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ON CONVERGENCE OF ORTHOGONAL SERIES OF BESSEL FUNCTIONS

by A. BENEDEK and R. PANZONE

ABSTRACT - SUMMARY. Let ν be a real number and $\{w_n(x)\}$ a system of solutions of Bessel'sequation $(xy')' + (k^2 x - \nu^2/x) y = 0$, $0 < x$, which satisfy a real boundary condition at $x = 1$:

(1) $$\alpha y(1) + \beta y'(1) = 0.$$

If it is orthogonal and complete with respect to the measure xdx, then it coincides with the Bessel or Dini system for $-1 < \nu < \infty$ or with one of the systems

(2) $$\begin{cases} v(a_n, x) = k\, a_n^{-\nu} J_\nu(a_n x) + a_n^\nu J_{-\nu}(a_n, x), & 1 > \nu > 0, -\infty < k < \infty, \\ v(a_n, x) = -(2/\pi)\, lg\,(ka_n)\, J_0(a_n x) + Y_0(a_n x), & \nu = 0, 0 < k < \infty, \end{cases}$$

where $\{a_n\}$ is the set of zeroes of a certain function associated to the system under consideration. Let be $\mu = (\nu - 1/2) \vee 0, 1 \geq \nu > 0$, and $1/2 - \beta + \mu < 1/p < 3/2 - \beta - \mu$, $1 < p < \infty$. If $\widetilde{S}_n(f, x)$ denotes the partial sum of the Fourier expansion of f with respect to the system (2) and the measure xdx, and $S_n(f, x)$ the partial sum of the expansion with respect to the system of Bessel functions satisfying (1) and of order $-\nu$, then

$$\int_0^1 | S_n(f, x) - \widetilde{S}_n(f, x) |^p x^{\beta p}\, dx \leq C_n \int_0^1 | f(x) | x^{\beta p}\, dx,$$

where $C_n = 0(1)\, n^{-2\nu}$ when $1 > \nu > 0$, $C_n = 0(1)/lg\, n$ if $\nu = 0$.

This is achieved with an estimation of the difference of the Dirichlet kernels.

1. Introduction.

Let us consider Bessel's equation

(1) $$(x\, y')' + (k^2 x - \nu^2/x)\, y = 0 \quad 0 < x, \nu \text{ real},$$

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in its standard form (Liouville's normal form):

$$z''(x) + \left(k^2 - \frac{\nu^2 - 1/4}{x^2}\right) z(x) = 0, \; z(x) = y(x)\sqrt{x}, \tag{2}$$

in the interval (0,1] with boundary condition $z(1) = 0$. An application to this equation of Weyl's theory of ordinary differential equations with singularities can be found in Titchmarsh's book [4], § 4.8.

It is shown there that if $\nu \geq 1$, this equation is an example of the so-called « limit-point case » of Weyl's theory, and the corresponding eigenfunction expansion of a function is its Fourier-Bessel series.

If $0 \leq \nu < 1$, except for $\nu = 1/2$ where the equation has no singularity, it is an example of the « limit-circle case ». In this case there are infinitety many systems of eigenfunctions, each one associated to a point of the limit-circle. It is proved that all these systems are complete in $L^2(0, 1)$ and for a large class of functions, dense in L^2, the corresponding series converge absolutely and uniformly.

In this paper we shall study the mean-convergence in certain L^p-spaces of these series, in case $\nu \in [0, 1)$, (cf. Th. 9 and its corollary). The case $\nu \geq 1$ has been considered already, cf. [6], [1] and [2]. We show that the mentioned eigenfunction expansions are equiconvergent in the mean with the Fourier-Bessel series of order $-\nu$.

In order to characterize all orthonormal systems of solutions of Bessel's equation (cf. Th. 10) and also to be as self-contained as possible we shall introduce the aforementioned systems in another way.

Let $L^p(a)$ be the family of functions p integrable with respect to the measure $x^a \cdot dx$ on the interval $(0, 1)$.

Let us consider the following problem. Let $u_n(x)$ be a solution of Bessels equation with $k = a_n$, $u_n(1) = 0$. When do the functions $u_n(x)$ form an orthogonal system in $L^2(1)$? Is this system complete in this space?

Since

$$\begin{cases} u_n(x) = A_n J_\nu(a_n x) + B_n J_{-\nu}(a_n x) & \text{if } 0 \leq \nu \neq \text{integer}, \\ u_n(x) = A_n J_\nu(a_n x) + B_n Y_\nu(a_n x) & \text{if } \nu = 0, \pm 1, \pm 2, \dots . \end{cases} \tag{3}$$

this problem consists in finding relations between A_n, B_n and a_n that assure the orthogonality with respect to the measure $x dx$ and such that

$$u_n(1) = 0. \tag{4}$$

Since the functions u_n must be in $L^2(1)$ if $B_n \neq 0$, we must necessarily have $0 \leq \nu < 1$. In this situation we must find for each $\nu \in [0, 1)$ all the orthogonal systems which are complete, and they are precisely those described above in the « limit-circle case ». One of these systems is the Bessel system corresponding to $A_n \equiv 0$ when $\nu > 0$.

The same problem can be posed changing only the boundary condition (4) into

(5) $$u_n'(1) - H\, u_n(1) = 0.$$

Then again an infinity of complete orthogonal systems can be found, one which is the Dini system (corresponding to $A_n \equiv 0$ if $\nu > 0$).

We shall show that, for fixed ν, the difference between the Diriclet kernels of the first (second) systems with the Dirichlet kernel of Bessel (Dini) system has an expression that assures equiconvergence of the corresponding series (cf. Th. 8 and 9).

2. Orthogonal solutions of Bessel' equation.

We shall suppose that the systems $\{u_n(x)\}$ *do not coincide* with the ordinary Bessel or Dini systems and call $D = \{z : 0 \leq \arg z < \pi\}$.

