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Two Point Boundary Value Problems for Operational Differential Equations.

H. O. FATTORINI (*)

§ 1. – Let E be a complex Banach space, A a linear operator with domain $D(A)$ dense in E and range in E , n a positive integer, α_0 and α_1 subsets of the set of integers $\{0, 1, \dots, n-1\}$, T a positive real number. We consider in the sequel the two-point boundary value problem

$$(1.1) \quad u^{(n)}(t) = Au(t) + f(t) \quad (0 \leq t \leq T)$$

$$(1.2) \quad u^{(j)}(0) = u_{0,j} (j \in \alpha_0) \quad u^{(k)}(T) = u_{1,k} (k \in \alpha_1)$$

Here $u_{0,j}$ and $u_{1,k}$ are elements of E ; the function $u(\cdot)$ is defined in $0 \leq t \leq T$, takes values in $D(A)$ and is n times continuously differentiable (in the sense of the norm of E), while f is a continuous, E -valued function defined in $0 \leq t \leq T$.

For the sake of future reference, we shall say that the boundary value problem (1.1), (1.2) satisfies condition \mathfrak{E} (existence) if

there exists a subspace D dense in E such that (1.1), (1.2) has a solution $u(\cdot)$ for any f continuous in $0 \leq t \leq T$ and any $u_{0,j}, u_{1,k} \in D (j \in \alpha_0, k \in \alpha_1)$.

We shall denote by \mathfrak{E}_0 the particular case of Condition \mathfrak{E} where $f = 0$.

On the other hand, (1.1), (1.2) satisfies Condition \mathfrak{CD} (Continuous Dependence) if

for any sequence $\{u_m(\cdot)\}$ of solutions of

$$u_m^{(n)}(t) = Au_m(t) + f_m(t)$$

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such that

$$f_m(t) \rightarrow 0 \quad \text{uniformly in } 0 \leq t \leq T$$

and such that

$$u^{(j)}(0) \rightarrow 0 \quad (j \in \alpha_0), \quad u^{(k)}(T) \rightarrow 0 \quad (k \in \alpha_1)$$

we have

$$u(t) \rightarrow 0 \quad \text{uniformly in } 0 \leq t \leq T.$$

We denote by \mathcal{CD}_0 (resp. \mathcal{CD}_1) the particular case of Condition \mathcal{CD} where $f \equiv 0$ (resp. $u^{(j)}(0) = u^{(k)}(T) = 0$, $j \in \alpha_0$, $k \in \alpha_1$). Note that both conditions \mathcal{CD}_0 and \mathcal{CD}_1 imply iniqueness of the solutions of (1.1), (1.2).

The problem of finding conditions on α_0 , α_1 , A in order that Conditions \mathcal{E} or \mathcal{CD} (or variants thereof) should hold has been considered by numerous authors, for systems of the type of (1.1), (1.2) or for systems that are more general in various senses and for various definitions of solutions. The reader is referred to [2], [4], [8], [9] and specially to [10] (where additional bibliography can be found) for a sampling of results of this type.

We consider in this paper a problem which is, in a sense, converse to the one outlined above, and which can be roughly formulated as follows. Assume one (or both) of Conditions \mathcal{E} and \mathcal{CD} or of their particular cases hold: what can we conclude about the sets α_0 and α_1 and the operator A ? We show in what follows that, under mild assumptions on A , if condition \mathcal{CD}_1 holds then

$$m_0 + m_1 \geq n,$$

where m_0 (resp. m_1) is the number of elements of α_0 (resp. α_1). On the other hand, if Conditions \mathcal{E}_0 and \mathcal{CD}_0 hold,

$$m_0 + m_1 \leq n.$$

These conclusions are not surprising if one considers the particular case where A reduces to multiplication by a complex number in onedimensional Banach space. A perhaps less evident fact is that, when \mathcal{E}_0 and \mathcal{CD} hold, unless A is *bounded* the boundary conditions (1.2) must be « evenly distributed » among the two end-points; precisely, we must have

$$] (n-1)/2 [\leq m_0, \quad m_1 \leq] (n+1)/2 [$$

where $]p[$ denotes the greatest integer less or equal than p . This kind of result is important in view of the fact that the study of the abstract boundary value

problem (1.1), (1.2) is motivated mainly by the particular case where A is a differential operator acting in some function space (more often than not, $L^2(\Omega)$ for some domain Ω). In this case A is in general unbounded and the result above rules out the differential operator interpretation.

Problems that are connected with the present one were considered in [11], [6] and [7]. In [11] and [6] the basic interval is $[0, \infty)$ rather than $[0, T]$ and the second set of boundary conditions (1.2) is replaced by a growth restriction at infinity, while in [7] solutions are singled out on the basis of boundary conditions at the origin plus some unspecified selection rule satisfying certain conditions. Some of the methods in this paper are very similar to those in [7]; the main idea is due to Radnitz ([11]) and consists on the construction of the resolvent of A by means of application of certain functionals to suitable « test functions »; this idea is combined here with a construction of approximate resolvents due to Chazarain ([3]).

§ 2. — We assume in the sequel that $\varrho(A) \neq \emptyset$, that is, that $R(\lambda; A) = (\lambda I - A)^{-1}$ exists for some complex λ . This is known to imply (see [5], Chapter VII) that A is closed; moreover, the domain $D(A^*)$ of A^* —the adjoint of A —is weakly dense in the dual space E^* , $\varrho(A^*) = \varrho(A^*)$ and $R(\lambda; A^*) = R(\lambda; A)^*$ for λ there.

Let ϱ, ω be positive numbers. We define $A(\varrho, \omega)$ as the set of all complex numbers μ such that

$$\operatorname{Re} \mu < \min(-\varrho \log |\mu|, -\omega)$$

If m, n are integers ($1 \leq m \leq n$), $A^*(n, m, \varrho, \omega)$ consists of all λ such that at least m of the n -th roots of λ lie in $A(\varrho, \omega)$.

