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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série*, tome 27, nº 4 (1973), p. 707-716

<http://www.numdam.org/item?id=ASNSP_1973_3_27_4_707_0>

# A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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## 1. Introduction.

Let $A$ be an elliptic operator acting on functions satisfying differential boundary conditions which cover $A$ (for definitions and descriptions of such problems see [4, 5]). The problem considered is to find a solution of $Au = g(x, u)$, where $g(x, t)$ is a given function. It is assumed that 0 is an eigenvalue of $A$.

Problems of this sort have been considered by Landesman-Lazer [2], Williams [3], Nirenberg [1] and Browder [7]. Landsman-Lazer [2] considered the Dirichlet problem for self adjoint second order operators. They assumed that the null space $N(A)$ of $A$ is of dimension one. The function $g(x, t)$, they considered was of the form $h(x) - g(t)$. Williams [3] extended their results to higher order self adjoint operators and allowed arbitrary dimension for $N(A)$. Browder [7] extended this to arbitrary self adjoint boundary value problems.

Nirenberg [1] was the first to remove the restriction of self adjointness. He assumed that $A$ has a unique continuation propery. He used degree of mapping and homotopy theory to solve the problem.

In the present paper we generalize the results of these authors in the following ways.

1. We weaken the hypotheses on the function $g(x, t)$.

2. We do not require unique continuation.

3. We do not require self adjointness.

4. Our hypotheses are expressed in terms of inequalities rather than degree of mapping or homotopy classes.

Pervenuto alla Redazione il 15 Novembre 1972.

The last statement is not to be interpreted to mean that such ideas are to be avoided. Indeed, they provide deep insights into our problem. We note merely that hypotheses in terms of inequalities are more easily verified in practice.

We also show the connection with our work and that of Nirenberg [1] by giving a slight generalization of his Theorem 1 with hypotheses expressed in terms of degree (Theorem 2.2).

I wish to thank L. Nirenberg for a very helpful conversation.

## 2. Description of Results.

We now state our main theorem. Let $\Omega$ be a bounded domain in $E^n$ with smooth boundary $\partial\Omega$. Let $L$ be linear properly elliptic partial differential operator of order $m$ with real valued coefficients which are smooth on $\overline{\Omega}$ (for all definitions see [5]). Let $\{B_j\}$ be a set of $\frac{1}{2}m$ differential boundary operators with real valued coefficients smooth on $\partial\Omega$ which covers $L$. Let $A$ be the operator $L$ acting on real valued functions satisfying.

$$B_j u = 0 \quad \text{on} \quad \partial\Omega, \quad 1 \leq j \leq \frac{1}{2} m. \tag{2.1}$$

When considered as an operator on $L^2(\Omega)$, $A$ is closable. Thus we may consider it closed. It is known that $A$ is a Fredholm operator, i. e., $R(A)$ is closed in $L^2(\Omega)$ and $R(A)^\perp$ and $N(A)$ are finite dimensional (cf. [5]). Moreover, the functions in these subspaces are smooth on $\overline{\Omega}$.

Let $g(x, t)$ be a real function which is measurable in $x$ for $x$ in $\Omega$ and continuous in $t$ for $t$ real. We assume that there is a function $\tilde{g}(x) \in L^1(\Omega)$ such that

$$|g(x,t)| \leq \tilde{g}(x), \quad x \in \Omega, \ t \text{ real}. \tag{2.2}$$

We also assume that there are functions $g_\pm(x) \in L^1(\Omega)$ such that

$$g(x,t) \to g_\pm(x) \text{ a. e. as } t \to \pm\infty. \tag{2.3}$$

Let $T$ be a map from $R(A)^\perp$ to $N(A)$. Define

$$M_T(z) = \int_{Tz > 0} g_+(x)\, z(x)\, dx + \int_{Tz < 0} g_-(x)\, z(x)\, dx$$

and

$$\varrho_T(c) = \sup_{z \in R(A)^\perp} \operatorname{meas} \left\{ x \in \Omega : \frac{|Tz(x)|}{|z(x)|} < c \right\}$$

(we define 0/0 as $\infty$). We have

THEOREM 2.1. *Assume that there is a linear map T such that*

$$M_T(z) > 0, \quad z \in R(A)^\perp \tag{2.4}$$

$$\varrho_T(c) \to 0 \text{ as } c \to 0. \tag{2.5}$$

*Then there is a* $u \in D(A)$ *such that*

$$Au = g(x, u). \tag{2.6}$$

REMARKS. 1. It follows from our method of proof that the solution of (2.6) that we find has no component in $N(A)$ outside the range of $T$.

