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The ergodic theory of positive operators on continuous functions

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THE ERGODIC THEORY OF POSITIVE OPERATORS ON CONTINUOUS FUNCTIONS

S. R. Foguel

Introduction.

We shall consider, in these notes, positive operators on the space of continuous and bounded functions over some topological space. Our main object is to establish results similar to results in ergodic theory. (In ergodic theory one studies positive contractions on L_∞ spaces whose adjoints act on L_1)) This necessitates the use of different tools, thus for instance, we were not able to find an analog to the Hopf Maximal Ergodic lemma, but did decompose the space into conservative and dissipative parts.

Most of the results appeared in various papers. We shall give references at the end of each chapter. One exception to this is the first chapter where some well known results in topology will be established. This was done for self completeness and to make it more accessible for readers whose main interest is in Probability Theory or Ergodic Theory.

I. Some Topology and Measure Theory.

(1) *Throughout these notes X will be a locally compact separable metric space.* Thus the space X satisfies :

(a) *X is normal.*

(b) *X is locally compact and σ compact* $\left(X = \bigcup_{n=1}^{\infty} X_n \text{ where } \{X_n\} \text{ is an increasing sequence of compact sets}\right)$.

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(c) *Every open set, A, is an F_σ set: $A = \bigcup_{n=1}^{\infty} B_n$ where B_n are closed sets.*

Note that $A = \cup (B_n \cup X_n)$ and the sets $B_n \cap X_n$ are compact.

If one defines the Borel sets to be the smalles σ field generated by open sets and the Baire sets to be the smallest σ field with regard to which every continuous function is meausurable, then (c) and the Urysohn lemma imply that every open set, and thus every Borel set, is a Baire set.

(2) *A non-negative function $f \geq 0$, will be called lower semi continuous (l. s. c.) if there exists an increasing sequence $\{f_n\}$, of continuous functions whose limit is f.*

If f is l. s. c. and α any real number then the set $\{x : f(x) > \alpha\}$ is open. Conversely, if for each α the set $\{x : f(x) > \alpha\}$ is open then f is l. s. c.:

First note that if A is an open set and 1_A denotes it characteristc function than 1_A is l. s. c.:

Put $A = \cup B_n$ where B_n are closed sets. Use Urysohn's lemma to find continuous functions g_n with $0 \leq g_n \leq 1$ $g_n(x) = 1$ if $x \in B_n$ and $g_n(x) = 0$ if $x \notin A$. Finally, $f_n = \max(g_1, \dots, g_n)$ is the required sequence. Now let f be a non-negative function such that $\{x : f(x) > \alpha\}$ is open for every α. For every rational r put $A_r = \{x : f(x) > r\}$ and an easy computation shows that $f = \sup_r r\, 1_{A_r}$. It is easy to show that the supremum of a countable collection of l. s. c. functions is again l. s. c..

(3) *Since the space X is nomal every continuous functional on $\mathbf{C}(X)$ (bounded continuous functions) is given by a finitely additive regular bounded measure:*

$$x^* f = \int f d\mu \qquad \|x^*\| = \text{total variation of } \mu.$$

See [1, IV. 6.2.].

Let us call a finitely additive measure μ such that $0 \leq \mu(A)$ for every Borel set A, *charge*. A charge that is countably additive will be called a *measure*. A charge, μ, is regular if and only if for every Borel set A

$$\mu(A) = \sup\{\mu(B) : B \subset A \text{ and } B \text{ is closed}\}.$$

If μ is a regular measure we can replace B by $B \cap X_n$ and thus

$$\mu(A) = \sup\{\mu(B) : B \subset A \text{ and } B \text{ is compact}\}.$$

(4) *A measure μ is always regular.*

See [3, II. 7.2]. This follows from the fact that every Borel set is a Baire set, and a measure on Baire sets is always regular.

A partial converse to this property is also valid.

A regular charge, μ, is countably additive on subsets of a compact set.
See [1, III. 5. 13].

(5) *Let μ be a charge then $\mu = \mu_0 + \mu_1$ where μ_0 is a measure and μ_1 is a pure charge namely: $0 \leq \lambda \leq \mu_1$ and λ is a measure implies $\lambda = 0$.*

See [2, page 52]. If μ is a regular charge then so are μ_0 and μ_1 since both are small whenever μ is small. Also:

a regular pure charge vanishes on every compact set:

The restriction of a regular charge to a compact set is a measure by (4) and thus must vanish by the definition of a pure charge.

(6) Throughout the paper P will be an operator on $\mathbf{C}(X)$ that satisfies:

(a) *If $f \geq 0$ then $Pf \geq 0$.*

(b) $P1 \leq 1$.

(c) *If μ is a measure so is $P^* \mu$.*

Note that P^* is defined on the space of regular charges. Condition (c) is necessary to avoid pathological examples such as Banach Limits.

Let δ_x be the unit measure at x. Then $P^* \delta_x$ is a measure denote

$$P^* \delta_x (A) = P(x, A) \; x \in X \; A \text{ a Borel set.}$$

THEOREM I.6.

(a) *The set function $P(x, \cdot)$ is a measure and $P(x, X) \leq 1$.*

(b) *The function $P(\cdot, A)$ is measurable for every Borel set A*

(c) *For every continuous bounded function f,*

$$Pf(x) = \int f(y) \, P(x, dy).$$

PROOF. (a) is obvious: $P^* \delta_x (X) = \delta_x (P1) \leq \delta_x (1) \leq 1$. Now if f is a bounded continuous function then

$$Pf(x) = P^* \delta_x (f) = \int f(y) \, P(x, dy)$$

Note: $\int f(y) \, P(x, dy)$ *extends $Pf(x)$ to every measurable non-negative function.*

Finally consider the class of functions, $0 \leq f$, such that $\int f(y) \, P(x, dy)$

is a measurable function of x. If $f \in \mathbf{C}(X)$ then Pf is continuous. Thus this class contains all continuous functions is additive and closed under monotone convergence. Therefore the class contains every Baire (Borel) measurable non-negative function. In particular $P(\cdot, A) = P1_A$ is measurable whenever A is a Borel set

(7) THEOREM 1.7. *If* $0 \leq f_n$ *are measurable functions and* $f_n \uparrow f$ *then* $Pf_n \uparrow Pf$. *In particular if* f *is l. s. c. then so is* Pf. *Thus if* A *is an open set then* $P1_A$ *is l. s. c.*: $\{x : P(x, A) > \alpha\}$ *is open.*

PROOF. The first part is just the Fatou's Lemma and the rest follows from the definition of l. s. c. functions and (2).

A converse to Theorem 1.6. is also valid: If P is an operator on $C(X)$ that satisfies 6 (a) and (b) and

(c′) If $f_n \downarrow 0$, $f_n \in \mathbf{C}(X)$, *then* $Pf_n(x) \to 0$ for every $x \in X$.

Then P satisfies (6). (c):

Let $P^* \mu = \mu_0 + \mu_1$ be the decomposition into a measure and a pure charge. Let g_n be a function with compact support such that $1_{X_n} \leq g_n \leq 1$. Put $f_n = \max(g_1, \ldots, g_n)$, then $f_n \uparrow 1$ and they have compact supports. Now $\mu_1(f_n) = 0$ by (5), thus

$$\mu_0(X) = \lim \mu_0(f_n) = \lim (P^* \mu)(f_n) = \lim \mu(Pf_n) = \mu(P1) = (P^* \mu)(X)$$

since μ is a measure and (c′). Thus $\mu_1(X) = 0$ and (c) holds.

(8) We will also consider semi groups of operators.

Let P_t *be a strongly continuous semi group of operators such that*:

(a) $P_0 = I$.

(b) *For every* $t > 0$ P_t *satisfies* 6. (a), (b) *and* (c).

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II. Decomposition into Conservative and Dissipative Parts.

(1) Let us define

$\mathfrak{A} = \{f : 0 \leq f \leq 1, f$ *is l. s. c.*, $Pf \leq f$ *and* $\lim P^n f(x) = 0$ *for every* $x \in X\}$.

(Note that $0 \in \mathfrak{A}$).

$$D_f = \{x : f(x) > 0\}$$

$$D = \cup \{D_f : f \in \mathfrak{A}\}.$$

The set D will be called the dissipative part. $C = X - D$.

The set C will be called the conservative part. Note that D is an open set.

LEMMA 1. *Let K be a compact subset of D. There exists a function $f \in \mathfrak{A}$ and a $\delta > 0$ such that $f \geq \delta 1_K$.*

PROOF. Since K is compact there exists n functions in $\mathfrak{A}$ such that

$$K \subset \bigcup_{i=1}^{n} D_{f_i} = D_{\frac{1}{n} \sum_{i=1}^{n} f_i}.$$

Put $f = \frac{1}{n} \sum_{i=1}^{n} f_i$ and observe that $K \subset \bigcup_{m=1}^{\infty} \left\{ x : f(x) > \frac{1}{m} \right\}$ and a finite union will suffice.

COROLLARY. *Let K be a compact subset of D then lim $P^n 1_K(x) = 0$ for every $x \in X$.*

PROOF. Using the notation of the above Lemma

$$P^n 1_K \leq \delta^{-1} P^n f \to 0.$$

COROLLARY. *Let μ be a measure and K a compact subset of D then* $\lim P^{*n} \mu(K) = 0$. *Thus if $\mu = P^* \mu$ then $\mu(D) = 0$.*

PROOF The first part follows from the first corollary and the Lebesgue Dominated Convergence Theorem. The second part follows from I. (3).