THEOREM 1. i) *Assume* $\nu \in (0, 1)$. *Let* $u_n(x) = A_n J_\nu(a_n x) + B_n J_{-\nu}(a_n x) \not\equiv 0$, $a_n \in D - \{0\}$, *verify a real homogeneous condition at* $x = 1$:

(7) $$\alpha\, u_n(1) + \beta\, u_n'(1) = 0\,; \quad \alpha^2 + \beta^2 \neq 0\,; \quad \alpha, \beta \text{ real.}$$

Then, the system $\{u_n\}$ *is orthogonal (with respect to* $x dx$) *if and only if*

(8) $$a_n^{2\nu} A_n / B_n = k, \qquad k \text{ a non-zero real constant}$$

ii) $\nu = 0$. Let $u_n(x) = A_n J_0(a_n x) + B_n Y_0(a_n x)$, $a_n \in D - \{0\}$, be $\not\equiv 0$ and verify (7).

Then the system $\{u_n\}$ is orthogonal iff

(9) $$A_n/B_n + (2 \lg a_n)/\pi = -(2 \lg k)/\pi, \quad k \text{ a real positive constant.}$$

In both cases a_n^2 *is real.*

PROOF. $u_n(x)$ is a solution of

(10) $$(x u_n')' + (a_n^2 x - \nu^2/x)\, u_n = 0 \text{ for } x > 0.$$

Then for $s > 0$

$$(a_n^2 - \overline{a}_m^2)\int_s^1 u_n(x)\,\overline{u}_m(x)\,xdx = (u_n \cdot x\overline{u}'_m - \overline{u}_m \cdot xu'_n)\,|_s^1 =$$

$$= s\,(-u_n(s)\,\overline{u}'_m(s) + \overline{u}_m(s)\,u'_n(s)).$$

Since

$$J_\nu(a \cdot s) = (as)^\nu/2^\nu\,\Gamma(1+\nu) + 0\,(s^{2+\nu}) \tag{11}$$

we get in case i) for $s \to 0$,

$$(a_n^2 - \overline{a}_m^2)\cdot \int_0^1 u_n\,\overline{u}_m\,xdx = 2\nu\,[a_n^\nu\,\overline{a}_m^{-\nu}\,A_n\,\overline{B}_m - \overline{a}_m^\nu\,a_n^{-\nu}\,\overline{A}_m\,B_n]/\Gamma(1+\nu)\,\Gamma(1-\nu) \tag{12}$$

The orthogonality therefore implies:

$$(a_n/\overline{a}_m)^\nu\,A_n\,\overline{B}_m = (\overline{a}_m/a_n)^\nu\,\overline{A}_m\,B_n\,. \tag{13}$$

B_n cannot vanish for any n. If not, by (13), B_m would vanish for all m contradicting our general assumption at the beginning of this paragraph (the same hold for A_m). So, (13) is equivalent to

$$a_n^{2\nu}\,A_n/B_n = \overline{a}_m^{2\nu}\,\overline{A}_m/\overline{B}_m \neq 0 \qquad \forall m, n.$$

Then obviously $a_n^{2\nu}\,A_n/B_n = k \neq 0$ and k real. Besides a_n^2 is real. In fact if $u_n(x)$ verifies (7) then from (12) we get

$$(a_n^2 - \overline{a}_n^2)\int_0^1 |\,u_n\,|^2\,xdx = 0,$$

impossible if a_n^2 is not real. Therefore, if $a_n \neq a_m$ then $a_n^2 \neq a_m^2 = \overline{a}_m^2$ and from (12) it follows that u_n is orthogonal to u_m with respect to xdx.

In case ii) recall that

$$Y_0(\lambda x) = (2/\pi)\,(\lg(x\lambda/2) - \gamma) + 0\,(x^2 \lg x), \qquad \gamma = \text{Euler's constant.} \tag{14}$$

So, instead of (12) we have

$$(15) \qquad (a_n^2 - \bar{a}_m^2)\int_0^1 u_n \bar{u}_m \, x dx =$$

$$= \frac{2}{\pi}\left\{\frac{2}{\pi} B_n \bar{B}_m \lg \frac{\bar{a}_m}{a_n} + \bar{A}_m B_n - A_n \bar{B}_m\right\}.$$

So, the orthogonality implies

$$B_n \left\{\frac{2}{\pi} \bar{B}_m \lg \bar{a}_m + \bar{A}_m\right\} = \bar{B}_m \left\{\frac{2}{\pi} B_n \lg a_n + A_n\right\}.$$

As before $B_n \neq 0$. Dividing this equality by $B_n \cdot \bar{B}_m$ we get:

$$(16) \qquad (2/\pi) \lg \bar{a}_m + \bar{A}_m/\bar{B}_m = (2/\pi) \lg a_n + A_n/B_n = (-2/\pi) \lg k,$$

where k is a real positive constant. The desired result follows as before. Q E D

Since $B_n \neq 0$, we may take for convenience $B_n = a_n^{\nu}$, $1 > \nu \geq 0$; in consequence the orthogonal systems of Theorem 1 become (cf. [4], § 4.8):

$$(17) \quad u_n(x) = \begin{cases} k\, a_n^{-\nu} J_\nu(a_n x) + a_n^{\nu} J_{-\nu}(a_n x), & \text{if } 1 > \nu > 0, \\ -(2/\pi) \lg (k a_n) J_0(a_n x) + Y_0(a_n x) = & \\ (-2/\pi) \lg (k/x) J_0(a_n x) + G(a_n x) & \text{if } \nu = 0, \end{cases}$$

where $G(y($ is the entire function defined by $G(y) = Y_0(y) - 2\pi^{-1}(\lg y) J_0(y)$ and the a_n' are the zeros of

$$(18) \quad \begin{cases} \psi(z) = \alpha\,(kz^{-\nu} J_\nu(z) + z^{\nu} J_{-\nu}(z)) + \beta\,(kz^{1-\nu} J_\nu'(z) + z^{1+\nu} J_{-\nu}'(z)) & \nu > 0, \\ \psi(z) = \alpha\left(-\frac{2}{\pi} \lg (kz) J_0(z) + Y_0(z)\right) + \beta\left(-\frac{2}{\pi} z \lg (kz) J_0'(z) + z Y_0'(z)\right), & \nu = 0, \end{cases}$$

which lie on the positive real axis or on the positive imaginary axis.

