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Linear Second Order Differential Equations with Discontinuous Coefficients in Hilbert Spaces.

LUCIANO DE SIMON - GIOVANNI TORELLI (*)

Introduction.

This work is concerned with some problems arising in the theory of linear abstract 2nd order differential equations on a finite interval $[0, T]$ of the « time line ». Our main aim is to give a contribution to the theory developed by some authors (Lions, Baiocchi and others), taking into account the case of equations affected by discontinuous coefficients. The results we have obtained can also be applied, for instance, to the Cauchy problem for a self-adjoint hyperbolic homogeneous linear partial differential equation of the form

$$\frac{\partial^2}{\partial t^2} u(x,t) - \sum_{1i,k}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x,t) \frac{\partial u}{\partial x_k}\right) = 0$$

in a cylinder $\Omega \times [0, T]$ (here, as usual, Ω is an open region of R^n and u vanishes, in a some sense, at the boundary of Ω) with coefficients a_{ik} continuous on x but not necessarily on t. We obtain for this case a further development of some earlier results (see, in particular, [3]).

To be more explicit in describing our problem, we anticipate a short outline of the situation. We suppose that V and H are a pair of real Hilbert spaces, with inner products $((,\cdot,))$ and $(,\cdot,)$ respectively, and with $V \subset H$ densely and continuously. Denoting by V' the (anti)dual space of V, normed in an obvious way, then there exists a canonical (one-to-one) continuous isomorphism Λ between V and V' such that, for every pair $u, v \in V$; $((u, v)) = (\Lambda u, v)$ (1).

Now, given a map $A(t)$ from $[0,T]$ to the Banach space of the linear symmetric continuous operators on V, we put, for every $v \in V$, $B(t)v = \Lambda A(t)v$

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(1) See, for further details, [4] and [5].

and consider the distributional equation

$$(*)\qquad \frac{d^2u(t)}{dt^2} - B(t)u(t) = 0\,, \quad u(0) = u^\circ\,, \quad \frac{du}{dt}(0) = u^1$$

(for example, if we are concerned with the n-dimensional wave equation, A is the Laplace operator).

Our main result is that if $t \mapsto A(t)$ is of strong bounded variation, then a theorem of existence and uniqueness holds. It is interesting to remark that the assumptions that $A(t)$ is of « positively bounded variation » and that $A(t)$ is of « negatively bounded variation » (in some sense which will be precised later) imply, respectively, the existence and the uniqueness for the solution of $(*)$.

We also obtain in this paper a generalisation of the classical « Gronwall lemma » which, at our knowledge, seems not yet quoted in the literature.

0. – Setting of the problem and preliminaries.

NOTATIONS. Let H and V be a pair of separable real Hilbert spaces, with $V \subset H$ algebraically and topologically and with V dense in H. The inner product, in H and V respectively, is indicated by $(,\cdot,)$ and $((,\cdot,))$ and the corresponding norms by $|\cdot|$ and $\|\cdot\|$.

$\mathcal{L}(V, V)$ denotes the Banach space of all continuous linear maps $A\colon V \to V$ equipped with the norm $\left|A\right| = \operatorname{Sup}_{\|x\|=1} \|Ax\|$. If φ is a map from a directed set D into $\mathcal{L}(V, V)$, we shall say that φ converges to $A \in \mathcal{L}(V, V)$ strongly if, $\forall u \in V$, φu converges to Au; uniformly if $\left|\varphi - A\right|$ converges to zero [2].

Let $[0, T]$ $(T > 0)$ be a generic but fixed interval, which will be the domain of the time variable t. Then we shall denote by $L^2(0, T; V)$ the Hilbert space of the (classes of) functions $u\colon [0, T] \to V$ which are Bochner square-summable, with the inner product

$$(u, v)_{L^2(0\,T:V)} = \int_0^T ((u(t), v(t)))\,dt\,;$$

the space $L^2(0, T; H)$ is defined in a similar way.

[2] Here we have indicated by φu and $\varphi - A$ the mappings $d \mapsto \varphi(d)u$, and $d \mapsto \varphi(d) - A$ $(d \in D)$.

Let also W be the space of the functions $u \in L^2(0, T; V)$ such that $du/dt \in L^2(0, T; H)$, with the inner product given by

$$(u, v)_W = (u, v)_{L^2(0,T;V)} + (u', v')_{L^2(0,T;H)} \left(u' = \frac{du}{dt}, \quad v' = \frac{dv}{dt}\right).$$

As it is well known, W is a Hilbert space. Furthermore, every $u \in W$, modified, if necessary, on a zero-measure set, is continuous as a map $[0, T] \to H$.

This situation can be realized, for instance, by assuming $H = L^2(\Omega)$, $V = H_0(\Omega)$ (Ω being an open region of $\boldsymbol{R}^n$) with the inner products

$$(u, v) = \int_\Omega u(x)v(x)\,dx\,; \qquad ((u, v)) = \int_\Omega \left(\sum_1^n {}_i \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right) dx\,.$$

Furthermore, let us suppose that a family $a(t; \cdot, \cdot)$ ($0 \leqslant t \leqslant T$) of continuous bilinear forms $V \times V \to \boldsymbol{R}$ is assigned, with the following properties:

(0.i) For every $u, v \in V$, the function $t \mapsto a(t; u, v)$ is measurable.

(0.ii) There exists an $M > 0$ such that, for every $t \in [0, T]$ and $u, v \in V$: $a(t; u, v) \leqslant M\|u\|\,\|v\|$.

(0.iii) $a(t; u, v) = a(t; v, u)$ for every $u, v \in V$ and $t \in [0, T]$.

(0.iiii) There exists $\alpha > 0$, such that $a(t; u, u) \geqslant \alpha\|u\|^2$ for every $u \in V$ and $t \in [0, T]$.

We point out that in our hypotheses the form $a(t; u, v)$, for every fixed t, generates a continuous symmetric linear operator $A(t)\colon V \to V$ defined by the relation

$$((A(t)u, v)) = a(t; u, v) \qquad u, v \in V\,.$$

Therefore, a map $t \mapsto A(t)$ of $[0, T]$ into $\mathcal{L}(V, V)$ is associated to the family of forms $a(t; u, v)$.

We recall now that a map f from $[0, T]$ to a Banach space B whatsoever is said (strongly) measurable if there exists a sequence of simple functions (i.e. finitely valued, constant over measurable sets and null outside a set of finite measure) which converges to f almost everywhere. Following [2], a map $f\colon [0, T] \to \mathcal{L}(V, V)$ will be called: uniformly measurable if it is measurable in the above sense; strongly measurable if, for every $u \in V$, the function $t \mapsto f(t)u$ is measurable as a map $[0, T] \to V$; weakly measurable if, for every $u, v \in V$, $((f(t)u, v))$ is (Lebesgue) measurable.

As it is well known, if f is almost-separably valued (i.e. f, after correction on a null set, has separable range) then the weak measurability implies the strong one.

Now, from the assumptions made on $a(t;\cdot,\cdot)$ it follows easily that:

(0.j) The map $t \mapsto A(t)$ is weakly, and hence (V is separable) strongly measurable.

(0.jj) There exists an $M > 0$ such that, for every $t \in [0, T]$ and for every $u, v \in V$, it is $|((A(t)u, v))| \leqslant M\|u\|\|v\|$, and therefore $|A(t)| \leqslant M$.

(0.jjj) For every $t \in [0, T]$, $A(t)$ belongs to the linear manifold $\mathcal{S} \subset \mathcal{L}(V, V)$ of the symmetric operators.

