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SCUOLA NORMALE SUPERIORE DI PISA
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S. FUČIK
J. NEČAS
J. SOUČEK
V. SOUČEK

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UPPER BOUND FOR THE NUMBER OF EIGENVALUES FOR NONLINEAR OPERATORS

S. Fučík, J. Nečas, J. Souček, V. Souček (Prague)

1. Introduction.

Let f and g be two nonlinear functionals defined on a real Hilbert space R. We consider the eigenvalue problem

$$\begin{cases} \lambda f'(u) = g'(u) \\ f(u) = r \end{cases} \tag{1.1}$$

$r > 0$ is a prescribed positive number, f' and g' denote Frechet derivatives of f and g, respectively).

Under some assumptions on f and g it is known that there exists an infinite number of points $u \in R$ (denote their set by U) and infinite number of eigenvalues $\lambda \in E_1$ (denote their set by Λ) satisfying (1.1).

For Hilbert spaces, this theorem is concluded in the book of M. A. Krasnoselskij [9]. For Banach spaces, such theorem was proved by E. S. Citlanadze [4], F. E. Browder [3] and S. Fučík-J. Nečas [7]. In all of these papers, some variant of the notion « the category of the set » in the sense of L. A. Ljusternik and L. Schnirelmann ([11], [12], [13]) is used.

We should remark that in the above cited papers the lower bound for the number of eigenvalues for the eigenvalue problem (1.1) is an immediate corollary of the fact that, so to say, the set of critical levels, i. e., the set $C = g(U)$ is infinite.

It is our object in this paper to prove that (under some reasonable assumptions) the set C of critical levels for the eigenvalue problem (1 1) is

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at most countable set, moreover, that $g(U) \cap \langle \varepsilon, \infty)$ is a finite set for each $\varepsilon > 0$ (see Section 5). This fact implies that the set Λ of eigenvalues of problem (1.1) is countable in the case of homogeneous functionals f and g (see Section 7).

For the nonlinear Sturm-Liouville equation

$$- (a \, | \, u' \, |^{p-2} \, u')' + b \, | \, u \, |^{p-2} \, u - \lambda \, c \, | \, u \, |^{p-2} \, u = 0$$

the above statement has been proved under suitable boundary conditions (see J. Nečas [18]). The same result for a fourth order ordinary differential equation is included in the work by A. Kratochvíl-J. Nečas [19].

In we consider homogeneous functionals f and g which are homogeneous with the same power we can apply the so called Fredholm alternative for nonlinear operators (see for instance J. Nečas [15], [20], [21], S. Fučík [5], [6] and M. Kučera [10]) obtaining that the operator $A_\lambda = \lambda f' - g'$ is onto for each $\lambda \in E_1$ except a countable set Λ (see Section 7). From these results it could seem that the homogeneous nonlinear eigenvalue problem has similar spectral properties as the linear case, for example that the range of $\lambda f' - g'$ is closed. However, it is shown on an example of a « nice » eigenvalue problem that this is not always true.

Finally, we apply our abstract results to the so called Lichtenstein and degenerated Lichtenstein integral equations (Section 8, 9) and to the Dirichlet boundary value problem for ordinary and partial differential equations (Section 10).

2. Analytic operators in Banach spaces.

Let X, Y be two complex Banach spaces, $D \subset X$ an open subset and $F : D \to Y$ a mapping.

a) The mapping F is said to be (G)-differentiable on D, if the limit

$$\lim_{\substack{\xi \to 0 \\ \xi \in \mathbb{C}}} \frac{F(x + \xi h) - F(x)}{\xi} = DF(x, h)$$

exists for each $x \in D$ and all $h \in X$, where $DF(x, \cdot)$ is a bounded linear operator.

b) The mapping F is said to be (F)-differentiable on D, if F is (G)-differentiable and

$$\lim_{h \to 0} \frac{\| F(x + h) - F(x) - DF(x, h) \|_Y}{\| h \|_X} = 0$$

locally uniformly on D. ($DF(x, \cdot) = F'(x)$ is called the Frechet derivative of F. Analogously one can define Frechet derivatives of higher orders).

c) The mapping F is said to be locally bounded on D, if for each $a \in D$ there exists $\delta > 0$ such that

$$\sup_{\{x \in D;\, \| x-a \| < \delta\}} \| F(x) \| < \infty.$$

d) The mapping F is said to be analytic on D, if F is (G)-differentiable and locally bounded on D.

PROPOSITION 1 (see [8, Theorem 3.17.1]). Let $F: D \longrightarrow Y$ be an analytic mapping. Then

(i) F is continuous on D,

(ii) F has Frechet derivatives $D^n F(x, \ldots)$ for arbitrary n and all $x \in D$,

(iii) for each $a \in D$ there exists $\delta > 0$ such that

$$F(x + h) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(x, h^n)$$

and the series on the right hand side is uniformly and absolutely convergent for $x \in \{y \in X;\ \| y - a \| < \delta\}$ and $\| h \| < \delta$ (h^n is the vector $[h, \ldots, h]$ with n components).

PROPOSITION 2 (see [8]). Let $\{F_k\}$ be a locally uniformly bounded sequence of analytic functions such that $\lim F_k = F$ on D.

Then F is an analytic mapping on D.

LEMMA 1. Let X, Y, Z be three complex Banach spaces, $D \subset X$ an open set, $F: D \longrightarrow Y$ an analytic mapping on D and $G: Y \longrightarrow Z$ an analytic mapping on Y.

Then the composition $G \circ F$ is an analytic mapping on D.