Now, from theorem 1 one readily obtains the following known resnlt:

COROLLARY 1. *The functions* $\{Y_\nu(a_n x);\ \alpha Y_\nu(a_n) + \beta a_n Y_\nu'(a_n) = 0\}$ *do not form an orthogonal system for any* $\nu \in [0, 1)$.

REMARK. Condition (13) (or (16)) may be regarded as a boundary condition at $x = 0$ since it assures the vanishing of (12) (or (15)).

$\psi(z)$ is an entire function of z and to each of its zeros in D we can associate a function of the orthogonal system even if $z = 0$ is a zero of ψ.

In fact, (17) for fixed x in $(0, 1]$ is an entire function of a_n. Besides, it is a solution of (10) for any complex number a_n. If $\psi(0) = 0$, making $a_n = 0$ in (17) we obtain a new function $E(x)$ that verifies (7) and (10).

Because $u_n(x) = E(x) + 0(x)$, we obtain instead of formulae (12) and (15):

$$a_n^2 \int_s^1 u_n(x)\, E(x)\, x dx = o(1) \qquad \text{for } s \longrightarrow 0.$$

Therefore, $E(x)$, the function associated to the zero $z = 0$ of ψ, is orthogonal to $\{u_n(x)\}$.

From now on, $\{a_n\}$ will be the set of zeros of ψ in D and the systems (17) *will include the functions obtained for $a_n = 0$ if $\psi(0) = 0$.*

3. Distribution of the eigenvalues.

We shall examine more closely the non positive zeroes of $\psi(z)$ which lie in D. First we state a well-known result ([4], ch. V).

THEOREM 2. *$\psi(z)$ has only a finite number of non-positive zeroes in D. They lie on the imaginary axis.*

PROOF. $a_n \neq 0$ implies a_n^2 is real (cf. th. 1). Therefore non-real zeroes are purely imaginary. It is easy to see that the entire function ψ (it) is even and its expansion about zero has real coefficients, which, except for a finite number, have the same sign. So, if $t \longrightarrow +\infty$, $|\psi(\mathrm{it})| \longrightarrow +\infty$ and $\psi(\mathrm{it})$ has no zeroes for t great enough. Q E D.

DEFINITION 1. *We shall call the boundary condition* (7) *simple if $\beta = 0$*, i. e. *if it is of the form*

$$u_n(1) = 0, \tag{19}$$

and mixed if $\beta \neq 0$, i. e *if it is of the form*

$$u_n'(1) - H u_n(1) = 0. \tag{20}$$

In case the boundary condition is simple, we can obtain more precise results about the zeroes of $\psi(z)$. The rest of this section is *not* used in the following ones). Let γ be Euler's constant.

THEOREM 3. *Under simple boundary condition $\psi(z)$ has at most one non-positive zero in D. More precisely: let $k_0 = -\Gamma((1+\nu)\,2^{2\nu}/\Gamma(1-\nu)$ if $1 > \nu > 0$ and $k_0 = e^{\gamma}/2$* if $\nu = 0$. *Then for $k > k_0$, $\psi(z)$ has only positive zeroes in D; for $k = k_0$, $a_0 = 0$ is the only zero on the imayinary axis; for $k < k_0$, there is one purely imaginary zero $a_0 = it$ with $t > 0$.*

To prove th. 3 we need an elenentary lemma.

LEMMA 1. *Let $\varphi(x) = \sum_{j=0}^{\infty} a_j x^j$ be an entire function such that for certain $j_0 > 0$, $a_0 < 0$, $a_j \le 0$ for $j < j_0$ and $a_{j_0} > 0$, $a_j \ge 0$ when $j > j_0$. Then $\varphi(x)$ has exactly one zero on $[0, \infty)$ and it is positive.*

PROOF. $\varphi(x) = 0$ is equivalent to

$$P(x) = \sum_{j=0}^{j_0-1} (-a_j)\, x^j = \sum_{j=j_0}^{\infty} a_j x^j = Q(x). \tag{21}$$

The assumptions yields $P(0) > 0 = Q(0)$ and $P(x) < Q(x)$ for x great enough.

Therefore there is at least one $x_0 > 0$ which verifies (21). Also

$$x_0 P'(x_0) \le (j_0 - 1)\, P(x_0) = (j_0 - 1)\, Q(x_0) < j_0\, Q(x_0) \le x_0\, Q'(x_0).$$

It follows $Q'(x_0) > P'(x_0)$. Therefore x_0 is unique.

PROOF OF THEOREM 3. In case $\beta = 0, \alpha = 1$,

$$\psi(it) = \sum_{j=0}^{\infty} (t^{2j}/j!\,2^{2j})\,[2^{\nu}/\Gamma(-\nu + j + 1) + k2^{-\nu}/\Gamma(\nu + j + 1)]. \tag{22}$$

Observe now that the bracket is positive for j great enough and if it is ≥ 0 for some j, it is > 0 for $j+1$. Therefore, if the first bracket

$$2^{\nu}/\Gamma(1-\nu) + k2^{-\nu}/\Gamma(1+\nu) \tag{23}$$

is negative we can apply lemma 1 to the function (22). Thus, there is one solution it_0 with $t_0 > 0$. If (23) equals zero the next brackets are positive, so 0 is the only solution on the imaginary axis. If (23) > 0, all coefficients are positive and there is no zero on the imaginary axis.

For $\nu = 0$

$$\psi(it) = (-2/\pi) \sum_{r=0}^{\infty} (t^{2r}/r\,!^2\, 2^{2r}) [\lg 2k + m(r+1)] \tag{24}$$

where $m(1) = -\gamma$, $m(r+1) = 1 + ... + 1/r - \gamma$, $\gamma =$ Euler's constant.

The preceding argument can be repeated. It is left to the reader. QED.

REMARK. In certain cases of mixed boundary conditions one could obtain similar results to theorem 3. For example for $\nu > 0$, $k \geq 0$, or for $k < 0$, $H \geq -\nu$, $\psi(z)$ has in D at most one zero on the imaginary axis.