(0.jjjj) There exists $\alpha > 0$ such that, for every $t \in [0, T]$ and for every $u \in V$, $((A(t)u, u)) \geqslant \alpha\|u\|^2$.

We remark also that, by virtue of our assumptions, there exists the strong integral of $A(t)$ on $[0, T]$; that is, for $\forall u \in V$, $A(t)u$ is (Bochner) summable on $[0, T]$. We shall denote with $\int_0^T A(t)\,dt$ the map $u \mapsto \int_0^T A(t)u\,dt$ which, as it is well known, belongs to $\mathcal{L}(V, V)$. Accordingly, we shall say that the sequence $\{A_n(t)\}$ converges to $A(t)$: strongly in $L^p(0, T; \mathcal{L}(V, V))$ ($p \geqslant 1$) if, for every $u \in V$, $\left(\int_0^T \|A_n(t)u - A(t)u\|^p dt\right)^{1/p} = \|A_n u - Au\|_{L^p(0,T;V)}$ converges to zero for $n \to +\infty$; uniformly in $L^p(0, T; \mathcal{L}(V, V))$ if $\left(\int_0^T |A_n(t) - A(t)|^p dt\right)^{1/p} = \|A_n - A\|_{L^p(0,T;\mathcal{L}(V,V))}$ converges to zero for $n \to +\infty$. Moreover, it can be proved that, for every $u(t); v(t) \in L^2(0, T; V)$, the maps $t \mapsto A(t)u(t)$ and $t \mapsto ((A(t)u(t), v(t)))$ are measurable (the former in the strong sense) (³).

Finally, let us point out that $|A(t)|$ is summable. In fact, by (0.j), for $\forall u \in V$, $\|A(t)u\|$ is measurable. Then, if $\{u_n\}$ is a countable dense subset of the unit sphere of V, $|A(t)|$ is the supremum of the countable set of the measurable functions $\|A(t)u_n\|$ and is, therefore, measurable. The summability follows from (0.jj).

We can now state the object of our work as follows:

(³) See, for further details, [5], pag. 17.

0.1 PROBLEM. Given $u^0 \in V$, $u^1 \in H$:

a) Find $u \in W$ in such a way that

$$(0.1.1) \qquad \begin{cases} u(0) = u^0 \\ \displaystyle\int_0^T \{(u'(t), v'(t)) - ((A(t)u(t), v(t)))\}\, dt = -(u^1, v(0)) \end{cases}$$

for every $v \in W$ such that $v(T) = 0$ (we shall call any such v a « test function »).

b) Give conditions for the uniqueness of the solution of *a*).

Formally, a solution u of (0.1.1) can be regarded as a solution (in the sense of the theory of distributions) of the linear abstract differential equation on $[0, T]$

$$\frac{d^2u}{dt^2} + B(t)u = 0$$

with initial conditions $u(0) = u^0$, $u'(0) = u^1$ (here, as explained in the introduction, it is $B(t) = \Lambda A(t)$).

At this point, for the sake of convenience and clarity, we recall several well known properties of the symmetric operators on a Hilbert space. Let K be a real Hilbert space with inner product $(,\cdot,)$ and let $\mathcal{S} \subset \mathcal{L}(K, K)$ be the linear manifold of the linear bounded symmetric operators on K. By setting, for $A, B \in \mathcal{L}(K, K)$, $A \geqslant B$ if and only if $(Ax, x) \geqslant (Bx, x)$ $\forall x \in K$, we define in $\mathcal{L}(K, K)$, and therefore in $\mathcal{S}$, a relation of partial order compatible with the linear structure of $\mathcal{S}$. Accordingly, an operator A is called positive if, $\forall x \in K$, $(Ax, x) \geqslant 0$. Note that, $\forall A \in \mathcal{S}$, A^2 is a positive operator. Furthermore, it is possible to define the « square root » of any positive symmetric operator and therefore the absolute value of every $A \in \mathcal{S}$ by $|A| = (A^2)^{\frac{1}{2}}$.

This allows us to split A canonically into the difference of two positive operators $A^+ = \frac{1}{2}(|A| + A)$, $A^- = \frac{1}{2}(|A| - A)$. It can be proved (4) that among all possible ways of writing A as the difference of two positive symmetric operators, the decomposition $A = A^+ - A^-$ is the most efficient one, in the sense that A^+ is the least positive operator which exceeds A and commutes with A, and A^- is the least positive operator which commutes with A and exceeds $-A$.

As previously observed, the function $t \mapsto A(t)$ takes his values in the manifold $\mathcal{S}$ of the symmetric operators on V; therefore the decomposition $A(t) = A^+(t) - A^-(t)$ holds.

(4) See, for example, [6], pages 274-75.

0.2. Definition. Let S be a generic finite subdivision of the interval $[t', t''] \subset [0, T]$ by the points $t' = t_0 < t_1 < t_2 \ldots < t_n = t''$.

We shall call positive variation of $A(t)$ on $[t', t'']$ the number ($\leqslant +\infty$):

$$V_A^+([t', t'']) = \operatorname*{Sup}_S \left(\sum_1^n {}_i \, \| [A(t_i) - A(t_{i-1})]^+ \| \right)$$

where the Sup is taken over all the finite subdivisions of $[t', t'']$.

If $V_A^+([t', t'']) < +\infty$, we shall say that $A(t)$ is of positively bounded variation (P.B.V.) on $[t', t'']$.

Likewise, the negative variation can be introduced, and the definition of an operator-valued function of negatively bounded variation (N.B.V.) follows in the same way.

0.3 Definition. The number ($\leqslant +\infty$):

$$V_A([t', t'']) = \operatorname*{Sup}_S \left(\sum_1^n {}_i \, \| A(t_i) - A(t_{i-1}) \| \right)$$

is called the (total) variation [5] of the function $A(t)$ over the interval $[t', t'']$. As before, it is understood that S runs on the set of all finite subdivisions of $[t', t'']$.

If $V_A([t', t'']) < +\infty$, we shall say that $A(t)$ is of bounded variation (B.V.) on $[t', t'']$.

If we indicate, for simplicity, by V^+, V^-, V the variations of $A(t)$ over an interval, it can be easily verified that the following inequalities hold:

$$V^+ + V^- \geqslant V \geqslant \max\{V^+, V^-\}$$

from which follows that $A(t)$ is of B.V. if and only if it is of positively and negatively bounded variation.

Furthermore, the variations we have introduced depend monotonically on t' and t''; hence there exist, for $0 < t' < t'' < T$, the left and right limits of $V([t', t''])$, both at t' and t''; the same fact holds also for V^+ and V^-.

In what follows, we shall denote right and left limits of a given function f at a point t by the symbols $f(t^+)$ and $f(t^-)$ respectively. Let us put now $v(t) = V_A[0, t])$: since $v(t)$ is a monotone non-decreasing function, there exists, for every continuous function $f(t)$, the Riemann-Stieltjes integral

([5]) More precisely: uniform variation.

$\int\limits_{[a,b]} f(t)\,dv(t)$ $(0 \leqslant a \leqslant b \leqslant T)$ (6). Moreover, if $a \leqslant c \leqslant T$, we mean that

$$\int\limits_{[a,c[} f(t)\,dv(t) = \begin{cases} 0 & \text{if } c = a \\ \lim\limits_{b \uparrow c} \int\limits_{[a,b]} f(t)\,dv(t) & \text{if } c > a \end{cases}$$

(it is not difficult to see that the above limit exists).

Clearly, this integral is a left-continuous function of $c > a$.

In the same way the integrals $\int\limits_{]a,c]} f(t)\,dv(t)$ and $\int\limits_{]a,c[} f(t)\,dv(t)$ can be defined. Accordingly, we shall write $V_A([t', t''[)$, $V_A(]t', t''])$, $V_A(]t', t''[)$ with an obvious meaning.

