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# SOME OPERATIONAL PROPERTIES OF THE H-R OPERATOR

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## 1. Introduction.

The work of this article is closely related to the study revealed by papers [1] and [2]. In [1] the author has studied some properties of what he called « H-R transform », which is associated with the concept of generalized derivative and integral. In fact the linear operator of the transform, or the H-R operator, may simply be considered as an integro differential operator of generalized order. Some writers refer to the H-R transform as the « fractional derivative and integral », [3], [4].

The H-R transform has been presented by the following:

DEFINITION. If $f(x)$ is a real valued function contained in $C^{(n)}$ on the interval $a \leqq x \leqq b$, and $\operatorname{Re}(\alpha + n) > 0$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha + n)} D_x^n \int_a^x (x - t)^{\alpha+n-1} f(t)\, dt, \qquad (n = 0, 1, 2, \ldots),$$

or

$$= \sum_{i=0}^{n-1} \frac{(x - a)^{\alpha+i}}{\Gamma(\alpha + i + 1)} f^{(i)}(a) + \frac{1}{\Gamma(\alpha + n)} \int_a^x (x - t)^{\alpha+n-1} f^{(n)}(t)\, dt,$$

$(n = 1, 2, \ldots)$, where $D_x^n$ is the nth derivative operator with respect to $x$, $f^{(n)} = D_x^n f$, $C^{(n)}$ is the set of functions with continuous nth derivative on $[a, b]$ and $\Gamma(x)$ is the conventional symbol representing Gamma function of $x$.

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The first form of the above definition can be written as

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} f.$$

Based on this definition, it has been shown in [1] that:

THEOREM 1. (i) If $f(x)$ is a function belonging to $C^{(m+n)}$ on $[a, b]$ and $\operatorname{Re}(\alpha + m) > 0$, then

$$D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{\alpha-n} f.$$

(ii) If $f(x)$ is a function in $C^{(0)}$ on $[a, b]$, then

$$\lim_{\alpha \to 0} \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{0} f = f(x).$$

From (i) and (ii), it follows that

$$\overset{x}{\underset{a}{I}}{}^{-n} f = D_x^n f = f^{(n)}(x).$$

THEOREM 2. If $f(x)$ and $g(x)$ satisfy the conditions of the definition and $A$ is a number, then

(i) $$\overset{x}{\underset{a}{I}}{}^{\alpha} A(f + g) = A \overset{x}{\underset{a}{I}}{}^{\alpha} f + A \overset{x}{\underset{a}{I}}{}^{\alpha} g.$$

(ii) If $f(x)$ is a polynomial of degree $n$ and $\operatorname{Re}(\alpha) > 0$, then

$$\overset{x}{\underset{a}{I}}{}^{\pm\alpha} f g = \sum_{p=0}^{n} \binom{\mp\alpha}{p} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{\mp\alpha+p} g.$$

(ii) represente a special case of a generalization of Leibnitz' rule of differentiation of product of two functions.

THEOREM 3. (i) If $x > a$ where a is a real number, and $\beta \neq -m$ (a negative integer), then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta},$$

for all values of $\alpha$ and $\beta$.

(ii) If $\beta = -n$ (a negative integer), $\operatorname{Re}(\alpha + m) > 0$ $(m = 0, 1, 2, \dots)$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha}(x-a)^{-n} = \frac{(-1)^{n-1}}{\Gamma(n)} D_x^{m+n+1}\left[\frac{(x-a)^{\alpha+m+1}\, ln\,(x-a)}{\Gamma(\alpha+m+2)} - \frac{(x-a)^{\alpha+m+1}}{\Gamma(\alpha+m+2)} + \right.$$

$$\left. + \frac{(x-a)^{\alpha+m+1}}{\Gamma(\alpha+m)} K(\alpha+m)\right],$$

for all values of $\alpha$, where $K(\alpha + m) = \int_0^1 u\,(1-u)^{\alpha+m-1}\, lnu\, du.$

**Theorem 4.** The relation

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\, \overset{x}{\underset{a}{I}}{}^{\beta} f = \overset{x}{\underset{a}{I}}{}^{\alpha+\beta} f,$$

holds if

(i) $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$ and $f(x)$ is in $C^{(0)}$ on $[a, b]$;

(ii) $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) \leqq 0$ or $\operatorname{Re}(\beta + m) > 0$ such that $\beta \neq -m$, a negative integer, and $f(x)$ is in $C^{(m)}$ on $[a, b]$;

(iii) $\operatorname{Re}(\alpha) \leqq 0$ or $\operatorname{Re}(\alpha + n) > 0$, $\operatorname{Re}(\beta) > 0$ and $f(x)$ is in $C^{(n)}$ on $[a, b]$;

(iv) $\operatorname{Re}(\alpha) \leqq 0$, $\operatorname{Re}(\beta) \leqq 0$, $\beta \neq -m$ and $f(x)$ is in $C^{(m+n)}$ on $[a, b]$.