(2) THEOREM 2.

$$P 1_D \leq 1_D.$$

Proof. It is anough to prove that $P(x, K) = 0$ for every $x \in C$ and K a compact subset of D. Using Lemma 1

$$P 1_K(x) \leq \delta^{-1} Pf(x) = 0$$

since $x \notin D_f$.

This Theorem serves to define restriction of P to C:

Let f_1 and f_2 be in $\mathbf{C}(X)$ and $f_1 = f_2$ ou C. Then $|f_1 - f_2| \leq M 1_D$ for some constant M and by Theorem 2 $Pf_1 = Pf_2$ on C too. Thus, for every $f \in \mathbf{C}(C)$ choose any extension $\tilde{f}$ to all of X and define the restriction of P to C by restricting $P\tilde{f}$ to C. Let us note at this point that if Y is any closed subset of X satisfying $P 1_{Y'} \leq 1_{Y'}$ then the operator P is defined on $\mathbf{C}(Y)$. We shall call the restricted operator a subprocess. In [3] it is proved that for any Borel set A there exists a minimal closed subset containing A and defining a subprocess.

(3) *The minimal subvariant majorant.*
Let $0 \leq g \leq 1$ and define inductively

$$g_0 = g \qquad g_n = \max(g_0, Pg_{n-1}).$$

By induction $0 \leq g \leq g_n \leq 1$ and the sequence is monotone:

$$g_{n+1} = \max(g_0, Pg_n) \geq \max(g_0, Pg_{n-1}) \geq g_n .$$

Denote by g_∞ the limit of this sequence. Then $g \leq g_\infty \leq 1$ and $Pg_\infty \leq g_\infty$.

Theorem 3. *Let $0 \leq g \leq 1$ be a measnrable function. There exists a function g_∞, such that $g \leq g_\infty \leq 1$ $Pg_\infty \leq g_\infty$ and if h is any measurable function with $g \leq h$, $Ph \leq h$ then $g_\infty \leq h$. If g is l. s. c. then so is g_∞.*

Proof. Let us prove by induction that $g_n \leq h$:

$$g_{n+1} = \max(g, Pg_n) \leq \max(h, Ph) = h.$$

Now if g is *l. s. c.* then, again by an induction argument, so are g_n and also $g_\infty = \sup\, g_n$.

(4) Let us study g_∞ in the case where $g = 1_A$ and A is an open set. Instead of g_∞ we shall use the notation i_A. Thus i_A is the minimal function that satisfies $1_A \leq i_A$ and $Pi_A \leq i_A$.

Let $T_{A'}$ be the operator on bounded measurable functions given by

$$T_{A'} f = 1_{A'} f$$

Then

$$g_N = \sum_{n=0}^{N} (T_{A'} P)^n 1_A :$$

$$g_{N+1} = \max (1_A , Pg_N)$$

$$= 1_A + 1_{A'} P \sum_{n=0}^{N} (T_{A'} P)^n 1_A .$$

Thus

$$i_A = \sum_{n=0}^{\infty} (T_{A'} P)^n 1_A .$$

Note that if $x \in A'$ then

$$i_A (x) = P \sum_{n=1}^{\infty} (T_{A'} \; P)^{n-1} 1_A (x) = Pi_A (x).$$

THEOREM 4. *If* $P1 = 1$ *then*

$$i_A (x) = 1 - \lim (T_{A'} \; PT_{A'})^n 1).$$

PROOF.

$$i_A = \lim_{N\to\infty} \sum_{n=0}^{N} (T_{A'}P)^n 1_A = \lim_{N\to\infty} \sum_{n=0}^{N} (T_{A'}P)^n (1 - T_{A'} \; 1) =$$

$$= \lim_{N\to\infty} \left[\sum_{n=0}^{N} (T_{A'} \; P)^n 1 - \sum_{n=0}^{N} (T_{A'} P)^n T_{A'} P1 \right] = 1 - \lim_{N\to\infty} (T_{A'} P)^{N+1} 1.$$

Now

$$(T_{A'} P)^{N+1} 1 = (T_{A'} PT_{A'})^N 1.$$

(5) Let A be an open set and put $j_A (x) = \lim\limits_{n\to\infty} P^n i_A (x)$. The limit exists since $P^{n+1} i_A (x) \leq P^n i_A (x)$.

The open set A will be called *inessential* if $j_A (x) = 0$ for all $x \in A$. Note that j_A is not necessarily l. s. c.

THEOREM 5. *If* $j_A (x) = 0$ *for every* $x \in A$ *then* $j_A = 0$.

PROOF. Since $Pj_A = j_A$ the function $i_A - j_A$ is subinvariant and is non negative. If $j_A (x) = 0$ for $x \in A$ then $i_A - j_A \geq 1_A$ and by minimality of i_A, $i_A - j_A \geq i_A$. Hence $j_A = 0$.

If A is an inessential set then $i_A \in \mathfrak{A}$ and thus $A \subset D$. Thus $\bigcup \{A : A$ is inessiental$\} \subset D$. Conversely if $x \in D$ let $f \in \mathfrak{A}$ with $f(x) = \alpha > 0$. Put

$A = \left\{x : f(x) > \frac{\alpha}{2}\right\}$ then $f \geq \frac{\alpha}{2} 1_A$ and by minimality of i_A we have $f \geq \geq \frac{\alpha}{2} i_A$. Finally, since $f \in \mathfrak{A}$ A is inessential and $x \in A$. Thus $D = \cup \{A : A$ is inessential$\}$. Actually we can replace the union by a union of countably many inessential sets:

By I. (1) (c) $D = \cup K_n$ where K_n are compact set. Now a finite union of inessential sets is again inessential, thus there exist inessential sets A_n $K_n \subset A_n$ and thus $D = \cup A_n$.

(6) Consider the closed sets $E_{j,n} = \left\{x : P^j i_{A_n}(x) \leq \frac{1}{2}\right\}$ then $\bigcup_j E_{j,n} = X$. Now $E_{j,n} - \text{int}\, E_{j,n}$ is a closed nowhere dense set and so $\bigcup_j (E_{j,n} - \text{int}.E_{j,n}) = F_n$ is a set of the first category. Also $1_{A_n \cap \text{int}.E_{j,n}} \leq 2\,(i_{A_n} - P^j i_{A_n})$ hence

$$\sum_{k=0}^{\infty} P^k 1_{A_n \cap \text{int}.E_{j,n}} \leq 2 \sum_{k=0}^{\infty} (P^k i_{A_n} - P^{k+j} i_{A_n}) \leq 2j.$$

Let us summarize:

THEOREM 6. *There exists a sequeuce of open sets.* $B_n \subset D$, *such that* $\sum_{k=0}^{\infty} P^k 1_{B_n}$ *is a bounded function and* $D - \bigcup_{n=1}^{\infty} B_n$ *is a set of the first category.*

PROOF. Arrange the sets $A_n \cap \text{int}. E_{j,n}$ in a sequence B_k. Now

$$D = \bigcup_{j,n} (A_n \cap E_{j,n}) \subset \bigcup_{j,n} (A_n \cap \text{int}.\, E_{j,n}) \bigcup_{j,n} (E_{j,n} - \text{int}\, E_{j,n}) = \cup B_k \cup (\cup F_n).$$

Since F_n are of the first category so is $\cup F_n$.

(7) We considered in (2) the restriction of the operator to $\mathbf{C}(C)$ a natural question is whether this restriction has no D part.

THEOREM 7. *If* C *is the closure of its interior then the operator* P *restricted to* C *has no dissipative part.*

PROOF. It is enough to show that for every open set A (open the relative topology) $j_A(x)$ does not vanish for every $x \in C$ (see (5)). Now let $A = B \cap C$ where B is open in X. Since A is not empty $B \cap \text{int}.\ C$ is not empty by the assumption of the Theorem. Now if $j_A(x) = 0$ $x \in C$ then $j_{B \cap \text{int}.C}(x) = 0$ for all $x \in C$ and by Theorem (5) also for every $x \in X$. Thus $B \cap \text{int}.\ C \subset D$ which is a contradiction.

(8) *Semi Groups.*

Let us consider semi groups as defined in I.8. Define

$$\mathfrak{A} = \{f : 0 \leq f \leq 1, f \text{ is } l.s.c., \ P_t f \leq f \text{ for } 0 \leq t$$

$$\text{and } \lim_{t \to \infty} P_t f(x) = 0 \text{ for every } 0 \leq t\}$$

D and C are defined as in (1).

All the results of sections (1) and (2) generalise to this case without any change in the proofs. The definition of the minimal subinvariant majorant has to be modified:

Define $g_0 = g$ and $g_{n+1} = \sup\{P_t g_n : 0 \leq t\}$. It is easy to show that $g \leq g_n \leq 1$ and g_n is an increasing sequence whose limit g_∞ is the smallest function that satisfies $g \leq g_\infty$ and $P_t g_\infty \leq g_\infty$ for every $0 \leq t$. Note that if g is *l.s.c.* so is g_∞. We did not find an analog for the explicit representation of i_A described in Section (4). Section (5) and (7) remain unchanged. In order to obtain the results of section (6), few observations are needed.

