k\to y$, $w_k\to w$ in $L^1(R)$.

We now may write out state equation (7.1) in the generalized form

$$\frac{dk}{\partial t}+Ak\ni h(t)$$

where A is as given above.

LEMMA 7.2. For $X = L^1(R)$, A is accretive, and $R(I + \lambda A) = X$ for $\lambda > 0$.

PROOF. The proof is an immediate consequence of Theorem 1.1 of [10].

We are almost at the point where we may apply Theorem 4.2 to gain existence and uniqueness of generalized solutions to our generalized model equation (7.4). The only point lacking is to show $\overline{D(A)} = L^1(R)$. This is done in the following lemma.

LEMMA 7.3. $\overline{D(A)} = L^1(R)$.

PROOF. We will follow an argument used by Benilan [11, Chap. II].

Let $\varphi\colon R \to R$ continuous and $\Psi \in L^1_{\mathrm{loc}}(\Omega)$ where Ω is open in R. We say y is a solution in the sense of Kruskov on Ω of $\varphi(y)_x = \Psi$ if there exists a function $y \in L^\infty(R)$ such that for all $f \in C_0^\infty(\Omega)$ such that $f > 0$ and all $C \in R$,

$$\int_R \operatorname{sign}_0\big(y(x) - c\big)\big\{\big(\varphi\big(y(x)\big) - \varphi(c)\big) f_x + \Psi f\big\}\, dx \geqslant 0 .$$

Again let $\varphi\colon R \to R$ continuous and define $D_\varphi = \{y \in L^1(R) \cap L^\infty(R);$ there exists Ψ, $\Psi \in L^1(R)$, so that y is a solution in the sense of Kruskov on R of $\varphi(y)_x = \Psi\}$.

Assume $\varphi(0) = 0$ and again that $\varphi\colon R \to R$ is continuous, then Benilan has shown in [11, Corollary 2.8] that $\overline{D_\varphi} = L^1(R)$ Now choosing $q = \varphi$ we see by our definition of $D(A_0)$ that $D_q \subset D(A_0)$. Thus $D_q \subset D(A_0) \subset D(A) \subset$ $\subset L^1(R)$ and since $\overline{D_q} = L^1(R)$ we have $\overline{D(A)} = L^1(R)$.

We may now apply Thm. 4.2 to conclude

THEOREM. 7.1. For $X = L^1(R)$ and $h\colon [0, T] \to X$ continuous there exists an evolution operator $U(t, s)$ on X for (7.4). Furthermore, if k is a strong solution of (7.4) then $k(x, t) = U(t, s)k_0(x)$, for $k_0 \in X$.

Having resolved the question of existence and uniqueness of generalized solution to our model (7.4) we are now in the position to take up some optimization problems. First let us consider the time optimization problem. Let us assume there is a desired set of traffic concentrations which we may desire to reach in minimum time. We may have our particular traffic concentration as our objective but we allow a set as our goal to permit certain tolerances. Let us call the desired set of traffic concentrations K and assuming K to be a closed subset of $L^1(R)$ we may apply Theorems 5.2 and 2.1 to conclude

THEOREM. 7.2. If the set of allowable traffic influxes and outfluxes V is a compact subset of $C([0, T], L^1(R))$ then the time optimal control problem has a solution.

Similarly we can also consider a cost optimization problem. Let us assume there is a traffic concentration $k_1(x)$, $k_1 \in L^1(R)$, which we can reach in some time $\tau > 0$. We may desire that the error between $k(x, t)$ and $k_1(x)$, measured in the appropriate $L^1(R)$ norm, be minimized uniformly in $0 \leqslant t \leqslant \tau$. That is we wish to minimize the cost functional

$$F(k) = \sup_{0 \leqslant t \leqslant \tau} \int_R |k(x, t) - k_1(x)| dx$$

where $k(x, \tau) = k_1(x)$. Since $F\colon C([0, \tau]; L^1(R)) \to R$ is continuous, we may immediately apply Theorems 5.2 and 3.1 to conclude

THEOREM. 7.3. If the set of allowable traffic influxed and outfluxes V is a compact subset of $C([0, T]; L^1(R))$, then the above cost optimization problem has a solution.

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