When $\beta = -m$ in (ii) and (iv), then

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\, \overset{x}{\underset{a}{I}}{}^{-m} f = \overset{x}{\underset{a}{I}}{}^{\alpha} f^{(m)} = \overset{x}{\underset{a}{I}}{}^{\alpha-m} f - \sum_{p=0}^{m-1} \frac{(x-a)^{\alpha-m+p}}{\Gamma(\alpha+p-m+1)} f^{(p)}(a).$$

Another property needs to be mentioned is that if $\beta \neq -m$ and $f(x)$ satisfies the conditions of (Theorem 4), then

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\, \overset{x}{\underset{a}{I}}{}^{\beta} f = \overset{x}{\underset{a}{I}}{}^{\beta}\, \overset{x}{\underset{a}{I}}{}^{\alpha} f,$$

for all values of $\alpha$ and $\beta$. If $\beta = -m$, then this relation holds if $f^{(\mu)}(a) = 0$ for $(\mu = 0, 1, 2, \dots, m-1)$.

The work of this paper deals mainly with operations of $\overset{x}{\underset{a}{I}}{}^{\alpha}$, the generalized integro-differential operator, on certain types of functions belonging to the set $T^{\alpha}$-Functions studied in [2]. So it may be useful to give definitions and some properties of functions related to such operations.

The function $E_\alpha(\lambda; x - a)$, simply denoted by $E_{\alpha;\lambda}$, has been defined by

$$E_{\alpha;\lambda} = E_\alpha(\lambda; x - a) = \sum_{n=1}^{\infty} \lambda^{n-1} \frac{(x-a)^{n\alpha-1}}{\Gamma(n\alpha)},$$

where $\lambda$ is a number. Also we have defined

$$S_{\alpha;\lambda} = S_\alpha(\lambda; x - a) = \frac{1}{2i}(E_{\alpha;i\lambda} - E_{\alpha;-i\lambda});$$

$$C_{\alpha;\lambda} = C_\alpha(\lambda; x - a) = \frac{1}{2}(E_{\alpha;i\lambda} + E_{\alpha;-i\lambda}), \quad i = (-1)^{1/2};$$

$$SH_{\alpha;\lambda} = SH_\alpha(\lambda; x - a) = \frac{1}{2}(E_{\alpha;\lambda} - E_{\alpha;\lambda});$$

$$CH_{\alpha;\lambda} = CH_\alpha(\lambda; x - a) = \frac{1}{2}(E_{\alpha;\lambda} + E_{\alpha;\lambda}).$$

It follows, from these definitions, that

$$S_{\alpha;\lambda} = \sum_{p=1}^{\infty} (-1)^{p+1} \lambda^{2p-1} \frac{(x-a)^{2p\alpha-1}}{\Gamma(2p\alpha)};$$

$$C_{\alpha;\lambda} = \sum_{p=1}^{\infty} (-1)^{p+1} \lambda^{2(p-1)} \frac{(x-a)^{\overline{2p-1}\,\alpha-1}}{\Gamma(\overline{2p-1}\,\alpha)};$$

$$SH_{\alpha;\lambda} = \sum_{p=1}^{\infty} \lambda^{2p-1} \frac{(x-a)^{2p\alpha-1}}{\Gamma(2p\alpha)};$$

$$CH_{\alpha;\lambda} = \sum_{p=1}^{\infty} \lambda^{2(p-1)} \frac{(x-a)^{\overline{2p-1}\,\alpha-1}}{\Gamma(\overline{2p-1}\,\alpha)}.$$