Define $E_{j,n}$ as in section (6) where P^j means P_j and note that:

$$\left\| \int_0^T P_t (i_A - P_j\, i_A)\, dt \right\| = \left\| \int_0^j P_t\, i_A\, dt \right\| \leq j.$$

Thus Theorem 6 should be rephrased:

THEOREM 8. *There exists a sequeuce of open sets B_n such that $\left\| \int_0^\infty P_t 1_{B_n} dt \right\|$ is finite and $D - \bigcup_{n=1}^{\infty}$ is a set of the first category.*

REFERENCES

Nost of the resnlt in this chapter appear in [1]. Theorem 6 was proved in [2] and Theorem 7 was proved (using a different method) in [3]. In Section (3) we followed closely [4] and Section (4) is essentially well known.

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III. Conservative operators.

(1) The operator P is called conservative if $D = \emptyset$ ($\mathfrak{A} = \{0\}$). We do not know whether the restriction of the process to C is actually conservative. A partial answer is given in Theorem 3.1 and 3.2 of [5].

THEOREM 1. *Let P be a conservative operator. If $f \in \mathbf{C}(X)$ and $Pf \leq f$ then $Pf = f$, also for every number a $P1_{\{f>a\}} = 1_{\{f>a\}}$.*

PROOF. First we may replace f by $f + M$ for any constant M, so let us assume that $f \geq 0$. Put $f_1 = f - Pf$ then $f_1 \geq 0$ and is continuous. (*Note*: If we would assume that $f \geq Pf$ and f is continuous, but not bounded then we would only know that Pf is l. s. c.). Put $g = \sum_{n=0}^{\infty} P^n f_1 \leq f$ then g is l. s. c. and $g \geq Pg \geq \ldots \geq P^k g \to 0$.

Finally define $h = \min(g, 1)$, then $Ph \leq 1$ $Ph \leq Pg \leq g$ so $Ph \leq h$ and $P^k h \leq P^k g \to 0$. Since h is l. s. c. and $0 \leq h \leq 1$ $h \in \mathbf{A}$ so $h = 0$. Thus $f_1 = 0$ or $f = Pf$. Now $f - a = (f-a)^+ - (f-a)^- = P(f-a) = P[(f-a)^+] - P[(f-a)^-]$ where $(f-a)^+ = \max\{f-a, 0\}$, $(f-a)^- = -\min\{f-a, 0\}$. Thus $P[(f-a)^+] \geq (f-a)^+$ and since $(f-a)^+$ is continuous the first part applies and we must have equality. Now for every integer n $P(\min\{n(f-a)^+, 1\}) \leq \min\{n(f-a)^+, 1\}$ and again by the first part equality must hold. Also, as $n \to \infty$ $\min\{n(f-a)^+, 1\}$ increases to $1_{\{f>a\}}$ which proves the last part of the Theorem.

(2) LEMMA 2. *Let P be a conservative operator. Let $0 \leq f$ satisfy $Pf \leq f$. For every $\delta > 0$ the set $\{x : f(x) - \lim P^n f(x) \geq \delta\}$ does not contain any open set.*

PROOF. The limit in question exists since $P^n Pf \leq P^n f$. Now if A is an open set contained in $\{x : f(x) - \lim P^n f(x) \geq \delta\}$ then $1_A \leq \delta^{-1}(f - \lim P^n f)$, but the right hand side is non-negative and subinvariant and by the minimality of i_A we have $i_A \leq \delta^{-1}(f - \lim^n f)$ but then $P^k i_A \leq \delta^{-1}(P^k f - \lim P^n f) \to 0$. (Note that we used the fact that $\lim P^n f$ is *invariant*). Since the operator P is conservative $A = \emptyset$.

(3) Let us first note that if A_1 and A_2 are open sets such that $A_1 - A_2$ does not contain any open set then $A_1 - \bar{A}_2 = \emptyset$ and $A_1 - A_2 \subset \bar{A}_2 - A_2$ thus $A_1 - A_2$ is no where dense since $\bar{A}_2 - A_2$ is such: any open set contained in $\bar{A}_2$ necessarily intersects A_2 and so int. $\bar{A}_2 - A_2$ is empty.

THEOREM 3. *The following conditions are equivalent*:

(a) *P is conservative*

(b) *If* $0 \leq f$ *is l. s. c. and* $Pf \leq f$ *then the set* $\{x : Pf(x) < f(x)\}$ *is of the first category.*

(c) *If* $0 \leq f$ *is l. s. c. then the set* $\left\{x : 0 < \sum_{n-0}^{\infty} P^n f(x) < \infty\right\}$ *is of the first category.*

(d) *For every non empty open set* U *the set* $U \cap \left\{x : \sum_{n=0}^{\infty} P^n 1_U(x) < \infty\right\}$ *if the first category.*

(e) *There does not exist a non empty open set* U *such that* $\sum_{n=0}^{\infty} P^n 1_U$ *is uniformly bonnded.*

PROOF. (a) $\Longrightarrow$ (b): It is enough to show that for any two rational numbers $0 \leq b < a$ the set $\{x : Pf(x) \leq b < a < f(x)\}$ is nowhere dense, but this follows from Lemma 2 and the above remark. Note that the set

$$\{x : \lim P^n f(x) < f(x)\} = \bigcup_{n=0}^{\infty} \{x : P^{n+1} f(x) < P^n f(x)\}$$

is again of the first category.

(b) $\Longrightarrow$ (c): Denote $f_1 = \min\left\{\sum_{n=0}^{\infty} P^n f, 1\right\} = \lim_{N \to \infty} \min\left\{\sum_{n=0}^{N} P^n f, 1\right\}$. Now

$$P\left(\min\left\{\sum_{n=0}^{N} P^n f, 1\right\}\right) \leq \min\left\{\sum_{n=1}^{N+1}, P^n f, 1\right\} \leq \min\left\{\sum_{n=0}^{N+1} P^n f, 1\right\}$$

thus $Pf_1 \leq f_1$. Also $P^k f_1(x) \to 0$ whenever $\Sigma P^n f(x) < \infty$ hence

$$\left\{x : 0 < \sum_{n=0}^{\infty} P^n f(x) < \infty\right\} \subset \{x : f_1(x) > 0, \lim P^n f_1(x) = 0\}$$

and by (b) this is a set of the first category

(c) $\Longrightarrow$ (d) obvious

(d) $\Longrightarrow$ (e) obvious

(e) $\Longrightarrow$ (a) by Theorem 6 of Chapter II.

(4) THEOREM 4. *The following conditions are equivalent*:

(a) *P is conservative*

(b) P^k *is conservative for every* k.

(c) P^k *is conservative for some* k.

PROOF. (a) ==> (b): Let $0 \leq f$ be l. s. c. and

$$0 \leq f - P^k f = (I - P)(I + Pf + \dots + P^{k-1} f)$$

since $I + Pf + \dots + P^{k-1} f$ is again l. s. c. and P is conservative the right hand side is zero except for a set of the first category and thus P^k is conservative by (b) of Theorem 3.

(b) ==> (c): obvious

(c) ==> (*a*): Note that for every open set U

$$\sum_{n=0}^{\infty} P^{nk} 1_U \leq \sum_{n=0}^{\infty} P^n 1_U \text{ and use (e) of Theorem 3.}$$

(5) Let A be an open set than i_A is l. s. c and $Pi_A \leq i_A$. Thus $j_A = \lim P^n i_A = i_A$ except for a set of the first category.

THEOREM 5. *Let P be a conservative operator. If A is an open set then the set $\{x : 0 < i_A(x) < 1\}$ is of the first category.*

PROOF. It is enough to prove that for every $\delta > 0, \varepsilon > 0$ the set $\{x : \delta < i_A(x) \leq 1 - \varepsilon\}$ is nowhere dense. Now this set is equal to the difference $\{x : i_A(x) > \delta\} - \{x : i_A(x) > i - \varepsilon\}$ and as in Section (3) it is enough to show that no open set is contained in the difference of these two open sets. Let B and open set and

$$B \subset \{x : \delta < i_A(x) \leq 1 - \varepsilon\} \subset \{x : \delta < i_A(x)\} \cap \{x : j_A(x) \leq 1 - \varepsilon\}$$

hence

$$1_B \leq \min \{\delta^{-1} i_A, \varepsilon^{-1}(1 - j_A)\}.$$

Now both functions are subinvariant as $Pj_A = j_A$ thus $i_B \leq \min \{\delta^{-1} i_A, \varepsilon^{-1} (1 - j_A)\}$. Let us show that $i_B(x) = 0$ for $x \in A$: if at some $x_0 \in A$ $i_B(x_0) > 0$ then $i_B(y) > 0$ in a neighborhood of x_0 and $1 - j_A(y) \geq \varepsilon i_B(y) > 0$ but $j_A(x) = i_A(x) = 1$ on a dense subset of A (see [4] Theorem 34, p. 200), which is a contradiction. Consider the function $i_A - \delta_{j_B}$: firstly, $i_A - \delta_{j_B} \geq i_A - \delta i_B \geq 0$.Also if $x \in A$ then $i_A(x) = 1$ and $j_B(x) \leq i_B(x) = 0$. Therefore $i_A - \delta j_B \geq 1_A$ and by minimality of i_A $i_A - \delta j_B \geq i_A$. Thus $j_B = 0$ and since P is conservative $B = \emptyset$. Since $i_A(x)$ can assume the values 0 and 1 only outside of a set of the first category, the same applies to $P^k i_A$ and to j_A.