2.1. LEMMA. *Assume Condition \mathcal{CD}_1 holds with $m_0, m_1 < n$. Then there exist nonnegative numbers ϱ, ω and analytic functions $\mathcal{R}_0(\lambda), \mathcal{R}_1(\lambda)$ with values bounded operators in E^* satisfying*

$$(2.1) \quad (\lambda I - A^*) \mathcal{R}_0(\lambda) = I(\lambda \in A^*(n, m_0 + 1, \varrho, \omega))$$

$$(2.2) \quad (\lambda I - (-1)^n A^*) \mathcal{R}_1(\lambda) = I(\lambda \in A^*(n, m_1 + 1, \varrho, \omega)).$$

There exists $\tau > 0$ such that

$$(2.3) \quad |\mathcal{R}_0(\lambda)| \leq K |\lambda|^{-\tau} \quad (\lambda \in A^*(n, m_0 + 1, \varrho, \omega))$$

$$(2.4) \quad |\mathcal{R}_1(\lambda)| \leq K |\lambda|^{-\tau} \quad (\lambda \in A^*(n, m_1 + 1, \varrho, \omega)).$$

Moreover \mathcal{R}_0 (resp. \mathcal{R}_1) is analytic in $\Lambda^*(n, m_0 + 1, \varrho, \omega)$ (resp. $\Lambda^*(n, m_1 + 1, \varrho\omega)$) except on a line Γ_0 (resp. Γ_1) that divides $\Lambda^*(n, m_0 + 1, \varrho, \omega)$ (resp. $\Lambda^*(n, m_1 + 1, \varrho\omega)$) in at most two connected pieces.

PROOF. Let $\lambda \in \Lambda^*(n, m_0 + 1, \varrho, \omega)$ (with ϱ, ω as yet undetermined) and let μ be the n -th root of λ with least positive argument that lies in $\Lambda(\varrho, \omega)$ (we count arguments, as usual, from $-\pi$ to π). Then

$$\mu, \mu\gamma, \dots, \mu\gamma^{m_0} \quad (\gamma = \exp(2\pi i/n))$$

are all n -th roots of λ and belong to $\Lambda(\varrho, \omega)$. Let now j_0 be an arbitrary element of ∂_0 (∂_0 denotes the complement of α_0 in $\{0, 1, \dots, n-1\}$), and let

$$\eta(t, \lambda) = \det \Theta(t, \lambda) / \det \Theta^{(j_0)}(0, \lambda),$$

where the index between parentheses indicates t -differentiation and Θ is the following $(m_0 + 1) \times (m_0 + 1)$ matrix:

$$\Theta(t, \lambda) = \begin{bmatrix} \mu^j & (\mu\gamma)^j & \dots & (\mu\gamma^{m_0})^j \\ \dots & \dots & \dots & \dots \\ \exp(\mu t) & \exp(\mu\gamma t) & \dots & \exp(\mu\gamma^{m_0} t) \end{bmatrix},$$

the index j roaming over α . It is not difficult to check (see [7] for details) that $\eta(t, \lambda)$ is well defined for all $\lambda \neq 0$ (in fact, $\Theta^{(j_0)}$ is the Vandermonde matrix corresponding to $\gamma^j (j \in \alpha)$ and γ^{j_0} and that, if we write

$$(2.5) \quad \eta(t, \lambda) = \sum_{r=0}^{m_0} c_r(\lambda) \exp(\mu\gamma^r t)$$

then

$$(2.6) \quad |c_r(\lambda)| = K_r |\mu|^{j_0} = K_r |\lambda|^{j_0/n} \quad (0 \leq r < m_0).$$

On the other hand, it follows immediately from the definition of η that

$$(2.7) \quad \eta^{(n)}(t, \lambda) = \lambda \eta(t, \lambda)$$

and that

$$(2.8) \quad \eta^{(j)}(0) = 0 \quad (j \in \alpha_1), \quad \eta^{(j_0)}(0) = 1.$$

We shall carry through the proof of Lemma 2.1 by means of the application of certain linear functionals to (vector multiples) of η . We proceed now to the construction of these functionals.

Let $\mathcal{C}(T; E)$ be the space of all continuous, E -valued functions defined in $0 \leq t \leq T$ endowed with its usual supremum norm, and let \mathcal{K} be the subspace of $\mathcal{C}(T; E)$ consisting of all functions of the form

$$f(t) = u^{(n)}(t) - Au(t)$$

where $u(\cdot)$ is n times continuously differentiable in $0 \leq t \leq T$, takes values in $D(A)$ and satisfies

$$u^{(j)}(0) = 0 \quad (j \in \alpha_0) \quad u^{(k)}(T) = 0 \quad (j \in \alpha_1).$$

Let u^* be an arbitrary element of E^* . Define a functional $f \rightarrow \mathcal{F}(u^*, f)$ in \mathcal{K} by means of the formula

$$\mathcal{F}(u^*, f(\cdot)) = \mathcal{F}(u^*, u^{(n)}(\cdot) - Au(\cdot)) = \langle u^*, R(\lambda_0; A) u^{(j_0)}(0) \rangle$$

where λ_0 is some fixed element of $\rho(A)$ and $\langle u^*, u \rangle$ denotes the value of u^* at u . Clearly, condition \mathcal{CD}_1 implies that \mathcal{F} is well-defined; moreover, it is linear in f and in u^* . Let $\{f_m\}$ be a sequence in \mathcal{K} converging to zero. There, according to condition \mathcal{CD}_1 , if

$$f_m(t) = u_m^{(n)}(t) - Au_m(t)$$

we have

$$u_m(t) \rightarrow 0$$

uniformly in $0 \leq t \leq T$. Setting $v_m = R(\lambda_0; A) u_m(t)$, we have

$$v_m^{(n)}(t) = R(\lambda_0; A) f_m(t) + R(\lambda_0; A) Au_m(t)$$

which means that $v_m^{(n)}(t) \rightarrow 0$ uniformly in $0 \leq t \leq T$. But