LEMMA 2. Let $D \subset X$ be an open set and f, $g: D \longrightarrow \mathbb{C}$ analytic functionals. Suppose that $f(x) \neq 0$ for $x \in D$.

Then g/f is an analytic functional on D. (Proofs of Lemmas 1 and 2 can be easily obtained from Proposition 1).

Let the mapping $F: D \longrightarrow Z$ be defined on an open set $D \subset X \times Y$. If there exists the limit

$$F'_y(x_0, y_0)\, h = \lim_{\xi \to 0} \frac{F(x_0, y_0 + \xi h) - F(x_0, y_0)}{\xi}$$

for all $h \in Y$ and $F'_y(x_0, y_0)$ is a bounded and linear operator from Y into Z, then $F'_y(x_0, y_0)$ is called the partial derivative by y of the mapping F.

LEMMA 3 (Implicit function theorem).

Let X, Y, Z be complex Banach spaces, $G \subset X \times Y$ an open set, $[x_0, y_0] \in G$. Let $F: G \longrightarrow Z$ be an analytic mapping such that $[F'_y(x_0, y_0)]^{-1}$ exists and $F(x_0, y_0) = \theta_z$.

Then there exist a neighborhood $U(x_0)$ in X of the point x_0 and a neighborhood $U(y_0)$ in Y of the point y_0 (such that $U(x_0) \times U(y_0) \subset G$) such that there exists one and only one mapping $y: U(x_0) \longrightarrow U(y_0)$ for which $F(x, y(x)) = \theta_z$ on $U(x_0)$. Moreover, y is an analytic mapping on $U(x_0)$.

PROOF. Denote

$$A: [x, y] \longmapsto y - [F'_y(x_0, y_0)]^{-1} \circ F(x, y).$$

The mapping A is analytic on G and the equation $F(x, y) = \theta_z$ is equivalent to the equation $y = A(x, y)$. Obviously

$$A'_y(x, y) = I - [F'_y(x_0, y_0)]^{-1} \circ F'_y(x, y) =$$

$$= [F'_y(x_0, y_0)]^{-1} \circ [F'_y(x_0, y_0) - F'_y(x, y)].$$

Hence it follows that there exists $r > 0$ and $q \in (0, 1)$ such that for

$$x \in \{x \in X;\ \| x - x_0 \| < r\} = U(x_0),$$

$$y_1, y_2 \in \{y \in Y;\ \| y - y_0 \| < r\} = U(y_0)$$

it is

$$\| A(x, y_1) - A(x, y_2) \|_Y \leq q \| y_1 - y_2 \|_Y.$$

According to the Banach contraction principle, for each $x \in U(x_0)$ there exists precisely one point $y(x) \in U(y_0)$ such that $F(x, y(x)) = \theta_z$ and $y(x)$ is the limit of succesive approximations $\{y_n(x)\}$, where $y_0(x) = y_0$, $y_{n+1}(x) = A(x, y_n(x))$. Thus $\{y_n\}$ is a locally uniformly bounded sequence of analytic mappings and in view of Proposition 2 the proof is complete.

3. Real-analytic operators.

Let X be a real Banach space. Then X can be isometrically imbedded into the complex Banach space $X + iX$.

DEFINITION. Let X and Y be two real Banach spaces, $D \subset X$ an open subset. The mapping $F: D \rightarrow Y$ is said to be real-analytic on D, if the following conditions are fulfilled:

(i) For each $x \in D$ there exist Frechet derivatives of arbitrary orders $D^n F(x, \dots)$.

(ii) For each $x \in D$ there exists $\delta > 0$ such that for all $h \in X$, $\|h\| < \delta$ it is

$$F(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(x, h^n)$$

(the convergence is locally uniform and absolute).

PROPOSITION 3 (see [1, Theorem 5.7]).

Let $F: D \rightarrow Y$ be a real-analytic operator on D. Then there exist an open set $Z \subset X + iX$ such that $D \subset Z$ and an analytic mapping $\widetilde{F}: Z \rightarrow Y + iY$ such that the restriction of $\widetilde{F}$ on D is F.

Analogously as in the complex case we can define the partial derivatives.

Using Proposition 3 we can formulate Lemmas 1-3 by saying « real-analytic » instead of « analytic ». The resulting lemmas will be called respectively Lemma 1R, 2R, 3R. Since Lemma 3R (Implicit function theorem) will be of great importance for further considerations we present it here in full:

LEMMA 3R (Implicit function theorem).

Let X, Y, Z be real Banach spaces, $G \subset X \times Y$ an open set, $[x_0, y_0] \in G$. Let $F: G \rightarrow Z$ be a real-analytic mapping such that $[F'_y(x_0, y_0)]^{-1}$ exists and $F(x_0, y_0) = \theta_z$.

Then there exist a neighborhood $U(x_0)$ in X of the point x_0 and a neighborhood $U(y_0)$ in Y of point y_0 (such that $U(x_0) \times U(y_0) \subset G$) such that there exists one and only one mapping $y: U(x_0) \rightarrow U(y_0)$ for which $F(x, y(x)) = \theta_z$ on $U(x_0)$. Moreover, y is a real-analytic mapping on $U(x_0)$.

Remark to the proof of Lemma 3R. If $\widetilde{F}$ is an extension of F in the sense of Proposition 3, one must show that assumptions of Lemma 3R imply the existence of $[\widetilde{F}'_y(x_0, y_0)]^{-1}$. This fact follows from

$$\widetilde{F}'_y(x_0, y_0)(h_1 + ih_2) = F'_y(x_0, y_0)\, h_1 + iF'_y(x_0, y_0)\, h_2 .$$

4. Critical levels for real-analytic functionals.

Let R be a real separable Hilbert space, $G \subset R$ an open set. Suppose that $f: G \rightarrow E_1$ is a real-analytic functional on G and denote for $x \in G$ by

$f'(x)$ its Frechet derivative. Let $B = \{x \in G; f'(x) = \theta\}$. If $\gamma \in f(B)$ then γ is called a critical level.