Other functions have been defined as follows:

$$T_{\alpha;\lambda} = S_{\alpha;\lambda} / C_{\alpha;\lambda}; \quad CT_{\alpha;\lambda} = C_{\alpha;\lambda} / S_{\alpha;\lambda};$$

$$SC_{\alpha;\lambda} = 1 / C_{\alpha;\lambda}; \qquad CS_{\alpha;\lambda} = 1 / S_{\alpha;\lambda};$$

$$TH_{\alpha;\lambda} = SH_{\alpha;\lambda} / CH_{\alpha;\lambda}; \quad CTH_{\alpha;\lambda} = CH_{\alpha;\lambda} / SH_{\alpha;\lambda};$$

$$SCH_{\alpha;\lambda} = 1 / CH_{\alpha;\lambda}; \qquad CSH_{\alpha;\lambda} = 1 / SH_{\alpha;\lambda}.$$

From the above definitions we have the following properties:

$$CH^2_{\alpha;\lambda} - SH^2_{\alpha;\lambda} = E_{\alpha;\lambda}\, E_{\alpha;-\lambda};$$

$$S^2_{\alpha;\lambda} + C^2_{\alpha;\lambda} = E_{\alpha;i\lambda}\, E_{\alpha;-i\lambda};$$

$$TH^2_{\alpha;\lambda} + E_{\alpha;\lambda}\, E_{\alpha;-\lambda}\, SCH^2_{\alpha;\lambda} = 1;$$

$$CTH^2_{\alpha;\lambda} - E_{\alpha;\lambda}\, E_{\alpha;-\lambda}\, CSH^2_{\alpha;\lambda} = 1;$$

$$SH_{\alpha;\lambda} + CH_{\alpha;\lambda} = E_{\alpha;\lambda};$$

$$CH_{\alpha;\lambda} - SH_{\alpha;\lambda} = E_{\alpha;-\lambda};$$

$$SH_{\alpha;-\lambda} = -\,SH_{\alpha;\lambda}; \quad CH_{\alpha;-\lambda} = CH_{\alpha;\lambda};$$

$$TH_{\alpha;-\lambda} = -\,TH_{\alpha;\lambda}; \quad S_{\alpha;-\lambda} = -\,S_{\alpha;\lambda};$$

$$C_{\alpha;-\lambda} = C_{\alpha;\lambda}; \qquad T_{\alpha;-\lambda} = -\,T_{\alpha;\lambda}.$$

In what follows, the valuse of $\alpha$ and conditions imposed on $f(x)$ will be similar to those mentioned by definition of $\overset{x}{\underset{a}{I}}{}^{\alpha} f$, and for which the operation has a meaning. Also when $F\left(\overset{x}{\underset{a}{I}}{}^{\alpha}\right) = \sum_{p=0}^{\infty} a_p \overset{x}{\underset{a}{I}}{}^{p\alpha}$, and $G\left(\overset{x}{\underset{a}{I}}{}^{\alpha}\right) = \sum_{q=0}^{\infty} b_q \overset{x}{\underset{a}{I}}{}^{q\alpha}$, where $F$ and $G$ belong to $T^{\alpha}$-Functions, then the product $FG$ in $T^{\alpha}$, (see [2]), where

$$FG = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_p\, b_q\, \overset{x}{\underset{a}{I}}{}^{\overline{p+q}\,\alpha} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_{p-q}\, b_q\, \overset{x}{\underset{a}{I}}{}^{p\alpha},$$

and consequently each one of these series is absolutely and uniformly convergent. Thus term by term integration and differentiation can be performed. Therefore, the operations carried out in the following sections are justified and permissible.

## 2. Some Operational Properties.

According to what has been written in the previous section we have

$$\overset{x}{\underset{a}{I}}{}^{-\alpha}\, \overset{x}{\underset{a}{I}}{}^{\alpha} f = f.$$

Therefore we have the following

**DEFINITION.**

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \left(1 \Big/ \overset{x}{\underset{a}{I}}{}^{-\alpha}\right) f = \left(\overset{x}{\underset{a}{I}}{}^{-\alpha}\right)^{-1} f.$$

(P·1): If $\lambda$ is a number, we have

$$\left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f = \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{p\alpha} f,$$

and in particular, when $f(x) \equiv 1$,

$$\left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} 1 = \sum_{p=0}^{\infty} \lambda^{p} \frac{(x-a)^{p\alpha}}{\Gamma(p\alpha + 1)}.$$

To show this, we have

$$\left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right) \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{p\alpha} f = \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{p\alpha} f - \sum_{p=0}^{\infty} \lambda^{p+1} \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f$$

$$= f(x),$$

from which we obtain (P·1). The other relation is obtained by putting $f(x) \equiv 1$ in (P·1).