2. L. Nirenberg has pointed out that is it not necessary to assume $T$ linear. It suffices to take it continuous and homogeneous.

3. When $A$ is self adjoint, $R(A)^\top = N(A)$. We then merely take $T$ the identity operator. Then (2.5) is trivially true and (2.4) reduces to the Landesman-Lazer [2] and Williams [3] condition.

4. If $A$ has a unique continuation property (i. e., if the only function in $N(A)$ which vanishes on a set of positive measure is $u \equiv 0$), then (2.5) holds. The proof is the same as that of Lemma 2 of [1].

5. The index of $A$ is defined as $i(A) = \dim N(A) - \dim R(A)^\perp$. If $i(A) < 0$, then hypothesis (2.4) cannot be satisfied for any linear $T$. For any such $T$ is not injective. Let $z_0$ be an element of $N(T)$ and pick $z$ so that

$$Q = \int_{Tz > 0} g_+(x) z_0(x) \, dx + \int_{Tz < 0} g_-(x) z_0(x) \, dx$$

is not zero. Then

$$M_T(z + a z_0) = M_T(z) + a Q$$

for any real $a$. By picking $a$ suitably we can make this quantity negative.

6. Suppose $i(A) \geqq 0$, and let $z_1, \dots, z_k$ be a basis for $R(A)^1$. Set $z = a_1 z_1 + \cdots + a_k z_k$, where $a = (a_1, \dots, a_k)$ is a real vector. Put

$$\varphi_j(a) = \int_{Tz>0} g_+(x) z_j(x)\,dx + \int_{Tz<0} g_-(x) z_j(x)\,dx, \quad 1 \leq j \leq k, \tag{2.7}$$

and $\varphi(a) = (\varphi_1(a), \dots, \varphi_k(a))$. Consider the mapping

$$\psi(a) = \varphi(a)/|\varphi(a)| \tag{2.8}$$

of $S^{k-1}$ into itself. Hypothesis (2.4) says that

$$a \cdot \varphi(a) = M_T(z) > 0.$$

If $\varphi(a)$ is continuous, this implies that the degree of the mapping $\psi$ is one. This shows the connection between our results and those of Nirenberg [1]. In fact a slight modification of our proof gives.

THEOREM 2.2. *If* (2.5) *holds, then the mapping* $\varphi(a)$ *given by* (2.7) *is continuous. If, in addition,* $\varphi(a) \neq 0$ *for* $a \neq 0$ *and the degree of the mapping* $\psi(a)$ *given by* (2.8) *is different from zero, then* (2.6) *has a solution.*

The proofs of Theorems 2.1 and 2.2 are given in Section 4.

## 3. A fixed point theorem.

We now prove an abstract theorem in Banach space which is used to obtain the solution to our problem. We follow the approach of Landesman-Lazer [2] and Williams [3]. Our method is slightly more general than required.

Let $X$ be a Banach space, and let $Z$ be a finite dimensional Hilbert space. Let $T$, $G$ and $H$ be compact mappings from $Z$ to $X$, $X$ to $X$, and $X$ to $Z$, respectively. Set

$$\tilde{\alpha} = \limsup_{\|z\| \to \infty} \|Tz\| / \|z\|$$

$$\tilde{\beta} = \limsup_{\|u\| \to \infty} \|Gu\| / \|u\|$$

$$\tilde{\gamma} = \limsup_{\|u\| \to \infty} \|Hu\| / \|u\|.$$

We have

THEOREM 3.1. *Assume that* $\tilde{\alpha}\tilde{\gamma}+\tilde{\beta}<1$ *and that*

$$2\,(z, H(Tz+Gu)) \geq \| H(Tz+Gu) \|^2 \tag{3.1}$$

*holds for* $\| z \|$ *large, uniformly in u. Then the mapping*

$$u^* = Tz + Gu, \quad z^* = z - Hu^* \tag{3.2}$$

*of* $X \times Z$ *into itself has a fixed point.*

PROOF. By hypothesis, there are numbers $\alpha$, $\beta$, $\gamma$, $R$ such that $\alpha\gamma+\beta<1$,

$$\| Tz \| \leq \alpha \| z \|, \quad \| z \| > R, \tag{3.3}$$

$$\| Gu \| \leq \beta \| u \|, \quad \| u \| > R, \tag{3.4}$$

$$\| Hu \| \leq \gamma \| u \| \quad \| u \| > R, \tag{3.5}$$

and (3.1) holds for $\| z \| > R$. Moreover, there are constants $K_i$ such that

$$\| Tz \| \leq K_1, \quad \| z \| \leq R,$$

and

$$\| Gu \| \leq K_2, \quad \| Hu \| \leq K_3, \quad \| u \| \leq R.$$

Set $\theta = (1-\beta)/\alpha$, and

$$M = \max\left\{\frac{K_1}{\alpha\theta}, \frac{K_2}{\beta}, \frac{K_3}{\gamma}, \frac{R}{\theta-\gamma}\right\}.$$