Now if $j_A(x) = \alpha > 0$ then

$$\sum_{n=k}^{\infty} P^n 1_A(x) \geq P^k \sum_{n=0}^{\infty} (T_{A'} P)^n 1_A(x) = P^k i_A(x) \geq \alpha$$

and

$$\sum_{n=0}^{\infty} P^n 1_A(x) = \infty.$$

If $i_A(x) = 0$ then

$$P^n 1_A(x) \leq P^n i_A(x) \leq i_A(x) = 0.$$

To summarize:
If $j_A(x) \neq 0$ then $\Sigma P^n 1_A(x) = \infty$ if $i_A(x) = 0$ then $\Sigma P^n 1_A(x) = 0$. The set $\{x : j_A(x) = 0 < i_A(x)\}$ is of the first category.

(6) Let us generalise the last statement of Theorem 1.

THEOREM 6. *Let P be a conservative operator and $0 \leq f$ a l. s. c. function with $Pf \leq f$. For every number $a \geq 0$ put $A = \{x : f(x) > a\}$ then: $\{x : i_A(x) \neq 1_A(x)\}$ is of the first category and if $B = \{x : i_A(x) > 0\}$ then $P1_B \leq 1_B$ and $B - A$ is a set of the first category.*

PROOF. Let $A_n = \{x : f(x) > a_n\}$ where a_n is a decreasing sequence whose limit is a. Then $A_n \subset A$ and $\bigcup A_n = A$. Now if $g_n = a_n^{-1} \min\{f, a_n\}$ then $g_n(x) = 1$ if $x \in A_n$ and $g_n(x) < 1$ if $x \notin A$. Also, $Pg_n \leq g_n$ since i_{A_n} assumes the values 0 and 1 only, outside of a set of the first category then except for a set of the first category $i_{A_n}(x) = 0$ if $x \notin A$. Finally $i_A = \lim i_{A_n} : i_A \geq i_{A_n}$ but $i_{A_n} \geq 1_{A_n}$ and the limit is subinvariant too. Thus $i_A = 1_A$ except for a set of the first category. Now put $B = \{x : i_A(x) > 0\}$ then $B - A$ is of the first category and $1_B = \lim n \min\left\{i_A, \frac{1}{n}\right\}$. The sequence is monotone and each function is subinvariant and so must be 1_B.

(7) *An example.* Let $\Phi(x)$ be a continuous map of X into itself and put $Pf(x) = f(\Phi(x))$. Then $P1 = 1$, hence by II. (4)

$$i_A(x) = 1 - \lim (T_{A'} P T_{A'})^n 1(x)$$

and it assumes the value 0 or 1 only. Put $i_A = 1_{\widetilde{A}}$ then $A \subset \widetilde{A}$ and $\Phi^{-1}(\widetilde{A}) \subset \widetilde{A}$. Finally $j_A = 1_{A^*}$ where $A^* = \bigcap_{n=1}^{\infty} \Phi^{-n}(\widetilde{A})$ and $\Phi^{-1}(A^*) = A^*$. Now if $x \in A^*$ then $j_A(x) = 1$ and by (5) $\Sigma P^n 1_A(x) = \infty$. If $x \notin A^*$ then for some integer

k $P^k i_A(x) = 0$ $(x \notin \Phi^{-k} \tilde{A})$) hence $\sum_{n=0}^{\infty} P^{n+k} 1_A(x) \leq \sum_{n=0}^{\infty} P^{n+k} i_A(x) = 0$ or

$$\sum_{n=0}^{\infty} P^n 1_A(x) = \sum_{n=0}^{k-1} P^n 1_A(x) < \infty.$$

We shall construct now a transformation Φ and a set A such that $\Phi^{-1}(A) \subset A$ (hence $A = \tilde{A}$) and $A - \Phi^{-1}(A)$ has the same cardinality as the continuum. This shows that Theorem 3 and Theorem 4 can not be improved: if $x \in A - \Phi^{-1}(A)$ then $\sum_{n=0}^{\infty} P^n 1_A(x) = 1$.

Put $X = [0, 1)$ and $\Phi(x) = \{3x\}$ ($3x$ minus the integral part of $3x$). One can describe X to be the unit circle and the transformation sends z to z^3 ($|z| = 1$). There exists an invariant measure that does not vanish on any open set (the Lebesgue measure) and thus the process is conservative. See II. (1). If we put $x = \cdot a_1 a_2 a_3 \ldots$ $\left(x = \Sigma \frac{a_i}{3^i}\ a_i = 0, 1, 2\right)$ then $\Phi(x) = \cdot a_2 a_3 \ldots$. Let B the Cantor set: those x' s such that $a_i \neq 1$ $i = 1, 2, 3, \ldots$ then $\Phi^{-1}(B) \supset B$ and $\Phi^{-1}(B) - B$ contains every fraction of the form $\cdot 1 a_2 a_3 \ldots a_i \neq 1$ $i \geq 2$. Thus $A = X - B$ is an open subinvarient set such that $A - \Phi^{-1}(A)$ has the continuum cardinality.

(8) *Semi groups.* Let P_t be a conservative semi group: $D = \Phi$. If for some $\tau > 0$ $P_\tau f \leq f$ where $0 \leq f \in \mathbf{C}(X)$

$$\int_0^T P_t(f - P_\tau f)(x)\, dt \leq \int_0^\tau P_t f(x)\, dt \leq \tau \|f\|.$$

Thus

$$g = \int_0^\infty P_t(f - P_\tau f)\, dt$$

is a bounded l. s. c. function and $g \geq P_s\, g \underset{s \to \infty}{\longrightarrow} 0$. Thus $\min(g, 1) \in \mathfrak{A}$ and so $g = 0$ or $P_\tau f = f$. Now if $P_\tau f \leq f$ for every $\tau \geq 0$ then equality holds and Theorem 1 follows with no change in the proof. In Lemma 2 one has to assume that $P_t f \leq f$ for all $t \geq 0$. The proof is again the same. Checking the proof of Theorem 3, one notes that if $0 \leq f$ is l. s. c. and $P_t f \leq f$ for $t \geq 0$ then for every t

$$\{x : P_t f(x) \leq b < a < f(x)\} \subset \{x : f(x) - \lim P_t f(x) \geq a - b\}$$

and the right hand side does not contain any open set. Thus these set are monotone in t hence

$$\bigcup_{t>0} \{x : P_t f(x) \leq b < a < f(x)\} = \bigcup_{n=1}^{\infty} \{x : P_n f(x) \leq b < a < f(x)\}.$$

is a set of the first category and so is

$$\bigcup_{t>0} \{x : P_t f(x) < f(x)\}.$$

Condition (c), (d), and (e) of Theorem 3 should be modified by replacing sums by integrals Theorems 5 and 6 are easily generalized too. Finally, the analog of Theorem 4 is

THEOREM 7. *The following conditions are equivalent:*

(a) $\{P_t\}$ *is conservative.*

(b) *For every* $\tau > 0$ *the operator* P_τ *is conservative.*

(c) *For some* $\tau > 0$ *the operator* P_τ *is conservative.*

PROOF. (a) $\Longrightarrow$ (b). Assume that $0 \leq f \leq 1$ is l. s. c. and $P_\tau f \leq f$. Define $g = \int_0^\tau P_t f dt$, which is again l. s. c. Now, if $0 < r < \tau$ then $(I - P_r) g =$

$$= (I - P_r) \int_0^\tau P_t f dt = \int_0^\tau P_t f dt - \int_r^{r+\tau} P_t f dt = \int_0^r P_t f dt - \int_\tau^{r+\tau} P_t f dt = \int_0^r P_t (I -$$

$- P_\tau) f dt \geq 0$ since $P_\tau f \leq f$. Now if $r \geq \tau$ then $\frac{r}{N} < \tau$ for some N so $P_{r/N} g \leq g$. Hence, $P_r g = (P_{r/N})^N g \leq g$ too. Thus, $P_r g \leq g$ for every r and, since $\{P_t\}$ is conservative, $\bigcup_{r>0} \{x : P_r g(x) < g(x)\}$ if of first category. But then $\{x : P_\tau f(x) < f(x)\}$ is also of the first category: If U is a nonempty open set contained in $\{P_\tau f(x) \leq b < a < f(x)\}$ then $f - P_\tau f \geq (a - b) 1_U$ and

$$(a - b) \int_0^r P_t 1_U(x) dt \leq \int_0^r P_t (f - P_\tau f)(x) dt = (I - P_r) g(x), \; r \leq \tau,$$

so $\int_0^\tau P_t 1_U(x) dt = 0$ except on a set of the first category which contradicts the continuity of P_t. Therefore, P_τ is conservative.

(b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a). If $\{P_t\}$ is not conservative, then there exists a non-empty open set U such that $\int_0^\infty P_t 1_U \, dt \leq M$ (analog of Theorem 3). Define $g = \int_0^\tau P_t \, 1_U \, dt$ which is l. s. c. and $\sum_{k=0}^{N-1} P_\tau^k g = \sum_{k=0}^{N-1} P_{\tau k} \, g = \int_0^{N\tau} P_t \, 1_U \, dt \leq M$. Since $U \neq \Phi$ the function $g \not\equiv 0$ and P_τ is not conservative.