For further considerations, the following proposition is fundamental. It is a complement of the well-known Morse-Sard theorem for real functions and their critical levels.

Proposition 4 (see [16]). Let f be a real-analytic function defined on an open subset $D \subset E_N$ (Euclidean N-space).

Then $f(B \cap K)$ is finite for every compact set $K \subset D$ and hence $f(B)$ is at most countable.

Corollary. Under the assumptions of Proposition 4, for each $x \in B$ there exists a neighborhood U of x such that $f(B \cap U)$ is a one-point set.

Proof. Suppose that there exists a sequence x_n, $x_n \to x$, $x_n \in B$, $f(x_n) \neq f(x)$ and $f(x_n) \to f(x)$. Then the set $\{f(x_n), n = 1, 2, ...\}$ is infinite and this is a contradiction to Prodosition 4.

In the sequel we wish to give an analogous assertion for the functionals in infinite dimensional Hilbert spaces.

Definition. Let f be a functional defined on an open subset G of a real Hilbert space R. We shall say that f satisfies Fredholm condition at a point $u_0 \in G$ if

(i) there exist the (G) derivatives $f'(u_0)$, $f''(u_0)$, where $f''(u_0)h = Df'(u_0, h)$,

(ii) the subspace $\{h \in R; f''(u_0)h = \theta\}$ has a finite dimension,

(iii) the set $f''(u_0)(R)$ is a closed subspace of R.

Theorem 1. Let $f: G \to E_1$ be a real-analytic functional on an open set $G \subset R$. Let us denote $B = \{x \in G; f'(x) = \theta\}$. Suppose that f satisfies Fredholm condition at a point $x_0 \in B$.

Then there exists a neighborhood $V(x_0) \subset G$ of x_0 such that $f(V(x_0) \cap B)$ contains only one point.

Corollary. Let $f: G \to E_1$ be a real-analytic functional on an open set $G \subset R$. Suppose that f satisfies Fredholm condition at each point of a set $M \subset G$.

Then the set $f(B \cap M)$ is at most countable.

PROOF OF THEOREM 1. Let us denote $f' = \Phi$, $\Phi'(x_0) = F$. Then $\Phi(x) = F(x - x_0) + \psi(x)$ where $\lim\limits_{x \to x_0} \dfrac{\| \psi(x) \|}{\| x - x_0 \|} = 0$. The set $Z = \{x \in R;\ Fx = \theta\}$ is a finite dimensional subspace of R. Let Y be such a closed subspace of R that $Y \oplus Z = R$. Obviously $F(Y) = Y$. Denote by P_Y the orthogonal projection on Y. The point $x = [y, z]$ belongs to B if and only if $\Phi(x) = \theta$ and thus

$$\theta = P_Y \Phi(x) = F(y - y_0) + \psi_1(y, z) = \mathcal{U}(y, z),$$

where $x_0 = [y_0, z_0]$, $\psi_1(y, z) = P_Y \psi(y, z)$. The assumptions of Lemma 3R hold, since $[\mathcal{U}'_y(y_0, z_0)]^{-1} = F^{-1}$ (for $(\psi_1)'_y(y_0, z_0) = \theta$). Hence there exist neighborhoods $U(y_0)$, $U(z_0)$ and a real-analytic operator $\omega : U(z_0) \to U(y_0)$ such that $\mathcal{U}(\omega(z), z) = \theta$ for all $z \in U(z_0)$.

Consequently, for arbitrary $x_1 \in B \cap [U(y_0) \times U(z_0)]$ there exists $z_1 \in U(z_0)$ such that $x_1 = [\omega(z_1), z_1]$. Define $g : U(z_0) \to E_1$ by $g(z) = f(\omega(z), z)$. According to Lemma 2R, g is a real-analytic functional on $U(z_0)$. For each $[\omega(z_1), z_1] \in B \cap [U(y_0) \times U(z_0)]$ it is $g'(z) = \theta_z$ (this immediately follows from differentiation of the composition of mappings) Denote $B_g = \{z \in U(x_0);\ g'(z) = \theta\}$. In view of Corollary of Proposition 4 there exists a neighborhood $U_1(z_0) \subset U(z_0)$ of the point z_0 such that $g(B_g \cap U_1(z_0)) = \{\gamma_0\}$. Let $x_1, x_2 \in B \cap [U(y_0) \times U_1(z_0)]$ and thus $x_i = [\omega(z_i), z_i]$, $z_i \in U_1(z_0)$, $f(x_i) = g(z_i) = \gamma_0$ for $i = 1, 2$. Hence the set $f(B \cap [U(y_0) \times U_1(z_0)])$ contains only one point.

PROOF OF COROLLARY. We proved that for each $x \in B \cap M$ there exists a neighborhood $U(x) \subset G$ such that $f(B \cap U(x))$ is a one-point set. Thus $B \cap M \subset \bigcup\limits_{x \in B \cap M} U(x)$ and since the space R is separable, there exists a sequence $\{x_n\} \subset B \cap M$ such that $M \cap B \subset \bigcup\limits_{n=1}^{\infty} U(x_n)$ and $f(B \cap M) \subset \bigcup\limits_{n=1}^{\infty} f(B \cap U(x_n))$. Thus $f(B \cap M)$ is at most countable.