Note also that for this kind of integrals the following addition rule can be easily proved $(a < c < b)$:

$$\int\limits_{[a,b[} f(t)\,dv(t) = \int\limits_{[a,c[} f(t)\,dv(t) + \int\limits_{[c,b[} f(t)\,dv(t) + f(c)\big(v(c) - v(c^-)\big)$$

from which it follows that our integral, regarded as a set function, is not additive in general.

Of course, if $v(t)$ is left-continuous (and generates therefore a measure on $[0, T]$) the additivity of the integral holds.

In a similar way we define $v^-(t) = V_A^-([0, t])$, $v^+(t) = V_A^+([0, t])$ and, accordingly, the Riemann Stieltjes integrals associated with the above functions.

We recall again that, if $A(t)$ is a function on $[0, T]$ with values in any Banach space normed by $|\cdot|$, then the following inequality holds:

$$|A(t'') - A(t')| \leqslant V_A([t', t'']) \qquad \forall t', t'' \in [0, T]$$

and therefore, at those points where V_A is continuous, $A(t)$ is too; at any rate, there exist everywhere both the left and the right limit of the variation. The set of points of discontinuity of $A(t)$ is, therefore, at most countable.

(6) We recall that the above integral is defined as the limit, for $|\pi| \to 0$, of the sums

$$\sum_0^{n-1}{}_k\, f(\tau_k)\big(v(t_{k+1}) - v(t_k)\big),$$

where

$$a = t_0 \leqslant \tau_0 \leqslant t_1 \leqslant \tau_1 \leqslant t_2 \leqslant \ldots \leqslant t_n = b\,, \qquad |\pi| = \max_k |t_{k+1} - t_k|\,.$$

1. – Some preparatory lemmas.

In this section we shall state some results which will be employed later in proving our main theorems.

1.1 LEMMA. *For every pair of linear continuous symmetric operators A, B on a real Hilbert space K, from $A \leqslant B$ follows $|A^+| \leqslant |B^+|$.*

PROOF. By setting

$$M_A = \operatorname*{Sup}_{|x|=1} (Ax, x), \qquad m_A = \operatorname*{Inf}_{|x|=1} (Ax, x)$$

(here $(,\cdot,)$ and $|\cdot|$ denote inner product and norm in K), it is well known that $|A^+| = \max\{M_A, 0\}$, $|A^-| = \max\{-m_A, 0\}$. Our assertion follows now from the fact that $A \leqslant B$ implies $M_A \leqslant M_B$ (7).

1.2 LEMMA. *Let K be a Hilbert space as in lemma 1.1, let $\mathcal{S} \subset \mathcal{L}(K, K)$ be the linear manifold of the linear bounded symmetric operators on K and assume that $\{A_m\}$ is a sequence in $\mathcal{S}$ wich converges strongly to the operator A. Then $\{|A_m|\}$ converges to $|A|$ strongly.*

PROOF. Let us first observe that, as it can be easily proved by recurrence on p, if $\{A_m\}$ is a sequence of symmetric operators which converges to A strongly, then, for every integer $p \geqslant 1$, A_m^p converges strongly to A^p.

As it is well known, there exists a sequence of polynomials $\{P_n(t)\}$ which converges to $|t|$ uniformly on every bounded set of $\mathbb{R}$. Then, for every $A \in \mathcal{S}$, the sequence $\{P_n(A)\}$ converges to $|A|$ in the strong and even in the uniform topology of $\mathcal{L}(K, K)$. Moreover, the convergence is uniform on every set of operators the spectrum of which lies in the same bounded set $B \subset \mathbb{R}$. Therefore, the convergence $P_n(A) \to |A|$ is uniform on every bounded set of $\mathcal{L}(K, K)$.

Now, the fact that $A_m \to A$ strongly as in our hypothesis implies, by the uniform boundedness principle, that the set $\{\|A_m\|\}$ $(m \in N)$ is bounded by some constant $C > 0$. On the other hand, according to our initial remark, for every $n \in N$ and $u \in K$ it is

$$\lim_m P_n(A_m)u = P_n(A)u\,.$$

(7) For further details, see [6], pages 259-277.

Then, as a consequence of the boundedness of the $\{\|A_m\|\}$, the double sequence $P_n(A_m)u$ $(m, n \in N)$ converges, uniformly on m, to $|A_m|u$: this assures that

$$\lim_m |A_m|u = \lim_n \lim_m P_n(A_m)u = \lim_n P_n(A)u = |A|u\,.$$

REMARK. The above argument can be adapted to prove that the lemma holds even when the convergence is understood in the uniform topology.

1.3 LEMMA. *Let $t \mapsto A(t)$ be a map from $\mathbb{R}$ to $\mathcal{S} \subset \mathcal{L}(K, K)$, strongly summable over an interval $[a, b]$. Then the following inequalities hold*:

$$\Big|\Big(\int_a^b A(t)\,dt\Big)^+\Big| \leq \int_a^b |A^+(t)|\,dt\,; \qquad \Big|\Big(\int_a^b A(t)\,dt\Big)^-\Big| \leq \int_a^b |A^-(t)|\,dt\,.$$

PROOF. Let $A_1 = \int_a^b A^+(t)\,dt$, $A_2 = \int_a^b A^-(t)\,dt$, $A_3 = \int_a^b A(t)\,dt$ and observe that A_1 and A_2 are positive operators. From lemma 1.2 it follows that, if $\{A_n(t)\}$ is a sequence of simple functions which converges strongly to $A(t)$ a.e., then $A_n^+(t)$ and $A_n^-(t)$ converge, respectively, to $A^+(t)$ and $A^-(t)$ a.e.. Therefore $A(t)$ is strongly measurable if and only if such are $A^+(t)$ and $A^-(t)$, and this guarantees that, in our assumptions, the first two integrals exist. Observe now that $A_3 = A_1 - A_2$, $A_1 \geq A_1 - A_2$, $A_1^+ = A_1$: we can therefore apply lemma 1.1, putting $A_3 = A$, $A_1 = B$ and so obtain

$$\Big|\Big(\int_a^b A(t)\,dt\Big)^+\Big| \leq \Big|\int_a^b A^+(t)\,dt\Big|\,. \tag{1.3.1}$$

Furthermore, from the above inequality if follows easily that

$$\Big|\Big(\int_a^b A(t)\,dt\Big)^-\Big| = \Big|\Big(\int_a^b - A(t)\,dt\Big)^+\Big| \leq \Big|\int_a^b \big(- A(t)\big)^+\,dt\Big| = \Big|\int_a^b A^-(t)\,dt\Big|\,. \tag{1.3.2}$$

Our assertion now can be obtained by replacing in the last integrals in (1.3.1) and in (1.3.2) $A^+(t)$ and $A^-(t)$ by their norms.

1.4 LEMMA. *Let $m\colon [t_0, t_1] \to R$ be a monotone, non-decreasing bounded function and $y(t)$ a map $[t_0, t_1] \to R$ non-negative, continuous, such that*

$$y(t) < y^0 + \int_{[t_0,t[} y(\tau)\,dm(\tau) \qquad \forall t \in [t_0, t_1]\,, \tag{1.4.1}$$

y^0 being a positive number, and let us agree that $m(t_0^-)$ means « $m(t_0)$ ». Then, for every $t \in [t_0, t_1]$, the inequality

$$y^0 + \int_{[t_0, t[} y(\tau)\, dm(\tau) \leqslant y^0 \exp\big(m(t^-) - m(t_0)\big) \quad (1.4.2)$$ [8]

holds.