(P·2):

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)^{-1} f = \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f.$$

This relation follows from

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right) \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f = \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{p\alpha} f - \sum_{p=0}^{\infty} \lambda^{p+1} \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f$$

$$= f(x).$$

Also we notice that

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right) f = \overset{x}{\underset{a}{I}}{}^{-\alpha} \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right) f.$$

(P·3):

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} 1 = \frac{x-a}{\alpha} E_{\alpha;\lambda}.$$

To show this, we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1=\overset{x}{\underset{a}{I}}{}^{\alpha}\sum_{n=1}^{\infty}n\,\lambda^{n-1}\overset{x}{\underset{a}{I}}{}^{\overline{n-1}\,\alpha}1$$

$$=\sum_{n=1}^{\infty}n\,\lambda^{n-1}\overset{x}{\underset{a}{I}}{}^{n\alpha}1$$

$$=\sum_{n=1}^{\infty}n\,\lambda^{n-1}\frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$=\frac{x-a}{\alpha}E_{\alpha;\lambda}.$$

(P·4):

$$\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1=\frac{\lambda}{\alpha}\left[\alpha\overset{x}{\underset{a}{I}}+(x-a)\right]E_{\alpha;\lambda}.$$

This may be shown as follows:

(i) $$\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1=\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1}1+\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1$$

$$=\overset{x}{\underset{a}{I}}\sum_{p=1}^{\infty}\lambda^{p}\frac{(x-a)^{p\alpha-1}}{\Gamma(p\alpha)}+\lambda\frac{x-\alpha}{\alpha}E_{\alpha;\lambda}$$

$$=\frac{\lambda}{\alpha}\left[\alpha\overset{x}{\underset{a}{I}}+(x-a)\right]E_{\alpha;\lambda}.$$

(ii) Also it can be obtained by direct expansion:

$$\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1=\sum_{n=1}^{\infty}n\,\lambda^{n-1}\frac{(x-a)^{\overline{n-1}\,\alpha}}{\Gamma(\overline{n-1}\,\alpha+1)}$$

$$=\sum_{n=0}^{\infty}\lambda^{n}\frac{(x-a)^{n\alpha}}{\Gamma(n\,\alpha+1)}+\sum_{n=1}^{\infty}n\,\lambda^{n}\frac{(x-a)^{n\alpha}}{\Gamma(n\,\alpha+1)}$$

$$=\frac{\lambda}{\alpha}\left[\alpha\overset{x}{\underset{a}{I}}+(x-a)\right]E_{\alpha;\lambda}.$$

(P·5):

$$\overset{x}{\underset{a}{I}}{}^{\alpha-1}\left(1-\lambda\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2}1=\overset{x}{\underset{a}{I}}{}^{\alpha-1}\sum_{p=1}^{\infty}p\,\lambda^{p-1}\overset{x}{\underset{a}{I}}{}^{\overline{p-1}\,\alpha}1$$

$$= \sum_{p=1}^{\infty} p\,\lambda^{p-1} \overset{x}{\underset{a}{I}}{}^{p\alpha-1}\, 1$$

$$= \sum_{p=1}^{\infty} p\,\lambda^{p-1} \frac{(x-a)^{p\alpha-1}}{\Gamma(p\alpha)} .$$

**(P·6): Extension of (P·3).**

**If n is a positive integer such that $n > 1$, then**

$$\overset{x}{\underset{a}{I}}{}^{\overline{n-1}\,\alpha} \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-n} 1 = \overset{x}{\underset{a}{I}}{}^{\overline{n-1}\,\alpha} \sum_{k=0}^{\infty} \lambda^k \frac{(n)_k}{\Gamma(k+1)} \overset{x}{\underset{a}{I}}{}^{k\alpha}\, 1$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k (n)_k}{\Gamma(k+1)} \overset{x}{\underset{a}{I}}{}^{\overline{n+k-1}\,\alpha}\, 1$$

$$= \frac{(x-a)^{\overline{n-1}\,\alpha}}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{\lambda^k\, \Gamma(n+k)\, (x-a)^{k\alpha}}{\Gamma(k+1)\, \Gamma(\overline{n+k-1}\,\alpha+1)} .$$