5. Main theorem.

THEOREM 2. Let us suppose:

(F1) f is a real-analytic functional on R,

(F2) $f(0) = 0$, $f(u) > 0$ for all $u \neq \theta$,

(F3) there exists a continuous and nondecreasing function $c_1(t) > 0$ for $t > 0$ such that for all $u, h \in R$

$$D^2 f(u, h, h) \geq c_1(f(u)) \| h \|^2,$$

(F4) the set $M_r(f) = \{x \in R\,;\ f(x) = r\}$ $(r > 0)$ is bounded,

(F5) the operator f' is a bounded mapping (i. e. it maps bounded subsets onto bounded subsets),

(F6)
$$\inf_{x \in M_r(f)} \langle f'(x), x \rangle \geq c_2 > 0,$$

(G1) g is a real-analytic functional on an open set $G \subset R$ such that $\{x \in R\,;\ f(x) \leq r\} \subset G$,

(G2) the derivative g' is strongly continuous (i. e. it maps weakly convergent sequences onto strongly convergent ones),

(G3) if $g(u) \neq 0$ then $g'(u) \neq \theta$.

Let us denote by B the set of all critical points of g with respect to $M_r(f)$, i. e. $B = \{x \in M_r(f)\,;$ there exists $\lambda \in E_1$ such that $g'(x) = \lambda f'(x)\}$.

Then the set $g(B)$ of all critical levels is at most countable and its only cumulation point can be zero.

PROOF. First, we can suppose $r = 1$, for if this is not the case we can consider the functional $f_1(x) = \dfrac{1}{r} f(x)$.

PART A. Let us suppose $u_0 \in B$, $g(u_0) \neq 0$.

I. From (G3) we have $g'(u_0) \neq \theta$ and also $f'(u_0) \neq \theta$ (for $g'(u_0) = \lambda f'(u_0)$). Then we can find a neighborhood U of u_0 such that $f(u) > 0$, $g(u) \neq 0$, $f'(u) \neq \theta$ for all $u \in U$. Let us define for $u \in U$

$$\beta(u) = \frac{\langle g'(u), f'(u) \rangle}{g(u) \, \| f'(u) \|^2}$$

and $\Phi(u) = \dfrac{g(u)}{[f(u)]^{\beta(u)}}$.

From Section 3 it follows that Φ is a real-analytic functional on U. Now we want to show that Φ satisfies Fredholm condition at the point u_0:

$$D\Phi(u_0, h) =$$
$$= \frac{Dg(u_0, h)[f(u_0)]^{\beta(u_0)} - g(u_0)[f(u_0)]^{\beta(u_0)} \left[D\beta(u_0, h) \lg f(u_0) + \dfrac{\beta(u_0)}{f(u_0)} Df(u_0, h) \right]}{[f(u_0)]^{2\beta(u_0)}}$$

and further (we have $f(u_0) = 1$)

$$D^2\Phi(u_0, h, k) = D^2 g(u_0, h, k) - \beta(u_0)\, Dg(u_0, h)\, Df(u_0, k) -$$

$$- \beta(u_0)\, Dg(u_0, k)\, Df(u_0, h) + g(u_0)\, \beta^2(u_0)\, Df(u_0, h)\, Df(u_0, k) -$$

$$- g(u_0)\, D\beta(u_0, h)\, Df(u_0, k) - g(u_0)\, D\beta(u_0, k)\, Df(u_0, h) -$$

$$- g(u_0)\, \beta(u_0)\, D^2 f(u_0, h, k) + g(u_0)\, \beta(u_0)\, Df(u_0, h)\, Df(u_0, k).$$

Now, let us denote $f' = F$, $g' = G$, $\beta' = B$, i. e. let F, G, B be mappings from U into R such that

$$Df(u, h) = \langle Fu, h \rangle, \quad Dg(u, h) = \langle Gu, h \rangle, \quad D\beta(u, h) = \langle Bu, h \rangle$$

for each $h \in R$. Then $\Phi''(u_0)\, h = \theta$ if and only if

$$g(u_0)\, \beta(u_0)\, DF(u_0, h) = DG(u_0, h) - \beta(u_0) \langle Gu_0, h \rangle Fu_0$$

$$- \beta(u_0) \langle Fu_0, h \rangle Gu_0 + g(u_0)\, \beta^2(u_0) \langle Fu_0, h \rangle Fu_0 - g(u_0) \langle Bu_0, h \rangle Fu_0$$

$$- g(u_0) \langle Fu_0, h \rangle Bu_0 + g(u_0)\, \beta(u_0) \langle Fu_0, h \rangle Fu_0 .$$

But $g(u_0) \neq 0$ and also $\beta(u_0) \neq 0$. Now, G is strongly continuous, hence its Frechet differential $DG(u_0, \cdot)$ is a linear bounded completely continuous operator (see [17, Chapter I, § 4]). Further, mappings of the type $\beta(u_0) \cdot \langle Gu_0, \cdot \rangle Fu_0$ have a one-dimensional range. Finally, for DF we have the condition

$$\langle DF(u_0, h), h \rangle \geq c_1(1)\, \| h \|^2.$$

Hence, the subspace of all solutions h has a finite dimension and the range is closed. This proves the Fredholm condition at the point u_0.

