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B. LEMAIRE

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# SADDLE-POINT PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS TO LINEAR QUADRATIC DIFFERENTIAL GAMES

B. LEMAIRE

**ABSTRACT.** We study existence, uniqueness and characterization for saddle-points of convex-concave functionals on Banach spaces and give examples involving partial differential equations. Then we consider differential two-person zero-sum games. The system is governed by an elliptic or parabolic linear differential equation with boundary conditions. The cost functional is quadratic and the two controls belong to subsets of Hilbert spaces.

## Introduction.

Let  $K_1$  and  $K_2$  be two sets and  $J$  a real function defined on  $K_1 \times K_2$ . Let us recall the definition of a saddle-point for  $J$  on  $K_1 \times K_2$ .

**DEFINITION.**  $(u_1, u_2) \in K_1 \times K_2$  is a saddle point for  $J$  on  $K_1 \times K_2$  iff

$$(0) \quad J(v_1, u_2) \leq J(u_1, u_2) \leq J(u_1, v_2), \quad \forall v_1 \in K_1, \quad \forall v_2 \in K_2.$$

The saddle-point problem is well known in the classical two-person zero-sum game theory. In this paper we study such a problem when  $J$  is a functional defined on a product of two real Banach spaces.

First we consider the case where  $K_1$  and  $K_2$  are closed convex subsets of real Hilbert spaces and  $J$  a quadratic functional (§ 1, 2, 3), then the case where  $K_1$  and  $K_2$  are closed convex subsets of reflexive real Banach spaces and  $J$  a convex-concave functional (§ 4). We prove that, under differentiability assumptions on  $J$ , the saddle-point problem is equivalent to a system of two coupled variational inequalities and we study this system independently of any optimization problem. We also give examples involving linear (§ 3) or non-linear monotone (§ 4) partial differential operators.

Then we apply the results of § 1, 2 to optimal control problems with two antagonistic controls, linear quadratic differential games for systems governed by elliptic (§ 5, 6, 7) or parabolic (§ 8, 9, 10, 11) equations, problems which, in our knowledge, have not much been entered upon before (see, for instance, PORTER [12], BENSOUSSAN [1], [2]). For each example we give a practical sufficient condition for the existence and uniqueness of one pair of optimal controls and a characterization with the help of the adjoint state using largely the methods of LIONS [7]. Some results of that book are thus extended to this game situation. In particular, we study the feed-back problem (§ 10).

This work constitutes the theoretical part of the differential games considered in the author thesis [3] and, in [4], we improve other MIN-MAX control problems also studied in that thesis. Numerical aspects are discussed in [5].

## § 1. Saddle-points of quadratic functionals.

### 1.1. Notations.

Let  $U_1$  and  $U_2$  be two real Hilbert spaces with norms indifferently noted  $\|\cdot\|$ .

Let us give :

three continuous bilinear forms

$$(i) \quad \begin{cases} a_i(u_i, v_i) \text{ on } U_i, \text{ symmetrical, } a_i(v_i, v_i) \geq \alpha_i \|v_i\|^2, \alpha_i \geq 0, i = 1, 2, \\ b(v_2, v_1) \text{ on } U_2 \times U_1; \end{cases}$$

$$(ii) \quad \begin{cases} \text{two continuous bilinear forms on } U_i, \\ v_i \rightarrow L_i(v_i), i = 1, 2; \end{cases}$$

(iii) two sets  $K_i$  closed convex in  $U_i$ ,  $i = 1, 2$ .

We consider the functional on  $U = U_1 \times U_2$  :

$$(1.1) \quad J(v_1, v_2) = a_1(v_1, v_1) - a_2(v_2, v_2) + 2\{b(v_2, v_1) - L_1(v_1) + L_2(v_2)\},$$

of which we look for the saddle-points on  $K_1 \times K_2$ , that is to say the pairs

$$(1.2) \quad u = (u_1, u_2) \in K_1 \times K_2 \quad \text{which verify (o).}$$

We denote by  $X$  the subset of  $U$  formed by the saddle-points of  $J$  on  $K_1 \times K_2$ .

1.2. Characterization of a saddle-point.

THEOREM 1.2. In order that  $u \in X$ , it is necessary and sufficient that

$$(1.3) \quad a_1(u_1, v_1 - u_1) + b(u_2, v_1 - u_1) \geq L_1(v_1 - u_1), \quad \forall v_1 \in K_1,$$

$$(1.4) \quad -b^*(u_1, v_2 - u_2) + a_2(u_2, v_2 - u_2) \geq L_2(v_2 - u_2), \quad \forall v_2 \in K_2,$$

where  $b^*(v_1, v_2) \triangleq b(v_2, v_1)$ .

*Proof.* We have

$$\begin{aligned} J(v_1, u_2) &= a_1(v_1, v_1) - 2\{-b(u_2, v_1) + L_1(v_1)\} + 2L_2(u_2) - a_2(u_2, u_2) \\ &= a_1(v_1, v_1) - 2\tilde{L}_1(v_1) + \gamma, \end{aligned}$$

where

$$\tilde{L}_1(v_1) = -b(u_2, v_1) + L_1(v_1),$$

$$\gamma = 2L_2(u_2) - a_2(u_2, u_2).$$

Consequently (cf. [7]), in order that  $u_1$  minimizes the quadratic form  $v_1 \rightarrow J(v_1, u_2)$  on the convex  $K_1$ , it is necessary and sufficient that

$$a_1(u_1, v_1 - u_1) \geq \tilde{L}_1(v_1 - u_1), \quad \forall v_1 \in K_1,$$

which is (1.3). Similar proof for (1.4), considering that  $u_2$  minimizes the quadratic form  $v_2 \rightarrow -J(u_1, v_2)$  on the convex  $K_2$ .

THEOREM 1.3. Characterization (1.3)(1.4) is equivalent to

$$(1.5) \quad a_1(v_1, v_1 - u_1) + b(v_2, v_1 - u_1) \geq L_1(v_1 - u_1), \quad \forall v_i \in K_i,$$

$$(1.6) \quad -b^*(v_1, v_2 - u_2) + a_2(v_2, v_2 - u_2) \geq L_2(v_2 - u_2), \quad i = 1, 2.$$

*Proof.* We consider the continuous bilinear form on  $U = U_1 \times U_2$  which, provided with the product norm  $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2$ , is a Hilbert space,

$$(1.7) \quad a(u; v) = a_1(u_1, v_1) + a_2(u_2, v_2) + b(u_2, v_1) - b(v_2, u_1).$$

Let us remark that  $a(u; v)$  is not, in general, symmetrical. We put

$$L = (L_1, L_2) \quad (L(v) = L_1(v_1) + L_2(v_2)).$$

Then  $L \in U'$ . At last put  $K = K_1 \times K_2$ . It is a closed convex set in  $U$ . It is easy to verify that (1.3)-(1.4) is then equivalent to

$$(1.8) \quad a(u; v - u) \geq L(v - u), \quad \forall v \in K.$$

But ([7]), (1.8) is equivalent to

$$(1.9) \quad a(v; v - u) \geq L(v - u), \quad \forall v \in K.$$

(1.9) is equivalent (easy verification) to (1.5)-(1.6).

## § 2. Coupled linear variational inequations.

2.1. *Statement of the problem.* We a priori consider the following problem.

Let  $\pi_i(u_i, v_i)$  be two continuous non necessarily symmetrical forms given on  $U_i$ ,  $i = 1, 2$ , and  $b(v_2, v_1)$ ,  $c(v_1, v_2)$ , two continuous bilinear forms given respectively on  $U_2 \times U_1$  and  $U_1 \times U_2$ ; we look for  $u_i \in K_i$ ,  $i = 1, 2$ , verifying

$$(2.1) \quad \pi_1(u_1, v_1 - u_1) + b(u_2, v_1 - u_1) \geq L_1(v_1 - u_1), \quad \forall v_1 \in K_1,$$

$$(2.2) \quad c(u_1, v_2 - u_2) + \pi_2(u_2, v_2 - u_2) \geq L_2(v_2 - u_2), \quad \forall v_2 \in K_2,$$

where  $L_i$  are continuous linear forms given on  $U_i$ ,  $i = 1, 2$ . We suppose that the forms  $\pi_i$  verify

$$(2.3) \quad \pi_i(v_i, v_i) \geq \beta_i \|v_i\|^2, \quad \beta_i \geq 0, \quad \forall v_i \in U_i, \quad i = 1, 2,$$

and that

$$(2.4) \quad b(v_2, v_1) + c(v_1, v_2) \geq 0, \quad \forall v_i \in U_i, \quad i = 1, 2.$$

So, the continuous bilinear form on  $U$ ,  $\pi(u, v)$ , defined in a similar way to (1.7), verifies

$$(2.5) \quad \pi(v; v) \geq \beta_1 \|v_1\|^2 + \beta_2 \|v_2\|^2 \geq \beta \|v\|^2, \quad \forall v \in U,$$

with

$$(2.6) \quad \beta = \inf(\beta_1, \beta_2) \geq 0.$$

2.2. *Convexity of the set of solutions.* Let  $X \subset K_1 \times K_2$  be the set of the solutions of (2.1)-(2.2).

**THEOREM 2.1.** *If  $X \neq \emptyset$ , then  $X$  is a closed convex set in  $U$ . More precisely  $X = X_1 \times X_2$  where  $X_i$  is a closed convex set in  $U_i$ ,  $i = 1, 2$ .*

*Proof.* We first remark that (2.1)-(2.2) is equivalent to

$$(2.7) \quad \pi(u; v - u) \geq L(v - u), \quad \forall v \in K, \quad u \in K,$$

where  $L$  is defined as in theorem 1.3  $X$  is therefore the set of the solutions of (2.7), and consequently ([7]), if it is not empty, it is closed and convex in  $U$ . But (2.7) is equivalent to

$$(2.8) \quad \pi(v; v - u) \geq L(v - u), \quad \forall v \in K, \quad u \in K,$$

that is to say to

$$(2.9) \quad \pi_1(v_1, v_1 - u_1) + b(v_2, v_1 - u_1) \geq L_1(v_1 - u_1), \quad \forall v_i \in K_i, \quad i = 1, 2,$$

$$(2.10) \quad c(v_1, v_2 - u_2) + \pi_2(v_2, v_2 - u_2) \geq L_2(v_2 - u_2).$$

Putting

$$u' = (u'_1, u'_2) \in X,$$

$$u'' = (u''_1, u''_2) \in X,$$

$u = \{\theta_1 u'_1 + (1 - \theta_1) u''_1, \theta_2 u'_2 + (1 - \theta_2) u''_2\}$ ,  $\theta_i \in [0, 1]$ ,  $i = 1, 2$ , we easily verify that  $u$  satisfies (2.9)-(2.10). Consequently,  $X$  contains the rectangle  $[u'_1, u''_1] \times [u'_2, u''_2]$ , and is therefore on the form  $X_1 \times X_2$ , where  $X_i$  is convex in  $U_i$ ,  $i = 1, 2$ .