PROOF. Let $g(t)$ indicate the function which appears on the left in (1.4.2) and observe that this function is left-continuous. For every $\bar{t} \in [t_0, t_1[$, there exists an $h > 0$ such that, for $t \in]\bar{t}, \bar{t} + h[$

$$g(t) \leqslant g(\bar{t}) \exp\big(m(t^-) - m(\bar{t}^-)\big)\,. \quad (1.4.3)$$

In fact, from $y(\bar{t}) < g(\bar{t})$, it follows, by the continuity of $y(t)$, that there exists an $h > 0$ such that $y(t) < g(\bar{t})$ for $t \in [\bar{t}, \bar{t} + h[$. Then

$$g(t) = y^0 + \int_{[t_0, t[} y(\tau)\, dm(\tau) \leqslant g(\bar{t}) + y(\bar{t})\big(m(\bar{t}) - m(\bar{t}^-)\big) + \int_{[\bar{t}, \bar{t}+h[} g(\bar{t})\, dm(\tau) \leqslant$$
$$\leqslant g(\bar{t})\big(1 + m(t^-) - m(\bar{t}^-)\big) \leqslant g(\bar{t}) \exp\big(m(t^-) - m(\bar{t}^-)\big)\,.$$

Let now $\bar{t}$ be the supremum of the set of those t such that (1.4.2) holds on $[t_0, t]$ and suppose that $\bar{t} < t_1$. This definition is correct, because (1.4.2) is true at t_0 and so, as we have shown, at least on a right neighbourhood of t_0 too.

For $t_0 < t < \bar{t}$ it is $g(t) \leqslant y^0 \exp\big(m(t^-) - m(t_0)\big)$, therefore, provided that $g(t)$ is left-continous, it follows, for $t \uparrow \bar{t}$, that $g(\bar{t}) \leqslant y^0 \exp\big(m(\bar{t}^-) - m(t_0)\big)$. But, as previously observed, there exists a number $h > 0$ such that, for $t \in]\bar{t}, \bar{t} + h[$, the inequality (1.4.3) holds, and therefore we have the estimates

$$g(t) \leqslant y^0 \exp\big(m(\bar{t}^-) - m(t_0)\big) \exp\big(m(t^-) - m(\bar{t}^-)\big) = y^0 \exp\big(m(t^-) - m(t_0)\big)$$

which assure that (1.4.2) is true on a whole right neighbourhood of $\bar{t}$. This contradicts the assumption $\bar{t} < t_1$; the assertion is so proved.

REMARK. The lemma holds, in particular, if m is a measure generating function (i.e. left-continuous everywhere). In this case the statement can also be proved, but in a more complicated way, with the hypothesis « $y(t)$ continuous » replaced by « $y(t)$ m-summable ».

[8] Note that for $t = t_0$ the set $[t_0, t_0[$ is empty; therefore it must be $y(t_0) < y_0$.

1.5 COROLLARY. *If* $y(t) \leqslant c \int_{[t_0, t[} y(\tau)\, dm(\tau)$ *with* $c > 0$, *all the other conditions of the lemma* (1.4) *being unchanged, then* $y(t) = 0$ *for every* $t \in [t_0, t_1]$.

PROOF. We have, for every $y^0 > 0$, $y(t) < y^0 + c \int_{[t_0, t[} y(\tau)\, dm(\tau)$ and hence $y(t) \leqslant y^0 \exp c\big(m(t^-) - m(t_0)\big)$ for every $t \geqslant t_0$. This fact, however, is true if and only if $y(t) = 0$ for every $t \in [t_0, t_1]$.

1.6 LEMMA. *Let* $m(t)$ *be monotone non-decreasing over the interval* $[a, b]$. *Then there exists, for every* $d > 0$, *a finite subdivision* $S = \{a = t_0 < t_1 < \ldots \ldots < t_n = b\}$ *of the interval* $[a, b]$ *such that*

$$m(t_{k+1}^-) - m(t_k^+) < d \qquad (k = 0, 1, \ldots, n-1)\,.$$

PROOF. Let us suppose that at every point of $[a, b]$ m does not have a jump larger than $d/2$. This is not restrictive, because in any case the number of those points where m has a jump larger than $d/2$ is finite. Therefore, if we consider a finite subdivision $S = \{a = s_0 < s_1 < \ldots < s_m = b\}$ consisting at least of all such points, it will be sufficient to prove that our assertion holds for every $[s_k, s_{k+1}]$ $(k = 0, 1, \ldots, m-1)$.

Suppose now that $[a, b]$ has not the property stated in the lemma. If c is the middle point of $[a, b]$, the property must not hold at least for one of the intervals $[a, c]$ and $[c, b]$; we call $[a_1, b_1]$ one of these two intervals in which the property is not true: clearly it is $m(b_1^-) - m(a_1^+) \geqslant d$. Subdividing $[a_1, b_1]$ by the middle point, we can find an interval $[a_2, b_2] \subset [a_1, b_1]$ for which the property is not valid, and so on. We obtain so two monotone sequences $\{a_n\}$ $\{b_n\}$, with $b_n - a_n = 2^{-n}(b-a)$. If $\bar{t}$ is the limit point to which both sequences converges, it must be $m(\bar{t}^+) - m(\bar{t}^-) \geqslant d$ and we have so a contradiction, from which our assertion follows.

2. – Preliminary results.

We begin with some lemmas, which we shall need later.

2.1 LEMMA. *Let* B *a Banach space normed by* $\|\cdot\|$ *and* $t \mapsto f(t)$ *a map* $[0, T] \to B$, *Bochner summable. Then there exists a sequence of step functions* $f_n(t)\colon [0, T] \to B$, *which converges to* $f(t)$ *in the* $L^1(0, T; B)$ *topology.*

PROOF. For fixed n, we consider the subdivision of the interval $[0, T]$ into 2^n equal subintervals by the points

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots < t_{2^n} = T$$

where $t_k = t_0 + k\delta_n$, $\delta_n = 2^{-n}T$ $(k = 0, 1, 2, \ldots, 2^n)$ and define

$$(2.1.1) \qquad f_n(t) = \frac{1}{\delta_n}\int_{t_k}^{t_{k+1}} f(\tau)\, d\tau = f_{n,k} \qquad \text{for } t_k \leqslant t < t_{k+1},$$

$$f_n(T) = f_{n,2^n-1}\,.$$

Obviously, the mapping $f(t) \mapsto f_n(t)$ is linear with respect to the argument $f(t)$.

Let us calculate the L^1-norm of the difference $f_n(t) - f(t)$. To this purpose, we extend the function $f(t)$ to the whole real line, with $f(t) = 0$ for $t \notin [0, T]$. Then we have the following estimates:

$$\int_0^T \|f(t) - f_n(t)\|\, dt = \sum_0^{2^n-1}{}_k \int_{t_k}^{t_{k+1}} \left\| f(t) - \frac{1}{\delta_n}\int_{t_k}^{t_{k+1}} f(\tau)\, d\tau \right\| dt =$$

$$= \sum_0^{2^n-1}{}_k \frac{1}{\delta_n}\int_{t_k}^{t_{k+1}} \left\| \int_{t_k}^{t_{k+1}} \big(f(\tau) - f(t)\big)\, d\tau \right\| dt \leqslant \frac{1}{\delta_n}\sum_0^{2^n-1}{}_k \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \|f(\tau) - f(t)\|\, d\tau\, dt \leqslant$$

$$\leqslant \frac{1}{\delta_n}\int_0^T dt \int_{t-\delta_n}^{t+\delta_n} \|f(\tau) - f(t)\|\, d\tau \leqslant \frac{1}{\delta_n}\int_0^T dt \int_{-\delta_n}^{\delta_n} \|f(t+z) - f(t)\|\, dz$$

with $t + z = \tau$.