4. Normalization of an auxiliary function.

Let us denote with v the following functions:

$$v(z, x) = kz^{-\nu} J_\nu(zx) + z^\nu J_{-\nu}(zx) \qquad \text{if } 0 < \nu < 1, \tag{25}$$

$$v(z, x) = (-2/\pi)(lg\, kz)\, J_0(zx) + Y_0(zx) \qquad \text{if } \nu = 0, \tag{26}$$

and with L the operator $L(u(x)) = \alpha\, u(1) + \beta\, u'(1)$. Then, according to our previous notation :

$$\begin{cases} \psi(z) = \alpha\, v(z, 1) + \beta\, (\partial v/\partial x)(z, 1) = L(v(z, x)), \\ u_n(x) = v(a_n, x). \end{cases} \tag{27}$$

Observe that $v\ (z, x)$ and $L(v)$ are even entire functions of z. Define for

$$0 \leq \nu < 1, z \in C, x > 0,$$

$$F(z, x) = L(z^\nu J_{-\nu}(z\, x))\, v(z, x) - \psi(z)\, z^\nu J_{-\nu}(zx). \tag{28}$$

Then it holds:

THEOREM 4. i) *if* $\nu > 0$

$$\int_0^1 F^2(z, x)\, x dx =$$

$$= (2k^2 z^{-2} \pi^{-1} \sin \nu\pi)\, [(\alpha^2 + \beta^2 (z^2 - \nu^2))\, \pi^{-1} \sin \nu\pi \; - \nu L(J_{-\nu}(zx))\; L(J_\nu(zx))]$$

ii) *if* $\nu = 0$

$$\int_0^1 F^2(z,t)\,tdt = 2z^{-2}\pi^{-2}[\alpha^2 + z^2\beta^2 - (L(J_0(zx)))^2].$$

Proof. Observe that $F(z,x)$ is for fixed complex z a solution of

$$(xu')' + (z^2 x - \nu^2/x)\,u = 0.$$

Then it is easily verified by derivation that

$$2z^2\int_x^1 F^2(z,t)\,tdt = [(xF')^2 + (x^2 z^2 - \nu^2)F^2]_x^1. \tag{29}$$

Besides, if $\nu > 0$, and $W(f,g)$ repsesents the wronskian of f and and g,

$$F(z,x) = k\,[L(J_{-\nu}(zx))\,J_\nu(zx) - L(J_\nu(zx))\,J_{-\nu}(zx)], \tag{30}$$

$$\begin{cases} F(z,1) = k\beta z\,W(J_\nu(z), J_{-\nu}(z)) = -2k\,\beta\pi^{-1}\sin\nu\pi \\ \dfrac{dF}{dx}(z,1) = k\alpha z\,W(J_{-\nu}(z), J_\nu(z)) = 2k\,\alpha\pi^{-1}\sin\nu\pi. \end{cases} \tag{31}$$

For small $x > 0$:

$$F(z,x) = k\,(a_\nu x^\nu - a_{-\nu}x^{-\nu}) + 0\,(x^{2-\nu}), \tag{32}$$

where $a_\nu = z^\nu L(J_{-\nu}(zx))/2^\nu\Gamma(1+\nu)$, while

$$x\cdot\frac{dF}{dx}(z,x) = k\nu\,(a_\nu x^\nu + a_{-\nu}x^{-\nu}) + 0\,(x^{2-\nu}). \tag{33}$$

Replacing (31), (32), (33) in (29) and letting $x \to 0$, we get

$$2z^2\int_0^1 F^2(z,t)\,tdt = k^2\,[(\alpha^2 + (z^2-\nu^2)\beta^2)(2\pi^{-1}\sin\nu\pi)^2 - 4\nu^2 a_\nu a_{-\nu}].$$

This proves i) since

$$\nu a_\nu a_{-\nu} = L(J_{-\nu}(zx))\,L(J_\nu(zx)/\Gamma(\nu)\,\Gamma(1-\nu) = L(J_{-\nu})\,L(J_\nu)\,\pi^{-1}\sin\nu\pi.$$

If $\nu = 0$,

$$F(z,x) = L(J_0)\,Y_0(zx) - L(Y_0)\,J_0(zx), \tag{34}$$

$$(35)\qquad \begin{cases} F(z,1) = \beta z\ W(Y_0(z), J_0(z)) = -\,2\beta/\pi, \\ F'(z,1) = \alpha z\ W(J_0(z), Y_0(z)) = 2\alpha/\pi, \end{cases}$$

and for samll $x > 0$;

$$(36)\qquad \begin{cases} F(z,x) = 0\ (lg\ x); \\ x \cdot \dfrac{dF}{dx}(z,x) = L(J_0(zx)) \cdot 2/\pi + 0\,(x\ lg\ x). \end{cases}$$

Replacing (35), (36) in (29) and letting $x \to 0$,

$$\int_0^1 F^2(z,t)\,tdt = 2z^{-2}\pi^{-2}[\alpha^2 + z^2\beta^2 - (L(J_0(zx)))^2].$$

ii) is thus proved. Q E D.

We need still another expression of the norm in theorem 4.

THEOREM 5. i) *if* $1 > \nu > 0$,

$$\int_0^1 F^2(z,x)\,xdx = kz^{-1}\pi^{-1}(\sin\nu\pi)\ W(L(z^\nu J_{-\nu}(zx)), L(v(z,x)));$$

ii) if $\nu = 0$,

$$\int_0^1 F^2(z,x)\,xdx = (z\pi)^{-1}\ W(L(J_0\,zx)), L(v(z,x))).$$

PROOF. Suppose $\nu > 0$ Then

$$(37)\qquad W(L(z^\nu J_{-\nu}(zx)), L(v(z,x))) = k\ W(z^\nu L(J_{-\nu}(zx)), z^{-\nu} L(J_\nu(zx))) =$$

$$= k\,\{W(L(J_{-\nu}(zx)), L(J_\nu(zx))) - 2\nu z^{-1} L(J_{-\nu}(zx)) \cdot L(J_\nu(zx))\}.$$