### 2.3. Results about existence and uniqueness.

**THEOREM 2.2 (Uniqueness).** *If  $\beta_i > 0$  ( $i = 1$  or  $2$ ), then  $X_i$  is reduced to one element.*

*Proof.* Let  $u', u'' \in X$  (supposed not empty). From (2.7), we have

$$(2.11) \quad \pi(u' ; v - u') \geq L(v - u'), \quad \forall v \in K,$$

$$(2.12) \quad \pi(u'' ; v - u'') \geq L(v - u''), \quad \forall v \in K.$$

Let us take  $v = u''$  in (2.11), and  $v = u'$  in (2.12), and add up. We have

$$- \pi(u' - u'' ; u' - u'') \geq 0;$$

which joined to (2.5), gives

$$0 \geq \pi(u' - u''; u' - u'') \geq \sum_{i=1}^2 \beta_i \|u_i' - u_i''\|^2,$$

which implies  $\beta_i \|u_i' - u_i''\|^2 = 0$ , therefore  $u_i' = u_i''$  if  $\beta_i > 0$ .

**THEOREM 2.3 (Existence).** *Under one of the following additional assumptions,  $X$  is not empty.*

- (i)  $\beta_i > 0$ ,  $i = 1$  and  $2$ ;
- (ii)  $\beta_i = 0$  and  $K_i$  bounded,  $i = 1$  and  $2$ ;
- (iii)  $\beta_i > 0$  and  $K_{3-i}$  bounded and  $\beta_{3-i} = 0$ ,  $i = 1$  or  $2$ .

*Proof.*

(i)  $\beta = \inf(\beta_1, \beta_2) > 0$ . We are in the coercive case for the problem (2.7). The results follows from [7].

(ii)  $\pi(u; v)$  is only non-negative. But  $K = K_1 \times K_2$  is bounded. The result still follows from [7].

(iii) Let us prove this for  $i = 1$ . (2.5) becomes

$$\pi(v; v) \geq \beta_1 \|v_1\|^2, \quad \forall v \in U.$$

Let

$$\pi_\varepsilon(u; v) = \pi(u; v) + \varepsilon((u_2, v_2)), \quad \varepsilon > 0.$$

$\pi_\varepsilon$  is a continuous bilinear form on  $U$ , and verifies

$$\pi_\varepsilon(v; v) \geq \alpha_\varepsilon \|v\|^2, \quad \alpha_\varepsilon = \inf(\beta_1, \varepsilon) > 0.$$

Therefore there exists a unique  $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}) \in K$  such that

$$(2.13) \quad \pi_\varepsilon(u_\varepsilon; v - u_\varepsilon) \geq L(v - u_\varepsilon), \quad \forall v \in K.$$

$u_{\varepsilon 2}$  is bounded (when  $\varepsilon \rightarrow 0$ ), since  $u_{\varepsilon 2} \in K_2$  which is bounded;  $u_{\varepsilon 1}$  is bounded, for (2.13) implies

$$\beta_1 \|u_{\varepsilon 1}\|^2 \leq \pi_\varepsilon(u_\varepsilon; u_\varepsilon) \leq \pi(u_\varepsilon; v) + \varepsilon((u_{\varepsilon 2}, v_2)) + L(u_\varepsilon - v) \leq c_1 \|u_{\varepsilon 1}\| + c_2,$$

with  $c_1, c_2$  positive constants. Then we deduce, as in [7], that  $u_\varepsilon$  strongly tends to  $u \in X$  of minimal norm, that is to say, from theorems 2.1 and 2.2,  $u = (u_1, u_2)$  with  $u_1 =$  the unique element of  $X_1$ , and  $u_2 =$  the element of minimal norm in  $X_2$ .

**REMARK 2.1.** Of course the results of this paragraph are valid in the situation of § 1.

REMARK 2.2.

(i) If  $K_1 = U_1$  then (2.1) is equivalent to

$$\Pi_1 u_1 + Bu_2 = L_1$$

where  $\Pi_1 \in \mathcal{L}(U_1, U_1')$ .  $B \in \mathcal{L}(U_2, U_1')$  are the operators defined by the forms  $\pi_1$  and  $b$ .

(ii) If  $K_1$  is a pointed convex cone, then (2.1) is equivalent to

$$(2.14) \quad \begin{cases} \pi_1(u_1, v_1) + b(u_2, v_1) \geq L_1(v_1) & \forall v_1 \in K_1 \\ \pi_1(u_1, u_1) + b(u_2, u_1) = L_1(u_1). \end{cases}$$

Of course, we have the two similar remarks for  $K_2$ .

### § 3. Examples.

EXAMPLE 3.1. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with regular boundary  $\Gamma$ . Let us take  $U_1 = U_2 = H^1(\Omega)$ , and

$$a_i(u_i, v_i) = \alpha_i \int_{\Omega} (u_i v_i + \text{grad } u_i \cdot \text{grad } v_i) dx, \quad i = 1, 2$$

$$b(v_2, v_1) = - \int_{\Omega} (v_2 v_1 + \lambda \text{grad } v_2 \cdot \text{grad } v_1) dx$$

$$L_i(v_i) = \int_{\Omega} f_i v_i dx + \int_{\Gamma} g_i v_i d\Gamma, \quad i = 1, 2$$

where  $\alpha_i > 0$  and  $\lambda \in \mathbb{R}$ ,  $f_i \in L^2(\Omega)$  and  $g_i \in H^{-\frac{1}{2}}(\Gamma)$ . Let us take  $K_1 = K_2 = K = \{v \mid v \in H^1(\Omega), \text{ trace of } v \text{ on } \Gamma \geq 0 \text{ a. e.}\}$ , which is closed convex in  $H^1(\Omega)$  ([7]). The functional  $J(v_1, v_2)$  defined as in (1.1) has a unique saddle-point. The interpretation methods used in [7] (chapter 1) applied to variational inequalities (1.5) and (1.6), show that the saddle-point of  $J$  is the unique solution of the coupled unilateral problem

$$\left. \begin{aligned} \alpha_1 (-\Delta u_1 + u_1) + \lambda \Delta u_2 - u_2 &= f_1 \\ -\lambda \Delta u_1 + u_1 + \alpha_2 (-\Delta u_2 + u_2) &= f_2 \end{aligned} \right\} \text{ in } \Omega,$$

$$u_i \geq 0 \quad \text{on } \Gamma,$$



$$\left. \begin{aligned} \varepsilon_i \lambda \frac{\partial u_{3-i}}{\partial n} + \alpha_i \frac{\partial u_i}{\partial n} - g_i &\geq 0 \\ u_i \left( \varepsilon_i \lambda \frac{\partial u_{3-i}}{\partial n} + \alpha_i \frac{\partial u_i}{\partial n} - g_i \right) &= 0 \end{aligned} \right\} \begin{array}{l} \text{on } \Gamma \quad i = 1, 2 \\ \varepsilon_1 = -1, \quad \varepsilon_2 = 1. \end{array}$$

REMARK 3.1. If  $\alpha_1 = \alpha_2 = 1$  and if  $\lambda = 0$ , we recover the example of unilateral problem (case of systems) given in [7] (§ 3.8, p. 30) connectionless with any optimization problem. In fact, the solution of this example appears here as the unique saddle-point of the functional

$$\begin{aligned} J(v_1, v_2) = & \sum_{i=1}^n \int_{\Omega} \left( \left| \frac{\partial v_1}{\partial x_i} \right|^2 - \left| \frac{\partial v_2}{\partial x_i} \right|^2 \right) dx + \int_{\Omega} (|v_1|^2 - |v_2|^2) dx - \\ & - 2 \left\{ \int_{\Omega} v_2 v_1 dx + \int_{\Omega} f_1 v_1 dx + \int_{\Gamma} g_1 v_1 d\Gamma - \int_{\Omega} f_2 v_2 dx - \int_{\Gamma} g_2 v_2 d\Gamma \right\} \end{aligned}$$

on  $K \times K$ .

EXAMPLE 3.2. Let us take up again, with the same notations, the point 2.2 of [7], chapter 2 control of elliptic variational problems, unconstrained case. The optimal control is given by the following rule:

(i) We solve the following problem which has a unique solution:

$$(3.1) \quad \begin{cases} C^* A C y - A^* p = C^* A z_d, \\ A y + B N^{-1} A_u^{-1} B^* p = f \end{cases}$$

(ii) the optimal control is given by

$$(3.2) \quad u = - N^{-1} A_u B^* p$$

(3.1) is a linear system in  $\{y, p\}$ , where the matrix of operators is non positive or non symmetrical. Consequently (3.1) does not correspond to a minimization problem. However, we shall see that the solution of (3.1) is a saddle-point of a certain functional.

In fact, let us consider the functional on  $V \times V$ :

$$(3.3) \quad \begin{aligned} J(y, p) = & (C^* A C y, y) - (B N^{-1} A_u^{-1} B^* p, p) \\ & - 2 \{(A y, p) + (C^* A z_d, y) + (f, p)\} \end{aligned}$$

where  $(\cdot, \cdot)$  denotes the duality between  $V'$  and  $V$ .  $C^* A C$  and  $B N^{-1} A_u^{-1} B^*$

are symmetrical operators and we have

$$(C^* A Cy, y) = |Cy|_{\mathcal{Q}}^2 \geq 0,$$

$$(BN^{-1} A_u^{-1} B^*p, p) = (N^{-1} A_u^{-1} B^*p, A_u^{-1} B^*p)_u \geq \nu |A_u^{-1} B^*p|_u^2 \geq 0.$$

Therefore functional (3.3) is, notations excepted, of type (1.1) and according to remark 2.2 (i), (3.1) is a necessary and sufficient condition for  $\{y, p\}$  be a saddle point of (3.3) on  $V \times V$ . In addition, we have there an example of a quadratic functional the second order parts of which are not coercive on  $V$ , and which however has a *unique saddle-point*.

#### § 4. Saddle-points of convex-concave functionals.

In this paragraph we consider a more general situation than in the previous ones.

##### 4.1. Assumptions and minimax theorem.

(4.1)  $K_i$  is a closed set in a real reflexive Banach space  $V_i$ , normed by  $\|\cdot\|_i$ ,  $i = 1, 2$ ;

Let us consider a functional  $J$  defined on  $V_1 \times V_2$ , verifying

(4.2)  $\forall v_2 \in K_2, v_1 \rightarrow J(v_1, v_2)$  is convex and lower semi continuous (l. s. c.);

(4.3)  $\forall v_1 \in K_1, v_2 \rightarrow J(v_1, v_2)$  is concave and upper semi continuous (u. s. c.);

(4.4)  $\left\{ \begin{array}{l} \text{There exists some } (v_1^0, v_2^0) \in K_1 \times K_2, \text{ such that} \\ J(v_1, v_2^0) - J(v_1^0, v_2) \rightarrow +\infty \text{ if } \|v_1\| + \|v_2\| \rightarrow +\infty \\ v_1 \in K_1, v_2 \in K_2. \end{array} \right.$

Assumption (4.4) is useless when  $K_1$  and  $K_2$  are both bounded. Then we can remind a theorem due to BENSOUSSAN [2]:

**THEOREM 4.1.** Under assumptions (4.1) to (4.4), there exists a saddle-point of  $J$  on  $K_1 \times K_2$ .

**REMARK 4.1.** Theorem 4.1 is in fact a corollary of the well-known KI-FAN SION minimax theorem [16]. Assumption (4.4) is used when at least

one of  $K_1$  or  $K_2$  is not bounded and thus is not necessarily compact (here for the weak topology). This theorem is also included in the general minimax theorems of MOREAU [11] or ROCKAFELLAR [15].