$$v_m(t) = \sum_{k=0}^{n-1} t^k v_m^{(k)}(0) + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} v_m^{(n)}(s) ds$$

which is easily seen to imply that

$$v_m^{(k)}(0) \rightarrow 0 \quad (0 \leq k \leq n).$$

This shows that \mathcal{F} is jointly continuous in f and u^* , that is that

$$(2.9) \quad |\mathcal{F}(u^*, f)| \leq C |u^*| |f| \quad ((u^*, f) \in E^* \times \mathcal{K})$$

for some $C > 0$. Let now $E^* \otimes \mathcal{C}(T; E)$ be the (algebraic) tensor product of E^* and $\mathcal{C}(T; E)$. Let $\chi \in E^* \otimes \mathcal{C}(T; E)$ and define

$$|\chi| = \inf \sum_{\nu} |u_{\nu}^*| |f_{\nu}|$$

the inf taken over all finite sequences $u_1^*, u_2^*, \dots, f_1, f_2, \dots$ such that

$$(2.10) \quad \chi = \sum_{\nu} u_{\nu}^* \otimes f_{\nu}.$$

Then (see [12] for this and related facts) 1.1 is a semi-norm in $E^* \otimes \mathcal{C}(T; E)$. If we now define

$$(2.11) \quad \mathcal{E}(\chi) = \sum_{\nu} \mathcal{F}(u_{\nu}^*, f_{\nu})$$

(the choice of the decomposition (2.9) does not alter the definition of \mathcal{E}) clearly,

$$(2.12) \quad |\mathcal{E}(\chi)| \leq C|\chi|$$

thus we can apply the Hahn-Banach theorem to extend \mathcal{E} to a linear functional $\tilde{\mathcal{E}}$ defined in all of $E^* \otimes \mathcal{C}(T; E)$ and satisfying (2.12) as well. If we now define

$$\tilde{\mathcal{F}}(u^*, f) = \tilde{\mathcal{E}}(u^* \otimes f),$$

then it is clear that $\tilde{\mathcal{F}}$ is defined in $E^* \times \mathcal{C}(T; E)$, bilinear and satisfies (2.9). We shall write simply \mathcal{F} instead of $\tilde{\mathcal{F}}$.

Let φ be a C^{∞} scalar function of t that equals 1 in $0 \leq t \leq T/3$ and vanishes in $2T/3 \leq t \leq T$. Define, for $u^* \in E^*$, $u \in E$,

$$\mathcal{M}(\lambda, \varphi, u^*, u) = \mathcal{F}(u^*, \eta(\cdot, \lambda) \varphi(\cdot) u),$$

where η is the function constructed previously. Clearly, \mathcal{M} is a bounded bilinear form in $E^* \times E$; on the other hand, making use of (2.7) and of (2.8) we obtain

$$(2.13) \quad \begin{aligned} \lambda \mathcal{M}(\lambda, \varphi, u^*, u) &= \mathcal{F}(u^*, \lambda \eta(\cdot, \lambda) \varphi(\cdot) u) \\ &= \mathcal{F}(u^*, \eta^{(n)}(\cdot, \lambda) \varphi(\cdot) u) \\ &= \mathcal{F}(u^*, (\eta(\cdot, \lambda) \varphi(\cdot))^{(n)} u - \eta(\cdot, \lambda) \varphi(\cdot) A u) \\ &\quad + \mathcal{F}(u^*, \eta(\cdot, \lambda) \varphi(\cdot) A u) - \mathcal{F}\left(u^*, \sum_{r=0}^{n-1} \binom{n}{r} \eta^{(r)}(\cdot, \lambda) \varphi^{(n-r)}(\cdot) u\right) \\ &= \langle u^*, R(\lambda_0; A) u \rangle + \mathcal{F}(u^*, \eta(\cdot, \lambda) \varphi(\cdot) A u) \\ &\quad - \mathcal{F}(u^*, \xi(\cdot, \lambda) u). \end{aligned}$$

Define a bounded operator $\mathcal{R}(\lambda, \varphi): E^* \rightarrow E^*$ by means of the formula

$$(2.14) \quad \langle \mathcal{R}(\lambda, \varphi) u^*, u \rangle = \mathcal{M}(\lambda, \varphi, u^*, u).$$

Then it is easy to see that (2.13) implies that $\mathcal{R}(\lambda, \varphi) E^* \subseteq D(A^*)$ and that

$$(2.15) \quad (\lambda I - A^*) \mathcal{R}(\lambda, \varphi) = R(\lambda_0; A) - \mathcal{Q}(\lambda, \varphi).$$

where $\mathcal{Q}(\lambda, \varphi): E^* \rightarrow E^*$ is defined as $\mathcal{R}(\lambda, \varphi)$ by means of the formula

$$(2.16) \quad \langle \mathcal{Q}(\lambda, \varphi) u^*, u \rangle = \mathcal{F}(u^*, \xi(\cdot, \lambda) u).$$

If $u \in D(A)$ we have (taking into account the fact that ξ vanishes identically near zero)

$$\begin{aligned} \mathcal{F}(u^*, \xi(\cdot, \lambda) Au) &= \mathcal{F}(u^*, \xi^{(n)}(\cdot, \lambda) - \xi(\cdot, \lambda) A u) \\ &\quad + \mathcal{F}(u^*, \xi^{(n)}(\cdot, \lambda) u) = \mathcal{F}(u^*, \xi^{(n)}(\cdot, \lambda) u) \end{aligned}$$