II. Let us denote $B_1 = B \cap U$, $B_2 = \{x \in U;\ \Phi'(x) = \theta\}$. It is obvious that $B_1 \subset B_2$.

We can use Theorem 1, which yields that there exists a neighborhood $V(u_0) \subset U$ such that the set $\Phi(V(u_0) \cap B_2)$ contains only one point. Now, $g = \Phi$ on $M_1(f)$, hence

$$g(V(u_0) \cap B_1) \subset \Phi(V(u_0) \cap B_2)$$

and the set $g(V(u_0) \cap B_1) = g(V(u_0) \cap B)$ contains again only one point.

PART B. Suppose $\gamma_n \in g(B)$, $\gamma_n \to \gamma \neq 0$. There exists a sequence $\{u_n\} \subset B$ such that $g(u_n) = \gamma_n$. We can suppose $u_n \rightharpoonup u_0$, i. e. u_n converge weakly to u_0, for $M_1(f)$ is a bounded set. From (F3) it follows that f is a convex functional, hence the set $\{x \in R; f(x) \leq 1\}$ is convex and closed, hence also weakly closed. We have $u_0 \in \{x \in R; f(x) \leq 1\} \subset G$ and from (G2) it follows that g is weakly continuous (see [17, Chap. I, § 4). Hence $\gamma_n = g(u_n) \rightarrow g(u_0) \neq 0$ (note here that $\gamma = g(u_0)$ must be a finite number) and from (G3) also $g'(u_0) \neq \theta$. There exist $\lambda_n \in E_1$ such that $g'(u_n) = \lambda_n f'(u_n)$. The sequence $\{1/\lambda_n\}$ is bounded, for if $\lambda_{n_k} \rightarrow 0$, then $g'(u_{n_k}) \rightarrow \theta = g'(u_0)$, which is a contradiction. Hence we can suppose that $1/\lambda_n \rightarrow \alpha \in E_1$ and then there exists $w \in R$ such that $f'(u_n) \rightarrow w$. From the following Lemma 4 we have $u_n \rightarrow u_0$, hence $u_0 \in B$, $g(u_0) \neq 0$ and Part A implies that $\gamma_n = \gamma$ for n sufficiently large.

PART C. From Part A it follows immediately that every nonzero critical level is isolated. Thus the set of all critical levels is at most countable and its only cumulation point can be zero.

LEMMA 4. Let the assumptions of Theorem 2 be fulfilled. Then the implication

$$u_n \in M_r(f), \quad u_n \rightharpoonup u_0, \quad f'(u_n) \rightarrow w \Longrightarrow u_n \rightarrow u_0$$

holds.

PROOF. We can write

$$\langle f'(u_n) - f'(u_0), u_n - u_0 \rangle = \int_0^1 \langle f''(u_0 + t(u_n - u_0))(u_n - u_0), u_n - u_0 \rangle \, dt$$
$$\geq K_n \| u_n - u_0 \|^2,$$

where $K_n = \int_0^1 c_1(f(u_0 + t(u_n - u_0)))\, dt > 0$. Suppose that $\liminf_{n \to \infty} K_n = 0$. Then there exists a subsequence n_j such that $K_{n_j} \rightarrow 0$ and (f being convex — see assumption (F3) — and continuous, thus being weakly lower-semicontinuous)

$$0 = \lim_{j \to \infty} K_{n_j} \geq \int_0^1 c_1(\liminf_{j \to \infty} f(u_0 + t(u_{n_j} - u_0)))\, dt \geq c_1(f(u_0)) \geq 0.$$

Thus $f(u_0) = 0$ (see (F3)) and $u_0 = \theta$ (see (F2)).

On the other hand, $\langle f'(u_n) - f'(u_0), u_n - u_0 \rangle = \langle f'(u_n), u_n \rangle \rightarrow 0$ and this is a contradiction to (F6). Thus $\liminf_{n \to \infty} K_n > 0$ and $u_n \rightarrow u_0$.

REMARK. If we not suppose condition (G3) then the following assertion can be proved:

Denote $K = g(\{x \in R;\ g'(x) = \theta\})$. Then each point from $g(B) - K$ is isolated. Thus if $K = \emptyset$ then the set $g(B)$ is finite and if K is countable then $g(B)$ is, too.

6. Set of critical levels.

THEOREM 3. Let f and g be two even real-analytic functionals defined on a real infinite-dimensional separable Hilbert space R. Let $r > 0$ and denote $M_r(f) = \{x \in R;\ f(x) = r\}$. Let the following assumptions be fulfilled:

(A) $f(x) = 0 \iff x = \theta$, $f(x) > 0$ for $x \neq \theta$, $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$.

(B) $\langle f'(x), x \rangle > 0$ for each $x \in R$, $x \neq \theta$.

(C) $\inf\limits_{x \in M_r(f)} \langle f'(x), x \rangle = c > 0$.

(D) f' and g' are bounded operators (i. e. they map bounded sets onto bounded sets).

(E) $g(x) \geq 0$ for each $x \in R$, $g(x) = 0 \iff x = \theta$.

(F) $g'(x) = \theta \iff x = \theta$, g' is a strongly continuous mapping.

(G) f' and g' are uniformly continuous in some neighborhood of $M_r(f)$ with respect to the $M_r(f)$ (i. e. for each $\eta > 0$ there exists $\delta > 0$ such that the inequalities

$$\| f'(x+h) - f'(x) \| \leq \eta, \quad \| g'(x+h) - g'(x) \| \leq \eta$$

hold for each $x \in M_r(f)$ and all $h \in R$ with $\|h\| \leq \delta$).

(H) There exists a continuous and nondecreasing function $c_1(t) > 0$ for $t > 0$ such that for all $u, h \in R$

$$D^2 f(u, h, h) \geq c_1(f(u)) \|h\|^2.$$

Then the set of all critical levels is an infinite sequence $\{\gamma_n\}$ and $\gamma_n \to 0$, $\gamma_n > 0$.