**THEOREM 4.2.** The set  $X \subset K_1 \times K_2$  of the saddle points of  $J$  is of the form  $X = X_1 \times X_2$  where  $X_1$  (resp.  $X_2$ ) is a closed convex set in  $K_1$  (resp. in  $K_2$ ).

**PROOF.**  $X$  is closed (this follows from the semi continuity assumptions).

If  $(u_1, u_2)$  and  $(u'_1, u'_2)$  belong to  $X$ , then the « rectangle »  $[u_1, u'_1] \times [u_2, u'_2]$  is contained in  $X$  because

$$(4.5) \quad J(\lambda_1 u_1 + (1 - \lambda_1) u'_1, v_2) \leq \lambda_1 J(u_1, v_2) + (1 - \lambda_1) J(u'_1, v_2) \leq \delta, \quad \forall v_2 \in K_2, \\ 0 \leq \lambda_1 \leq 1,$$

$$(4.6) \quad \delta \leq \lambda_2 J(v_1, u_2) + (1 - \lambda_2) J(v_1, u'_2) \leq J(v_1, \lambda_2 u_2 + (1 - \lambda_2) u'_2), \quad \forall v_1 \in K_1, \\ 0 \leq \lambda_2 \leq 1,$$

where

$$\delta = J(u_1, u_2) = J(u'_1, u'_2).$$

**THEOREM 4.3.** If in assumption (4.2) we suppose  $v_1 \rightarrow J(v_1, v_2)$  strictly convex and if we have the similar modification in (4.3), then  $X_1$  (resp.  $X_2$ ) defined in theorem 4.2 is reduced to one element.

**PROOF.** From theorem 4.2,  $(u_1, u_2) \in X$  and  $(u'_1, u'_2) \in X$  implies  $(u_1, u'_2) \in X$ . So  $u_1$  and  $u'_1$  minimize both the strictly convex function  $v_1 \rightarrow J(v_1, u'_2)$  on  $K_1$ . Therefore  $u_1 = u'_1$ .

Likewise we have  $u_2 = u'_2$ .

#### 4.2. Characterization of a saddle point.

We still suppose that  $J$  is defined on  $V_1 \times V_2$  and is convex-concave, i. e.

$$v_1 \rightarrow J(v_1, v_2) \quad \text{is convex} \quad \forall v_2 \in V_2 \\ v_2 \rightarrow J(v_1, v_2) \quad \text{is concave} \quad \forall v_1 \in V_1.$$

Consequently,  $J$  has partial directional differentials

$$J'_1(v_1, v_2; w_1) = \lim_{\lambda \rightarrow 0^+} \frac{J(v_1 + \lambda w_1, v_2) - J(v_1, v_2)}{\lambda} \\ J'_2(v_1, v_2; w_2) = \lim_{\lambda \rightarrow 0^+} \frac{J(v_1, v_2 + \lambda w_2) - J(v_1, v_2)}{\lambda}.$$

Whatever be  $v_1, w_1 \in V_1$  and  $v_2, w_2 \in V_2$  ([17]).

**THEOREM 4.2.** If  $K_i$  is a convex set in  $V_i$ ,  $i = 1, 2$ , if  $J$  is convex-concave on  $V_1 \times V_2$ , a necessary and sufficient condition in order that  $(u_1, u_2) \in K_1 \times K_2$  be a saddle-point of  $J$  on  $K_1 \times K_2$  is

$$(4.7) \quad J'_1(u_1, u_2; v_1 - u_1) \geq 0, \quad \forall v_1 \in K_1, \quad u_1 \in K_1,$$

and

$$(4.8) \quad J'_2(u_1, u_2; v_2 - u_2) \leq 0, \quad \forall v_2 \in K_2, \quad u_2 \in K_2.$$

**PROOF.** It is well known that (4.7) (resp. (4.8)) is necessary and sufficient for the convex function  $v_1 \rightarrow J(v_1, u_2)$  attain its infimum on  $K_1$  at  $u_1$  (resp. the concave function  $v_2 \rightarrow J(u_1, v_2)$  attain its supremum on  $K_2$  at  $u_2$ ).

**REMARK 4.2.** (4.7) and (4.8) is equivalent to

$$(4.9) \quad J'_1(u_1, u_2; v_1 - u_1) - J'_2(u_1, u_2; v_2 - u_2) \geq 0 \quad \forall (v_1, v_2) \in K_1 \times K_2 \\ (u_1, u_2) \in K_1 \times K_2.$$

**COROLLARY 4.1.** If  $J$  has partial Gateaux derivatives

$$J'_i(v_1, v_2) \in V'_i \quad \forall (v_1, v_2) \in V_1 \times V_2, \quad i = 1, 2$$

where  $V'_i$  denotes the topological dual of  $V_i$ , then (4.7) (4.8) becomes

$$(4.11) \quad \langle J'_1(u_1, u_2), v_1 - u_1 \rangle_1 \geq 0, \quad \forall v_1 \in K_1, \quad u_1 \in K_1$$

$$(4.12) \quad \langle J'_2(u_1, u_2), v_2 - u_2 \rangle_2 \leq 0, \quad \forall v_2 \in K_2, \quad u_2 \in K_2$$

where  $\langle \cdot, \cdot \rangle_i$  denotes the duality between  $V'_i$  and  $V_i$ .

**PROOF.** We have

$$J'_i(u_1, u_2; w_i) = \langle J'_i(u_1, u_2), w_i \rangle_i.$$

**REMARK 4.3.** Under the assumptions of corollary 4.1, let us put  $V = V_1 \times V_2$  provided with the product topology,

$$u = (u_1, u_2), \quad v = (v_1, v_2), \quad K = K_1 \times K_2,$$

$$G(u) = (J'_1(u_1, u_2), -J'_2(u_1, u_2)) \in V'.$$

From corollary 4.1 and remark 4.2, the saddle-point problem for  $J$  is then equivalent to the variational inequation

$$(4.10) \quad \langle G(u), v - u \rangle \geq 0, \quad \forall v \in K, u \in K.$$

It has been proved by ROCKAFELLAR [14] that  $G$  is a monotone operator. Therefore under convenient assumptions on the derivatives of  $J$  we can apply the results about this class of non-linear inequations (see for instance LIONS [9]).

REMARK 4.4. If  $V_1$  and  $V_2$  are Hilbert spaces and if  $J$  is a quadratic functional (cf. 1.1) then it is convex-concave (strictly convex-concave and the condition (4.4) is satisfied if the forms  $a_i$  are coercive). In addition  $J$  is differentiable and we have

$$(4.13) \quad J_1'(v_1, v_2) = 2(A_1 v_1 + B v_2 - L_1)$$

$$(4.14) \quad J_2'(v_1, v_2) = 2(B^* v_1 - A_2 v_2 + L_2)$$

where  $A_i, B, B^*$  denote the continuous linear operators defined by the forms  $a_i, b, b^*$  (adjoint of  $b$ ). Therefore, the results of this paragraph contain those of § 2 in the case where the forms  $\pi_i$  are symmetrical.

#### 4.3. *Non quadratic example.*

Let, for  $i = 1, 2$ ,  $K_i$  being still a closed convex set in  $V_i$  real reflexive Banach space,  $v_i \rightarrow J_i(v_i)$  be a functional on  $V_i$  verifying

$$(4.15) \quad J_i \text{ is strictly convex and continuous,}$$

$$(4.16) \quad \lim_{\substack{\|v_i\| \rightarrow \infty \\ v_i \in K_i}} \frac{J_i(v_i)}{\|v_i\|} = +\infty, \text{ useless if } K_i \text{ is bounded.}$$

Let

$$(4.17) \quad (v_1, v_2) \rightarrow b(v_2, v_1) \text{ be a continuous bilinear form on } V_2 \times V_1,$$

$$(4.18) \quad L_i \in V_i', \quad i = 1, 2.$$

We take

$$(4.19) \quad J(v_1, v_2) = J_1(v_1) - J_2(v_2) + b(v_2, v_1) - L_1(v_1) + L_2(v_2).$$

PROPOSITION 4.1. The saddle-point problem for the functional (4.19) has, under assumptions (4.15) to (4.18) a unique solution.

PROOF.  $J$  is of course strictly convex-concave, l. s. c.-u. s. c.. In addition we have

$$(4.20) \quad J(v_1, v_2) \geq \|v_1\|_1 \left( \frac{J_1(v_1)}{\|v_1\|_1} - (M\|v_2\|_2 + \|L_1\|_1^*) \|v_1\| \right) - J_2(v_2)$$

where, from (4.17),  $M > 0$  is such that  $|b(v_2, v_1)| \leq M\|v_1\|_1\|v_2\|_2$ . (4.20) implies, from (4.16),

$$\lim_{\substack{\|v_1\|_1 \rightarrow \infty \\ v_1 \in K_1}} J(v_1, v_2) = +\infty, \quad \forall v_2 \in K_2.$$

In the same way

$$\lim_{\substack{\|v_2\|_2 \rightarrow \infty \\ v_2 \in K_2}} J(v_1, v_2) = -\infty, \quad \forall v_1 \in K_1.$$

Therefore condition (4.4) is also satisfied and we can apply theorem 4.1.

Now let us suppose  $J_i$  be  $G$ -differentiable and let  $J'_i(v_i)$  be its Gateaux derivative. Then  $J$  defined by (4.19) has partial Gateaux derivatives

$$(4.21) \quad J'_1(v_1, v_2) = J'_1(v_1) + Bv_2 - L_1$$

$$(4.22) \quad J'_2(v_1, v_2) = B^*v_1 - J'_2(v_2) + L_2$$

where  $B \in \mathcal{L}(V_2, V'_1)$  is defined by  $\langle Bv_2, v_1 \rangle_1 = b(v_2, v_1)$  and where  $B^* = \text{adjoint of } B, B^* \in \mathcal{L}(V_1, V'_2)$ .

From corollary 4.1, the saddle-point  $(u_1, u_2)$  of  $J$  on  $K_1 \times K_2$  (cf. proposition 4.1) is characterized by

$$(4.23) \quad \langle J'_1(u_1) + Bu_2 - L_1, v_1 - u_1 \rangle_1 \geq 0, \quad \forall v_1 \in K_1, u_1 \in K_1$$

$$(4.24) \quad \langle B^*u_1 - J'_2(u_2) + L_2, v_2 - u_2 \rangle_2 \leq 0, \quad \forall v_2 \in K_2, u_2 \in K_2.$$

REMARK 4.5. The quadratic functionals (1.1) belong to the class of functionals (4.19).