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We followed [2] throughout most of this chapter. Theorems 3 and 4 were first proved in [3] by a different method. Theorem 5 appeared in [1] with a different proof Theorem 6 was proved in [3]. The example was constructed by S. Horowitz and the author. Theorem 4 appeared in [5] and Theorem 7 is due to M. Lin.

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IV. Existence of Invariant Measures.

(1) Let μ be an invariat charge: $P^* \mu = \mu$. Let $\mu = \mu_0 + \mu_1$ be the decomposition into a measure, μ_0, and a pure charge, μ_1, see I (5). Now $\mu = P^* \mu = P^* \mu_0 + P^* \mu_1$ and $P^* \mu_0$ is a measure while $P^* \mu_1$ may be decomposed again into a measure plus a pure charge. Thus $P^* \mu_0 \leq \mu_0$ and $P^* \mu_1 \geq \mu_1$. Therefore $P^* \mu_1 (X) \geq \mu (X)$ but equality must hold since $P^* \mu_1 (X) = \langle \mu_1, P1 \rangle \leq \langle \mu_1, 1 \rangle = \mu_1 (X)$. Hence $P^* \mu_1 (A) = \mu_1 (A)$ for every Borel set A.

THEOREM 1. *If μ is an invariant charge and $\mu = \mu_0 + \mu_1$ where μ_0 is a measure and μ_1 a pure charge, then both are invariant too.*

(2) Let A be a compact set. Define

$$A_{N, \delta} = \left\{ x : \frac{1}{N} \sum_{n=1}^{N} P^n 1_A (x) > \delta \right\}.$$

There are two exclusive possibilities : either

(a) $$\left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| \xrightarrow[N \to \infty]{} 0$$

or

(b) There exists a $\delta > 0$ and an infinite sequence N_i such that $A_{N_i, \delta} \neq \Phi$.

Let us consider the case (b):

Choose $x_i \in A_{N_i, \delta}$ and put $\lambda_i = \frac{1}{N_i} \sum_{n=1}^{N_i} P^{*n} \delta_{x_i}$. Then λ_i are measures $\lambda_i (X) \leq 1$ and

$$\lambda_i (A) = \left\langle \delta_{x_i}, \frac{1}{N_i} \sum_{n=1}^{N_i} P^n 1_A \right\rangle \geq \delta.$$

Let λ be any weak $*$ limit of the sequence (considered as functionals over $C(X)$). Then if f is any continuous functions one can find an index i such that $| \langle \lambda, f \rangle - \langle \lambda_i, f \rangle | < \varepsilon$, $| \langle \lambda, Pf \rangle - \langle \lambda_i, Pf \rangle | < \varepsilon$ and also $\langle \lambda_i, f \rangle - - \langle \lambda_i, Pf \rangle | \leq \| \lambda_i - \lambda_i \| \leq \frac{2}{N_i}$. Hence λ is an invariant charge. Put $\lambda = = \lambda_0 + \lambda_1$ as in (1) then λ_0 in an invariant measure. Finally if f is any continuous function with compact support and $f \geq 1_A$ then

$$\langle \lambda_0, f \rangle = \langle \lambda, f \rangle \geq \delta \text{ since } \langle \lambda_1, f \rangle = 0.$$

Now λ_0 is a regular measure hence $\lambda_0(A) \geq \delta$. Let us summarize:

THEOREM 2. *Let A be a compact set. Either* $\left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| \to 0$ *or there exists an invariant measure* λ, with $\lambda(A) > 0$.

COROLLARY. *Let A be a compact set. The following are equivalent:*

(a) $\liminf \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \not\equiv 0$.

(b) $\limsup \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \not\equiv 0$.

(c) $\limsup \left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| \neq 0$.

(d) *There exists an invariant measure μ with* $\mu(A) > 0$.

(e) *There exists a measure τ, with* $\liminf P^{*n} \tau(A) > 0$.

(f) *There exists a measure τ, with* $\liminf \frac{1}{N} \sum_{n=1}^{N} P^{*n} \tau(A) > 0$.

(g) *There exists a measure τ, with* $\limsup \frac{1}{N} \sum_{n=1}^{N} P^{*n} \tau(A) > 0$.

PROOF. It is clear that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Now (c) $\Longrightarrow$ (d) by the Theorem. Also (d) $\Longrightarrow$ (a) by the Ergodic Theorem since the sequence $\frac{1}{N} \sum_{n=1}^{N} P^n 1_A$ converges almost everywhere, with respect to μ, to a limit g such that $\int g d\mu = \int 1_A \, d\mu = \mu(A) \neq 0$ so $g \not\equiv 0$. Now clearly (d) $\Longrightarrow$ (e) (choose $\tau = \mu$) and (e) $\Longrightarrow$ (f) $\Longrightarrow$ (g). Finally, if (g) holds then so does (c): otherwise

$$\frac{1}{N} \sum_{n=1}^{N} P^{*n} \tau(A) = \left\langle \tau, \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\rangle \leq \| \tau \| \left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1 \right\| \to 0.$$

(3) Define

$$\mathfrak{A}_1 = \left\{ A : A \text{ is an open set and } \left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| \to 0 \right\}.$$

Clearly $\mathfrak{A}_1$ is closed under finite unions. Put

$$A_0 = \bigcup \{ A : A \in \mathfrak{A}_1 \}.$$

THEOREM 3. *every invariant measure vanishes on A_0. There exists an invariant measure that does not vanish on any open set that intersects $X - A_0$.*

PROOF. Let μ be an invariant measure and K a compact subset of A_0. Then $K \subset A$ for some $A \in \mathfrak{A}_1$ hence

$$\left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_K \right\| \leq \left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| \to 0 \text{ and}$$

$$\mu(K) = \left\langle \mu, \frac{1}{N} \sum_{n=1}^{N} P^n 1_K \right\rangle = 0.$$

Let U be any open set such that $U \cap A_0' \neq \Phi$. Find an open set, V, with compact closure such that $V \subset \overline{V} \subset U$ and $V \cap A_0' \neq \Phi$. By Theorem 2 there exists an invariant measure μ such that $\mu(\overline{V}) \neq 0$ (otherwise $V \subset A_0$). Hence $\mu(U) > 0$ too. Let U_n be a basis for the neighborhoods of A_0'. For each U_n define μ_n to be the invariant measure with $\mu_n(U_n) > 0$. Then $\Sigma 2^{-n} \mu_n$ is an invariant measure that does not vanish on any open set that intersects A_0'.

(4) If $A \in \mathfrak{A}_1$ then the sets $\{x : P^k 1_A(x) > \varepsilon\} = A_{k,\varepsilon}$ are open sets too and $1_{A_{k,\varepsilon}} \leq \varepsilon^{-1} P^k 1_A$. Thus

$$\left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_{A_{k,\varepsilon}} \right\| \leq \varepsilon^{-1} \left\| \frac{1}{N} \sum_{n=1}^{N+k} P^n 1_A \right\| \to 0.$$

Hence $A_{k,\varepsilon} \in \mathfrak{A}_1$ too and thus $P^k 1_A(x) = 0$ if $x \notin A_0$. Since A_0 is a countable union of sets in $\mathfrak{A}_1$ $P^k 1_{A_0}(x) = 0$ if $x \notin A_0$ too or $P 1_{A_0} \leq 1_{A_0}$.

This last inequality implies that the process P can be restricted to A_0'. Note $D \subset A_0$.

(5) *Semi groups.* Theorem 2 is valid for semi groups too: one has to replace $A_{N,\delta}$ by $A_{\tau,\delta} = \left\{x : \frac{1}{\tau} \int_0^{\tau} P_t 1_A(x)\, dt > \delta\right\}$ and if $A_{\tau_i,\delta} \neq \emptyset$ for some $\tau_i \to \infty$ then put $\lambda_i = \frac{1}{\tau_i} \int_0^{\tau_i} P_t \delta_{x_i}\, dt$ where $x_i \in A_{\tau_i,\delta}$. The integral exists at least in the weak sense. Thus $\lambda_i(A) \geq \delta$ and any weak limit of λ_i is an invariant charge that does not vanish on A. The rest of the proof is identical.

Theorem 3 and 4 follow in the same way : we have to replace sums by integrals.

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V. The existence of an invariant σ-finite measure.

(1) Throughout this chapter we shall assume :

(*a*) *There exists a continuous non-negative function g, with compact support such that*

$$\sum_{n=0}^{\infty} P^n g(x) = \infty \text{ for every } x \in X.$$

Let us denote the continuous functions with compact support by $C_0(X)$. Let A be any compact set. Since $A \subset \bigcup_{N=0}^{\infty} \left\{x : \sum_{n=0}^{N} P^n g(x) > 1\right\}$ then a finite union already covers A or $1_A \leq \sum_{n=0}^{N} p^n g$. Thus

LEMMA 1. *Assume condition (a). If* $0 \leq f \in C_0(X)$ *then* $f \leq \|f\| \sum_{n=1}^{N} P^n g$ *for some integer N where N depends only on the support of f.*

(2) Let f be as in Lemma 1; then

$$\|f\|^{-1} \sum_{j-0}^{K} P^j f \leq \sum_{j=0}^{K} \sum_{n=0}^{N} P^{n+j} g = \sum_{n=0}^{N} \sum_{j=0}^{K} P^{n+j} g \leq$$

$$(N+1) \sum_{j=0}^{K} P^j g + [P^{K+1} g + (P^{K+1} g + P^{K+2} g) + \dots + (P^{K+1} g + \dots + P^{K+N} g)]$$

$$\leq (N+1) \sum_{j=0}^{K} P^j g + N^2 \|g\|.$$

Given any compact set A. Then $\sum_{j=0}^{K} P^j g \to \infty$ uniformly on A. Consider the set of measures m such that $m(A) \geq \delta > 0$ for a fixed $\delta > 0$. Now, $\sum_{j=0}^{K} \langle m, P^j g \rangle \to \infty$ uniformly for all such measures. Thus, by Lemma 1:

LEMMA 2. *Assume condition (a) and let* $m_K(A) \geq \delta$ *for a fixed compact set A and a fixed constant* $\delta > 0$. *Then*

$$\limsup \frac{\sum_{j=0}^{K} \langle m_K, P^j f \rangle}{\sum_{j=0}^{K} \langle m_K, P^j g \rangle} \leq (N+1) \|f\|$$

where N depends on the support of f only.