PROOF. The upper bound of critical levels is shown in Theorem 2. The lower bound is shown in [7, Theorem 2].

Remark. Without difficulties we can prove the same result about the lower bound for critical levels in the case of a functional g defined on an open set G such that $G \supset \{x \in R\,;\, f(x) \leq r\}$.

7. Spectral properties of homogeneous operators.

A number $\lambda_0 \in E_1$ is said to be an eigenvalue number for problem (1.1) if there exists a point $u_0 \in R$ satisfying (1.1). Theorem 3 gives us the final result about critical levels but it say nothing about the set of eigenvalues. In the case of homogeneous functionals f and g, i. e. if there exist $a > 0$, $b > 0$ such that

$$f(tu) = t^a f(u), \tag{7.1}$$

$$g(tu) = t^b g(u) \tag{7.2}$$

for each $t > 0$ and all $u \in R$, it holds

$$\langle g'(u), u \rangle = bg(u), \quad \langle f'(u), u \rangle = af(u)$$

and thus the eigenvalues have the form $\lambda = \frac{b}{a}\,\frac{g(u)}{r}$ where u are eigenvectors and $\gamma = g(u)$ their critical levels. Thus $\gamma = r\,\frac{a}{b}\,\lambda$.

Theorem 4. Let the functionals f and g be defined on the whole space R. Suppose that the assumptions of Theorem 3 are fulfilled and, moreover, let the relations (7.1), (7.2) hold.

Then the set of eigenvalues contains exactly countable number of positive eigenvalues and zero is the only cumulation point of this set.

In the sequel we suppose $a = b$, for these cases the so called Fredholm alternative for nonlinear operators is true (see [5], [6], [15], [20], [21]). Hence we obtain

Theorem 5. Let the assumptions of Theorem 4 be fulfilled.

Then if λ is not an eigenvalue (according to Theorem 4, this occurs for all real numbers except a countable number of them) the operator $A_\lambda = \lambda f' - g'$ maps R onto R.

From the assertions of Theorems 4 and 5 it could seem that in the homogeneous case the spectral properties are the same as in the linear case. The following example shows that in the nonlinear case an unusual behaviour can occur.

EXAMPLE. In the two dimensional Euclidean space, set

$$f(x, y) = \frac{1}{4}(x^2 + y^2)^2,$$

$$g(x, y) = \frac{1}{4}(2x^4 + y^4).$$

Functionals f and g satisfy all assumptions of Theorems 4 and 5. The eigenvalue problem

$$\begin{cases} \lambda f'(u) = g'(u) \\ f(u) = 1 \end{cases}$$

has the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = \frac{2}{3}$ (by the Ljusternik-Schnirelmann process we obtain only λ_1, λ_3).

CASE $\lambda_1 = 2$. The range of the operator $A_{\lambda_1} = \lambda_1 f' - g'$ is

$$\{[x, y] \in \mathrm{E}_2 ; \; y^2 - 2x^2 \geq 0\},$$

i. e. a closed set.

CASE $\lambda_2 = 1$. The range of the operator A_{λ_2} is the whole plane.

CASE $\lambda_3 = \frac{2}{3}$. The range of the operator A_{λ_3} is

$$(E_2 - \{[x, y] ; \; 2y^2 - y^2 = 0\}) \cup [0, 0],$$

i. e. it is not closed.

8. Linchtenstein integral equations.

Denote $\langle 0, 1 \rangle^n = \langle 0, 1 \rangle \times \ldots \times \langle 0, 1 \rangle$ (n-times). Let $K_n : \langle 0, 1 \rangle^{n+1} \to E_1$ ($n = 1, 2, \ldots$) be continuous and symmetric functions in all variables and denote

$$g_n(u) = \frac{1}{n+1} \int_0^1 \ldots \int_0^1 K_n(s, t_1, \ldots, t_n)\, u(s)\, u(t_1) \ldots u(t_n)\, ds\, dt_1 \ldots dt_n$$

for $u \in L_2 \langle 0, 1 \rangle$ (the real space of real measurable square integrable functions). We can extend g_n to $\tilde{g}_n$ where $\tilde{g}_n$ is a complex functional on the complex space $L_2 \langle 0, 1 \rangle + iL_2 \langle 0, 1 \rangle = \widetilde{L}_2 \langle 0, 1 \rangle$ defined by the following way:

$$\widetilde{g}_n(z) = \frac{1}{n+1} \int_0^1 \dots \int_0^1 K_n(s, t_1, \dots, t_n)\, z(s)\, z(t_1) \dots z(t_n)\, ds\, dt_1 \dots dt_n .$$

The functional g_n has at each point $z \in \widetilde{L}_2 \langle 0, 1 \rangle$ the Gâteaux derivative

$$\langle h, \widetilde{g}(z) \rangle = \int_0^1 \overline{h(s)} \left(\int_0^1 \dots \int_0^1 K_n(s, t_1, \dots, t_n)\, z(t_1) \dots z(t_n)\, dt_1 \dots dt_n \right) ds$$

and $\widetilde{g}_n$ is continuous (and thus locally bounded) on $\widetilde{L}_2 \langle 0, 1 \rangle$. Thus $\widetilde{g}_n$ is an analytic functional and g_n is a real-analytic functional. If we suppose

$$\sum_{n=1}^{\infty} \left(\int_0^1 \dots \int_0^1 [K_n(s, t_1, \dots, t_n)]^2\, ds\, dt_1 \dots dt_n \right)^{1/2} < \infty \tag{8.1}$$