Now observe that $L(J_\nu(zx)) = \alpha J_\nu(z) + \beta z J_\nu'(z)$, and its derivative

$$(38)\qquad \frac{d}{dz}(L\,J_\nu) = \alpha J_\nu'(z) + \beta(\nu^2/z - z)\,J_\nu(z).$$

Therefore

$$W(L(J_{-\nu}(zx)), L(J_\nu(zx))) = W(J_{-\nu}, J_\nu)(\alpha^2 + \beta^2(z^2 - \nu^2)) =$$

$$= 2z^{-1}\pi^{-1}(\sin\ \nu\pi)(\alpha^2 + \beta^2(z^2 - \nu^2)).$$

Replacing in (37) we get

$$(39) \qquad W(L(z^\nu J_{-\nu}(zx)), L(v(z, x))) =$$

$$= k\{2z^{-1}\pi^{-1}(\alpha^2 + \beta^2(z^2 - \nu^2))(\sin\nu\pi) - 2\nu z^{-1} L(J_{-\nu}) L(J_\nu)\}.$$

In view of theorem 4, i) follows Analogously, if $\nu = 0$,

$$W(LJ_0, Lv) = W(L(J_0(zx)), L(Y_0(zx))) - 2z^{-1}\pi^{-1}(L(J_0(zx)))^2 =$$

$$= W(J_0, Y_0)(\alpha^2 + \beta^2 z^2) - 2(LJ_0)^2/\pi z = 2\{\alpha^2 + \beta^2 z^2 - (LJ_0)^2\}/\pi z.$$

And from theorem 4, ii) follows. Q E D.

As a corollary of the preceding results we can obtain:

THEOREM 6. *The zeroes of $\psi(z)$ and $L(z^\nu J_{-\nu}(zx))$ are distinct. Except for $z = 0$ they are simple. If $z = 0$ is a zero of any of them, it is of second order.*

PROOF. Let $\nu > 0$. Suppose some $z_0 \neq 0$ is a zero of $\psi(z)$ or $L(z^\nu J_{-\nu}(zx))$ of order greater than one or it is a zero of both. Then

$$W(L(z^\nu J_{-\nu}(zx)), \psi(z))(z_0) = 0$$

and in view of theorem 5, $F(z_0, x) \equiv 0$ By (30) this implies

$$(40) \qquad L(J_{-\nu}(z_0 x)) = L(J_\nu(z_0 x)) = 0.$$

Using (39), it follows $\alpha^2 + \beta^2(z_0^2 - \nu^2) = 0$. Taking into account (38) and (40), we also conclude that $(dL(J_\nu)/dz)(z_0) = (dL(J_{-\nu})/dz)(z_0) = 0$, therefore z_0 is at least a double zero of $L(z^{-\nu} J_\nu)$ and $L(z^\nu J_{-\nu})$.

Therefore, at least it a double zero of ψ and a triple zero of $W(L(z J_{-\nu}), \psi)$. This is a contradiction since in $z_0 \neq 0$ (39) has a zero of first order Suppose now 0 is a zero of $\psi(z)$ and $L(z^\nu J_{-\nu}(zx))$, or it is a zero of order greater than two of either one. Because of the evenness of these functions $W(L(z^\nu J_{-\nu}), \psi)$ has at $z = 0$ a zero of order greater than one.

But this is in contradiction with next formula which is obtained from (39):

$$W\,(L\,(z^{\nu}\,J_{-\nu}),\psi) = kz\pi^{-1}\,(\sin\nu\pi)\,((\alpha+\beta)^2+(1-\nu^2)\,\beta^2)/(1-\nu^2)+0\,(z^3).$$

5. Estimates for the Dirichlet kernels.

We shall compare in this section the Dirichlet kernel of the system $\{v\,(a_n\,,x)\}$ and that of the system $\{\tilde{v}\,(b_n\,,x) = b_n^{\nu}\,J_{-\nu}\,(b_n\,x)\}$ when both satisfy the same boundary condition (7) at $x=1$. That is $\{a_n\} = \{z \in D\,;\,\psi\,(z) = 0\}$ and $\{b_n\} = \{z \in D\,;\,L\,(z^{\nu}\,J_{-\nu}\,(zx)) = 0\}$.

We shall call $\varphi\,(z) = L\,(z^{\nu}\,J_{-\nu}\,(zx))$, in order to shorten the formulae.

The zeroes of φ tend asymptotically to differ by π and the difference between the zeroes of φ and ψ tends to zero. So, there exist a sequence $\{A_n\}$ of real numbers tending to ∞, such that: no zero of φ or ψ falls in $(A_n-\pi/8,\,A_n+\pi/8)$ and $(A_n\,,A_{n+1})$ contains exactly one zero of φ and ψ. In consequence, $A_n \sim n\pi$. *We suppose such a sequence chosen for every fixed* $k,\alpha,\beta,\nu \in [0,1)$.

DEFINITION 2. *We shall call the* n^{th} *Dirichlet kernel of the system* $\{v\,(a_n\,;x)\}$, *the function*

$$D_n\,(x,y) = \sum_{\mathrm{Re}\,a_k < A_n} v\,(a_k\,,x)\,v\,(a_k\,,y)\Big/\int_0^1 v^2\,(a_k\,,x)\,xdx.$$

Analogously, the function

$$\tilde{D}_n\,(x,y) = \sum_{\mathrm{Re}\,b_k < A_n} \tilde{v}\,(b_k\,,x)\,\tilde{v}\,(b_k\,,y)\Big/\int_0^1 \tilde{v}^2\,(b_k\,,x)\,xdx,$$

we shall call the n^{th} *Dirichlet kernel of the system* $\{\tilde{v}\,(b_k\,,x)\}$.

THEOREM 7. *If* $F\,(z,x)$ *is given by* (28) *and* C_1 *is the half-circle* $z = A_n\,e^{i\Phi}$, $\Phi \in (-\pi/2,\pi/2)$, *then*

$$(41)\qquad D_n\,(x,y) - \tilde{D}_n\,(x,y) = (\gamma/2\pi\,i)\int_{C_1} F\,(z,x)\,F\,(z,y)\,(\varphi\,(z)\cdot\psi\,(z))^{-1}\,zdz,$$

where $\gamma = \pi/k\,\sin\nu\pi$ *if* $\nu>0$, $=\pi$ *if* $\nu=0$.