EXAMPLE. Let  $\Omega$  be an open bounded set in  $\mathbb{R}^n$  with boundary  $\Gamma$ . Let  $p_1 \geq 2$  and  $p_2 \geq 2$ . Let us take

$$V_1 = L^{p_1}(\Omega) \cap H_0^1(\Omega), \quad V_2 = W_0^{1,p_2}(\Omega).$$

Normed by  $\|v_1\|_1 = \|v_1\|_{L^{p_1}} + \|v_1\|_{H_0^1}$ , where

$$(4.25) \quad \|w\|_{H_0^1} = \left( \sum_{i=1}^n \|D_i w\|_{L^2}^2 \right)^{\frac{1}{2}}, \quad D_i w = \frac{\partial w}{\partial x_i},$$

$V_1$  is a reflexive Banach space and  $V_1 \subset L^{p_1}(\Omega)$ ,  $V_1 \subset H_0^1(\Omega)$  with continuous injections,  $V_2$  normed by

$$\|v_2\|_2 = \|v_2\|_{W_0^{1,p_2}} = \left( \sum_{i=1}^n \|D_i v_2\|_{L^{p_2}}^{p_2} \right)^{1/p_2}$$

is also a reflexive Banach space and, as  $L^{p_2}(\Omega) \subset L^2(\Omega)$  because  $\Omega$  is bounded and  $p_2 \geq 2$ ,  $V_2 \subset H_0^1(\Omega)$  with continuous injection for the norm (4.25).

Let us take

$$J_1(v_1) = \frac{1}{p_1} \|v_1\|_{L^{p_1}}^{p_1}, \quad J_2(v_2) = \frac{1}{p_2} \|v_2\|_{W_0^{1,p_2}}^{p_2}$$

$$b(v_2, v_1) = \sum_{i=1}^n \int_{\Omega} D_i v_1 D_i v_2 dx.$$

Let  $f_i \in L^{p_i'}(\Omega)$ ,  $\frac{1}{p_i'} + \frac{1}{p_i} = 1$ , let us take  $L_i(v_i) = \int_{\Omega} f_i v_i dx$ .

So, we take

$$(4.26) \quad J(v_1, v_2) = \frac{1}{p_1} \|v_1\|_{L^{p_1}}^{p_1} - \frac{1}{p_2} \|v_2\|_{W_0^{1,p_2}}^{p_2} + \sum_{i=1}^n \int_{\Omega} D_i v_1 D_i v_2 dx -$$

$$- \int_{\Omega} f_1 v_1 dx + \int_{\Omega} f_2 v_2 dx.$$

Assumptions (4.15) to (4.18) are satisfied. Particularly,  $J_i$  is strictly convex since  $L^{p_i}(\Omega)$  and  $W_0^{1,p_i}(\Omega)$  are uniformly convex spaces and  $p_i \geq 2$ . In addition  $J_i$  is  $G$ -derivable and it is proved (see for instance [9]) that

$$J_1'(v_1) = |v_1|^{p_1-2} v_1 \in L^{p_1'}(\Omega) \subset V_1'$$

$$J_2'(v_2) = - \sum_{i=1}^n D_i |D_i v_2|^{p_2-2} D_i v_2 \in W^{-1,p_2'} = V_2'.$$

So, the saddle-point of (4.26) exists and is unique and is characterized by

$$(4.27) \quad \int_{\Omega} |u_1|^{p_1-2} u_1 (v_1 - u_1) dx + b(u_2, v_1 - u_1) \geq \int_{\Omega} f_1 (v_1 - u_1) dx$$

$$\forall v_1 \in K_1, u_1 \in K_1,$$

and

$$(4.28) \quad -b(v_2 - u_2, u_1) - \sum_{i=1}^n \int_{\Omega} D_i |D_i u_2|^{p_i-2} D_i u_2 (v_2 - u_2) dx \geq \\ \geq \int_{\Omega} f_2 (v_2 - u_2) dx, \quad \forall v_2 \in K_2, u_2 \in K_2.$$

If  $K_i = V_i, i = 1, 2$ , coupled variational inequations (4.27), (4.28) become equations and are equivalent to

$$(4.29) \quad \begin{cases} |u_1|^{p_1-2} u_1 - \Delta u_2 = f_1 \\ \Delta u_1 - \sum_{i=1}^n D_i |D_i u_2|^{p_i-2} D_i u_2 = f_2 \text{ in the sens of distributions in } \Omega. \\ u_1 = u_2 = 0 \text{ on } \Gamma. \end{cases}$$

§ 5. Control of variational elliptic problems with to antagonistic controls.

5.1. Statement of the problem.

The situation is that of [7] chapter 2.

Let  $V$  and  $H$  be to real Hilbert spaces. We denote by  $\| \cdot \|$  (resp.  $| \cdot |$ ) the norm in  $V$  (resp.  $H$ ) and by  $((, ))$  (resp.  $(, )$ ) the associated scalar products. We suppose.

(5.1)  $V \subset H$  algebraically and topologically, and  $V$  dense in  $H$ , so that, identifying  $H$  to its dual, we can write

(5.2)  $V \subset H \subset V'$  algebraically and topologically, where  $V' = \text{dual of } V$ , each space being dense in the following one.

Let

$$(5.3) \quad \begin{cases} a(\varphi, \psi) \text{ be a continuous bilinear form on } V, \text{ coercive :} \\ a(\varphi, \varphi) \geq \alpha \| \varphi \|^2, \quad \forall \varphi \in V, \alpha > 0. \end{cases}$$

Let  $L$  be an element of  $V'$ . We also denote  $(, )$  the duality between  $V'$  and  $V$ .

Then we know ([7]), that there exists a unique  $y \in V$  such that

$$(5.4) \quad a(y, \psi) = (L, \psi) \quad \forall \psi \in V.$$



The form  $a(\varphi, \psi)$  defines an operator  $A \in \mathcal{L}(V, V')$  by

$$(5.5) \quad a(\varphi, \psi) = (A\varphi, \psi),$$

and (5.4) is equivalent to

$$(5.6) \quad Ay = L.$$

The considered control problem is then the following one :

Let us give the Hilbert spaces  $\mathcal{U}_i$  of controls and the operators

$$(5.7) \quad B_i \in \mathcal{L}(\mathcal{U}_i, V'), \quad i = 1, 2.$$

Consider a system governed, by the operator  $A$ . For each pair

$$(u_1, u_2) \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2,$$

the state of the system is given by  $y$ , solution of

$$Ay = L + B_1 u_1 + B_2 u_2, \quad y \in V.$$

$y$  depends of  $u_1$  and  $u_2$ ; we write  $y(u_1, u_2)$ . Therefore

$$(5.8) \quad Ay(u_1, u_2) = L + B_1 u_1 + B_2 u_2,$$

that defines  $y(u_1, u_2)$  uniquely.

We then give the observation

$$(5.9) \quad z(u_1, u_2) = Cy(u_1, u_2),$$

where  $C \in \mathcal{L}(V, \mathcal{H})$ ,  $\mathcal{H}$  real Hilbert space.

At last we give

$$(5.10) \quad \begin{cases} N_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{U}_i), N_i \text{ symmetrical and verifying :} \\ (N_i u_i, u_i) \geq v_i \|u_i\|^2, \quad v_i \geq 0, \quad i = 1, 2. \end{cases}$$

To each pair of controls  $(u_1, u_2)$  is associated the value of the cost function

$$(5.11) \quad J(u_1, u_2) = \|Cy(u_1, u_2) - z_d\|_{\mathcal{H}}^2 + (N_1 u_1, u_1) + (N_2 u_2, u_2),$$

where  $z_d$  is given in  $\mathcal{H}$ .

Let

$$(5.12) \quad \mathcal{U}_{ad}^i = \text{closed convex set in } \mathcal{U}_i, \quad i = 1, 2.$$

sets of admissible controls.

PROBLEM. Find  $(u_1, u_2)$ , saddle-point of  $J$  on  $\mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2$ .

5.2. Results about existence and uniqueness. From (5.8), the mapping

$$(u_1, u_2) \rightarrow y(u_1, u_2)$$

is affine and continuous from  $\mathcal{U}_1 \times \mathcal{U}_2$  to  $V$ . So there exists  $G_i \in \mathcal{L}(\mathcal{U}_i, V)$  such that

$$(5.13) \quad y(u_1, u_2) - y(0, 0) = G_1 u_1 + G_2 u_2,$$

where  $G_i u_i$  is the solution of

$$(5.14) \quad Ay_i = B_i u_i, \quad y_i \in V, \quad i = 1, 2.$$

Let us write  $J(u_1, u_2)$  in the form

$$J(u_1, u_2) = \|Cy(u_1, u_2) - y(0, 0) + Cz_d - z_d\|_{\mathcal{H}}^2 + (N_1 u_1, u_1) - (N_2 u_2, u_2).$$

Let us take

$$(5.15) \quad a_i(u_i, v_i) = \varepsilon_i (CG_i u_i, CG_i v_i)_{\mathcal{H}} + (N_i u_i, v_i), \quad \varepsilon_1 = 1, \quad \varepsilon_2 = -1,$$

$$(5.16) \quad b(u_2, u_1) = (CG_2 u_2, CG_1 u_1)_{\mathcal{H}},$$

$$(5.17) \quad L_i(v_i) = \varepsilon_i (z_d - Cy(0, 0), CG_i v_i)_{\mathcal{H}}.$$

The forms  $a_i$  and  $b$  are bilinear and continuous on  $\mathcal{U}_i$  and  $\mathcal{U}_2 \times \mathcal{U}_1$  respectively and we have

$$(5.18) \quad J(u_1, u_2) = a_1(u_1, u_1) - a_2(u_2, u_2) + 2\{b(u_2, u_1) - L_1(u_1) + L_2(u_2)\} + \|Cy(0, 0) - z_d\|_{\mathcal{H}}^2.$$

$J(u_1, u_2)$  is then, with an excepted constant which plays no roll in the saddle-point problem, of type (1.1). Then we can apply theorems 2.1, 2.2, 2.3, if however we suppose

$$(5.19) \quad r_2 \geq \|CG_2\|_{(\mathcal{U}_2, \mathcal{H})}^2.$$

In fact, since  $\|CG_1 v_1\|_{\mathcal{H}}^2 \geq 0$ , we have, from (5.10),

$$a_1(v_1, v_1) \geq r_1 \|v_1\|^2, \quad \forall v_1 \in \mathcal{U}_1.$$

In the other hands,

$$a_2(v_2, v_2) = (N_2 v_2, v_2) - \|CG_2 v_2\|_{\mathcal{Q}}^2 \geq (\nu_2 - \|CG_2\|^2) \|v_2\|^2, \quad \forall v_2 \in \mathcal{U}_2.$$

Therefore, if (5.19) is fulfilled,  $a_i(v_i, v_i)$  satisfies the assumption of type (2.3).