then the series $\sum_{n=1}^{\infty} \widetilde{g}_n(z) = \widetilde{g}(z)$ is convergent and bounded in the unit ball $\widetilde{K}_1 = \{z \in \widetilde{L}_2 \langle 0, 1 \rangle ;\ \| z \| < 1\}$. Thus the functional $\widetilde{g}$ is analytic (see Proposition 2), the functional $g(u) = \sum_{n=1}^{\infty} g_n(u)$ is real-analytic in the unit ball $K_1 = \{u \in L_2 \langle 0, 1 \rangle ;\ \| u \| < 1\}$ and $g'(u) = \sum_{n=1}^{\infty} g'_n(u)$ is a strongly continuous mapping on $\overline{K}_r = \{u \in L_2 \langle 0, 1 \rangle ;\ \| u \| \leq r\}$ $(r < 1)$ (see [17, Theorem 21.2]).

The operator $g' : L_2 \langle 0, 1 \rangle \to L_2 \langle 0, 1 \rangle$ is called the Lichtenstein operator and the integral equation $\lambda u(s) = g'(u)$ is called the Lichtenstein integral equation.

Denote $f : u \mapsto \frac{1}{2} \langle u, u \rangle$. It is true that f is also a real analytic functional.

Assertion 1. Suppose
(i) $K_{2n} \equiv 0$ for each $n = 1, 2, \dots$,
(ii) $g_n(u) \geq 0$ for each $n = 1, 2, \dots$ and all $u \in K_1$.
Consider the eigenvalue problem

$$\begin{cases} \lambda u = g'(u) \\ f(u) = r. \end{cases} \tag{8.2}$$

Then the set of critical levels of this problem is at most countable and its only cumulation point can be zero. Suppose moreover that

(iii) there exists a positive integer n_0 such that $g_{2n_0+1}(u) > 0$ for each $u \in K_1$, $u \neq \theta$.

Then according to Assertion 1 (upper bound) and the result of E. S. Citlanadze [4] (lower bound) we have

ASSERTION 2. The set of critical levels of the Lichtenstein integral equation is a sequence of nonzero numbers which is convergent to zero.

The same result (under the same assumptions) can be obtained for the equation

$$\lambda \langle u, u \rangle^p u(s) = \sum_{n=1}^{\infty} \int_0^1 \dots \int_0^1 K_n(s, t_1, \dots, t_n) u(t_1) \dots u(t_n) \, dt_1 \dots dt_n \tag{8.3}$$

$$\| u \| = r \quad (r < 1),$$

where p is a positive integer.

REMARK. If we suppose instead of (8.1) the inequality

$$\sum_{n=1}^{\infty} n! \left(\int_0^1 \dots \int_0^1 [K_n(s, t_1, \dots, t_n)]^2 \, ds \, dt_1 \dots dt_n \right)^{1/2} < \infty \tag{8.4}$$

then we can solve (by means of our abstract main theorem) equations (8.2) and (8.3) for arbitrary $r \in (0, \infty)$.

9. Degenerated Lichtenstein integral equations.

First we consider the equation

$$\lambda u(s) = \int_0^1 \dots \int_0^1 K_{2n+1}(s, t_1, \dots, t_n) u(t_1) \dots u(t_{2n+1}) \, dt_1 \dots dt_{2n+1}, \tag{9.1}$$

where the function K_{2n+1} has the properties from Section 8. Let $g_{2n+1}(u) > 0$ for $u \neq \theta$. In view of Theorem 4 we have

ASSERTION 3. The eigenvalue problem

$$\begin{cases} \lambda u = g'_{2n+1}(u) \\ \| u \| = 1 \end{cases} \tag{9.2}$$

has an exactly countable set of positive eigenvalues, their cumulation point being zero.

Assertion 4. The eigenvalue problem

$$\left\{\begin{array}{l}\lambda \langle u, u \rangle^n u = g'_{2n+1}(u) \\ \| u \| = 1\end{array}\right. \tag{9.3}$$

has an exactly countable set of positive eigenvalues $\{\lambda_n\}$, their cumulation point being zero.

Moreover, the operator $A_\lambda u = \lambda \langle u, u \rangle^n u - g'_{2n+1}(u)$ for $\lambda \notin \{\lambda_n\} \cup \{0\}$ maps $L_2 \langle 0, 1 \rangle$ onto $L_2 \langle 0, 1 \rangle$.

10. Differential equations.

Let Ω be a fixed bounded domain in Euclidean N-space E_N with the boundary $\partial\Omega$. Denote by $\overset{\circ}{W}{}_2^{(m)}(\Omega)$ the well-known Sobolev space (for definition and properties see [14]).

We consider the weak solution of the equation

$$\left\{\begin{array}{l}\lambda(-1)^m \Delta^m u + h(u) = 0 \\ D^\alpha u = 0 \text{ on } \partial\Omega \text{ for } |\alpha| \leq m-1,\end{array}\right. \tag{10.1}$$

i. e., we seek a function $u \in \overset{\circ}{W}{}_2^{(m)}(\Omega)$ such that for each $v \in \overset{\circ}{W}{}_2^{(m)}(\Omega)$ the relation

$$\lambda \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx = \int_\Omega h(u(x))\, v(x)\, dx \tag{10.2}$$

holds.

We suppose that h is continuous on E_1, $h(0) = 0$, $h(t)\cdot t > 0$ for $t \neq 0$.