(3) Let A be a fixed compact set $\delta > 0$ a constant m_K a sequence of measures such that $m_K(A) \geq \delta$. Choose a subsequence N_j of the integers and define.

$$\nu(f) = \underset{j \to \infty}{\mathrm{LIM}} \frac{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n f \rangle}{\sum_{n=0}^{N_j} \langle m_{N_j}, p^n g \rangle}$$

where LIM denote a Banach limit. By Lemma 2 ν is well defined. Since ν is a positive functional on $C_0(X)$ there exists a set function μ such that

1. μ is a finite measure (non-negative) on every compact set.

2. If $f \in C_0(X)$ then $\langle \nu, f \rangle = \int f d\mu$.

3. $\int g d\mu = 1$.

A set function satisfying 1 will be called a Boral measure. Note that μ is a σ-finite measure. The existence of such a representation is proved in [2, p. 247].

(Let $X = UX_n$ where X_n are open sets and $\bar{X}_n$ are compact sets. The functional ν restricted io $\bar{X}_n$ is clearly a measure and μ is the limit of these restrictions).

Let $0 \leq f \in C_0(X)$. Then $0 \leq Pf \in C(X)$. Choose a sequence $0 \leq f_n \in$ $\in C_0(X)$ $f_n \leq Pf$ then

$$\frac{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n f_r \rangle}{\sum_{n-0}^{N_j} \langle m_{N_j}, P^n g \rangle} \leq \frac{\sum_{n=0}^{N_j} \langle m_{N_j}, P^{n+1} f \rangle}{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n g \rangle} \leq$$

$$\leq \frac{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n f \rangle}{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n g \rangle} + \frac{\|f\|}{\sum_{n=0}^{N_j} \langle m_{N_j}, P^n g \rangle}$$

and as $N_j \to \infty$ we obtain $\langle \nu, f_r \rangle \leq \langle \nu, f \rangle$ for every integer r.

Thus, $\int f_r \, d\mu \leq \int f \, d\mu$. Therefore, $\int Pf \, d\mu \leq \int f d\mu$ for every $0 \leq f \in C_0(X)$. Now

$$\left\langle \mu - P^* \mu, \sum_{n=0}^{N} P^n g \right\rangle = \langle \mu, g \rangle - \langle \mu, P^{N+1} g \rangle \leq \langle \mu, g \rangle < \infty$$

and since $\sum_{n=0}^{\infty} P^n g \equiv \infty$ we must $\mu = P^* \mu$.

THEOREM 3. *Assume condition (a). The there exists an invariant Borel measure.*

(4) Let again the sequence m_K satisfy $m_K(A) \geq \delta$. If there exists a unique invariant Borel measure μ with $\int g d\mu = 1$ then the limit in Lemma 2 does not depend on the subsequence N_j, thus

$$\lim_{N \to \infty} \frac{\sum_{n=0}^{N} \langle m_N, P^n f \rangle}{\sum_{n=0}^{N} \langle m_N, P^n g \rangle} = \int f \, d\mu.$$

Otherwise, we could find a subsequence that converges to $\alpha \neq \int f \, d\mu$ and repeating the above argument we would find a second invariant Borel measure. Let us summarize.

THEOREM 4. *Assume condition (a), and the invariant Borel measure is unique. Let m_N be a sequence of measures such that $m_N(A) \geq \delta$ for a fixed compact set A and a constant $\delta > 0$. Then*

$$\lim_{N \to \infty} \frac{\sum_{n=0}^{N} \langle m_N, P^n f \rangle}{\sum_{n=0}^{N} \langle m_N, P^n g \rangle} = \int f \, d\mu, \quad f \in C_0(X).$$

Moreover

$$\frac{\sum_{n=0}^{N} \langle m, P^n f \rangle}{\sum_{n=0}^{N} \langle m, P^n g \rangle}$$

converges uniformly for the collection of measures m with $m(A) \geq \delta$.

Proof. Only the last part should be proved. If the convergence is not uniform we could find a sequence N_j measures m_j such that $m_j(A) \geq \delta$ but

$$\left| \frac{\sum_{n=0}^{N_j} \langle m_j, P^n f \rangle}{\sum_{n=0}^{N_j} \langle m_j, P^n g \rangle} - \int f \, d\mu \right| \geq \varepsilon$$

for some fixed $\varepsilon > 0$. Use the first part of the theorem to obtain a contradiction.

In particular, $\dfrac{\sum_{n=0}^{N} P^n f(x)}{\sum_{n=0}^{N} P^n g(x)}$ cenverges to $\int f \, d\mu$ and the convergence is uniform over compact sets.

(5) Let us consider the Assumption of uniqueness introduced in Theorem 4. If there is no finite invariant measure (namely, either no invariant Borel measure, or the unique invariant Borel measure is only σ-finite) then by Theorem 2 of IV $\left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_K \right\| \xrightarrow[N \to \infty]{} 0$ for every compact set K.

If there is a unique Borel measure λ such that $\lambda(X) = 1$ then whenever $0 \leq f, g \in C_0(X)$ then $h = f - \dfrac{\int f \, d\lambda}{\int g \, d\lambda} g \in \overline{(I - P) C(X)}$: If $\langle \mu, (I - P) \cdot C(X) \rangle = 0$ then $\mu = a\lambda + \mu_1$ where μ_1 is a pure charge, so $\langle \mu, h \rangle = 0$ since h has compact support and $\langle \lambda, h \rangle = 0$. Thus, the Hahn Banach Theorem applies. A standard argument shows that in this case

$$\left\| \frac{1}{N} \sum_{n=1}^{N} P^n f \left(f - \frac{\int f \, d\lambda}{\int g \, d\lambda} g \right) \right\| \xrightarrow[N \to \infty]{} 0.$$

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VI. The strong ratio limit theorem.

(1) Throughout this chapter we shall assume

(a) $P1 = 1$.

(b) There exists a set A with compact closure such that $i_A \equiv 1$.

By Theorem 1 of Chapter II, condition (b) is equivalent (provided (a) is assumed) to:

(b') $\lim_{n \to \infty} (T_{A'}\ PT_{A'})^n\ 1 = 0$.

Also, if $A_1 \supset A$ then $i_{A_1} \geq i_A$. Thus, we may replace A, with no loss of generality, by any bigger set. It will be convenient to assume that the set A is an open set.

Let α be a continuous function with compact support such that $1_A \leq \alpha \leq 1$ put $\beta = 1 - \alpha$. Now $\Sigma P^n 1_A \equiv \infty$ by the remarks at the end of Section 5 of Chapter III. Thus $\Sigma P^n \alpha \equiv \infty$ and by the previous Chapter there exists a Borel invariant measure μ with $\langle \mu, \alpha \rangle = 1$ (clearly, $\mu(A) > 0$).

Theorem 1. *Assume conditions* (a) *and* (b). *If* α *is a continuous function with compact support and* $1_A \leq \alpha \leq 1$ *then there exists a Borel measure* μ, *such that* $\langle \mu, Pf \rangle = \langle \mu, f \rangle$ *for every* $0 \leq f$, *and* $\langle \mu, \alpha \rangle = 1$.

(2) For any continuous function h define $T_h f = hf$ which is again a Markov operator on $\mathbf{C}(X)$.

If μ_n are σ-finite measures with $0 \leq \mu_n \leq \mu_{n+1} \leq \mu$ then their limit ν is again a measure: Let $\bigcup_{k=k}^{\infty} A_k = A$ if $\nu(A) < \infty$ then the measures μ_n restricted to A are finite and $\nu\left(\bigcup_{k=1}^{N} A_k\right) \to \nu(A)$ since the limit of a bounded sequence of finite measure is a finite measure. If $\nu(A) = \infty$ then for every M, $\mu_j(A) > M$ for j large enough. Thus, $\nu\left(\bigcup_{k=1}^{N} A_k\right) \geq \mu_j\left(\bigcup_{k=1}^{N} A_k\right) > M$ if N is chosen correctly. Therefore $\nu\left(\bigcup_{k=1}^{N} A_k\right) \xrightarrow[N \to \infty]{} \infty$.