But, from (5.3) (5.4) and (5.14) we have

$$\alpha \|y_2\|^2 \leq a(y_2, y_2) = (B_2 u_2, y_2) \leq \|B_2 u_2\|_{V'} \|y_2\|$$

and therefore

$$(5.20) \quad \|CG_2\|_{\mathcal{L}(\mathcal{U}_2, \mathcal{Q})} \leq (\|C\|_{\mathcal{L}(V, \mathcal{Q})} \|B_2\|_{\mathcal{L}(\mathcal{U}_2, V')})/\alpha \triangleq \mathfrak{G}.$$

Then we have the following result:

**THEOREM 5.1.** Under one of the following assumptions, the set  $X$  of optimal pairs (saddle-points of  $J$ ) is not empty, and of the form  $X_1 \times X_2$  where  $X_i$  is a not empty closed convex set in  $\mathcal{U}_i$ :

- (i)  $\nu_1 > 0$  and  $\nu_2 > \mathfrak{G}^2$ .
- (ii)  $\nu_1 = 0$ ,  $\nu_2 = \mathfrak{G}^2$ ,  $\mathcal{U}_{ad}^2$  bounded  $i = 1, 2$ ,
- (iii)  $\nu_1 > 0$ ,  $\nu_2 = \mathfrak{G}^2$ ,  $\mathcal{U}_{ad}^2$  bounded
- (iv)  $\nu_1 = 0$ ,  $\mathcal{U}_{ad}^1$  bounded,  $\nu_2 > \mathfrak{G}^2$ .

In addition, in the case (i),  $X_i$  is reduced to one element  $i = 1, 2$ ; in the case (iii) (resp. (iv))  $X_1$  (resp.  $X_2$ ) is reduced to one element.

### 5.3 Characterizing the optimal controls.

Let us rewrite here the relations (1.3)-(1.4). We get

$$(5.21) \quad (C(G_1 u_1 + G_2 u_2 + y(0, 0)) - z_a, CG_1(v_1 - u_1))_{\mathcal{Q}} + (N_1 u_1, v_1 - u_1) \geq 0.$$

$$(5.22) \quad (C(G_1 u_1 + G_2 u_2 + y(0, 0)) - z_a, CG_2(v_2 - u_2))_{\mathcal{Q}} - (N_2 u_2, v_2 - u_2) \leq 0.$$

As in [7] let us introduce  $O^* \in \mathcal{L}(\mathcal{Q}', V')$  the adjoint of  $O$ ,  $A =$  canonical isomorphism from  $\mathcal{H}$  onto  $\mathcal{H}'$ , and  $A^* =$  adjoint of  $A$ . (5.21)-(5.22) is equivalent (according to (5.13)) to

$$(5.23) \quad (O^* A(Cy(u_1, u_2) - z_a), y(v_1, u_2) - y(u_1, u_2)) + (N_1 u_1, v_1 - u_1) \geq 0,$$

$$(5.24) \quad (O^* A(Cy(u_1, u_2) - z_a), y(u_1, v_2) - y(u_1, u_2)) - (N_2 u_2, v_2 - u_2) \leq 0.$$

For a pair of controls  $(v_1, v_2)$ , define the adjoint state  $p(v_1, v_2) \in V$  by

$$(5.25) \quad A^* p(v_1, v_2) = C^* A (Cy(v_1, v_2) - z_d).$$

(5.23)-(5.24) is then equivalent to

$$(5.26) \quad (p(u_1, u_2), B_1(v_1 - u_1))_{V V'} + (N_1 u_1, v_1 - u_1) \geq 0,$$

$$(5.27) \quad (p(u_1, u_2), B_2(v_2 - u_2))_{V V'} - (N_2 u_2, v_2 - u_2) \leq 0.$$

Let  $B_i^* \in \mathcal{L}(V, \mathcal{U}_i)$  be the adjoint of  $B_i$ , and  $A_i$  the canonical isomorphism from  $\mathcal{U}_i$  onto  $\mathcal{U}_i'$ . (5.26) (5.27) is equivalent to

$$(5.28) \quad (B_1^* p(u_1, u_2) + A_1 N_1 u_1, v_1 - u_1)_{\mathcal{U}_1' \mathcal{U}_1} \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1,$$

$$(5.29) \quad (B_2^* p(u_1, u_2) - A_2 N_2 u_2, v_2 - u_2)_{\mathcal{U}_2' \mathcal{U}_2} \leq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2.$$

Then we have the theorem :

**THEOREM 5.2.** *Under the assumptions of theorem 5.1, the set  $X$  is characterized by*

$$(5.30) \quad (B_1^* p(u_1, u_2) + A_1 N_1 u_1, u_1) = \inf_{v_1 \in \mathcal{U}_{ad}^1} (B_1^* p(u_1, u_2) + A_1 N_1 u_1, v_1).$$

$$(5.31) \quad (B_2^* p(u_1, u_2) - A_2 N_2 u_2, u_2) = \sup_{v_2 \in \mathcal{U}_{ad}^2} (B_2^* p(u_1, u_2) - A_2 N_2 u_2, v_2),$$

**REMARK 5.1.** If  $\mathcal{U}_{ad}^1$  is a pointed closed convex cone, we can show as in [7], that (5.30) is equivalent to

$$(5.32) \quad \begin{cases} u_1 \in \mathcal{U}_{ad}^1, \\ (B_1^* p(u_1, u_2) + N_1 u_1, v_1)_{\mathcal{U}_1' \mathcal{U}_1} \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1, \\ (B_1^* p(u_1, u_2) + N_1 u_1, u_1) = 0. \end{cases}$$

We have the same remark for  $\mathcal{U}_{ad}^2$ .

§ 6. **First applications. Distributed observation.**

6.1. *System governed by a Dirichlet problem; the two controls are distributed in  $\Omega$ .*

We take  $V = H_0^1(\Omega)$ , and  $H = L^2(\Omega)$  where  $\Omega$  is an open bounded set in  $\mathbb{R}^n$  with regular boundary  $\Gamma$ .  $A$  is the elliptic operator of second order,

$$(6.1) \quad A\varphi = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \varphi \right) + \alpha_0 \varphi,$$

defined by the form  $a(\varphi, \psi)$  which verifies

$$(6.2) \quad a(\varphi, \varphi) \geq \alpha \|\varphi\|^2, \quad \forall \varphi \in H_0^1(\Omega), \quad \alpha > 0.$$

(We take up again, with the same notations, the point 2.1 of [43], p. 55), choose

$$(6.3) \quad \mathcal{U}_i = H (= L^2(\Omega)), \quad \text{therefore } A_i = \text{identity},$$

$$(6.4) \quad B_i = \text{identity},$$

$$(6.5) \quad \left\{ \begin{array}{l} C = \text{injection from } V \text{ into } H (\mathcal{C} = H, \text{ therefore } A = \text{identity}), \\ L \in V' \text{ defined by } L(\varphi) = \int_{\Omega} f \varphi \, dx, \text{ where } f \text{ is given in } L^2(\Omega). \end{array} \right.$$

Therefore the state  $y(u_1, u_2)$  is given, by the solution of the Dirichlet problem

$$(6.6) \quad \left\{ \begin{array}{l} Ay(u_1, u_2) = f + u_1 + u_2, \\ y(u_1, u_2) \in H_0^1(\Omega), \end{array} \right.$$

and we look for the saddle-points of

$$(6.7) \quad \int_{\Omega} |y(v_1, v_2) - z_a|^2 \, dx + (N_1 v_1, v_1) - (N_2 v_2, v_2)$$

on  $\mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2$ , where  $\mathcal{U}_{ad}^i =$  closed convex set in  $\mathcal{U}^i (= L^2(\Omega))$ .

Let us see what becomes condition (5.19) in the present situation: operator  $G_2$  is defined by the Dirichlet problem

$$(6.8) \quad (Az = u_2, z \in H_0^1(\Omega)) \iff (z = G_2 u_2),$$

according to (6.3), (6.5), we must find an overestimation of  $\|G_2\|_{\mathcal{L}(H,H)}^2$ . (6.8) is equivalent to

$$(6.9) \quad a(z, \varphi) = \int_{\Omega} u_2 \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega).$$

Let us take  $\varphi = z$  in (6.9). (6.2) and the Schwarz inequality give

$$(6.10) \quad \alpha |z|^2 \leq \alpha \|z\|^2 \leq a(z, z) = \int_{\Omega} u_2 z \, dx \leq |u_2| |z|.$$

Then

$$(6.11) \quad |G_2 u_2| \leq \frac{1}{\alpha} |u_2|,$$

and

$$(6.12) \quad \|CG_2\|_{\mathcal{L}(\mathcal{U}_2, \mathcal{U})} = \|G_2\|_{\mathcal{L}(H, H)} \leq \frac{1}{\alpha}.$$

Consequently, (5.19) will be verified if

$$(6.13) \quad r_2 \geq \frac{1}{\alpha^2}.$$

Then we can apply theorems 5.1 and 5.2. Under the assumption (for example)

$$r_1 > 0 \quad \text{and} \quad r_2 > \frac{1}{\alpha^2},$$

the set of saddle-points is not empty, and is reduced to one element characterized by

$$(6.14) \quad \left\{ \begin{array}{l} Ay(u_1, u_2) = f + u_1 + u_2 \text{ in } \Omega, \quad y(u_1, u_2) = 0 \text{ on } \Gamma, \\ A^*p(u_1, u_2) = y(u_1, u_2) - z_d \text{ in } \Omega, \quad p(u_1, u_2) = 0 \text{ on } \Gamma, \\ \int_{\Omega} (p(u_1, u_2) + N_1 u_1)(v_1 - u_1) \, dx \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1, \\ \int_{\Omega} (p(u_1, u_2) - N_2 u_2)(v_2 - u_2) \, dx \leq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2, \\ (u_1, u_2) \in \mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2. \end{array} \right.$$

EXAMPLE 6.1. *Unconstrained case* :  $\mathcal{U}_{ad}^i = \mathcal{U}_i$ ,  $i = 1, 2$ . The two last conditions become

$$\begin{cases} p(u_1, u_2) + N_1 u_1 = 0, \\ p(u_1, u_2) - N_2 u_2 = 0. \end{cases}$$

Then we can eliminate  $u_1, u_2$ , and the optimal controls are given by the following rule :

(i) We solve the partial differential equations system :

$$(6.15) \quad \begin{cases} Ay + N_1^{-1} p - N_2^{-1} p = f \text{ in } \Omega, \\ A^* p - y = -z_a \text{ in } \Omega, \\ y = 0, p = 0 \text{ on } \Gamma; \end{cases}$$

(ii) Then

$$(6.16) \quad \begin{cases} u_1 = -N_1^{-1} p, \\ u_2 = N_2^{-1} p. \end{cases}$$

*Particular case* :  $N_1 = N_2 = N$ . The optimal controls are given by the rule :

(i) We solve

$$\begin{cases} Ay = f \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma; \end{cases}$$

(ii) We solve

$$\begin{cases} A^* p - y = -z_a \text{ in } \Omega, \\ p = 0 \text{ on } \Gamma; \end{cases}$$

(iii) Then

$$u_1 = -u_2 = -N^{-1} p.$$

The « value » of the game is then

$$J(u_1, u_2) = \int_{\Omega} |y - z_a|^2 dx,$$

and can be obtained without computing the optimal controls.