Case 1. Suppose that $2m < N$ and g is a polynomial of degree $k < \frac{N+2m}{N-2m}$. The functional $g: u \longmapsto \int_0^1 \int_\Omega h(tu(x))\, u(x)\, dt\, dx$ is a real-analytic weakly continuous functional, and g' is strongly continuous with respect to the compact imbedding $\overset{\circ}{W}{}_2^{(m)}(\Omega) \subset L_2(\Omega)$ with $\frac{1}{2} > \frac{1}{2} - \frac{m}{N}$. Analogously $f: u \longmapsto \frac{1}{2}\langle u, u \rangle$ in $\overset{\circ}{W}{}_2^{(m)}(\Omega)$ is a real-analytic functional. Then our abstract theorems may be applied.

CASE 2. Let $a(x) > 0$ be a continuous function on the real line. Suppose $2m > N$ and let $\widetilde{h}$ be an analytic function on $A = \{z\,;\, z = x + iy,$ $x \in E_1,\ |y| < a(x)\}$. Denote by h the restriction of $\widetilde{h}$ on E_1. The functional

$$g : u \longmapsto \int_0^1 \int_\Omega h(tu(x))\, u(x)\, dt\, dx$$

is real-analytic and weakly continuous and its derivative is strongly continuous with respect to compact imbedding $\overset{\circ}{W}{}_2^{(m)}(\Omega) \subset C(\bar{\Omega})$ and the abstract theorems may be applied again.

CASE 3. Suppose $2m = N$ and let $\widetilde{h}$ be an entire function of the type zero. Let h be the restriction of $\widetilde{h}$ on the real line. Functional g is real-analytic and weakly continuous with respect to the compact imbedding of $\overset{\circ}{W}{}_2^{(m)}(\Omega)$ into the space of John-Nirenberg and thus our abstract theorems may be applied to the eigenvalue problem (10.1). Namely, we have for $u \neq \theta$

$$\int_\Omega e^{\frac{\vartheta}{\|u\|_{2,m}} |u(x)|}\, dx \leq c_1, \quad \vartheta > 0.$$

Hence it follows that g is weakly continuous and g' is strongly continuous.

REFERENCES

[1] A. ALEXIEWICZ, W. ORLICZ : *Analytic operations in real Banach spaces,* Studia Math. XIV, 1954, 57-78.

[2] M. S. BERGER : *An eigenvalue problem for nonlinear elliptic partial differential equations,* Trans. Amer. Math. Soc., 120, 1965, 145-184.

[3] F. E. BROWDER : *Infinite dimensional manifolds and nonlinear elliptic eigenvalue problems,* Annals of Math. 82, 1965, 459-477.

[4] E. S. CITLANADZE : *Existence theorems for minimax points in Banach spaces* (in Russian) Trudy Mosk. Mat. Obšč. 2, 1953, 235-274.

[5] S. FUČÍK : *Fredholm alternative for nonlinear operators in Banach spaces and its applications to the differential and integral equations,* Čas. pro pěst. mat. 1971, No. 4.

[6] S. FUČÍK : *Note on the Fredholm alternative for nonlinear operators,* Comment. Math. Univ. Carolinae 12, 1971, 213-226.

[7] S. FUČÍK - J. NEČAS : *Ljusternik-Schnirelmann theorem and nonlinear eigenvalue problems,* Math. Nachr. 53, 1972, 277-289

[8] E. HILLE - R. S. PHILIPS : *Functional analysis and semigroups,* Providence 1957.

[9] M. A. KRASNOSELSKIJ : *Topological methods in the theory of nonlinear integral equations,* Pergamon Press, N. Y. 1964.

[10] M. KUČERA : *Fredholm alternative for nonlinear operators,* Comment. Math. Univ. Carolinae 11, 1970, 337-363.

[11] L. A. LJUSTERNIK : *On a class of nonlinear operators in Hilbert space* (in Russian) Izv. Ak. Nauk SSSR, ser. mat., No 5, 1939, 257-264.

[12] L. A. LJUSTERNIK - L. G. SCHNIRELMANN : *Application of topology to variational problems* (in Russian) Trudy 2. Vsesojuz. mat. sjezda 1, 1935, 224-237.

[13] L. A. LJUSTERNIK - L. G. SCHNIRELMANN : *Topological methods in variational problems and their application to the differential geometry of surface* (in Russian) Uspechi Mat. Nauk II, 1947, 166-217.

[14] J. NEČAS : *Les méthodes directes en théorie des équations elliptiques,* Academia, Praha 1967.

[15] J. NEČAS : *Sur l'alternative de Fredholm pour les operateurs non linéaires avec applications aux problèmes aux limites,* Ann. Scuola Norm. Sup. Pisa, XXIII, 1969, 331-345.

[16] J. SOUČEK - V. SOUČEK : *The Morse-Sard theorem for real-analytic functions,* Comment. Math. Univ. Carolinae, 13, 1972, 45-51.

[17] M. M. VAJNBERG : *Variational methods for the study of nonlinear operators,* Holden-Day, 1964.

[18] J. NEČAS: *On the discreteness of the spectrum of nonlinear Sturm-Liouville equation* (in Russian) Dokl. Akad. Nauk SSSR, 201, 1971, 1045-1048.

[19] A. KRATOCHVÍL - J. NEČAS: *On the discreteness of the spectrum of nonlinear Sturm-Liouville equation of the fourth order* (in Russian) Comment. Math. Univ. Carolinae, 12, 1971, 639-653.

[20] J. NEČAS: *Fredholm alternative for nonlinear operators and applications to partial differential equations and integral equations* (to appear).

[21] J. NEČAS: *Remark on the Fredholm alternative for nonlinear operators with application to integrat equations of generalized Hammerstein type* (to appear).