Put now $\langle \eta_k, f \rangle = \langle \mu, (PT_\beta)^k f \rangle$ (then $\langle \eta_{k+1}, f \rangle = \langle \mu, (PT_\beta)^{k+1} f \rangle \leq \leq \langle \mu, P(PT_\beta)^k f \rangle = \langle \mu, (PT_\beta)^k f \rangle \leq \langle \mu, f \rangle$ for every $0 \leq f$. Thus, the sequence $0 \leq \mu_k = \mu_k = \mu - \eta_k$ is increasing and bounded by μ. Note that

$$\begin{aligned}\langle \mu_k , Pf \rangle &= \langle \mu, f \rangle - \langle \eta_k , Pf \rangle \\ &= \langle \mu, f \rangle - \langle \mu, (PT_\beta)^k Pf \rangle \\ &= \langle \mu, f \rangle - \langle \mu, (PT_\beta)^{k+1} f \rangle - \langle \mu, (PT_\beta)^k PT_\alpha f \rangle \\ &\leq \langle \mu_{k+1} , f \rangle .\end{aligned}$$

Thus, if $\lim \mu_k = \nu$ then $\langle \nu, Pf \rangle \leq \langle \nu, f \rangle$. Use the argument of Section (3), Chapter V to conclude that $\langle \nu, Pf \rangle = \langle \nu, f \rangle$. Now put $\eta = \mu - \nu$ $\langle \eta, f \rangle = = \lim \langle \mu, (PT_\beta)^k f \rangle$ then the restriction of η, to any set with finite μ measure, is a finite measure. Also $\eta(A) = 0$ and since $\langle \eta, Pf \rangle = \langle \eta, f \rangle$ whenever $\langle \mu, f \rangle < \infty$ we obtain $\langle \eta, f \rangle = 0$ whenever f has compact support (A is an open set and $\Sigma P^n 1_A \equiv \infty$). Thus, $\langle \eta, \alpha \rangle = 0$ and also $\langle \eta, P^r \alpha \rangle = 0$.

LEMMA 2. *Assume conditions* (*a*) *and* (*b*) *Then*

(*) $\lim_{n \to \infty} \langle \mu, PT_\beta)^n P^r \alpha \rangle = 0, \; r = 0, 1, 2, \ldots$ *and if* $0 \leq f \leq \sum_{r=0}^{R} P^r \alpha$

then

(**) $$\langle \mu, f \rangle = \lim_{N \to \infty} \sum_{n=0}^{N} \langle \mu, T_\alpha (PT_\beta)^n f \rangle .$$

PROOF. We proved (*) above. Now $\sum_{n=0}^{N} \langle \mu, T_\alpha (PT_\beta)^n f \rangle = \sum_{n=0}^{N} \langle \mu, (I - T_\beta)(PT_\beta)^n f \rangle$ and since $\langle \mu, Pg \rangle = \langle \mu, g \rangle$ the above sum is equal to

$$\sum_{n=0}^{N} \langle \mu, (PT_\beta)^n f \rangle - \sum_{n=0}^{N} \langle \mu, (PT_\beta)^{n+1} f \rangle = \langle \mu, f \rangle - \langle \mu, (PT_\beta)^{N+1} f \rangle .$$

Thus,

$$\langle \mu, f \rangle - \sum_{n=0}^{N} \langle \mu, T_\alpha (PT_\beta)^n f \rangle = \langle \mu, (PT_\beta)^{N+1} f \rangle \leq \sum_{r=0}^{R} \langle \mu, (PT_\beta)^{N+1} P^r \alpha \rangle .$$

Each term in this sum tends to zero by (*).

(3) Let us assume in this section:

(*c*) There exists a finite measure m, such that

$$\lim_{n \to \infty} \frac{\langle m, P^{n+1} T_\alpha h \rangle}{\langle m, P^n \alpha \rangle} = \langle \mu, h\alpha \rangle$$

for every $0 \leq h \in \mathbf{C}(X)$.

If one considers a discrete space X and A is the atom $\{i\}$ while m is the Dirac measure at $\{i\}$ the condition (c) reads

$$\lim_{n \to \infty} \frac{p_{ii}^{(n+1)}}{p_{ii}^{(n)}} = 1.$$

Define

$$M = \{ f : 0 \leq f, \langle \mu, f \rangle < \infty \text{ and } \lim_{n \to \infty} \frac{\langle m, P^n f \rangle}{\langle m, P^n \alpha \rangle} = \langle \mu, f \rangle \}.$$

Clearly, M is a cone. Also $PM \subset M$ since $\dfrac{\langle m, P^{n+1} \alpha \rangle}{\langle m, P^n \alpha \rangle} \longrightarrow 1$ by (c).

Finally, condition (c) means that $T_\alpha h \in M$ whenever $0 \leq h \in \mathbf{C}(X)$. Consider $(PT_\beta)^N P^r \alpha$:

If $N = 0$ $P^r \alpha \in M$. because $\alpha \in M$. Let us prove by induction that $(PT_\beta)^N P^r \alpha \in M$:

$$(PT_\beta)^{N+1} P^r \alpha = P(I - T_\alpha)(PT_\beta)^N P^r \alpha = P[(PT_\beta)^N P^r \alpha - T_\alpha (PT_\beta)^N p^r \alpha\}.$$

The expression in the brackets belongs to M since $T_\alpha h \in M$ for every $h \in \mathbf{C}(X)$ and $(PT_\beta)^N P^r \alpha \in M$ by the induction hypothesis.

THEOREM 3. *Assume conditions* (a), (b), *and* (c). *Let*

$$0 \leq f \leq \sum_{r=0}^{R} P^r \alpha, f \in \mathbf{C}(X): \textit{ then } \lim_{n \to \infty} \frac{\langle m, P^n f \rangle}{\langle m, P^n \alpha \rangle} = \langle \mu, f \rangle.$$

REMARK. By definition $1_A \leq \alpha \leq 1$. Put $A_R = \left\{ x : \sum_{r=0}^{R} P^r \alpha > 1 \right\}$ then condition (c) reads:

If $f \in \mathbf{C}(X)$ is supported on A then $f \in M$ while the conclusion of the theorems reads:

If $f \in \mathbf{C}(X)$ is supported on A_r then $f \in M$.

Since $\bigcup_{R=0}^{\infty} A_R = X$ and the sets A_R are open we conclude: *every continuous function with compact support belongs to M.*

In this discrete case:

If $\lim_{n \to \infty} \dfrac{P_{ii}^{(n+1)}}{P_{ii}^{(n)}} = 1$ then $\lim \dfrac{P_{ij}^{(n)}}{P_{ii}^{(n)}} = \dfrac{\mu_j}{\mu_i}$ where $\langle \mu_k \rangle$ is the σ-finite invariant measure.

PROOF. Note first that for any $0 < K < n$,

$$P^n = P^{n-1}(PT_\alpha) + P^{n-1}(PT_\beta) =$$

$$P^{n-1}(PT_\alpha) + P^{n-2}(PT_\alpha)(PT_\beta) + P^{n-2}(PT_\beta)^2 =$$

$$\ldots = P^{n-1}(PT_\alpha) + P^{n-2}(PT_\alpha)(PT_\beta) + \ldots$$

$$+ P^{n-K}(PT_\alpha)(PT_\beta)^{K-1} + P^{n-K}(PT_\beta)^K .$$

Thus,

$$\frac{\langle m, P^n f\rangle}{\langle m, P^n \alpha\rangle} = \langle m, P^n \alpha\rangle^{-1} \sum_{j=0}^{K-1} \langle m, P^{n-j} T_\alpha (PT_\beta)^j f\rangle +$$

$$\langle m, P^n \alpha\rangle^{-1} \langle m, P^{n-K}(PT_\beta)^K f\rangle.$$

Fix K and let $n \to \infty$. Since

$$\frac{\langle m, P^{n-j} T_\alpha g\rangle}{\langle m, P^n \alpha\rangle} = \frac{\langle m, P^{n-j} T_\alpha g\rangle}{\langle m, P^{n-j} \alpha\rangle} \frac{\langle m, P^{n-j} \alpha\rangle}{\langle m, P^n \alpha\rangle}$$

by (*c*) each term in the first sum tends to the limit $\dfrac{\langle \mu, T_\alpha (PT_\beta)^j f\rangle}{\langle \mu, \alpha\rangle}$ while the last lerm is bounded by $\displaystyle\sum_{r=0}^{R} \frac{\langle m, P^{n-K}(PT_\beta)^K P^r \alpha\rangle}{\langle m, P^n \alpha\rangle}$.

Now,

$$\lim_{n\to\infty} \frac{\langle m, P^{n-K}(PT_\beta)^K P^r \alpha\rangle}{\langle m, P^n \alpha\rangle} = \lim_{n\to\infty} \frac{\langle m, P^n (PT_\beta)^K P^r \alpha\rangle}{\langle m, P^n \alpha\rangle} = \langle \mu, (PT_\beta)^K P^r \alpha\rangle$$

since $(PT_\beta)^K P^r \alpha \in M$. Thus

$$\limsup_{n\to\infty} \frac{\langle m, P^{n-K}(PT_\beta)^K f\rangle}{\langle m, P^n \alpha\rangle} \leq \sum_{r=0}^{R} \langle \mu, (PT_\beta)^K P^r \alpha\rangle$$

and this tends to zero as $K \to \infty$ by (*) of Lemma 2. Finally

$$\lim_{K\to\infty} \lim_{n\to\infty} \langle m, P^n \alpha\rangle^{-1} \sum_{j=0}^{K-1} \langle m, P^{n-j} T_\alpha (PT_\beta)^j f\rangle =$$

$$\lim_{K\to\infty} \sum_{j=0}^{K-1} \langle \mu, T_\alpha (PT_\beta)^j f\rangle = \langle \mu, f\rangle$$

the last conclusion by (**) of Theorem 2.