This is immediately seen to imply that $\mathcal{Q}(\lambda; \varphi) E^* \subseteq D(A^*)$; more precisely, that, if $\tilde{\mathcal{Q}}(\lambda, \varphi) = (\lambda I - A^*) \mathcal{Q}(\lambda, \varphi)$,

$$(2.17) \quad \begin{aligned} \langle \tilde{\mathcal{Q}}(\lambda, \varphi) u^*, u \rangle &= \langle (\lambda_0 I - A^*) \mathcal{Q}(\lambda, \varphi) u^*, u \rangle \\ &= \mathcal{F}(u^*, (\lambda_0 \xi(\cdot, \lambda) - \xi^{(n)}(\cdot, \lambda)) u). \end{aligned}$$

We proceed now to estimate the right-hand side of (2.16). In view of (2.5) and (2.6) we have

$$(2.18) \quad |\eta^{(r)}(t, \lambda)| \leq K |\lambda|^{(j_0+r)/n} \max_{0 \leq p \leq m_0} \exp[\operatorname{Re}(\mu \gamma^p) t] \quad (0 \leq t \leq T)$$

for some $K > 0$. Taking into account this and (once again) the fact that ξ vanishes identically near zero, we obtain

$$|\lambda_0 \xi(t, \lambda) - \xi^{(n)}(t, \lambda)| \leq K' |\lambda|^{-\sigma} \leq K' \omega^{-\sigma} \quad (0 \leq t \leq T)$$

where K' is another constant and

$$\sigma = \frac{\rho T}{3n} - \frac{2n+1+j_0}{n}.$$

Accordingly, taking ϱ such that $\sigma > 0$ and ω large enough we may assume that

$$(2.19) \quad |\tilde{\mathcal{Q}}(\lambda, \varphi)| < \beta < 1 \quad (\lambda \in \mathcal{A}^*(n, m_0 + 1, \varrho, \omega)).$$

Hence $(I - \tilde{\mathcal{Q}}(\lambda, \varphi))^{-1} = \sum_{r=0}^{\infty} \tilde{\mathcal{Q}}(\lambda, \varphi)^r$ exists. Pre-multiplying now (2.15) by $(\lambda_0 I - \mathcal{A}^*)$ and post-multiplying it by $(I - \tilde{\mathcal{Q}}(\lambda, \varphi))^{-1}$ we obtain (2.1) with

$$(2.20) \quad \mathcal{R}_0(\lambda) = (\lambda I - \mathcal{A}^*) \mathcal{R}(\lambda, \varphi) (I - \tilde{\mathcal{Q}}(\lambda, \varphi))^{-1}$$

in $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$. The estimate (2.3) follows immediately from (2.5), (2.6), (2.15), (2.16), (2.19) and (2.20).

It remains to settle the analyticity question. Observe first that, ϱ being fixed, $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ is *connected* ω large enough. This can be seen as follows. Let π be the line issuing from the origin and tangent to the curve

$$(2.21) \quad \varkappa = \{\mu; \operatorname{Re} \mu = -\varrho \log |\mu|; \operatorname{Im} \mu < 0\}$$

(which is convex seen from the left half-plane) and let P be its point of intersection with (2.21). If $\operatorname{Re} \mu = -\omega$ cuts \varkappa in a point to the left of P , then no line issuing from the origin can cut the boundary of $\mathcal{A}(\varrho, \omega)$ more than once. But this is easily seen to imply that

$$(2.21) \quad \text{if } \lambda \in \mathcal{A}^*(n, m_0 + 1, \varrho, \omega) \text{ and } \tau \geq 1 \text{ then } \tau\lambda \in \mathcal{A}^*(n, m_0 + 1, \varrho, \omega).$$

On the other hand, it is plain that if $\lambda_1, \lambda_2 \in \mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ and $|\lambda_1| = |\lambda_2|$ then the arc of circumference joining λ_1 and λ_2 (centered at the origin) belongs as well to $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$. This combined with (2.21) shows that any two points in $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ can be joined by a curve contained there.

Let Γ_0 be the set of all complex λ such that one of the n -th roots of λ lies in the piece of the boundary of $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ lying in the upper half-plane. The curve Γ_0 may or may not belong to $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$; in case it does is a line of discontinuity of η , for as λ traverses Γ_0 counterclockwise the group of n -th roots of λ used to define η changes. On the other hand, it is plain that a point λ moving along a circle can cross Γ_0 only once in a rotation of 2π , so that Γ_0 can divide $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ in at most two connected pieces, \mathcal{A}_1 and \mathcal{A}_2 .

Now, it is immediate from its definition that η and any of its t -derivatives are analytic (uniformly with respect to t) in $\mathcal{A}_1, \mathcal{A}_2$, so that the last conclusion

of Lemma 2.1 that refers to \mathcal{R}_0 follows from known results on operator-valued analytic functions.

To prove the assertions on \mathcal{R}_1 , we only have to observe that the change of variable $t \rightarrow T - t$ transforms the boundary-value problem (1.1), (1.2) into

$$\begin{aligned} u^{(n)}(t) &= (-1)^n \mathcal{A}u(t) + f(T - t) \\ u^{(k)}(0) &= (-1)^j u_{1,k} \quad (k \in \alpha_1), \quad u^{(j)}(T) = (-1)^j u_{0,j} \quad (j \in \alpha_0) \end{aligned}$$

to which problem we can apply all the previous arguments.