Now, we consider first the case in which $f(t)$ is constant $(= u)$ on a measurable set $E \subset [0, T]$ and null outside E, that is, if χ_E is the characteristic function of the set E, $f(t) = u\chi_E(t)$. Under this assumption it follows trivially that:

$$f_n(t) = \frac{1}{\delta_n}\, u \operatorname{mis}\{[t_k, t_{k+1}] \cap E\} \qquad \text{for } t_k \leqslant t \leqslant t_{k+1}$$

and

$$\int_0^T \|f(t) - f_n(t)\|\, dt \leqslant \|u\| \int_0^T \frac{1}{\delta_n}\left\{ \int_{-\delta_n}^{\delta_n} |\chi_E(t+z) - \chi_E(t)|\, dz \right\} dt \leqslant$$

$$\leqslant \|u\|\, \frac{1}{\delta_n}\int_{-\delta_n}^{\delta_n} dz \int_0^T |\chi_E(t+z) - \chi_E(t)|\, dt\,.$$

But $\lim\limits_{z\to 0}\int_0^T |\chi_E(t+z) - \chi_E(t)|\, dt = 0$; thus, provided that $-\delta_n \leqslant z \leqslant \delta_n$ and that, for $n \to +\infty$, $\delta_n \to 0$, our assertion follows. Because of the linearity of the mapping defined by (2.1.1), the assertion holds also for every simple function $f(t)$ (i.e. a sum of finite number of functions of the type first considered; we recall that the set of all such simple functions is dense in $L^1(0, T; B)$).

In the general case, we observe that, as a simple computation shows, the following inequality holds for $f_n(t)$ given by (2.1.1):

$$\|f_n\|_{L^1(0,T;B)} \leqslant \|f\|_{L^1(0,T;B)} . \tag{2.1.2}$$

In fact, (2.1.1) yields

$$\|f_n(t)\| \leqslant \frac{1}{\delta_n} \int_{t_k}^{t_{k+1}} \|f(t)\| dt \qquad \text{for } t_k \leqslant t < t_{k+1}$$

from which, by integrating over $[0, T]$:

$$\int_0^T \|f_n(t)\| dt \leqslant \frac{1}{\delta_n} \sum_0^{2^n-1}{}_k \delta_n \int_{t_k}^{t_{k+1}} \|f[t)\| dt = \int_0^T \|f(t)\| dt .$$

Now, by (2.1.2), the mappings $T_n: f(t) \mapsto f_n(t)$ are bounded uniformly on n; moreover, on the dense set S of the simple functions the sequence $\{T_n\}$ converges to the identity map. But this fact implies that $\{T_n\}$ converges to the identity everywhere, that is $\lim_n f_n(t) = f(t)$ for each $f(t) \in L^1(0, T; B)$.

2.2 Corollary. *If the map $F: [0, T] \to \mathfrak{L}(V, V)$ is strongly summable, then the sequence $\{F_n(t)\}$ associated to $F(t)$ by formula* (2.1.1) *converges L^1-strongly to $F(t)$.*

In fact, by the above lemma, for every $u \in V$, $F_n(t)u$ converges to $F(t)u$ in the norm of $L^1(0, T; V)$.

We remark that, if we assume as $F(t)$ the map $t \mapsto A(t)$ with the properties (0.j) ... (0.jjjj), then each $A_n(t)$ verifies this assumptions too.

2.3 Lemma. *If $A(t): [0, T] \to \mathfrak{L}(V, V)$ is strongly summable and is of positively bounded variation on $[0, T]$, then the same property is true for the $A_n(t)$ associated to $A(t)$ by formula* (2.1.1) *of lemma* 2.1, *and*

$$V^+_{A_n}([0, T]) \leqslant V^+_A(]0, T[) \qquad \forall n \in N .$$

The same statement holds also for the negative and the total variations.

PROOF. Let, as in lemma 2.1,

$$\delta_n = 2^{-n}T, \qquad t_k = k\delta_n \qquad (k = 0, 1, 2, \ldots, 2^n).$$

Since $A_n(t)$ is a step function, it is not difficult to see that, if $\tau_0, \tau_1, \ldots, \tau_{2^n-1}$ are points such that $t_k < \tau_k < t_{k+1}$ for $k = 0, 1, \ldots 2^n - 1$, the positive variation of $A_n(t)$ is given by

$$V^+_{A_n} = \sum_0^{2^n-1}{}_k \left|(A_n(\tau_{k+1}) - A_n(\tau_k))^+\right| = \sum_0^{2^n-1}{}_k \left|(A_{n,k+1} - A_{n,k})^+\right| =$$

$$= \sum_1^{2^n-1}{}_k \left|\left(\frac{1}{\delta_n}\int_{t_k}^{t_{k+1}} A(t)\,dt - \frac{1}{\delta_n}\int_{t_{k-1}}^{t_k} A(t)\,dt\right)^+\right| = \sum_1^{2^n-1}{}_k \left|\left(\frac{1}{\delta_n}\int_{t_{k-1}}^{t_k} (A(t+\delta_n) - A(t))\,dt\right)^+\right| =$$

$$= \sum_1^{2^n-1}{}_k \left|\left(\frac{1}{\delta_n}\int_0^{\delta_n} \{A(t_k+\xi) - A(t_{k-1}+\xi)\}\,d\xi\right)^+\right|.$$

But, by virtue of lemma 1.3, each integral of the sum is dominated by

$$\frac{1}{\delta_n}\int_0^{\delta_n} \left|(A(t_k+\xi) - A(t_{k-1}+\xi))^+\right| d\xi$$

and hence

$$V^+_{A_n} \leqslant \sum_1^{2^n-1}{}_k \frac{1}{\delta_n}\int_0^{\delta_n} \left|(A(t_k+\xi) - A(t_{k-1}+\xi))^+\right| d\xi =$$

$$= \frac{1}{\delta_n}\int_{]0,\delta_n[} \left\{\sum_1^{2^n-1}{}_k \left|(A(t_k+\xi) - A(t_{k-1}+\xi))^+\right| d\xi \leqslant \frac{1}{\delta_n}\int_{]0,\delta_n[} V^+_A(]0, T[)\right\} d\xi =$$

$$= V^+_A(]0, T[).$$

3. – Uniqueness of the solution.

It is well known that, under suitable hypotheses of regularity for the map $t \mapsto A(t)$, the equation (0.1.1) cannot have more than one solution. For instance, this fact was proved by J.L. Lions under the assumption that $A(t)$ has a continuous derivative. As we shall see, the techniques employed by the above author, duly improved, can be applied to our more general situation. We remark, also, that the uniqueness follows from the only fact that $A(t)$ is of negatively bounded variation.

Before to state our first lemma, let us recall that $\mathcal{S} \subset \mathcal{L}(V, V)$ is the manifold of the symmetric operators.

3.1 LEMMA. *Let $\{A_n(t)\}$ be a sequence of operator-valued functions: $[0, T] \to \mathcal{S}$, which converges strongly to $A(t)$ in $L^2(0, T; \mathcal{L}(V, V))$ and suppose that $\{u_n(t)\}$ is a sequence which converges to $u(t)$ in $L^2(0, T; V)$. Assume also that $|A_n(t)| \leqslant K$ for some $K > 0$. Then, for every $v \in L^2(0, T; V)$*

$$\lim_n \int_0^T ((A_n(t) u_n(t), v(t))) dt = \int_0^T ((A(t) u(t), v(t))) dt .$$

PROOF. We prove first that

$$\lim_n \int_0^T ((A_n(t) u(t), v(t))) dt = \int_0^T ((A(t) u(t), v(t))) dt .$$

To this end we observe that, for every $u \in L^2(0, T; V)$, $A_n(t) u(t)$ converges to $A(t) u(t)$ in $L^2(0, T; V)$. In fact, this property is trivially true when u belongs to the dense set of the simple functions. Furthermore, every $T_n: u(t) \mapsto A_n(t) u(t)$ is a bounded (by K) endomorphism of $L^2(0, T; V)$. Hence the property holds for every $u \in L^2(0, T; V)$.