PROOF. Let C_2 be the segment $(iA_n, -iA_n)$ indented by semicircles of radius ε centered at the zeroes of $\varphi(z)$ and $\psi(z)$ which lie on the imaginary axis, in such a way that the closed curve $C_1 + C_2$ contains only the zeroes of φ and ψ which lie in D. We have by theorem 5

$$(42) \qquad (\gamma/2\pi i) \int_{C_1+C_2} F(z, x)\, F(z, y)\, (\varphi(z)\, \psi(z))^{-1}\, z dz =$$

$$= (2\pi i)^{-1} \int_{C_1+C_2} F(z, x)\, F(z, y)\, W(\varphi, \psi) \left[\varphi \cdot \psi \cdot \int_0^1 F^2(z, x)\, x dx\right]^{-1} dz =$$

$$= (2\pi i)^{-1} \int_{C_1+C_2} \left\{F(z, x)\, F(z, y) / \int_0^1 F^2(z, x)\, x dx\right\} \{\psi'/\psi - \varphi'/\varphi\}\, dz.$$

The only singularities of the integrend which lie on $C_1 + C_2$ are the zeroes of φ and ψ which lie in D. These, by theorem 6, are distinct and of first order except when the origin is one of them. In the latter case it is of second order. Let be $n(z) = 1$ if $z \neq 0$, $= 2$ if $z = 0$. Applying the theorem of residues to (42) we get,

$$(43) \qquad (\gamma/2\pi i) \int_{C_1+C_2} F(z, x)\, F(z, y)\, (\varphi \cdot \psi)^{-1}\, z dz =$$

$$= \sum_{\mathrm{Re}\, a_k < A_n} \left\{F(a_k, x)\, F(a_k, y) / \int_0^1 F^2(a_k, x)\, x dx\right\} n(a_k) -$$

$$- \sum_{\mathrm{Re}\, b_k < A_n} \left\{F(b_k, x)\, F(b_k, y) / \int_0^1 F^2(b_k, x)\, x dx\right\} n(b_k).$$

Since F, φ and ψ ere all even functions, the integrand in (43) is odd.
Besides $C_2 \cap \{|z| > \varepsilon\}$ is symmetric about the origin. So

$$(\gamma/2\pi i) \int_{C_2} F(z, x)\, F(z, y)\, (\varphi \cdot \psi)^{-1}\, z dz = (\gamma/2\pi i) \int_{C_\varepsilon} F(z, x)\, F(z, y)\, (\varphi \cdot \psi)^{-1}\, z dz,$$

where C_ε is the half circle $\{z\,;\, z = \varepsilon\, e^{i\Phi}, \pi/2 < \Phi < \pi\, 3/2\}$.

$$(44)\qquad (\gamma/2\pi i)\int_{C_1} F(z,x)\,F(z,y)\,(\varphi\cdot\psi)^{-1} z\,dz =$$

$$= \sum_{\operatorname{Re} a_k < A_n} \left\{ F(a_k, x)\,F(a_k, y)\Big/\int_0^1 F^2\cdot x\,dx \right\} (n(a_k) - 1) -$$

$$- \sum_{\operatorname{Re} b_k < A_n} \left\{ F(b_k, x)\,F(b_k, y)\Big/\int_0^1 F^2\cdot x\,dx \right\} (n(b_k) - 1).$$

Substracting (44) from (43) we get (41) Q E D.

Theorem 8. *i) if* $1 > \nu > 0, 0 < x < 1, 0 < y < 1, \mu = (\nu - 1/2) \vee 0,$

$$| D_n(x,y) - \widetilde{D}_n(x,y) | =$$

$$= 0(1)\, A_n^{-2\nu} (xy)^{-1/2} \{(x+y)^{-1} \wedge A_n\} \{1 + (A_n x)^{-\mu}\} \{1 + (A_n y)^{-\mu}\},$$

ii) *if* $\nu = 0$

$$|D_n(x,y) - \widetilde{D}_n(x,y)| = 0(1)(\sqrt{xy}\cdot \lg A_n)^{-1} \{(x+y)^{-1} \wedge A_n\}.$$

First we state an auxiliary result.

Lemma 2. *Let* $z \in C_1, 0 < x < 1, \mu = (\nu - 1/2) \vee 0.$ *Then for fixed* ν, $0 \leq \nu < 1,$

$$F(z,x) = 0(1)\,|\alpha + \beta z| \cdot |z|^{-1} x^{-1/2} \cdot e^{(1-x)|\operatorname{Im} z|} \{1 + |zx|^{-\mu}\}.$$

Proof. From $J_\nu(z) = (H_\nu^{(1)}(z) + H_\nu^{(2)}(z))/2$; $Y_\nu(z) = (H_\nu^{(1)}(z) - H_\nu^{(2)}(z))/2i$; $J_{-\nu}(z) = \{e^{i\nu\pi} H_\nu^{(1)}(z) + e^{-i\nu\pi} H_\nu^{(2)}(z)\}/2$ and (30) if $\nu > 0$, and (34) if $\nu = 0$, we obtain

$$(45)\qquad F(z,x) = k_\nu \{H_\nu^{(1)}(zx)\cdot L(H_\nu^{(2)}(zx)) - H_\nu^{(2)}(zx)\cdot L(H_\nu^{(1)}(zx))\},$$

where k_ν is a conatant different of zero : $k_0 = 1/2i, k_\nu = (\sin \nu\pi)/2i$ if $\nu > 0$.