2.2. REMARK. It is not clear whether we can conclude, on the basis of the assumptions of Lemma 2.1 that $\mathcal{R}_0 = R(\lambda; \mathcal{A})$, $\mathcal{R}_1 = R(\lambda; (-1)^n \mathcal{A})$; however, we can prove

2.3. COROLLARY. *Let the assumptions in Lemma 2.1 be satisfied.*

(a) *Suppose that*

$$(2.21) \quad \varrho(\mathcal{A}) \cap \mathcal{A}^*(n, m_0 + 1, \varrho, \omega) \neq \emptyset.$$

Then

$$(2.22) \quad \mathcal{A}^*(n, m_0 + 1, \varrho, \omega) \subseteq \varrho(\mathcal{A}) \quad \text{and} \quad \mathcal{R}_0(\lambda) = R(\lambda; \mathcal{A}) \quad \text{there}$$

(b) *The same conclusions (replacing \mathcal{A} by $(-1)^n \mathcal{A}$) hold for m_1, \mathcal{R}_1*

PROOF: Assume (2.21) holds; since $\varrho(\mathcal{A})$ is open, we can suppose that

$$(2.23) \quad \varrho(\mathcal{A}) \cap \mathcal{A}_1 \neq \emptyset \quad \text{or} \quad \varrho(\mathcal{A}) \cap \mathcal{A}_2 \neq \emptyset$$

where $\mathcal{A}_1, \mathcal{A}_2$ are the two connected pieces of $\mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$ determined by Γ_0 . Assume the first relation in (2.23) holds. Then—by uniqueness of the resolvent— $\mathcal{R}_0(\lambda) = R(\lambda; \mathcal{A}^*)$ in $\varrho(\mathcal{A}) \cap \mathcal{A}^*(n, m_0 + 1, \varrho, \omega)$; in particular,

$$\mathcal{R}_0(\lambda)(\lambda I - \mathcal{A}^*)u^* = u^*(u^* \in D(\mathcal{A}^*))$$

for $\lambda \in \varrho(\mathcal{A}) \cap \mathcal{A}_1$ and, a fortiori, by analytic continuation, for all $\lambda \in \mathcal{A}_1$: Accordingly,

$$\mathcal{R}_0(\lambda) = R(\lambda; \mathcal{A}^*)(\lambda \in \mathcal{A}_1).$$

As λ approaches Γ_0 , η and all of its t -derivatives remain bounded; thus the same is true of $R(\lambda; \mathcal{A}^*)$. But $|R(\lambda; \mathcal{A}^*)| \rightarrow \infty$ as λ approaches a point in

$\sigma(A^*)$, so we conclude that $\Gamma_0 \in \rho(A^*)$. This, in turn implies that $\rho(A^*) \cap \Lambda_2 \neq \emptyset$ and we can repeat the previous reasoning for Λ_2 , thus proving the Corollary. Naturally, the argument runs along the same lines if the second relation in (2.23) holds or if the proof of (b) is concerned.

2.4. REMARK. Note that the assumptions in Corollary 2.3 are satisfied if $\lambda I - A^*$ is one-to-one for some $\lambda \in \Lambda^*$. In fact, assume $D(A^*)$ endowed with its usual graph norm. The operator

$$(\lambda I - A^*): D(A^*) \rightarrow E^*$$

is then 1-1 and onto (the latter because of 2.1). By the closed graph theorem, $(\lambda I - A^*)^{-1}: E^* \rightarrow D(A^*)$ must be bounded. But there $(\lambda I - A^*)^{-1}$ is also bounded as an operator from E^* into E^* , that is $\lambda \in \rho(A^*)$.

Note also that the argument in Corollary 2.3 can be used to prove the following result: let $\mathcal{R}(\lambda)$ be an analytic operator-valued function defined in a connected set Ω such that $\Omega \cap \rho(A) \neq \emptyset$. Assume $\mathcal{R}(\lambda)E \subseteq D(A)$ and $(\lambda I - A)\mathcal{R}(\lambda) = I(\lambda \in \Omega)$. There $\Omega \subseteq \rho(A)$ and $\mathcal{R}(\lambda) = R(\lambda; A)$ there.

§ 3. - The main result here is

3.1. THEOREM. Assume Condition \mathcal{CD}_1 holds. Then

$$(3.1) \quad m_0 + m_1 \geq n.$$

The proof will be carried out by application of the functional \mathcal{F} to a « test function » different from η . It is constructed as follows. Assume (3.1) is false, that is, that $m_0 + m_1 < n$; without loss of generality we may suppose that $m_0 + m_1 = n - 1$. Let μ be an arbitrary n -th root of λ , and let $\Delta(t, \lambda)$ be the $n \times n$ matrix

$$(3.2) \quad \Delta(t, \lambda) = \begin{bmatrix} \mu^j & (\mu\gamma)^j & \dots & (\mu\gamma^{n-1})^j \\ \dots & \dots & \dots & \dots \\ \exp(\mu t) & \exp(\mu\gamma t) & \dots & \exp(\mu\gamma^{n-1} t) \\ \dots & \dots & \dots & \dots \\ \mu^k \exp(\mu T) & (\mu\gamma)^k \exp(\mu\gamma T) & \dots & (\mu\gamma^{n-1})^k \exp(\mu\gamma^{n-1} T) \end{bmatrix}$$

where $j \in \alpha_1$, $k \in \alpha_2$ and define

$$(3.3) \quad \zeta(t, \lambda) = \det \Delta(t, \lambda) / \det \Delta^{(0)}(0, \lambda)$$

where j_0 is, as in Section 2, a fixed element of $\tilde{\alpha}$. Since $\Delta^{(j_0)}(0, \lambda)$ does not vanish identically, $\zeta(t, \lambda)$ exists for every complex λ except for a sequence

$$Z = \{\lambda_m\}$$

of zeros of $\Delta^{(j_0)}(0, \lambda)$. In the complement of Z , ζ satisfies

$$\begin{aligned} \zeta^{(n)}(t, \lambda) &= \lambda \zeta(t, \lambda), \\ \zeta^{(j)}(0, \lambda) &= 0 \quad (j \in \alpha_1) & \zeta^{(k)}(T, \lambda) &= 0 \quad (k \in \alpha_2) \\ \zeta^{(j_0)}(0, \lambda) &= 1 \end{aligned}$$