(4) In the rest of this chapter we shall replace condition (*c*) by some other conditions. If A is not an atom condition (*c*) may be very difficult to verify. Let us continue to assume (*a*) and (*b*) and:

(*d*) There exists a finite measure m, such that

$$\lim_{n\to\infty} \frac{\langle m, P^{n+1} T_\alpha h \rangle - \langle m, P^n T_\alpha h \rangle}{\langle m, P^n \alpha \rangle} = 0$$

for every $0 \leq h \in \mathbf{C}(X)$.

Note that again $\lim \dfrac{\langle m, P^{n+1} \alpha \rangle}{\langle m, P^n \alpha \rangle} = 1$.

Define

$$M_1 = \left\{ f : 0 \leq f, f \in \mathbf{C}(X) \text{ and } \lim_{n\to\infty} \frac{\langle m, P^{n+1} f \rangle - \langle m, P^n f \rangle}{\langle m, P^n \alpha \rangle} = 0 \right\}.$$

Again, M_1 is linear $PM_1 \subset M_1$ and $(PT_\beta)^n PT_\alpha g \in M_1$ for every $0 \leq \leq g \in \mathbf{C}(X)$.

Note that

$$P^n T_\alpha \sum_{k=0}^{K} (PT_\beta)^k PT_\alpha g = P^n (I - T_\beta) \sum_{k=0}^{K} (PT_\beta)^k PT_\alpha g =$$

$$= P^n \sum_{k=0}^{K} (PT_\beta)^k PT_\alpha g - P^{n-1} \sum_{k=1}^{K+1} (PT_\beta)^k PT_\alpha g =$$

$$= P^n \sum_{k=0}^{K} (PT_\beta)^k PT_\alpha g - P^n \sum_{k=1}^{K+1} (PT_\beta)^k PT_\alpha g + (P^n - P^{n-1}) \sum_{k=1}^{K+1} (PT_\beta)^k PT_\alpha g \leq$$

$$\leq P^{n+1} T_\alpha g + (P^n - P^{n-1}) \sum_{k=1}^{K+1} (PT_\beta)^k PT_\alpha g .$$

Since the last term belongs to M_1 one gets:

Lemma 4. *Assume conditions* (*a*), (*b*), *and* (*d*). *Let* n_i *be any subsequence of the integers and*

$$\langle \tau, f \rangle = \mathrm{LIM} \frac{\langle m, P^{n_i} \alpha f \rangle}{\langle m, P^{n_i} \alpha \rangle}$$

where LIM *is a Banach limit and* $0 \leq f \in \mathbf{C}(X)$. *Then*

$$\langle \tau, \sum_{k=0}^{K} (PT_\beta)^k PT_\alpha f \rangle \leq \langle \tau, f \rangle.$$

Proof. It is enough to use the above computation and

$$\lim \frac{\langle m, P^{n_i+1}\alpha f\rangle - \langle m, P^{n_i}\alpha f\rangle}{\langle m, P^{n_i}\alpha\rangle} = 0$$

by condition (*d*).

(5). Let us assume conditions (*a*) and (*b*) in this section. Let $E = \{x : \alpha(x) > 0\}$; then E is an open set with compact closure. Define the operators on $\mathbf{C}(\bar{E})$, $P_N = \sum_{n=0}^{N} (PT_\beta)^n PT_\alpha$ and note that

$$P_N 1 = P_N 1_E = \sum_{n=0}^{N} (PT_\beta)^n P(1 - T_\beta 1) = 1 - (PT_\beta)^{N+1} 1 \leq 1.$$

Thus, the sequence $P_N 1$ is monotone and by Dini's Theorem it converges uniformly on $\bar{E}$ to 1. Now if $f \in \mathbf{C}(\bar{E})$ and $|f| \leq 1$ then

$$|P_{N+K}f - P_N f| \leq \sum_{n=N+1}^{N+K} (PT_\beta)^n PT_\alpha |f| \leq \|f\| \sum_{n=N+1}^{N+K} (PT_\beta)^n PT_\alpha 1 =$$

$$= \|f\| ((PT_\beta)^{N+1} 1 - (PT_\beta)^{N+K+1} 1) \xrightarrow[N \to \infty]{} 0.$$

Hence P_N converges in the operator norm topology. Let us denote its limit by P_∞ and summarize.

Theorem 5. *The sequence of Markov operators P_N, on $\mathbf{C}(\bar{E})$, converges in the uniform topology to $P_\infty = \sum_{n=0}^{\infty} (PT_\beta)^n PT_\alpha$. $P_\infty 1_E = P_\infty 1 = 1$.*

Proof. $P_\infty 1 = \lim_{N \to \infty} P_N 1 = 1.$

(6). Let us return now to the functional τ defined in Section (4). Since τ is defined on $\mathbf{C}(\bar{E})$, τ is given by a measure on $\bar{E}$ to be denoted by the same letter. Now clearly τ is a bounded functional. Since $\| P_N - P_\infty \| \to 0$, as operators on $\mathbf{C}(\bar{E})$, we can conclude from Lemma 4:

Lemma 6. $\langle \tau, P_\infty f\rangle = \langle \tau, f\rangle$ *for every* $0 \leq f \in \mathbf{C}(\bar{E})$.

Proof. From Lemma 4 we get $\langle \tau, P_\infty f\rangle \leq \langle \tau, f\rangle$ but $\langle \tau, P_\infty 1\rangle = \langle \tau, 1\rangle$. Hence equality holds for every $f \in \mathbf{C}(X)$.

(7) Put
$$\langle \eta, f \rangle = \sum_{n=0}^{\infty} \langle \tau, (PT_\beta)^n f \rangle$$
and note that
$$\sum_{n=0}^{\infty} (PT_\beta)^n P^r \alpha \leq r, \qquad r = 1, 2, \ldots$$

Let us see that by induction. If $r = 1$ then

$$\sum_{n=0}^{N} (PT_\beta)^n P\alpha = \sum_{n=0}^{N} (PT_\beta)^n P(1-\beta) = \sum_{n=0}^{N} (PT_\beta)^n 1 - \sum_{n=0}^{N} (PT_\beta)^{n+1} 1 =$$
$$= 1 - (PT_\beta)^{N+1} 1 \xrightarrow[N \to \infty]{} 1 .$$

Now
$$\sum_{n=0}^{N} (PT_\beta)^n P^{r+1} \alpha = \sum_{n=0}^{N} (PT_\beta)^n (PT_\alpha + PT_\beta) P^r \alpha \leq$$
$$\leq \sum_{n=0}^{N} (PT_\beta)^n P\alpha + \sum_{n=0}^{N} (PT_\beta)^{n+1} P^r \alpha \leq 1 + r$$

where we used $T_\alpha P^r \alpha \leq \alpha$ and the induction hypothesis. Thus $\langle \eta, P^r \alpha \rangle < \infty$ and since $\Sigma P^r \alpha = \infty$ the set function η is finite on every compact set. As in Section (2), it is easy to see that η is a σ-additive set function. Thus η is a Borel measure. Finally

$$\langle \eta, Pf \rangle = \sum_{n=0}^{\infty} \langle \tau, (PT_\beta)^n Pf \rangle =$$
$$= \sum_{n=0}^{\infty} \langle \tau, (PT_\beta)^n PT_\beta f \rangle + \sum_{n=0}^{\infty} \langle \tau, PT_\beta)^n PT_\alpha f \rangle =$$
$$= \langle \eta, f \rangle - \langle \tau, f \rangle + \langle \tau, P_\infty f \rangle = \langle \eta, f \rangle.$$

Note also that
$$\langle \eta, \alpha \rangle = \langle \eta, P\alpha \rangle = \sum_{n=0}^{\infty} \tau, (PT_\beta)^n P\alpha \rangle = \langle \tau, 1 \rangle = 1$$

and if f is supported on A (thus $\beta f = 0$) then $\langle \eta, f \rangle = \langle \tau, f \rangle$.

THEOREM 7. *Assume conditions (a), (b), and (d). If the Borel measure μ is a unique invariant Borel measure then*

$$\lim_{n \to \infty} \frac{\langle m, P^{n+1} f \rangle}{\langle m, P^n \alpha \rangle} = \langle \mu, f \rangle$$

for every $0 \leq f \in \mathbf{C}(X)$ which is supported on A.

PROOF. Assume, to the contrary, that $\lim_{i \to \infty} \frac{\langle m, P^{n_i+1} f \rangle}{\langle m, P^{n_i} \alpha \rangle} = a \neq \langle \mu, f \rangle$. By (d) $\lim \frac{\langle m, P^{n_i} f \rangle}{\langle m, P^{n_i} \alpha \rangle} = a$ too. Now, use the above construction to define η which is an invariant. Borel measure and $\langle \eta, \alpha \rangle = 1$. Since the invariant measure is unique $\langle \eta, f \rangle = \langle \mu, f \rangle$ but

$$\langle \eta, f \rangle = \langle \tau, f \rangle = \mathrm{LIM} \frac{\langle m, P^{n_i} f \rangle}{\langle m, P^{n_i} \alpha \rangle} = \lim_{i \to \infty} \frac{\langle m, P^{n_i} f \rangle}{\langle m, P^{n_i} \alpha \rangle} \neq \langle \mu, f \rangle.$$

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