The asymptotic formulae for $H_\nu^{(j)}(z)$ yield

$$(46)\qquad \begin{cases} H_\nu^{(1)}(z) = 0(1)\, e^{iz}/\sqrt{z}\,; & dH_\nu^{(1)}/dz = 0(1)\, e^{iz}/\sqrt{z}, \\ H_\nu^{(2)}(z) = 0(1)\, e^{-iz}/\sqrt{z}\,; & dH_\nu^{(2)}/dz = 0(1)\, e^{-iz}/\sqrt{z}, \end{cases}$$

uniformly in $|z| > 1, -\pi/2 \leq \arg z \leq \pi/2$. Therefore on C_1,

$$L(H_\nu^{(1)}(zx)) = 0(1)(\alpha + \beta z)\, e^{iz}/\sqrt{z},$$

(47)

$$L(H_\nu^{(2)}(zx)) = 0(1)(\alpha + \beta z)\, e^{-iz}/\sqrt{z}.$$

Using (46) and (47), we get from (45)

(48) $$F(z, x) = |\alpha + \beta z|\, e^{(1-x)|\operatorname{Im} z|} \cdot 0(1/|z|\sqrt{x}),$$

whenever $z \in C_1$ and $|zx| = A_n x > 1$.

If $|zx| < 1$, then for $j = 1,2$,

$$H_\nu^{(j)}(zx) = 0(1)\, e^{\pm izx}/\, |zx|^{\nu \vee 1/2}.$$

This combined with (47) yields:

(49) $$F(z, x) = |\alpha + \beta z|\, e^{(1-x)|\operatorname{Im} z|} \cdot 0(1/|z| \cdot |zx|^{(\nu-1/2)\vee 0} \cdot \sqrt{x}),$$

for $z \in C_1$, $x < 1/A_n$. From (48) and (49) we get the thesis Q E D.

PROOF OF THEOREM 8. By theorem 7, it is enough to estimate the integral in (41) Let $\nu > 0$. Then, for $|z| > 1$,

$$\varphi(z) = (\alpha J_{-\nu} + z\beta J'_{-\nu})\, z^\nu =$$

$$= (z^\nu \sqrt{2/\pi z})\{\alpha \cos(z + \nu\pi/2 - \pi/4) - \beta z \sin(z + \nu\pi/2 - \pi/4) + (\alpha + \beta z) 0 (e^{|\operatorname{Im} z|}/z)\}.$$

By the choice of A_n, we have for $z \in C_1$

(48) $$|\varphi(z)| \geq M|\alpha + z\beta|\, |z|^{\nu-1/2} \cdot \exp|\operatorname{Im} z|,$$

with a certain positive constant M.

Analogously

(49) $$|\psi(z)| \geq M|\alpha + \beta z|\, |z|^{\nu-1/2} \cdot \exp|\operatorname{Im} z|,\ z \in C_1.$$

Using the estimates (48), (49) and lemma 2, we get from (41),

(50) $$|D_n(x, y) - \widetilde{D}_n(x, y)| =$$

$$= 0\,(1) \int_{C_1} e^{-(x+y)\,|\,\mathrm{Im}\, z\,|} (1 + |\,zx\,|^{-\mu})\,(1 + |\,zy\,|^{-\mu}) \cdot |\,xy\,|^{-1/2}\,|\,z\,|^{-2\nu}\,|\,dz\,| =$$

$$= 0\,(1)\,(xy)^{-1/2}\,A_n^{-2\nu}\,(1 + (A_n\,x)^{-\mu})\,(1 + (A_n\,y)^{-\mu})$$

$$\int_{-\pi/2}^{\pi/2} A_n \cdot \exp\,[-\,(x+y)\,A_n\,|\,\sin\Phi\,|]\,d\Phi =$$

$$= 0\,(1)\,A_n^{-2\nu}\,(xy)^{-1/2}\,\{1 + (A_n\,x)^{-\mu}\}\,\{1 + (A_n\,y)^{-\mu}\}\,\{A_n \wedge 1/(x+y)\}.$$

If $\nu = 0$, (48) still holds and instead of (49) we have,

$$|\,\psi(z)\,| \geq C\,|\,\alpha + \beta z\,|\,(\lg|\,z\,|)\,|\,z\,|^{-1/2} \exp|\,\mathrm{Im}\,z\,|\,,\; z \in C_1\,.$$

Then instead of (50) we have,

$$|\,D_n(x,y) - \widetilde{D}_n(x,y)\,| = 0\,(1)\,(xy)^{-1/2}\,(\lg A_n)^{-1}\,\{A_n \wedge 1/(x+y)\}. \qquad \text{Q E D.}$$

6. Equiconvergence properties of the orthogonal systems.

Let us call

$$S_n(f, x) = \int_0^1 D_n(x,y)\,f(y)\,y\,dy,\; \widetilde{S}_n(f, x) = \int_0^1 \widetilde{D}_n(x,y)\,f(y)\,y\,dy,$$

the partial sums of the expansion of $f(x)$ in the systems $\{v\}$ and $\{\widetilde{v}\}$, respectively.

We have the following result.

THEOREM 9. *Let be* $\mu = (\nu - 1/2) \vee 0$. *If*

(51) $$1/2 - \beta + \mu < 1/p < 3/2 - \beta - \mu\,; \quad 1 < p < \infty$$

then

(52) $$\int_0^1 |\,S_n(f,x) - \widetilde{S}_n(f,x)\,|^p\,x^{\beta p}\,dx \leq C_n \int_0^1 |\,f(x)\,|^p\,x^{\beta p}\,dx,$$

where $C_n = C A_n^{-2\nu}$ *if* $1 > \nu > 0$, $C_n = C\,(lg\,A_n)^{-1}$ if $\nu = 0$, *with* C *independent of* n *and* f.

(In different formulae C may have different values).

PROOF. We separate two cases. Assume $0 \leq \nu \leq 1/2$. Then $\mu = 0$ and by theorem 8:

$$\Delta = |\, D_n(x, y) - \widetilde{D}_n(x, y)\,| \cdot y \leq C_n (y/x)^{1/2} \cdot 1/(x+y).$$

To prove (52) it is therefore enough to prove:

$$\int_0^1 \left| \int_0^1 (y/x)^{1/2} x^\beta f(y)/(x+y)\, dy \right|^p dx \leq C \int_0^1 |f(y)\, y^\beta|^p\, dy. \tag{53}$$

But this follows from lemma 8 of [3] for

$$-1/p < \beta - 1/2 < 1 - 1/p; \quad 1 < p < \infty, \tag{54}$$

which is just (51) for $\mu = 0$.