Operating exactly as in Section 2 with η we obtain

$$\lambda \mathcal{F}(u^*, \zeta(\cdot, \lambda) u) = \langle u^*, R(\lambda_0; A) u \rangle + \mathcal{F}(u^*, \zeta(\cdot, \lambda) Au)$$

so that, if we define a bounded operator

$$\delta(\lambda): E^* \rightarrow E^* \quad \text{for} \quad \lambda \notin Z$$

by

$$\langle \delta(\lambda) u^*, u \rangle = \mathcal{F}(u^*, \zeta(\cdot, \lambda) u),$$

then $\delta(\lambda) E^* \subseteq D(A^*)$ and $(\lambda I - A^*) \delta(\lambda) = R(\lambda_0; A^*)$. Accordingly, if we set $\mathcal{R}(\lambda) = (\lambda_0 - \lambda) \delta(\lambda) + R(\lambda_0; A^*)$, $\mathcal{R}(\lambda) E^* \subseteq D(A^*)$ and

$$(\lambda I - A^*) \mathcal{R}(\lambda) = I.$$

This time, however, we can conclude that

$$\mathcal{R}(\lambda) = R(\lambda; A^*) \quad (\lambda \notin Z);$$

in fact, since $\varrho(A)$ is open $\varrho(A) \cap (\mathbf{C} \setminus Z) = \emptyset$ and Remark 2.4 applies to the connected domain $\mathbf{C} \setminus Z$.

We show now that $\varrho(A)$ actually coincides with \mathbf{C} . Let $\lambda_1 \in Z$. Clearly, we must have

$$(3.6) \quad |\zeta(t, \lambda)| = 0(|\lambda - \lambda_1|^{-p})$$

(p a nonnegative integer) for λ near λ_1 . It follows from the definition of $R(\lambda; A^*)$ in terms of $\zeta(t, \lambda)$ that an estimate of the type of (3.6) holds as

well for $R(\lambda; A^*)$. Accordingly, the (possible) singularity of $R(\lambda; A^*)$ at $\lambda = \lambda_1$ can be only a pole. Since $R(\lambda; A^*) = R(\lambda; A)^*$, the same is true of $R(\lambda; A)$. This is known to imply ([5], Ch VII) that λ_1 is an eigenvalue of A . If u is any eigenvector of A corresponding to λ_1 , and $\mu^n = \lambda_1$, let

$$u(t) \left(= \sum_{r=0}^{n-1} c_r \exp [\mu \gamma^r t] \right) u$$

Then $u(\cdot)$ is a solution of (1.1) with $f = 0$. But since $\Delta^{(j)}(0, \lambda_1) = 0$, it is easy to see that we can find coefficients c_0, \dots, c_{n-1} not all zero and such that

$$u^{(j)}(0) = 0 \quad (j \in \alpha_0), \quad u^{(k)}(T) = 0 \quad (k \in \alpha_1)$$

which obviously violates Condition \mathcal{CD}_1 . Accordingly, $R(\lambda; A)$ is regular at $\lambda = \lambda_1$: Since λ_1 is an arbitrary element of Z ,

$$(3.7) \quad \sigma(A^*) = \emptyset.$$

We estimate now $R(\lambda; A^*)$. Observe first that, for any t $\det \Delta(t, \lambda)$ is single-valued (the substitution of μ by another n -th root of λ merely causes a cyclic change in the columns of $\Delta(t, \lambda)$). Then $\det \Delta(t, \lambda)$ is entire, and, for any $\varepsilon > 0$

$$(3.8) \quad |\det \Delta(t, \lambda)| \leq K \exp [(T|\lambda|^{(1/n)+\varepsilon})]$$

uniformly in $0 \leq t \leq T$. The same reasoning can be applied to any t -derivative of Δ ; in particular, $\det \Delta^{(j)}(0, \lambda)$ satisfies an inequality of the type of (3.8); a fortiori, it is of exponential type. We can then apply Corollary 3.7.3 in [1] to deduce the existence of a sequence $\{\varrho_m\}$ of density 1 (i.e., such that

$$\varrho_m = m(1 + o(m)) \text{ as } m \rightarrow \infty)$$

and of a sequence $\{\varepsilon_m\}$ of positive numbers tending to zero such that

$$(3.9) \quad |\det \Delta^{(j)}(0, \lambda)| \geq \exp [-\varepsilon_m \varrho_m] (|\lambda| = \varrho_m, m = 1, 2, \dots)$$

Combining (3.8) and (3.9) we obtain that, for any $\varepsilon > 0$

$$(3.10) \quad |\zeta(t, \lambda)| \leq K \exp [(T\varrho_m^{(1/n)+\varepsilon} + \varepsilon_m \varrho_m)] < \\ < K \exp (\varepsilon'_m \varrho_m) \quad (|\lambda| = \varrho_m, m = 1, 2, \dots)$$

where $\varepsilon'_m \rightarrow 0$ as $m \rightarrow \infty$. Then,

$$|R(\lambda; A^*)| \leq K' \exp(\varepsilon'_m \varrho_m) \quad (|\lambda| = \varrho_m, m = 1, 2, \dots).$$

By the maximum modulus theorem,

$$(3.11) \quad |R(\lambda; A^*)| \leq K' \exp(\tilde{\varepsilon}_m |\lambda|) \quad (\varrho_{m-1} < |\lambda| \leq \varrho_m)$$

where

$$\tilde{\varepsilon}_m = \max(\varepsilon_m, \varepsilon_{m-1}) \varrho_m / \varrho_{m-1}.$$

But $\varrho_m / \varrho_{m-1} \sim n / (n+1) \rightarrow 1$ as $n \rightarrow \infty$ so that $R(\lambda; A^*)$ is of exponential type zero.