EXAMPLE 6.2. Let us take now

$$(6.17) \quad \mathcal{U}_{ad}^i = \{v_i \mid v_i \geq 0 \text{ a. e. in } \Omega\}, \quad i = 1, 2.$$

according to remark 5.1, we get

$$(6.18) \quad \begin{cases} u_1 \geq 0 \text{ in } \Omega, & u_2 \geq 0 \text{ in } \Omega, \\ p(u_1, u_2) + N_1 u_1 \geq 0 \text{ in } \Omega, & p(u_1, u_2) - N_2 u_2 \leq 0 \text{ in } \Omega, \\ u_1 (p(u_1, u_2) + N_1 u_1) = 0 \text{ in } \Omega, & u_2 (p(u_1, u_2) - N_2 u_2) = 0. \end{cases}$$

We can eliminate one of the two controls, for example  $u_1$ , as the following :

$$(6.19) \quad \begin{cases} Ay - f - u_2 \geq 0 \text{ in } \Omega, \\ A^*p - y = -z_d \text{ in } \Omega, \\ p + N_1 (Ay - f - u_2) \geq 0 \text{ in } \Omega, \\ (Ay - f - u_2)(p + N_1 (Ay - f - u_2)) = 0 \text{ in } \Omega, \\ u_2 \geq 0 \text{ in } \Omega, \\ p - N_2 u_2 \leq 0 \text{ in } \Omega, \\ u_2 (p - N_2 u_2) = 0 \text{ in } \Omega, \\ y = 0, p = 0 \text{ on } \Gamma, \end{cases}$$

and then  $u_1 = Ay - f - u_2$ .

We shall see, in 6.4, how we can eliminate the two controls when  $N_i = \nu_i I$  ( $I = \text{identity}$ ).

EXAMPLE 6.3.  $\mathcal{U}_{ad}^1 = \mathcal{U}_1 (= L^2(\Omega))$ ;  $\mathcal{U}_{ad}^2$  as in (6.17). The optimal controls are obtained as the following

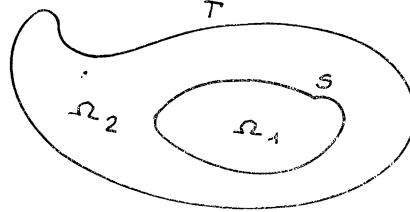
We solve

$$(6.20) \quad \begin{cases} Ay + N_1^{-1} p - f \geq 0 \text{ in } \Omega, \\ A^*p - y = -z_d \text{ in } \Omega, \\ p - N_2 (Ay + N_1^{-1} p - f) \leq 0 \text{ in } \Omega, \\ (Ay + N_1^{-1} p - f)(p - N_2 (Ay + N_1^{-1} p - f)) = 0 \text{ in } \Omega, \\ y = 0, p = 0 \text{ on } \Gamma, \end{cases}$$

then  $u_1 = -N^{-1}p$ , and  $u_2 = Ay + u_1 - f$ .



6.2. *Dirichlet problem, variant.* Let us suppose  $\Omega = \Omega_1 \cup \Omega_2 \cup S$  (see fig.)



We take

$$\mathcal{U}_i = L^2(\Omega_i), \quad A_i = \text{identity},$$

$$B_i u_i = \begin{cases} u_i & \text{in } \Omega_i, \\ 0 & \text{in } \Omega_{3-i}, \end{cases}$$

$$\mathcal{H} = L^2(\Omega_2), \quad A = \text{identity}.$$

We then observe in  $\Omega_2$ ,

$$z(u_1, u_2) = My(u_1, u_2)$$

$M = \text{characteristic function of } \Omega_2.$

So the state is given by the solution of the problem

$$\begin{cases} Ay(u_1, u_2) = f + u_1 & \text{in } \Omega_1, \\ Ay(u_1, u_2) = f + u_2 & \text{in } \Omega_2, \\ y(u_1, u_2) = 0 & \text{on } \Gamma. \end{cases}$$

The cost function is

$$(6.22) \quad J(v_1, v_2) = \int_{\Omega_1} |y(v_1, v_2) - z_d|^2 dx + (N_1 v_1, v_1) - (N_2 v_2, v_2),$$

with  $z_d$  given in  $L^2(\Omega_2)$ .

$G_2$  is defined by the problem

$$(6.23) \quad \begin{cases} Az = 0 & \text{in } \Omega_1, \\ Az = u_2 & \text{in } \Omega_2, \\ z = 0 & \text{on } \Gamma, \end{cases}$$

or, which is equivalent,

$$a(z, \varphi) = \int_{\Omega_1} u_2 \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega).$$

From which we deduce

$$\alpha \|z\|_2^2 \leq \alpha \|z\|^2 \leq \alpha \|z\|^2 \leq a(z, z) = \int_{\Omega_1} u_2 z \leq \|u_2\|_2 \|z\|_2,$$

where  $|\cdot|_i$  denotes the norm in  $L^2(\Omega_i)$ .

We have still the overestimation (6.12), and the condition (6.13) is still sufficient for (5.19) to be satisfied. Applying theorems 5.1 et 5.2, under the assumption

$$\nu_1 > 0 \quad \text{and} \quad \nu_2 > \frac{1}{\alpha^2}.$$

We get the conditions which characterize the unique optimal pair :

$$(6.24) \quad \left\{ \begin{array}{l} Ay(u_1, u_2) = f + u_1 \text{ in } \Omega_1, \\ Ay(u_1, u_2) = f + u_2 \text{ in } \Omega_2, \\ A^*p(u_1, u_2) = 0 \text{ in } \Omega_1, \\ A^*p(u_1, u_2) = y(u_1, u_2) - z_d \text{ in } \Omega_2, \\ y(u_1, u_2) = 0, \quad p(u_1, u_2) = 0 \text{ on } \Gamma, \\ \int_{\Omega_1} (p(u_1, u_2) + N_1 u_1)(v_1 - u_1) \, dx \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1 \\ \int_{\Omega_2} (p(u_1, u_2) - N_2 u_2)(v_2 - u_2) \, dx \leq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2, \\ u_1 \in \mathcal{U}_{ad}^1, \quad u_2 \in \mathcal{U}_{ad}^2. \end{array} \right.$$

EXAMPLE 6.4. We can take again  $\mathcal{U}_{ad}^i$  as in 6.1. For example,  $\mathcal{U}_{ad}^i = \mathcal{U}_{ad}$  gives :

$$(6.25) \quad \left\{ \begin{array}{l} Ay + N_1^{-1} p_1 = f \text{ in } \Omega_1, \\ Ay - N_2^{-1} p_2 = f \text{ in } \Omega_2, \\ A^*p - y = 0 \text{ in } \Omega_1, \\ A^*p - y = -z_d \text{ in } \Omega_2, \\ y = 0, p = 0 \text{ on } \Gamma, \\ u_1 = -N_1^{-1} p_1, u_2 = N_2^{-1} p_2, \end{array} \right.$$

where  $p_i$  denotes the restriction of  $p$  to  $\Omega_i$ .

### 6.3. System governed by a Neumann problem.

We take  $V = H^1(\Omega)$ ,  $H = L^2(\Omega)$ . The operator  $A$  is defined as 6.1. For the same type of observation, we can consider several variants according to the nature of the controls (distributed or frontier). The situation is the following one:

$$\mathcal{U}_1 = L^2(\Gamma), \quad A_1 = \text{identity},$$

$$B_1 \text{ defined by } B_1 u_1(\varphi) = \int_{\Gamma} u_1 \varphi d\Gamma,$$

$$\mathcal{U}_2 = L^2(\Omega), \quad A_2 = \text{identity}, \quad B_2 = \text{identity},$$

$$C = \text{injection from } V \text{ into } H.$$

Let us take  $L$  by

$$(6.26) \quad (L, \varphi) = \int_{\Omega} f \varphi dx + \int_{\Gamma} g \varphi d\Gamma, \quad f \in L^2(\Omega), \quad g \in H^{-\frac{1}{2}}(\Gamma).$$

The state  $y(u_1, u_2)$  is then given by the solution of

$$(6.27) \quad \left\{ \begin{array}{l} Ay(u_1, u_2) = f + u_2 \text{ in } \Omega \\ \frac{\partial y}{\partial \nu_A} = g + u_1 \text{ on } \Gamma. \end{array} \right.$$

The cost function is given as in (6.7).

The operator  $G_2$  is defined by the problem

$$(6.28) \quad \begin{cases} Az = u_2, & \text{in } \Omega, \\ \frac{\partial z}{\partial \nu_A} = 0, & \text{on } \Gamma, \end{cases}$$

that is to say  $a(z, \varphi) = \int_{\Omega} u_2 \varphi \, dx, \forall \varphi \in H^1(\Omega)$ . Then

$$\|CG_2\| \leq \frac{1}{\alpha}.$$

Therefore, if  $\nu_1 > 0$  and  $\nu_2 > \frac{1}{\alpha^2}$ , the optimal pair  $(u_1, u_2)$  exists and is unique, and is given by the solution of

$$(6.29) \quad \begin{cases} Ay(u_1, u_2) = f + u_2 \text{ in } \Omega, & A^*p(u_1, u_2) = y(u_1, u_2) - z_a \text{ in } \Omega, \\ \frac{\partial v}{\partial \nu_A} = g + u_1 \text{ on } \Gamma, & \frac{\partial p}{\partial \nu_{A^*}}(u_1, u_2) = 0 \text{ on } \Gamma, \\ \int_{\Gamma} (p(u_1, u_2) + N_1 u_1)(v_1 - u_1) \, d\Gamma \geq 0, & \forall v_1 \in \mathcal{U}_{ad}^1, u_1 \in \mathcal{U}_{ad}^1, \\ \int_{\Omega} (p(u_1, u_2) - N_2 u_2)(v_2 - u_2) \, dx \leq 0, & \forall v_2 \in \mathcal{U}_{ad}^2, u_2 \in \mathcal{U}_{ad}^2. \end{cases}$$

EXAMPLE 6.5. Let us take

$$\mathcal{U}_{ad}^1 = \{v_1 \mid v_1 \in L^2(\Gamma), v_1 \geq 0 \text{ a. e. on } \Gamma\},$$

$$\mathcal{U}_{ad}^2 = \mathcal{U}_2 (= L^2(\Omega)).$$

Then we can eliminate  $u_1$  and  $u_2$ . The rule is:

We solve

$$Ay - N_2^{-1}p = f \text{ in } \Omega, \quad A^*p - y = -z_a \text{ in } \Omega,$$

$$\frac{\partial y}{\partial \nu_A} - g \geq 0 \text{ on } \Gamma, \quad \frac{\partial p}{\partial \nu_{A^*}} = 0 \text{ on } \Gamma,$$

$$p|_{\Gamma} + N_1 \left( \frac{\partial y}{\partial \nu_A} \Big|_{\Gamma} - g \right) \geq 0, \quad \left( \frac{\partial y}{\partial \nu_A} \Big|_{\Gamma} - g \right) \left( p|_{\Gamma} + N_1 \left( \frac{\partial y}{\partial \nu_A} \Big|_{\Gamma} - g \right) \right) = 0,$$