Setting now $A_n(t) u(t) = \overline{u}_n(t)$, $A(t) u(t) = \overline{u}(t)$, it is $\lim_n \|\overline{u}_n - \overline{u}\|_{L^2(0,T;V)} = 0$. from which follows that $(\overline{u}_n, v)_{L^2(0,T;V)}$ converges to $(\overline{u}, v)_{L^2(0,T;V)}$, namely our first assertion.

In the general case, by using the inner products in $L^2(0, T; V)$ we shall write:

$$(A_n u_n, v) - (Au, v) = \big((A_n - A) u_n, v\big) + \big(A(u_n - u), v\big) = \tag{3.1.1}$$
$$= \big(u_n, (A_n - A) v\big) + (u_n - u, Av) .$$

Now, provided that the convergent sequence $\{u_n\}$ is bounded in $L^2(0, T; V)$, there exists a constant $C > 0$ such that

$$|(u_n, (A_n - A) v)| \leqslant \|u_n\|_{L^2(0,T;V)} \|(A_n - A) v\|_{L^2(0,T;V)} \leqslant C \|(A_n - A) v\|_{L^2(0,T;V)} .$$

Then our assertion is true, because (the absolute value of) the difference at the first member of (3.1.1) is dominated by

$$C \|(A_n - A) v\|_{L^2(0,T;V)} + \|u_n - u\|_{L^2(0,T;V)} \|Av\|_{L^2(0,T;V)} .$$

3.2 LEMMA. *Assume that the map* $t \mapsto A(t)$, *satisfying conditions* (0.j) ... (0.jjjj), *be of* N.B.V. *and let* $\varphi: [0, T] \to V$ *be continuous and such that* $\varphi' \in L^2(0, T; V)$. *Then, for each* $s \in [0, T]$:

$$2\int_0^s ((A(t)\varphi(t), \varphi'(t)))dt \leqslant M\|\varphi(s)\|^2 - \alpha\|\varphi(0)\|^2 + \int_{]0,s[} \|\varphi(t)\|^2 dv^-(t).$$

PROOF. As in lemma 2.1, consider the sequence of step functions $A_n: [0, s] \to \mathcal{L}(V, V)$ defined by:

$$A_n(t) = \frac{2^n}{s}\int_{t_k}^{t_{k+1}} A(t)\,dt = A_{n,k} \quad \text{for} \quad \frac{k}{2^n}s = t_k \leqslant t < t_{k+1} = \frac{k+1}{2^n}s$$
$$(k = 0, 1, 2, \ldots, 2^n - 1)$$

and observe that, as we have shown in lemma 3.1,

$$\int_0^s ((A(t)\varphi(t), \varphi'(t)))dt = \lim_n \int_0^s ((A_n(t)\varphi(t), \varphi'(t)))dt.$$

Because the hypotheses made on $\varphi(t)$, we can now calculate

$$\int_0^s ((A_n(t)\varphi(t), \varphi'(t)))dt$$

by integrating by parts over each sub-interval $[t_k, t_{k+1}[$. A simple computation yields:

$$2\int_0^s ((A_n(t)\varphi(t), \varphi'(t)))dt = 2\sum_0^{2^n-1}{}_k \int_{[t_k, t_{k+1}[} ((A_{n\,k}\varphi(t), \varphi'(t)))\,dt =$$
$$= \sum_0^{2^n-1}{}_k ((A_{n,k}\varphi(t_{k+1}), \varphi(t_{k+1}))) - \sum_0^{2^n-1}{}_k ((A_{n,k}\varphi(t_k), \varphi(t_k)))$$

and hence

$$2\int_0^s ((A_n(t)\varphi(t), \varphi'(t)))dt + ((A_{n,0}\varphi(0), \varphi(0))) - ((A_{n,n-1}\varphi(s), \varphi(s))) =$$
$$= -\sum_1^{2^n-1}{}_k (((A_{n,k} - A_{n,k-1})\varphi(t_k), \varphi(t_k))).$$

Let us denote by θ the last sum. Then

$$\theta = -\sum_{1}^{2^n-1}{}_k\big(\big((A_{n,k} - A_{n,k-1})\varphi(t_k), \varphi(t_k)\big)\big) =$$

$$= -\sum_{1}^{2^n-1}{}_k \frac{1}{\delta_n}\left(\left(\left(\int_0^{\delta_n}\Big(A(t_k+\xi) - A(t_{k-1}+\xi)\Big)\varphi(t_k)\,d\xi\right), \varphi(t_k)\right)\right) =$$

$$= \frac{1}{\delta_n}\int_0^{\delta_n}\left\{\sum_{1}^{2^n-1}{}_k\big(\big(-\,(A(t_k+\xi) - A(t_{k-1}+\xi)\,\varphi(t_k), \varphi(t_k)\big)\big)\right\}d\xi \leqslant$$

$$\leqslant \frac{1}{\delta_n}\int_0^{\delta_n}\left\{\sum_{1}^{2^n-1}{}_k\big(\big((A(t_k+\xi) - A(t_{k-1}+\xi))^-\varphi(t_k), \varphi(t_k)\big)\big)\right\}d\xi \leqslant$$

$$\leqslant \frac{1}{\delta_n}\int_0^{\delta_n}\left\{\sum_{1}^{2^n-1}{}_k\big|(A(t_k+\xi) - A(t_{k-1}+\xi))^-\big|\,\|\varphi(t_k)\|^2\right\}d\xi \leqslant$$

$$\leqslant \frac{1}{\delta_n}\int_0^{\delta_n}\left\{\sum_{1}^{2^n-1}{}_k\big((v^-(t_k+\xi) - v^-(t_{k-1}+\xi)\big)\|\varphi(t_k)\|^2\right\}d\xi\,.$$

From the definition of Riemann-Stieltjes integral it follows that, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that [9]:

$$\sum_{1}^{2^n-1}{}_k\big(v^-(t_k+\xi) - v^-(t_{k-1}+\xi)\big)\|\varphi(t_k)\|^2 \leqslant \int_{\xi}^{t_{2^n-1}+\xi}\|\varphi(t)\|^2\,dv^-(t) + \varepsilon \leqslant$$

$$\leqslant \int_{]0,s[}\|\varphi(t)\|^2\,dv^-(t) + \varepsilon$$

provided that $\max|t_k - t_{k-1}| \leqslant \delta$.

(9) In fact, if f and v are a pair of functions, defined over an interval $[a, b]$ the former continuous, the latter bounded and non-decreasing, then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any subdivision $\pi = \{c = \tau_0 < \tau_1 < \ldots < \tau_m = d\}$ of any interval $[c, d] \subset [a, b]$ such that $|\pi| = \max|\tau_i - \tau_{i-1}| \leqslant \delta$ it is

$$\sum_{1}^{m}{}_k f(\theta_k)\big(v(\tau_k) - v(\tau_{k-1})\big) \leqslant \int_c^d f(t)\,dv(t) + \varepsilon$$

where $\tau_{k-1} \leqslant \theta_k \leqslant \tau_k$.