The result just obtained is now combined with those in Section 2 as follows. In view of (2.1), (2.2) and uniqueness of the resolvent, we must have

$$(3.12) \quad \mathcal{R}_0(\lambda) = R(\lambda; A^*) \quad (\lambda \in A^*(n, m_0 + 1, \varrho, \omega))$$

$$(3.13) \quad \mathcal{R}_1(\lambda) = R((-1)^n \lambda; A^*) \quad (\lambda \in A^*(n, m_1 + 1, \varrho, \omega)).$$

Assume n even, $m_0 < (n-2)/2$. Then it is not difficult to see that there exists $a > 0$ such that

$$(3.14) \quad \{\lambda; |\lambda| \geq a\} \subset A^*(n, m_0 + 1, \varrho, \omega).$$

Making use of (3.12), (2.3) and of Liouville's theorem we see that $R(\lambda; A^*)$ must be identically zero, which is absurd. The same conclusion holds $m_1 < (n-2)/2$ (we use (3.13) and (2.4) instead of (3.12) and (2.3) in this case). If $m_0, m_1 \geq (n-2)/2$, as $m_0 + m_1 = n-1$ we must have $m_0 = m_1 = (n-2)/2$, so that $m_0 + 1 = n/2$. Now, a moment's consideration shows that if $n = 4k$ and $\beta > 0$, there exists $a > 0$ such that

$$(3.15) \quad \{\lambda; |\arg \lambda| \geq \beta, |\lambda| \geq a\} \subset A^*(n, m_0 + 1, \varrho, \omega).$$

If we take $\beta < \pi/2$, the fact that $R(\lambda; A^*)$ is of exponential type and (3.3) we can apply the Phragmén-Lindelöf theorem ([1], §1.4) to show that $R(\lambda; A^*)$ is bounded, and we deduce again a contradiction from Liouville's theorem. The case $n = 4k + 2$ is treated in the same way, but replacing (3.15) by

$$(3.16) \quad \{\lambda; \pi - |\arg \lambda| \geq \beta; |\lambda| \geq a\} \subset A^*(n, m_0 + 1, \varrho, \omega).$$

Consider now the case n odd. If $m_0 < (n-1)/2$ then (3.14) holds for some $a > 0$ and the proof ends as before. If $m_0, m_1 \geq (n-1)/2$ we must have $m_0 = m = (n-1)/2$, so that $m_0 + 1 = m_1 + 1 = (n+1)/2$. If $n = 3k$ it is not difficult to see that, for every $\tau > 0$ there exists $a > 0$ such that

$$(3.17) \quad \{\lambda; |\arg \lambda| \leq \pi - \beta; |\lambda| \geq a\} \subseteq A^*(n, m_0 + 1, \varrho, \omega) = A^*(n, m_1 + 1, \varrho, \omega).$$

The proof ends now as in the previous case, but this time making use of both (3.12) and (3.13), and then (2.3), the Phragmén-Lindelöf theorem, and Liouville's theorem. The case $n = 3k + 2$ is treated exactly in the same way, but here (3.17) is replaced by

$$\{\lambda; |\arg \lambda| \geq \pi + \beta; |\lambda| \geq a\} \subseteq A^*(n, m_0 + 1, \varrho, \omega) = A^*(n, m_1 + 1, \varrho, \omega).$$

3.2. THEOREM. Assume Conditions \mathcal{E} and \mathcal{CD}_0 hold. Then

$$(3.18) \quad m_0 + m_1 \leq n.$$

PROOF. Assume (3.18) does not hold, and let j_0 be a fixed element of α_0 . Let $u \in D$ and denote by $u(t)$ the solution of (1.1) with $f = 0$,

$$\begin{aligned} u^{(j)}(0) &= 0 \quad (j \in \alpha_0, j \neq j_0), & u^{(j_0)}(0) &= (-1)^{j_0-1} u \\ u^{(k)}(T) &= 0 \quad (k \in \alpha_1). \end{aligned}$$

Define an operator $S(t)$ by the formula

$$(3.19) \quad S(t)u = u(t) \quad (0 \leq t \leq T).$$

Then it is easy to see that Condition \mathcal{CD}_0 implies that each $S(t)$ is bounded in D and can thus be extended to a bounded operator from E into itself; moreover, $t \rightarrow S(t)u$ is continuous in $0 \leq t \leq T$ for any $u \in E$ and $\{S(t); 0 \leq t \leq T\}$ is uniformly bounded in $0 \leq t \leq T$. Let now $u^{**} \in E^{**}$, $f \in \mathcal{C}(T, E^*)$. Define

$$(3.20) \quad \mathcal{F}(u^{**}, f) = \int_0^T \langle (S(t))^{**} u^{**}, R(\lambda; A^*) f^*(s) \rangle ds.$$

Since $S^{**}(\cdot)u^{**}$ is weakly measurable, (3.20) makes sense. Assume now $u^*(\cdot)$ is a n times continuously differentiable function with values in $D(A^*)$, such that

$$u^{*(j)}(0) = 0 \quad (n-1-j \in \tilde{\alpha}_1), \quad u^{*(k)}(T) = 0 \quad (n-1-k \in \tilde{\alpha}_2)$$

and let

$$f^*(t) = u^{*(n)}(t) - (-1)^n A^* u^*(t).$$

Taking into account the definition of $S(\cdot)$ we obtain, integrating by parts in (3.20)

$$(3.21) \quad \mathcal{F}(u, f^*) = \langle u, R(\lambda_0; A^*) u^{*(n-1-j)}(0) \rangle.$$

It is easy to see that (3.21) actually holds for $u^{**} \in E^{**}$; this follows from the fact ([5], Chapter II) that any element $u^{**} \in E^{**}$ can be approximated by a sequence $\{u_k\}$ of elements in E —or in D , since the latter is dense in E —such that $|u_k| < |u^{**}|$ in the weak $\sigma(E^{**}, E^*)$ topology.

Having constructed the functional \mathcal{F} with respect to $\tilde{\alpha}_0 = \{j; n-1-j \in \tilde{\alpha}_0\}$ (which has $n-m_0$ elements) and $\tilde{\alpha}_1 = \{k; n-1-k \in \tilde{\alpha}_1\}$ (which has $n-m_1$ elements) the proof continues exactly like that of Theorem 3.1.