Then

$$u_1 = \frac{\partial y}{\partial v_A} |_{\Gamma} - g, \quad u_2 = N_2^{-1} p.$$

EXAMPLE 6.6.

$$\mathcal{U}_{ad}^1 = \{v_1 \mid v_1 \geq 0 \text{ on } \Gamma\}, \quad \mathcal{U}_{ad}^2 = \{v_2 \mid v_2 \geq 0 \text{ in } \Omega\}.$$

We solve

$$\left\{ \begin{array}{l} Ay = f \geq 0 \text{ in } \Omega, \quad A^*p - y = -z_d \text{ in } \Omega, \\ \frac{\partial y}{\partial v_A} - g \geq 0 \text{ on } \Gamma, \quad \frac{\partial p}{\partial v_{A^*}} = 0 \text{ on } \Gamma, \\ p |_{\Gamma} + N_1 \left( \frac{\partial y}{\partial v_A} |_{\Gamma} - g \right) \geq 0, \quad p - N_2 (Ay - f) \leq 0, \\ \left( \frac{\partial y}{\partial v_A} |_{\Gamma} - g \right) \left( p |_{\Gamma} - N_1 \left( \frac{\partial y}{\partial v_A} |_{\Gamma} - g \right) \right) = 0, \quad (Ay - f) (p - N_2 (Ay - f)) = 0, \end{array} \right.$$

and the optimal controls are given by

$$u_1 = \frac{\partial y}{\partial v_A} |_{\Gamma} - g, \quad u_2 = Ay - f.$$

6.4. *The case*  $N_i = v_i I$ . Conditions (5.28)-(5.29) become

$$(6.30) \quad \left\{ \begin{array}{l} (A_1^{-1} B_1^* p(u_1, u_2) + v_1 u_1, v_1 - u_1)_{\mathcal{U}_1} \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1, \\ (A_2^{-1} B_2^* p(u_2, u_2) - v_2 u_2, v_2 - u_2)_{\mathcal{U}_2} \geq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2, \end{array} \right.$$

which is equivalent, from the properties of the projection operator (denoted here by  $P_i$ ) onto the closed convex  $\mathcal{U}_{ad}^i$ , and if  $v_k > 0$ ,  $k = 1, 2$ , to

$$(6.31) \quad \left\{ \begin{array}{l} u_1 = P_1 \left( -\frac{1}{v_1} A_1^{-1} B_1^* p(u_1, u_2) \right), \\ u_2 = P_2 \left( \frac{1}{v_2} A_2^{-1} B_2^* p(u_1, u_2) \right). \end{array} \right.$$

Let us take up again example 6.2. The optimal controls are given by

$$u_1 = \left( -\frac{p}{v_1} \right)^+ = \frac{p^-}{v_1}, \quad u_2 = \frac{p^+}{v_2},$$

where

$$\xi^+(x) = \begin{cases} \xi(x), & \text{iff } \xi(x) \geq 0, \\ 0, & \text{iff } \xi(x) < 0, \end{cases} \quad \xi^-(x) = \begin{cases} 0, & \text{iff } \xi(x) > 0, \\ -\xi(x), & \text{iff } \xi(x) \leq 0, \end{cases}$$

$(y, p)$  being the solution of the non linear problem

$$(6.32) \quad \begin{cases} Ay - \frac{p^-}{r_1} - \frac{p^+}{r_2} = f \text{ in } \Omega, \\ A^*p - y = -z_d \text{ in } \Omega, \\ y = 0, p = 0 \text{ on } \Gamma. \end{cases}$$

Let us take again example 6.6. The optimal controls are given by

$$u_1 = \frac{p^-}{r_1}|_{\Gamma}, \quad u_2 = \frac{p^+}{r_2}$$

$(y, p)$  being the solution of

$$(6.33) \quad \begin{cases} Ay - \frac{p^+}{r_2} = f \text{ in } \Omega, \\ A^*p - y = -z_d \text{ in } \Omega, \\ \frac{\partial y}{\partial \nu_A} - \frac{p^-}{r_1} = g, \quad \frac{\partial p}{\partial \nu_{A^*}} = 0, \text{ on } \Gamma, \end{cases}$$

REMARK 6.1. We can also eliminate one of the two controls, for example  $u_1$ , in the state equation, which leads to an optimal control problem (non linear if  $\mathcal{U}_{ad}^1 \neq \mathcal{U}_1$ ) with control  $u_2$  and state the pair  $(y, p)$ ;  $u_1$  is then given by (6.31).

## § 7. Frontier observation.

### 7.1. System governed by a Neumann problem.

We take the situation of point 6.3. The state is then given by

$$(7.1) \quad \begin{cases} Ay(v_1, v_2) = f + u_2 \text{ in } \Omega, f, v_2 \in L^2(\Omega), \\ \frac{\partial y}{\partial \nu_A}(v_1, v_2) = g + u_1 \text{ on } \Gamma, g, v_1 \in L^2(\Gamma), \end{cases}$$

$y(v_1, v_2) \in H^1(\Omega)$ . Therefore  $y(v_1, v_2)|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ , and if  $M \in L^\infty(\Gamma)$ , we can consider the cost function,  $z_d$  being given in  $L^2(\Gamma)$ ,

$$(7.2) \quad J(v_1, v_2) = \int_\Gamma |My(v_1, v_2) - z_d|^2 d\Gamma + (N_1 v_1, v_1) - (N_2 v_2, v_2),$$

then  $C: y \rightarrow My|_\Gamma$ .

The mapping  $y \rightarrow y|_\Gamma$  being linear and continuous from  $H^1(\Omega)$  onto  $H^{\frac{1}{2}}(\Gamma)$ , there exists a constant  $\gamma_0 > 0$  such that

$$\|c\| \leq \gamma_0 \sup_\Gamma \text{ess } |M|.$$

As in 6.3. we get  $\|G_2\| \leq \frac{1}{\alpha}$  and therefore the existence and uniqueness of the optimal couple, if  $r_1 > 0$  and  $r_2 > (\gamma_0 \sup_\Gamma \text{ess } |M|/\alpha)^2$ . The adjoint state is defined by

$$(7.3) \quad \begin{cases} A^* p(u_1, u_2) = 0 & \text{in } \Omega, \\ \frac{\partial p}{\partial v_{A^*}} = M(My(u_1, u_2) - z_d) & \text{on } \Gamma. \end{cases}$$

The optimal controls are given as in 6.3 (solution of (6.29)) excepted for the adjoint state, given by (7.3).

### 7.2. *System governed by a Dirichlet problem. Distributed controls.*

The situation is that of 6.1. The state is given by

$$(7.4) \quad \begin{cases} Ay(v_1, v_2) = f + v_1 + v_2 & \text{in } \Omega, f, v_1, v_2 \in L^2(\Omega), \\ y(v_1, v_2) = 0 & \text{on } \Gamma. \end{cases}$$

If we suppose the coefficients of  $A$  regular enough in order that  $y \in H^2(\Omega)$ , we can define  $\frac{\partial v}{\partial v_A} \in L^2(\Gamma)$  in fact,  $\frac{\partial v}{\partial v_A} \in H^{\frac{1}{2}}(\Gamma)$ , [10] and

$$(7.5) \quad \left\| \frac{\partial y}{\partial v_A} \right\|_{L^2(\Gamma)} \leq \gamma_1 \|y\|_{H^2(\Omega)}, \quad \gamma_1 > 0.$$

We then take the cost function

$$(7.6) \quad J(v_1, v_2) = \left\| \frac{\partial y}{\partial v_A}(v_1, v_2) - z_d \right\|_{L^2(\Gamma)}^2 + (N_1 v_1, v_1) - (N_2 v_2, v_2)$$

where  $z_d$  is given in  $H^{\frac{1}{2}}(\Gamma)$  (cf. remark 7.1 hereafter). Therefore  $\mathcal{H} = L^2(\Gamma)$ . As  $A$  is an isomorphism from  $H^2(\Omega) \cap H_0^1(\Omega)$  onto  $L^2(\Omega)$ , we can take in fact,  $V = H^2(\Omega) \cap H_0^1(\Omega)$  and  $C$  is then the trace mapping  $y \rightarrow \frac{\partial y}{\partial v_A}$ .

Therefore  $\|C\| \leq \gamma_1$  and  $G_2 = A^{-1}$ . Consequently if

$$v_2 \geq \gamma_1^2 \|A^{-1}\|_{\mathcal{L}(L^2(\Omega), H^2(\Omega))}^2,$$

condition (5.19) will be verified, and if  $v_1 > 0$  and  $v_2 > \gamma_1^2 \|A^{-1}\|^2$ , the optimal pair exists and is unique, and is characterized by

$$(7.7) \quad \left( \frac{\partial y}{\partial v_A}(u_1, u_2) - z_d, \frac{\partial y}{\partial v_A}(v_1, u_2) - \frac{\partial y}{\partial v_A}(u_1, u_2) \right)_{L^2(\Gamma)} + (N_1 v_1, v_1 - u_1) \geq 0,$$

$$\forall v_1 \in \mathcal{U}_{ad}^1,$$

$$(7.8) \quad \left( \frac{\partial y}{\partial v_A}(u_1, u_2) - z_d, \frac{\partial y}{\partial v_A}(u_1, v_2) - \frac{\partial y}{\partial v_A}(u_1, u_2) \right)_{L^2(\Gamma)} - (N_2 u_2, v_2 - u_2) \leq 0,$$

$$\forall v_2 \in \mathcal{U}_{ad}^2.$$

got from (5.21), (5.22).

The adjoint state is the unique solution, in  $H^1(\Omega)$ , of

$$(7.9) \quad \begin{cases} A^* p(u_1, u_2) = 0 & \text{in } \Omega, \\ p(u_1, u_2) = - \left( \frac{\partial y}{\partial v_A} - z_d \right) & \text{on } \Gamma. \end{cases}$$

Let us remark that (7.9) defines  $p(u_1, u_2) \in H^1(\Omega)$  as the solution of

$$(7.10) \quad a(\varphi, p - \xi) = -a(\varphi, \xi), \quad \forall \varphi \in H_0^1(\Omega), p - \xi \in H_0^1(\Omega)$$

where  $\xi \in H^1(\Omega)$  is a raising up of  $z_d - \frac{\partial y}{\partial v_A} (\in H^{\frac{1}{2}}(\Gamma))$ .