**(P·7):**

$$\left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f = \sum_{p=0}^{\infty} \lambda^p \overset{x}{\underset{a}{I}}{}^{p\alpha} f$$

$$= \sum_{p=0}^{\infty} \frac{\lambda^p}{\Gamma(p\alpha)} \int_a^x (x-t)^{p\alpha-1} f(t)\, dt$$

$$= \lambda \int_a^x \left[ \sum_{p=1}^{\infty} \frac{\lambda^{p-1}}{\Gamma(p\,\alpha)} (x-t)^{p\alpha-1} \right] f(t)\, dt$$

$$= \lambda \int_a^x E_\alpha(\lambda\, ;\, x-t)\, f(t)\, dt.$$

**(P·8):**

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f = \sum_{p=0}^{\infty} \lambda^p \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f$$

$$= \sum_{p=0}^{\infty} \frac{\lambda^p}{\Gamma(\overline{p+1}\,\alpha)} \int_a^x (x-t)^{\overline{p+1}\,\alpha-1} f(t)\, dt$$

$$= \int_a^x \left[ \sum_{p=0}^{\infty} \frac{\lambda^p}{\Gamma(\overline{p+1}\,\alpha)} (x-t)^{\overline{p+1}\,\alpha-1} f(t) \right] dt$$

$$= \int_a^x E_\alpha(\lambda\, ;\, x-t)\, f(t)\, dt.$$

From (P·7) and (P·8) we may conclude that

(P·9):

$$\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f=\lambda^{k}\,\overset{x}{\underset{a}{I}}{}^{k\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f,$$

for $(k=0,1,2,\ldots)$.

(P·10):

$$\begin{aligned}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f &=\sum_{p=0}^{\infty}(p+1)\,\lambda^{p}\,\overset{x}{\underset{a}{I}}{}^{p\alpha} f\\ &=\sum_{p=0}^{\infty}\frac{(p+1)\,\lambda^{p}}{\Gamma(p\alpha)}\int_{a}^{x}(x-t)^{p\alpha-1} f(t)\,dt\\ &=\int_{a}^{x}\left[\sum_{p=1}^{\infty}\frac{(p+1)\,\lambda^{p}}{\Gamma(p\alpha)}(x-t)^{p\alpha-1} f(t)\right]dt\\ &=\frac{\lambda}{\alpha}\int_{a}^{x} f(t)\,D_{x}\left[(x-t)\,E_{\alpha}(\lambda\,;x-t)\right]dt+\\ &\qquad\qquad+\lambda\int_{a}^{x}E_{\alpha}(\lambda\,;x-t)\,f(t)\,dt.\end{aligned}$$

(P·11):
Writing

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f=\overset{x}{\underset{a}{I}}{}^{\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} f,$$

and using property (P·9), then we would have

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f=\lambda^{-1}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f.$$

In general we find that

$$\overset{x}{\underset{a}{I}}{}^{n\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f=\lambda^{-n}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-2} f,$$

and

$$\overset{x}{\underset{a}{I}}{}^{n\alpha}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-p} f=\lambda^{-n}\left(1-\lambda\,\overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-p} f,$$

where $(n=0,1,2,\ldots)$ and $(p=1,2,3,\ldots)$.

3. **Operations on some Special $T^{\alpha}$-Functions**

(P·12):

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} E_{\alpha;\lambda} = \sum_{p=2}^{\infty} \lambda^{p-1} \frac{(x-a)^{\overline{p-1}\,\alpha-1}}{\Gamma(\overline{p-1}\,\alpha)}$$

$$= \lambda \sum_{p=1}^{\infty} \frac{\lambda^{p-1}}{\Gamma(p\alpha)} (x-a)^{p\alpha-1}$$

$$= \lambda E_{\alpha;\lambda}.$$

Consequently we have

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right) E_{\alpha;\lambda} = 0.$$

Also if we let

$$H\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}\right) = \sum_{k=0}^{N} a_k \overset{x}{I}{}^{\overline{k-n}\,\alpha},$$

where $N \leqq n$ and $(a_k)$ is a sequence of numbers, we would have

$$H\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}\right) E_{\alpha;\lambda} = H(\lambda) E_{\alpha;\lambda},$$

where

$$H(\lambda) = \sum_{k=0}^{N} a_k \lambda^{n-k}.$$

(P·13):