Assume that

$$\| u \| \leq M, \quad \| z \| \leq \theta M. \tag{3.6}$$

If $\| u \| > R$, then

$$\| u^* \| \leq \| Tz \| + \| Gu \| \leq \alpha\,\theta\, M + \beta\, M = M$$

when $\| z \| > R$. On the other hand, if $\| u \| \leq R$, then

$$\| u^* \| \leq \alpha\,\theta\, M + K_2 \leq M$$

when $\| z \| > R$, and

$$\| u^* \| \leq K_1 + K_2 \leq M$$

when $\| z \| \leq R$. Thus in any case $\| u^* \| \leq M$.

Next, if $\| z \| > R$, then

$$\| z^* \|^2 = \| z \|^2 - 2\ (z, Hu^*) + \| Hu^* \|^2 \leq \| z \|^2.$$

If $\| z \| \leq R$, then

$$\| z^* \| \leq R + \gamma M \leq (\theta - \gamma) M + \gamma M = M$$

when $\| u^* \| > R$, and

$$\| z^* \| \leq R + K_3 \leq (\theta - \gamma)\ M + \gamma M = M$$

when $\| u^* \| \leq R$. Thus $\| z^* \| \leq \theta M$ in any event. It therefore follows that the mapping (3.2) is compact from the closed convex subset (3.6) of $X \times Z$ into itself. It thus follows that it has a fixed point. This completes the proof.

## 4. The proofs.

We now give the proof of Theorem 2.1. Let $P$ be the projection of $L^2(\Omega)$ onto $R(A)^{\perp}$. From the theory of elliptic boundary value problems, there is an operator $S$ from $R(A)$ into $N(A)^{\perp}$ which is an inverse of $A$. Moreover, for each $s \geq 0$ and $p > 1$ there is a constant $C$ such that

$$\| Sh \|_{s,p} \leq C \| h \|_{s-m,p}, \quad h \perp R(A). \tag{4.1}$$

(cf. [6]). By definition, for $a > 0$

$$\| h \|_{-a,p} = \sup_{\varphi} \frac{|(h, \varphi)|}{\| \varphi \|_{a,p'}}.$$

Thus if $h \in L^1(\Omega)$ and $p' > n/a$, we have

$$\| h \|_{-a,p} \leq C \| h \|_1. \tag{4.2}$$

Now $P$ is of the form

$$P h = \Sigma (h, z_k)\, z_k, \tag{4.3}$$

where the $z_k$ form an orthonormal basis for $R(A)^{\perp}$. Since the $z_k$ are smooth in $\bar{\Omega}$, we see that $P$ maps $L^1(\Omega)$ into $L^\infty(\Omega)$. Hence the operator $S(I - P)$ maps $L^1(\Omega)$ into $L^p(\Omega)$ for some $p > 1$. We take this as our Banach space $X$ in applying Theorem 3.1. Note too that $S(I - P)$ is a compact operator from $L^1(\Omega)$ to $X$. We take $Z = R(A)^{\perp}$ and consider the mapping (3.2) with

$$Gu = S(I - P)\, g(x, u), \quad Hu = Pg(x, u). \tag{4.4}$$

By (4.1) - (4.3) we see that

$$\| Gu \| + \| Hu \| \leq K, \qquad u \in X. \tag{4.5}$$

Clearly $T$, $G$, $H$ are compact operators. Since $T$ is bounded, $\tilde{\alpha} < \infty$. By (4.5), $\tilde{\beta} = \tilde{\gamma} = 0$. The only hypothesis of Theorem 3.1 which has not yet been verified is (3.1). We proceed to verify it. In fact, we shall show that there is a constant $b > 0$ such that

$$(z, H(Tz + Gu)) \geq b \| z \|, \quad z \in Z \tag{4.6}$$

holds for $\| z \|$ sufficiently large uniformly in $u$. By (4.5), this implies (3.1).