§ 4.

4.1. THEOREM. Assume Conditions \mathcal{E} and \mathcal{CD} hold for (1.1), (1.2). Assume, further, that (a) n is even and

$$(4.1) \quad m_0 < (n-2)/2 \quad \text{or} \quad m_1 < (n-2)/2$$

or that (b) n is odd and

$$(4.2) \quad m_0 < (n-1)/2 \quad \text{or} \quad m_1 < (n-1)/2$$

Then A must be bounded.

PROOF. Let $\lambda \in C$, $\mu^n = \lambda$. Consider the $n \times n$ matrix

$$(4.3) \quad \mathcal{E}(\lambda) = \begin{bmatrix} \mu^j & & (\mu\gamma)^j & \dots & (\mu\gamma^{n-1})^j \\ \dots & \dots & \dots & \dots & \dots \\ \mu^k \exp(\mu T) & (\mu\gamma^k) \exp(\mu T) & \dots & (\mu\gamma^{n-1})^k \exp(\mu^{n-1} T) & \dots \end{bmatrix}$$

where $n-1-j \in \tilde{\alpha}_0$, $n-1-k \in \tilde{\alpha}_1$: Plainly \mathcal{E} is entire and does not vanish identically, so that it can only have a countable set of zeros

$$(4.4) \quad Z = \{\lambda_1, \lambda_2, \dots\}.$$

Let λ_m be one of these zeros, $\mu_m^n = \lambda_m$: Then we can find coefficients

c_1, c_2, \dots, c_n , not all zero and such that if

$$\zeta(t) = \sum_{r=0}^{n-1} c_r \exp(\mu_m \gamma^r t)$$

we have

$$\zeta^{(j)}(0) = 0 \quad (n-1-j \in \tilde{\alpha}_1), \quad \zeta^{(k)}(T) = 0 \quad (n-1-k \in \tilde{\alpha}_1).$$

Since ζ is not identically zero, there must exist some j_0 , $n-1-j_0 \in \alpha_0$ such that $\zeta^{(n-1-j_0)}(0) \neq 0$; multiplying ζ by a constant if necessary we may assume that

$$\zeta^{(j_0)}(0) = (-1)^{j_0-1}.$$

Let $S(\cdot)$ be the operator constructed in the proof of Theorem 3.2, and let

$$Ru = \int_0^x \zeta(t) S(t) u dt.$$

A simple integration by parts shows that $(\lambda_m I - A)Ru = u$ for $u \in D$; this and the fact that A is closed and D dense show that $(\lambda_m I - A)E \subseteq D(A)$ and

$$(\lambda_m I - A)\mathcal{R} = I.$$

This and a simple duality argument show that $(\lambda_m I - A^*)$ is one-to-one (in fact, $\langle u^*, (\lambda_m I - A)v \rangle$ must vanish for any eigenvector u^* of A^* corresponding to λ_m).

Observe next that \mathcal{E} is of order $1/n$; then according to [1], 2.9.2 the set of zeros Z must be infinite; in particular, we must have zeros λ_m of Z of arbitrarily large module. Assume n even and $m_0 < (n-2)/2$. Then (as indicated in (3.14))

$$\{\lambda; |\lambda| > a\} \subseteq A^*(n, m_0 + 1, \varrho, \omega).$$

Since $\lambda_m \in A^*(n, m_0 + 1, \varrho, \omega)$ for m large enough, there exist points therein where $\lambda I - A$ is one-to-one; then Remark 2.4 applies to show that

$$\{\lambda; |\lambda| > a\} \subseteq \varrho(A)$$

and that

$$|R(\lambda: A)| = o(1) \quad \text{as} \quad |\lambda| \rightarrow \infty$$

which is known to imply that A^* —thus A —is bounded. The case $m_1 < (n-2)/2$ can be handled through a change of t by $-t$ like the one used in the proof of Theorem 2.1. The proof runs also along similar lines in the case n odd.

REFERENCES

- [1] R. P. BOAS, *Entire functions*, Academic Press, New York, 1954.
- [2] L. CATTABRIGA, *Un problema al contorno per una equazione parabolica di ordine dispari*, Ann. Scuola Norm. Sup. Pisa, **13** (1954), 163-203.
- [3] J. CHAZARAIN, *Problèmes de Cauchy Abstraits et Applications à Quelques Problèmes Mixtes*, Jour. Funct. Anal., **7** (1971), 386-446.
- [4] G. DA PRATO, *Weak solutions for linear abstract differential equations in Banach spaces*, Advances in Math., **5** (1970), 181-245.
- [5] N. DUNFORD - J. T. SCHWARTZ, *Linear operators*, part I, Interscience, New York, 1958.
- [6] H. O. FATTORINI - A. RADNITZ, *The Cauchy problem with incomplete initial data*, Mich. Math. Jour., **18** (1972), 291-320.
- [7] H. O. FATTORINI, *The underdetermined Cauchy problem in Banach spaces*, Math. Ann. (1973).
- [8] P. GRISVARD, *Equations operationnelles abstraits dans les espaces de Banach et problèmes aux limites dans des ouverts cylindriques*, Ann. Scuola Norm. Sup. Pisa, **21** (1967), 307-347.
- [9] J. KADLEC, *On the solution of the first boundary value problem for a generalized heat flow equation in classes of functions having fractional derivatives with respect to time*, (Russian), Czech. Jour. Math., **91** (1966), 91-113.
- [10] J. L. LIONS - E. MAGENES, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.
- [11] A. RADNITZ, *The incomplete Cauchy problem in Banach spaces*, Dissertation, University of California, Los Angeles, 1970.
- [12] R. SCHATTEN, *A theory of cross-spaces*, Ann. of Math. Studies n. 26, Princeton University Press, Princeton, 1950.