We have thus, for every $\varepsilon > 0$, $\theta \leqslant \int_{]0,s[} \|\varphi(t)\|^2 dv^-(t) + \varepsilon$ and then

$$\theta \leqslant \int_{]0,s[} \|\varphi(t)\|^2 dv^-(t) .$$

Finally, taking into account (0.jj), (0.jjjj), we obtain the inequality:

$$2\int_0^s \big(\big(A_n(t)\varphi(t), \varphi'(t)\big)\big)dt \leqslant M\|\varphi(s)\|^2 - \alpha\|\varphi(0)\|^2 + \int_{]0,s[} \|\varphi(t)\|\, dv^-(t) .$$

Now, if $n \to \infty$, our assertion follows easily by lemma 2.1.

3.3 Theorem. *If $t \mapsto A(t)$ is of* N.B.V. *on* $[0, T]$, *then the problem* 0.1 *has at most one solution.*

Proof. Clearly, it suffices to show that (0.1) with zero initial data has, in W, only the solution $u = 0$.

Let $u \in W$ be some solution of (0.1.1) with $u^0 = u^1 = 0$ and define

$$\varphi_s(t) = \begin{cases} -\int_t^s u(\sigma)\, d\sigma & \text{for } t < s \\ 0 & \text{for } t \geqslant s . \end{cases}$$

It's easy to see that, for any $s \in [0, T]$, $\varphi_s \in L^2(0, T; V)$, $\varphi_s' = u\chi_s \in \in L^2(0, T; V)$ (10), $\varphi_s(T) = 0$; then each φ_s is a test function. By putting φ_s in (0.1.1) it follows

$$\int_0^s \big(\big(A(t)\varphi_s'(t), \varphi_s(t)\big)\big)dt = \int_0^s \big(u'(t), u(t)\big)\, dt$$

and hence, provided that the operator $A(t)$ is symmetric:

$$2\int_0^s \big(\big(A(t)\varphi_s(t) ,\ \varphi_s'((t))\big)\big)dt = |u(s)|^2 \geqslant 0 .$$

(10) We denote by χ_s the characteristic function of $[0, s]$.

On the other side we have, by lemma 3.2

$$2\int_0^s ((A(t)\varphi_s(t), \varphi_s'(t)))dt \leqslant M\|\varphi_s(s)\|^2 - \alpha\|\varphi_s(0)\|^2 + \int_{]0,s[} \|\varphi_s(t)\|^2 dv^-(t).$$

Therefore, observing that $\varphi_s(s) = 0$, we can state that, for a suitable constant $k > 0$, this inequality holds:

$$\|\varphi_s(0)\|^2 \leqslant \frac{k}{2} \int_{]0,s[} \|\varphi_s(t)\|^2 dv^-(t) .$$

By setting $w(t) = \int_0^t u(\sigma)\, d\sigma$ and writing $\varphi_s(t) = w(t) - w(s)$ it follows that:

$$\|w(s)\|^2 \leqslant k\int_{]0,s[} \{\|w(t)\|^2 + \|w(s)\|^2\} dv^-(t) =$$

$$= k\|w(s)\|^2\{v^-(s^-) - v^-(0^+)\} + k \cdot \int_{]0,s[} \|w(t)\|^2 dv^-(t).$$

Now, by choosing s_0 such that $v^-(s_0^-) - v^-(0^+) \leqslant 1/2k$, we obtain, for all $s \in [0, s_0]$, an inequality of the form:

$$\|w(s)\|^2 \leqslant 2k \int_{]0,s[} \|w(t)\|^2 dv^-(t) \leqslant 2k \int_{[0,s[} \|w(t)\|^2 dv^-(t) \qquad (k > 0).$$

This fact, by corollary 1.5, implies that $w = 0$ on $[0, s_0]$. By repeating this argument starting from s_0, and so on, it can be proved, in a finite number of steps, as lemma 1.6 assures, that w and hence u vanish on the whole of $[0, T]$.

4. – Existence of solutions.

Let us consider, first, the case in which $A(t)$ is a step function. Under this assumption, one can easily obtain the following result.

4.1 LEMMA. *Let all the hypotheses* (0.j) ... (0.jjjj) *be verified and assume, moreover, that $A(t)$ be a step function* (*and left-continuous*) *on* $[0, T]$; *then the problem* 0.1 *has a unique solution, for which the following estimates hold*:

$$\text{(4.1.1)} \qquad \left.\begin{array}{l} \|u(t)\|^2 \\ \\ |u'(t)|^2 \end{array}\right\} \leqslant \{M\alpha^{-1}\|u^0\|^2 + \alpha^{-1}|u^1|^2\} \exp\left(\alpha^{-1} V_A^+([0, t])\right) \qquad t \in [0, T].$$

Moreover, the mappings $u: [0, T] \to V$, $u': [0, T] \to H$, are continuous.

PROOF. Observe, first, that if the function $A(t)$ is constant ($=A$) on $[0, T]$, then the assertion of the lemma can be trivially proved by standard arguments. Furthermore, the following energy-equality is true for $t\in[0, T]$:

$$((A(t)u(t), u(t)))+|u'(t)|^2=((A(0)u(0),u(0)))+|u'(0)|^2. \tag{4.1.2}$$

Now, according to our hypothesis, put $A(t)=A_k\in\mathcal{L}(V, V)$ for $t\in[t_k, t_{k+1}[$ $(k=0, 1, \ldots, n-1)$. Let us consider, over each interval $[t_k, t_{k+1}]=I_k$, the Cauchy problem for the «restriction» of (0.1.1) to I_k, with initial data $u(t_k)=u_k^0$, $u'(t_k)=u_k^1$ (here the initial values u_k^0 and u_k^1 are vectors of V and H, chosen in a quite arbitrary way). For convenience, this «restricted problem» will be referred to in the sequel as the $(0.1)_k$ problem.

Since $A(t)$ is constant on I_k, it follows, by virtue of the above mentioned results, that each Cauchy problem we have considered has a unique solution $u_k(t)\colon I_k\to V$, with u_k and u_k' continuous, respectively, in the V and H topology. From this fact it can be deduced, by simple computations, that a function $u\in W$ is a solution of (0.1) over the whole interval $[0, T]$ if and only if the restriction of u to I_k satisfies (over I_k) the problem $(0.1)_k$, with initial data given by the values assumed by u_{k-1} and u_{k-1}' at $t=t_k$ and, obviously, for $t=0$, by the vectors u^0 and u^1, that is:

$$u_k^0=u_{k-1}(t_k), \quad u_k^1=u_{k-1}'(t_k) \quad (k=1, 2, \ldots, n-1)$$
$$u_0^0=u^0, \quad u_0^1=u^1.$$

By (4.1.2) we obtain the relation

$$((A(t)u(t), u(t)))+|u'(t)|^2=((A_k u_k^0, u_k^0))+|u_k'|^2$$

from which, if m is the largest integer such that $t_m<t$:

$$((A(t)u(t), u(t)))+|u'(t)|^2-((A_0u^0, u^0))-|u^1|^2=$$
$$=\sum_0^{m-1}{}_k(((A_{k+1}-A_k)u_{k+1}^0, u_{k+1}^0))=\sum_0^{m-1}{}_k(((A_{k+1}-A_k)^+u_{k+1}^0, u_{k+1}^0))-$$
$$-\sum_0^{m-1}{}_k(((A_{k-1}-A_k)^-u_{k+1}^0, u_{k+1}^0)).$$

It follows, by virtue of (0.jj), (0.jjjj):

$$\|u(t)\|^2 \leqslant M\alpha^{-1}\|u^0\|^2 + \alpha^{-1}|u^1|^2 + \alpha^{-1}\sum_{0}^{m-1}{}_k \left\|(A_{k+1}-A_k)^+\right\| \|u^0_{k+1}\|^2$$

$(t_m \leqslant t \leqslant t_{m+1})$, or, equivalently

$$\|u(t)\|^2 \in M\alpha^{-1}\|u^0\|^2 + \alpha^{-1}|u^1|^2 + \alpha^{-1}\int_{[0,t[} \|u(\tau)\|^2 \, dv^+(\tau) .$$

Hence, by lemma 1.4 we have

$$\|u(t)\|^2 \leqslant \{M\alpha^{-1}\|u^0\|^2 + \alpha^{-1}|u^1|^2\} \exp\left(\alpha^{-1} V_A^+([0, t])\right).$$

With the same argument it can be proved that the above estimate holds also for $|u'(t)|^2$.