Let $u, z$ and $\varepsilon > 0$ be given. Define $h(x)$ to equal $g_+(x)$ when $Tz(x) > 0$, to equal $g_-(x)$ when $Tz(x) < 0$ and to vanish when $Tz(x) = 0$. Since $\tilde{g} \in L^1(\Omega)$, there is a $\delta > 0$ such that

$$\int_W \tilde{g}(x)\, dx < \varepsilon/6 \tag{4.7}$$

for any subset $W$ of $\Omega$ having measure $m(W)$ less than $\delta$. Set $v(x) = Gu$. By (4.5), the $L^1(\Omega)$ norm of $v$ is bounded by a coustant $\sigma$ independent of $u$. Thus the set $W_1$ of points where $| v(x) | > 3\sigma/\delta$ has measure less than $\delta/3$. By (2.5) there is a constant $N$ independent of $z$ such that the set $W_2$ where

$$\frac{| Tz(x) |}{| z(x) |} < \frac{1}{N} \tag{4.8}$$

has measure less than $\delta/3$. There is also a set $W_3$ of measure less than $\delta/3$ and a constant $J$ such that

$$| g(x, t) - g_{\pm}(x) | < \varepsilon/3m(\Omega) \tag{4.9}$$

holds for $\pm t > J$ and $x \in \Omega / W_3$. Set $W = \cup W_i$ and

$$L = \frac{6}{\varepsilon} \int_\Omega \tilde{g}(x)\, dx. \tag{4.10}$$

Then $m(W) < \delta$. Let $D$ be the set of those $x \in \Omega \setminus W$ such that

$$\| Tz(x) | < \| z \|_\infty / LN.$$

Finally, set $E = \Omega \setminus (D \cup W)$. Then

$$\int_\Omega | [g(x, u^*) - h(x)] z(x) | \, dx \le \int_W + \int_D + \int_E.$$

Now by (4.1)

$$\int_W \le 2 \| z \|_\infty \int_W \tilde{g} \, dx < \frac{\varepsilon}{3} \| z \|_\infty .$$

Moreover,

$$\int_D \le \frac{2}{L} \| z \|_\infty \int_\Omega \tilde{g} \, dx < \frac{\varepsilon}{3} \| z \|_\infty ,$$

since $| z(x) | < \| z \|_\infty \setminus L$ on $D$. Now take

$$\| z \|_\infty \ge LN \left( J + \frac{3\sigma}{\delta} \right) \tag{4.11}$$

(note that none of these quantities depend on $z$ or $u$). Then on $E$

$$| Tz + v | \ge | Tz | - | v | \ge \frac{\| z \|_\infty}{LN} - \frac{3\sigma}{\delta} \ge J .$$

Thus by (4.9)

$$\int_E \le \frac{\varepsilon}{3m(\Omega)} \| z \|_\infty \, m(\Omega) = \frac{\varepsilon}{3} \| z \|_\infty .$$

Thus we have shown that

$$(g(x, t\, Tz + v), z) \longrightarrow M_I(z) \text{ as } t \longrightarrow \infty \tag{4.12}$$

uniformly in $u$ and $z$ provided $\| z \|$ is bounded. Since the left hand side of (4.12) is a continuous function of $z$ for each $t$ and closed bounded sets in $Z$ are compact, it follows that $M_T(z)$ is a continuous function of $z$. Thus there is a constant $b > 0$ such that

((4.13) $\qquad M_T(z) \ge 2\, b \| z \|, \qquad z \in Z.$

This combined with (4.12) gives

$$(g(x, Tz + Gu), z) \geq b \| z \|, \qquad z \in Z,$$

when $\| z \|$ is sufficiently large, uniformly in $u$. This is precisely (4.1). The proof is complete.

PROOF OF THEOREM 2.2. We follow the reasoning of [1]. We have already shown that $\varphi(a)$ is continuous. Now consider the mapping

$$V[u, a] = [\tilde{u}, \tilde{a}],$$

where

$$\tilde{u} = u - G(u + \Sigma a_i T z_i)$$

and

$$\tilde{a}_j = (g(x, u + \Sigma a_i T z_i), z_j), \ 1 \leq j \leq k.$$

One checks easily that $V[u, a] \neq 0$ for $\| u \| + \| a \|$ sufficiently large. In fact by (4.5) $\| \tilde{u} \| \to \infty$ as $\| u \| \to \infty$, while we have shown that

$$\tilde{a} \to \varphi(a) \text{ as } \| a \| \to \infty$$

uniformly in $u$ for $\| u \|$ bounded. Now the mapping $V$ can be continuously deformed into

$$V_0[u, a] = [u, \varphi(a)]$$

without having the mapping vanish for $\| u \| + \| a \|$ large. Thus the degree of mapping of $V$ is the same as that of $V_0$. But the degree of the latter is the same as that of $\psi$ given by (2.8). It remains only to note that $V[u, a] = 0$ implies that $u + \Sigma a_i T z_i$ is a solution of (2.6). This completes the proof.

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