Assume now that $1/2 < \nu < 1$. Then $\mu > 0$ and by theorem 8

$$\Delta \leq C_n (y/x)^{1/2} \{1 + (A_n x)^{-\mu}\} \{1 + (A_n y)^{-\mu}\} \{A_n \wedge 1/(x+y)\}. \tag{55}$$

Considering the cases where $A_n x > 1$ and $A_n y > 1$, and where at least one of these inequalities does not hold, we obtain from (55) that:

$$\Delta \leq C_n \sqrt{\frac{y}{x}} \left\{ \frac{4}{x+y} + \frac{\{1 + (A_n x)^{-\mu}\}\{1 + (A_n y)^{-\mu}\}}{(1 + A_n x)(1 + A_n y)} \cdot 4A_n \right\}. \tag{56}$$

Because of (53) and (54) we see from (56) that to obtain (52) it only remains to show next inequality:

$$\int_0^1 \left| x^\beta \int_0^1 \sqrt{\frac{y}{x}} \frac{\{1 + (A_n x)^{-\mu}\}\{1 + (An\, y)^{-\mu}\}}{(1 + A_n x)(1 + A_n y)} f(y)\, A_n\, dy \right|^p dx \leq$$

$$\leq C \int_0^1 |f(y)|^p\, y^{\beta p}\, dy,$$

or equivalently,

$$(57)\qquad \int_0^{A_n} dx \left| \int_0^{A_n} \left(\frac{x}{y}\right)^{\beta-1/2} \frac{(1+x^{-\mu})(1+y^{-\mu})\, y^\beta}{(1+x)(1+y)} f\left(\frac{y}{A_n}\right) dy \right|^p \leq$$

$$\leq C \int_0^{A_n} \left| f\left(\frac{y}{A_n}\right) y^\beta \right|^p dy.$$

Now observe that if (51) holds, then $x^{\beta-1/2}(1+x^{-\mu})(1+x)^{-1} \in L^p$ on $(0,\infty)$ and $y^{1/2-\beta}\ (1+y^{-\mu})(1+y)^{-1} \in L^q(0,\infty)$, $1/p+1/q=1$. Therefore (57) is consequence of Hölder's inequality. Q E D.

In [2] it is proved for the system $\{\tilde{v}(b_n, x)\}$ that whenever (51) holds,

$$\lim_{n\to\infty} \int_0^1 |\tilde{S}_n(f,x) - f(x)|^p x^{\beta p}\, dx = 0 \qquad \forall f \in L^p(\beta p).$$ Then, we obtain:

COROLLARY. *If* (51) *holds, then*

$$(58)\qquad \lim_{n\to\infty} \int_0^1 |S_n(f,x) - f(x)|^p x^{\beta p}\, dx = 0 \qquad \forall f \in L^p(\beta p).$$

This implies in particular the $L^2(1)$-completeness of these systems, a result which is well-known ([4], ch. 2). Now we can round up the contents of theorem 1. We shall use the following terminology. Two systems of functions $\{w_n(x)\}$ and $\{u_n(x)$ are called equivalent when $w_n = k_n u_n$, the k_n, $n = 1,2,\ldots$, being nonzero complex constants.

THEOREM 10. *For real* ν *let* $\{w_n(x)\}$ *be an orthogonal system of solutions of Bessel's equation* (1) *which satisfy the real boundary condition* (7).

If it is complete in $L^2(1)$, *then it is equivalent to a real system which coincides either with the Bessel or Dini system for* $-1 < \nu < \infty$, *or with one the systems* $\{v(a_n, x)\}$ *for* $\nu \in [0,1)$.

PROOF. Suppose $w_n(x)$ satisfies $(xu')' + (x\lambda_n - \nu^2/x)\, u_n(x) = 0$ and let $a_n \in D$ be such that $a_n^2 = \lambda_n$. Then $w_n = A_n J_\nu(a_n x) + B_n J_{-\nu}(a_n x)$, $a_n \neq 0$, or $w_n = Ax^\nu + Bx^{-\nu}$ if $a_n = 0$. If $\{w_n\}$ is not equivalent to the Bessel or Dini system, we get by theorem 1, *i*) that it must be equivalent to one of the systems $\{v_n(a_n, x)\}$. Similar considerations hold for the case $\nu = 0$. Q E D.

REMARK 1). It can be proved that Corollary to theorem 9 is best possible in the sense that if for some p and β, $1 \leq p \leq \infty$, $-\infty < \beta < \infty$,

$$\lim_{n \to \infty} \| (S_n(f, x) - f(x)) x^\beta \|_p = 0 \qquad \forall f \text{ such that } f x^\beta \in L^p$$

holds, then (51) also holds (cf. [2], theorem 6).

2) It easily follows from theorem 8 that if (51) holds then

$$| S_n(f, x) - \widetilde{S}_n(f, x) | \leq C_n K(x) \left(\int_0^1 | f(x) x^\beta |^p dx \right)^{1/p},$$

where $K(x)$ is a finite function in $(0, 1)$ wich depends only of ν, β and p. This implies the pointwise equiconvergence.

REFERENCES

[1] BENEDEK A. and PANZONE R., *On mean convergence of Fourier-Bessel series of negative order,* Studies in App. Math. Vol. 4 n^0 3, (1971), 281-292.

[2] BENEDEK A. and PANZONE R., *Mean convergence of Bessel and Dini series, Notices* AMS, 18, n^0 6, (1971), p. 951 (to appear Rev. UMA, vol. 26, (1972)).

[3] MUCKENHOUPT B., *Mean convergence of Hermite and Laguerre series, II, TAMS,* 147, (1970), 433-460.

[4] TITCHMARSH, E. C., *Eigenfunction expansions I,* Oxford, (1962).

[5] WATSON, G. N., *A treatise on the theory of Bessel functions,* Cambridge, (1952).

[6] WING, G. M., *The mean convergence of orthogonal series,* Amer. J. of Math., LXXII, (1950), 792-808.

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