By applying Theorem 2 (ii) we find that

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)(x-a) E_{\alpha;\lambda} = (x-a) \overset{x}{\underset{a}{I}}{}^{-\alpha} E_{\alpha;\lambda} + \alpha \overset{x}{\underset{a}{I}}{}^{1-\alpha} E_{\alpha;\lambda} - \lambda (x-a) E_{\alpha;\lambda}$$

$$= \alpha \lambda \overset{x}{\underset{a}{I}} E_{\alpha;\lambda}.$$

It cas also be shown that for $k \geqq 1$,

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)(x-a)^k E_{\alpha;\lambda} = \lambda \Gamma(\alpha+1) \Gamma(k+1)$$

$$\sum_{p=1}^{k} \frac{(x-a)^{k-p} \overset{x}{\underset{a}{I}}{}^{p} E_{\alpha;\lambda}}{\Gamma(\alpha-p+1)\Gamma(k-p+1)\Gamma(p+1)}.$$

(P·14):

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^2 (x-a)\, E_{\alpha\,;\,\lambda}=0,$$

for the left hand side, according to (P·13), is equal to

$$\alpha\lambda\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)\overset{x}{\underset{a}{I}}\, E_{\alpha\,;\,\lambda}=\alpha\lambda\left(\lambda\,\overset{x}{\underset{a}{I}}\, E_{\alpha\,;\,\lambda}-\lambda\,\overset{x}{\underset{a}{I}}\, E_{\alpha\,;\,\lambda}\right)$$

$$=0.$$

Also we find that

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^2 (x-a)^2\, E_{\alpha\,;\,\lambda}=\lambda^2\alpha^2\,\Gamma(3)\,\overset{x}{\underset{a}{I}}{}^2\, E_{\alpha\,;\,\lambda}\,.$$

This property can be easily obtained by applying to the right-hand side of

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^2 (x-a)^2\, E_{\alpha\,;\,\lambda}=\left(\overset{x}{\underset{a}{I}}{}^{-2\alpha}-2\,\lambda\,\overset{x}{\underset{a}{I}}{}^{-\alpha}+\lambda^2\right)(x-a)^2\, E_{\alpha\,;\,\lambda}\,,$$

THEOREM 2 (ii). In general, for $k\geqq 2$, we have

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^2 (x-a)^k\, E_{\alpha\,;\,\lambda}=\lambda^2\,\Gamma^2(\alpha+1)\,\Gamma(k+1)\times$$

$$\times\sum_{p=1}^{k}\sum_{r=1}^{k-p}\frac{(x-a)^{k-p-r}\,\overset{x}{\underset{a}{I}}{}^{p+r}E_{\alpha\,;\,\lambda}}{\Gamma(\alpha-r+1)\,\Gamma(\alpha-p+1)\,\Gamma(p+1)\,\Gamma(r+1)\,\Gamma(k-p-r+1)}.$$

(P·15):

In a similar way as above it has been found that:

(i) $\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^3 (x-a)^3\, E_{\alpha\,;\,\lambda}=\lambda^3\,\alpha^3\,\Gamma(4)\,\overset{x}{\underset{a}{I}}{}^3\, E_{\alpha\,;\,\lambda}$;

(ii) for $k\geqq 3$,

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}-\lambda\right)^3 (x-a)^k\, E_{\alpha\,;\,\lambda}=\lambda^3\,\Gamma^3(\alpha+1)\,\Gamma(k+1)\times$$

$$\times\sum_{p_1=1}^{k}\sum_{p_2=1}^{k-p_1}\sum_{p_3=1}^{k-p_1-p_2}\frac{(x-a)^{k-p_1-p_2-p_3}\,\overset{x}{\underset{a}{I}}{}^{p_1+p_2+p_3}\,E_{\alpha\,;\,\lambda}}{\Gamma\left(k-\sum_{r=1}^{3}p_r+1\right)\prod_{r=1}^{3}\Gamma(\alpha-p_r+1)\,\Gamma(p_r+1)}\,;$$

A generalization of the above established formulas may be given by

(P·16) :

(A) For $k < n$,

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)^n (x-a)^k E_{\alpha;\lambda} = 0 ;$$