We are now able to state the main result of this section.

4.2 THEOREM. *Assume that all the hypotheses* (0.j) ... (0.jjjj) *are verified. If the map* $t \mapsto A(t)$ *is of positively bounded variation, then there exists at least one solution* u *for the problem* 0.1. *Moreover,* u *and* u' *are weakly continuous as mappings from* $[0, T]$ *to* V *and* H *respectively.*

PROOF. Let $\{A_n(t)\}$ be the sequence of the operator-valued step function defined as in lemma 2.1 and let u_n be the (unique) solution of

$$(4.2.1) \qquad \begin{cases} \displaystyle\int_0^T \{(u'(t), v'(t)) - ((A_n(t)u(t), v(t)))\}\, dt = -(u^1, v(0)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every test function } v \in W \\ u(0) = u^0 \qquad u^0 \in V, \qquad u^1 \in H . \end{cases}$$

By lemma 4.1, there exists a number K, which depends neither from t nor from n, such that

$$\|u_n(t)\| \leqslant K, \qquad |u_n'(t)| \leqslant K .$$

I we denote by θ_δ the continuous map $[0, T] \to [0, 1]$ such that

$$\theta_\delta(t) = \begin{cases} 1 & \text{for } \tau' + \delta \leqslant t \leqslant \tau'' - \delta \qquad (0 < 2\delta < \tau'' - \tau') \\ 0 & \text{for } t \notin]\tau', \tau''[\\ \text{affine elsewhere } (^{11}), \end{cases}$$

(11) I.e. of the form $\theta_\delta(t) = at + b$.

then, for every fixed $v \in V$, the map $t \to \theta_\delta(t)v$ is, of course, a test function of W. By putting this function in (4.2.1), we obtain

$$\frac{1}{\delta}\int_{\tau'}^{\tau'+\delta}(u_n'(t), v)\,dt - \frac{1}{\delta}\int_{\tau''-\delta}^{\tau''}(u_n'(t), v)\,dt = \int_{\tau'}^{\tau''}\theta_\delta(t)\big((A_n(t)u_n(t), v)\big)\,dt$$

and, for $\delta \to 0$,

$$\big(u_n'(\tau'), v\big) - \big(u_n'(\tau''), v\big) = \int_{\tau'}^{\tau''}\big((A_n(t)u_n(t), v)\big)dt\,.$$

We recall now that, by (4.1.1), there exists a bounded set $U \subset V$ such that $u_n(t) \in U$ for every n and t; furthermore, for every n and t, $A_n(t)$ belongs to a bounded set of $\mathfrak{L}(V, V)$. Therefore, for a suitable constant $K' > 0$, we must have

$$|(u_n'(\tau''), v) - u_n'(\tau'), v)| \leqslant K'\|v\|\,|\tau'' - \tau'|\,.$$

The set $\{u_n'\}$ $(n \in N)$ consists then of equicontinuous functions, with respect to the weak topology of H. We shall now state that also the functions u_n $(n \in N)$ are weak-equicontinuous in V. In fact, consider the operator Λ defined by

$$(u, \Lambda v) = \big((u, v)\big). \tag{4.2.2}$$

Clearly, the domain $\mathfrak{D}\Lambda$ of Λ is a dense subspace of V; taking $v \in \mathfrak{D}\Lambda$ and applying (4.2.2) it follows that:

$$|\big((u_n(\tau'') - u_n(\tau'), v)\big)| = |\big(u_n(\tau'') - u_n(\tau'), \Lambda v\big)|$$
$$\int_{\tau'}^{\tau''}|(u_n'(t), \Lambda v)|\,dt \leqslant K\int_{\tau'}^{\tau''}|\Lambda v|\,dt = K|\Lambda v|\,|\tau'' - \tau'|\,.$$

The above estimate holds, clearly, for every $v \in \mathfrak{D}\Lambda$. Thanks to the density of $\mathfrak{D}\Lambda$, we can conclude, with standard arguments, that the $\{u_n\}$ $(n \in N)$ are weakly equicontinuous.

Therefore, by the well known lemma of Ascoli-Arzelà, there exists in $\{u_n\}$ a subsequence, which we also call $\{u_n\}$, and a differentiable map $u(t)$: $[0, T] \to V$, with $u'(t): [0, T] \to H$, such that:

$$\begin{cases} \lim\limits_n u_n(t) = u(t) & \text{uniformly in the weak topology of } V \\ \lim\limits_n u_n'(t) = u'(t) & \text{uniformly in the weak topology of } H \end{cases} \tag{4.2.3}$$

and, furthermore, such that u_n and u_n' converge, respectively to u and u' in the weak topologies of $L^2(0, T; V)$ and $L^2(0, T; H)$.

Clearly, u and u' are weakly continuous. Our purpose, now, is to prove that $u(t)$ is a solution of (0.1.1). To this end, taking into account the fact that $u_n(t)$ is a solution of (4.2.1), we have

$$\int_0^T \Big((u_n'(t), v'(t)) - ((A_n(t) u_n(t), v(t))) \Big)\, dt = -(u^1, v(0)), \qquad u_n(0) = u^0 .$$

But, by (4.2.3)

$$\lim_n \int_0^T (u_n'(t), v'(t))\, dt = \int_0^T (u'(t), v'(t))\, dt ;$$

furthermore, all the functions $A_n(t)$ being bounded in norm by the constant M which appears in (0.jj), we have by lemma 3.1

$$\lim_n \int_0^T ((A_n(t) u_n(t), v(t)))dt = \int_0^T ((A(t) u(t), v(t)))dt .$$

The proof of the theorem is then complete.

The results of sections 3 and 4 give finally the theorem:

4.3 THEOREM. *If the mapping $t \mapsto A(t)$ is of bounded variation on $[0, T]$, then the problem* 0.1 *has one and only one solution.*

In fact, this statement is a direct consequence of theorems 3.2 and 4.2.

REMARK 1. Under the hypothesis of theorem 4.3, the assumption that V is separable (which allowed us to state the measurability of $A(t)$) can be deleted. In fact, as we have observed, in this case $A(t)$ is continuous almost everywhere and hence almost-separable valued. This condition, however, assures that $A(t)$ is strongly measurable.

REMARK 2. Clearly, problem (0.1) does not change if the function $A(t)$ is modified only on a null set. Therefore, if we call «essential variation» of $A(t)$ the number $eV_A = \operatorname{Inf} V_{A^*}$, where $A^*(t)$ runs over the whole class of equivalence to which $A(t)$ belongs, our results hold under the assumption that eV_A (respectively eV_A^+ and eV_A^-) is finite. In particular, the inequality (4.1.1) can be improved by

$$\|u(t)\|^2 \leqslant \big(\alpha^{-1}(M\|u^0\|^2 + |u^1|^2\big) \exp\big(\alpha^{-1} eV_A^+([0, t])\big) .$$

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