(B) For $k \geqq n$, we have

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)^n (x-a)^k E_{\alpha;\lambda} = \lambda^n \Gamma^n(\alpha+1)\,\Gamma(k+1) \times$$

$$\times \sum_{p_1=1}^{k} \sum_{p_2=1}^{k-p_1} \sum_{p_3=1}^{k-p_1-p_2} \cdots \sum_{p_n=1}^{k-p_1-p_2-\ldots-p_{n-1}}$$

$$\frac{(x-a)^{k-p_1-p_2-\ldots-p_n} \overset{x}{\underset{a}{I}}{}^{p_1+p_2+\ldots+p_n} E_{\alpha;\lambda}}{\Gamma\left(k - \sum_{r=1}^{n} p_r + 1\right) \prod_{r=1}^{n} \Gamma(\alpha - p_r + 1)\,\Gamma(p_r+1)},$$

from which have for $n = k$,

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - \lambda\right)^k (x-a)^k E_{\alpha;\lambda} = \lambda^k \alpha^k \Gamma(k+1) \overset{x}{\underset{a}{I}}{}^{k} E_{\alpha;\lambda} .$$

(P·17) :

From properties of the functions $S_{\alpha;\lambda}$, $C_{\alpha;\lambda}$, $SH_{\alpha;\lambda}$, and $CH_{\alpha;\lambda}$ we have the following :

(i) $\overset{x}{\underset{a}{I}}{}^{-\alpha-1} \left(\overset{x}{\underset{a}{I}}{}^{-2\alpha} - \lambda^2\right)^{-1} 1 = CH_{\alpha;\lambda}$ ;

(ii) $\lambda \overset{x}{\underset{a}{I}}{}^{2\alpha-1} \left(1 - \lambda^2 \overset{x}{\underset{a}{I}}{}^{2\alpha}\right)^{-1} 1 = SH_{\alpha;\lambda}$ ;

(iii) $\overset{x}{\underset{a}{I}}{}^{-\alpha-1} \left(\overset{x}{\underset{a}{I}}{}^{-2\alpha} + \lambda^2\right)^{-1} 1 = C_{\alpha;\lambda}$ ;

(iv) $\lambda \overset{x}{\underset{a}{I}}{}^{2\alpha-1} \left(1 + \lambda^2 \overset{x}{\underset{a}{I}}{}^{2\alpha}\right)^{-1} 1 = S_{\alpha;\lambda}$ .

It would be sufficient to show the validity of the first two relations as the others follows in a similar way. We have

$$\underset{a}{\overset{x}{I}}{}^{-\alpha-1}\left(\underset{a}{\overset{x}{I}}{}^{-2\alpha}-\lambda^{2}\right)^{-1}1=\underset{a}{\overset{x}{I}}{}^{\alpha-1}\left(1-\lambda^{2}\,\underset{a}{\overset{x}{I}}{}^{2\alpha}\right)^{-1}1$$

$$=\sum_{n=0}^{\infty}\lambda^{2n}\,\underset{a}{\overset{x}{I}}{}^{\overline{2n+1}\,\alpha-1}\,1$$

$$=\sum_{n=0}^{\infty}\lambda^{2n}\,\frac{(x-a)^{\overline{2n+1}\,\alpha-1}}{\Gamma(\overline{2n+1}\,\alpha)}$$

$$=CH_{\alpha\,;\,\lambda}\,.$$

Also we have

$$\lambda\,\underset{a}{\overset{x}{I}}{}^{2\alpha-1}\left(1-\lambda^{2}\,\underset{a}{\overset{x}{I}}{}^{2\alpha}\right)^{-1}1=\lambda\,\underset{a}{\overset{x}{I}}{}^{-1}\left(\underset{a}{\overset{x}{I}}{}^{-2\alpha}-\lambda^{2}\right)^{-1}1$$

$$=\sum_{p=1}^{\infty}\lambda^{2p-1}\,\frac{(x-a)^{2p\alpha-1}}{\Gamma(2p\alpha)}$$

$$=SH_{\alpha\,;\,\lambda}\,.$$

## 4. Some Applications.

The operational properties of the integro differential operator of generalized order may be used to deal with solving differential or integro differential equations of generalized order in a similar way the ordinary differential operators are used in solving ordinary linear differential equations. This may be explained by the following:

EXAMPLE 1. Consider the differential equation of generalized order $\alpha > 0$,

(i) $$\underset{a}{\overset{x}{I}}{}^{-\alpha}\,y-\lambda\,y=f(x),$$

where $\lambda$ is a parameter and $f(t)$ is integrable function on $a<\overset{x}{t}<b$. This equation can be written as

$$y-\lambda\,\underset{a}{\overset{x}{I}}{}^{\alpha}\,y=\underset{a}{\overset{x}{I}}{}^{\alpha}f+C,$$

where $C$ is an arbitrary constant. Now if we put the equation in the form

$$\left(1-\lambda\,\underset{a}{\overset{x}{I}}{}^{\alpha}\right)y=\underset{a}{\overset{x}{I}}{}^{\alpha}f+C,$$

then

$$y(x) = \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} \overset{x}{\underset{a}{I}}{}^{\alpha} f + \left(1 - \lambda \overset{x}{\underset{a}{I}}{}^{\alpha}\right)^{-1} C$$

$$= \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{\overline{p+1}\,\alpha} f + \sum_{p=0}^{\infty} \lambda^{p} \overset{x}{\underset{a}{I}}{}^{p\alpha} C$$

$$= \sum_{p=1}^{\infty} \lambda^{p-1} \overset{x}{\underset{a}{I}}{}^{p\alpha} f + C \sum_{p=0}^{\infty} \lambda^{p} \frac{(x-a)^{p\alpha}}{\Gamma(p\alpha+1)}$$

$$= \int_{a}^{x} E_{\alpha}(\lambda\,;\,x-t)\,f(t)\,dt + C\,\lambda\,\overset{x}{\underset{a}{I}}\,E_{\alpha\,;\,\lambda}\,.$$

Thus, denoting $y(x)$ by $Y_{\alpha}(x)$ we would obtain the solutions

(ii) $$Y_{\alpha}(x) = \overset{x}{\underset{a}{I}}\,E_{\alpha\,;\,\lambda} f + C\,\lambda\,\overset{x}{\underset{a}{I}}\,E_{\alpha\,;\,\lambda}\,.$$

REMARK. It may be interesting to note that when $\alpha = 1$, equation (i) is reduced to $y' - \lambda y = f(x)$, and (ii) is reduced to

$$Y_{1}(x) = \int_{a}^{x} e^{\lambda(x-t)} f(t)\,dt + C\,e^{\lambda x}\,.$$

EXAMPLE 2. Solution of Equations with Constant Coefficients. Let

(iii) $$H\left(\overset{x}{\underset{a}{I}}{}^{\alpha}\right) y = \sum_{k=0}^{n} a_{k}\, \overset{x}{I}{}^{\overline{k-n}\,\alpha}\, y = 0.$$

In order to solve this equation, we notice that if we assume that the equation has solutions of the form $E_{\alpha\,;\,\lambda}$, then we have

(iv) $$H\left(\overset{x}{\underset{a}{I}}{}^{-\alpha}\right) E_{\alpha\,;\,\lambda} = H(\lambda),$$

where $H(\lambda) = 0$ is similar to the characteristic equation associated with an ordinary linear differential equation. Thus $E_{\alpha\,;\,\lambda}$ will be a solution of the differential equation of generalized order (iii) if $\lambda$ is chosen as a root of the equation

$$H(\lambda) = \sum_{k=0}^{n} a_{k}\,\lambda^{n-k} = 0.$$

If this equation has $n$-distinct real roots, say $\lambda_1, \lambda_2, \dots, \lambda_n$, then the general solution $Y_a(x)$ may be given by

$$Y_a(x) = \sum_{p=1}^{n} C_p E_{a;\lambda_p}; \tag{v}$$

where $C_p$, $p = 1, 2, \dots, n$, are arbitrary constants.

The same form of general solution may be obtained if $2k < n$ of the roots are complex and it may be expressed in terms of functions of the forms $S_{a;\lambda}$, and $C_{a;\lambda}$.

If the equation has $n$-repeated roots, say $\lambda = r$, then the equation may take the form

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - r\right)^n y = 0.$$

By (P·16) we have

$$\left(\overset{x}{\underset{a}{I}}{}^{-\alpha} - r\right)^n (x-a)^k E_{a;r} = 0,$$

for $(k = 0, 1, 2, \dots, n-1)$. Therefore $Y_a(x)$ may take the form

$$Y_a(x) = \left[\sum_{k=0}^{n-1} C_k (x-a)^k\right] E_{a;r},$$

where the $C_k$'s are arbitrary constants.

Other cases related to the roots of the characteristic equation may be dealt with in a similar way.

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