Then let  $\varphi \in H^2(\Omega)$ . Let us multiply the first equation (7.9) by  $\varphi$  and apply the Green formula (or rather the definition of  $\frac{\partial p}{\partial \nu_{A^*}} \in H^{-3/2}(\Gamma)$ ):

$$(7.12) \quad 0 = \int_{\Omega} A^* p(u_1, u_2) \varphi \, dx = - \int_{\Gamma} \frac{\partial p}{\partial \nu_{A^*}}(u_1, u_2) \varphi \, d\Gamma + \\ + \int_{\Gamma} p(u_1, u_2) \frac{\partial \varphi}{\partial \nu_A} \, d\Gamma + \int_{\Omega} p(u_1, u_2) A \varphi \, dx.$$

Let us take successively in (7.12)

$$\varphi = y(v_1, u_2) - y(u_1, u_2),$$

$$\varphi = y(u_1, v_2) - y(u_1, u_2).$$

We get, according to the second conditions (7.4) and (7.9),

$$(7.13) \quad \int_{\Omega} p(u_1, u_2) (Ay(v_1, u_2) - Ay(u_1, u_2)) \, dx = \\ \left( \frac{\partial y}{\partial \nu_A}(u_1, u_2) - z_d, \frac{\partial y}{\partial \nu_A}(v_1, u_2) - \frac{\partial y}{\partial \nu_A}(u_1, u_2) \right)_{L^2(\Gamma)},$$

$$(7.14) \quad \int_{\Omega} p(u_1, u_2) (Ay(u_1, v_2) - Ay(u_1, u_2)) \, dx = \\ \left( \frac{\partial y}{\partial \nu_A}(u_1, u_2) - z_d, \frac{\partial y}{\partial \nu_A}(u_1, v_2) - \frac{\partial y}{\partial \nu_A}(u_1, u_2) \right)_{L^2(\Gamma)},$$

which, joined to (7.7)-(7.8), and according to the first equation (7.4), gives finally

$$(7.15) \quad \begin{cases} \int_{\Omega} p(u_1, u_2) (v_1 - u_1) \, dx + (N_1 v_1, v_1 - u_1) \geq 0, & \forall v_1 \in \mathcal{U}_{ad}^1, \\ \int_{\Omega} p(u_1, u_2) (v_2 - u_2) \, dx - (N_2 u_2, v_2 - u_2) \leq 0, & \forall v_2 \in \mathcal{U}_{ad}^2. \end{cases}$$

Therefore the optimal pair is given by the solution of

$$(7.16) \quad \left\{ \begin{array}{l} Ay = f + u_1 + u_2 \text{ in } \Omega, \quad A^*p = 0 \text{ in } \Omega \\ y = 0 \quad \text{on } \Gamma, \quad p = z_d - \frac{\partial y}{\partial \nu_A} \text{ on } \Gamma \\ \int_{\Omega} (p + N_1 u_1)(v_1 - u_1) dx \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1, u_1 \in \mathcal{U}_{ad}^1, \\ \int_{\Omega} (p - N_2 u_2)(v_2 - u_2) dx \leq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2, u_2 \in \mathcal{U}_{ad}^2. \end{array} \right.$$

REMARK 7.1. The datum of  $z_d$  in  $H^{\frac{1}{2}}(\Gamma)$  is a little restrictive since we observe in  $L^2(\Gamma)$ . If  $z_d \in L^2(\Gamma)$ , result (7.16) holds but  $p(u_1, u_2)$  must be defined by transposition (see [7] chap. 2, points 4.2 and 4.3).

§ 8. Control of systems governed by an operational differential equation of first order with two antagonistic controls.

8.1. Statement of the problem.

We give two real Hilbert spaces  $V$  and  $H$  as in point 5, and with the same notations:  $V'$  dual of  $V$ ,  $H'$  identified to  $H$  and therefore

$$V \subset H \subset V'.$$

The variable  $t$  denotes the time. We suppose that  $t \in ]0, T[$ ,  $T$  finite fixed. We give a family of bilinear continuous forms on  $V$

$$\varphi, \psi \rightarrow a(t; \varphi, \psi), \quad \forall t \in ]0, T[.$$

Suppose

$$(8.1) \quad \left\{ \begin{array}{l} \forall \varphi, \psi \in V, t \rightarrow a(t; \varphi, \psi) \text{ measurable on } ]0, T[ \text{ and} \\ |a(t; \varphi, \psi)| \leq C \|\varphi\| \|\psi\|, \end{array} \right.$$

$$(8.2) \quad \left\{ \begin{array}{l} \text{there exists } \lambda \text{ such that} \\ a(t; \varphi, \varphi) + \lambda \|\varphi\|^2 \geq \alpha \|\varphi\|^2, \quad \alpha > 0, \quad \forall \varphi \in V, t \in ]0, T[ \end{array} \right.$$

Then we can define a family of operators (cf. [8]).

$$A(t) \in \mathcal{L}(L^2(0, T; V), L^2(0, T; V'))$$

by

$$(8.3) \quad a(t; \varphi, \psi) = (A(t) \varphi, \psi), \quad t \in ]0, T[,$$

the brackets denoting the duality between  $V'$  and  $V$ .

Now we denote by  $\mathcal{U}_k$ ,  $k = 1, 2$  the real Hilbert spaces of controls and we give

$$(8.4) \quad B_k \in \mathcal{L}(\mathcal{U}_k, L^2(0, T; V')), \quad k = 1, 2.$$

Let  $\mathcal{F}$  and  $y_0$  given,  $\mathcal{F} \in L^2(0, T; V')$ ,  $y_0 \in H$ . Denote by  $y(v_1, v_2)$  the solution, which exists and is unique under assumptions (8.1)-(8.2) ([7], [8]), of

$$(8.5) \quad \begin{cases} \frac{dy}{dt} + A(t)y = \mathcal{F} + B_1 v_1 + B_2 v_2, \\ y(0) = y_0, \\ y \in L^2(0, T; V), \end{cases}$$

where  $\frac{dy}{dt}$  = derivative in the sens of distributions on  $]0, T[$  with values in  $V$ .  $y(v_1, v_2)$  (or  $y(t; v_1, v_2)$  or  $y(x, t; v_1, v_2)$  in applications) is the state of the system governed by the problem (8.5).

Define the observation by

$$(8.6) \quad \mathcal{C} \in \mathcal{L}(W(0, T); \tilde{\mathcal{H}})$$

where  $\tilde{\mathcal{H}}$  is a real Hilbert space, the space of observations, and

$$W(0, T) = \left\{ \varphi / \varphi \in L^2(0, T; V), \frac{d\varphi}{dt} \in L^2(0, T; V') \right\},$$

space to which belongs in fact  $y(v_1, v_2)$  ([7]).

Then let the cost function be

$$(8.7) \quad J(v_1, v_2) = \| \mathcal{C} y(v_1, v_2) - z_a \|_{\tilde{\mathcal{H}}}^2 + (N_1 v_1, v_1) - (N_2 v_2, v_2)$$

where

$$(8.8) \quad \begin{cases} N_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{U}_k) \text{ with } (N_k v_k, v_k)_k \geq \nu_k \|v_k\|_k^2, \quad \nu_k \geq 0 \\ \text{and symmetrical, } k = 1, 2. \end{cases}$$

and  $z_a$  given in  $\tilde{\mathcal{H}}$ .

Then take, as in point 5,

$$\mathcal{U}_{ad}^k = \text{closed convex set in } \mathcal{U}_k .$$

and look for  $(u_1, u_2)$  saddle-point of  $J$  on  $\mathcal{U}_{ad}^1 \times \mathcal{U}_{ad}^2$ .

8.2. Results about existence and uniqueness.

The mapping  $(v_1, v_2) \rightarrow y_1(v_1, v_2)$  is affine and continuous from  $\mathcal{U}_1 \times \mathcal{U}_2$  into  $W(0, T)$  since the solution of (8.5) depends continuously from the data ([8]). Therefore there exist  $G_k \in \mathcal{L}(\mathcal{U}_k, W(0, T))$  and  $\zeta \in W(0, T)$  such that

$$(8.9) \quad y(v_1, v_2) = G_1 v_1 + G_2 v_2 + \zeta$$

where  $G_k$  is defined by  $y_k = G_k v_k$ ,  $y_k$  solution of

$$(8.10) \quad \begin{cases} \frac{dy_k}{dt} + A(t) y_k = B_k v_k, \\ y_k(0) = 0, \\ y_k \in L^2(0, T; V), \end{cases}$$

and where  $\zeta$  is the solution of

$$(8.11) \quad \begin{cases} \frac{d\zeta}{dt} + A(t) \zeta = \mathcal{F}, \\ \zeta(0) = y_0, \\ \zeta \in L^2(0, T; V). \end{cases}$$

The cost function (8.7) is then written

$$(8.12) \quad J(v_1, v_2) = \| E_1 v_1 + E_2 v_2 + \xi \|_{\tilde{\mathcal{H}}}^2 + (N_1 v_1, v_1) - (N_2 v_2, v_2)$$

with

$$(8.13) \quad E_k = C G_k, \quad \xi = C \zeta - z_d .$$

As in point 5, we can apply theorems 2.1, 2.2, 2.3 if we suppose

$$(8.14) \quad v_2 \geq \| C G_2 \|_{\mathcal{L}(\mathcal{U}_2, \tilde{\mathcal{H}})}^2 .$$

The problem is now to find, as in point 5, a sufficient condition on the data in order that (8.14) be satisfied. So we consider the following situation :

Let  $\mathcal{H}$  be a real Hilbert space,  $C \in \mathcal{L}(L^2(0, T; V), \mathcal{H})$ ,  $D \in \mathcal{L}(H, H)$  and  $\beta, \gamma$  two constants  $\geq 0$  with  $\beta + \gamma > 0$ . We take  $\mathcal{H} = \tilde{\mathcal{H}} \times H$  and we define  $\mathcal{C}$  by

$$\mathcal{C}: y \rightarrow (\sqrt{\beta} Cy, \sqrt{\gamma} Dy(T)).$$

$y(T)$  has sens since  $y \in W(0, T)$  because it is proved ([10])

$$(8.15) \quad \left\{ \begin{array}{l} W(0, T) \subset \mathcal{C}^0([0, T]; H) \stackrel{\Delta}{=} \text{space of continuous functions from } [0, T] \\ \text{into } H, \text{ the inclusion being topological.} \end{array} \right.$$

We take  $z_a = (\sqrt{\beta} z_{a_1}, \sqrt{\gamma} z_{a_2})$  where  $z_{a_1} \in \mathcal{H}$ ,  $z_{a_2} \in H$ .

Then, (8.6) is satisfied and (8.7) is written

$$(8.16) \quad J(v_1, v_2) = \beta \|Cy(v_1, v_2) - z_{a_1}\|_{\mathcal{H}}^2 + \gamma \|Dy(T; v_1, v_2) - z_{a_2}\|_H^2 + \\ + (N_1 v_1, v_1) - (N_2 v_2, v_2).$$

As we can have «  $\beta = 1$  and  $\gamma = 0$  » or «  $\beta = 0$  and  $\gamma = 1$  », this allows to study simultaneously a total observation on  $]0, T[$  and a final observation.

Then we have the following result

**PROPOSITION 8.1.** Let us put

$$(8.17) \quad \mathfrak{G}^2 = \|B_2\|_{\mathcal{L}(\mathcal{U}_2, L^2(0, T; V))}^2 \left( \frac{\beta}{\alpha^2} \|C\|_{\mathcal{L}(L^2(0, T; V), \mathcal{H})}^2 + \frac{\gamma}{\alpha} \|D\|_{\mathcal{L}(H, H)}^2 \right) e^{2\lambda T}$$

where  $\lambda$  is the constant which appears in (8.2). Then

$$(8.18) \quad \|e G_2\|_{\mathcal{L}(\mathcal{U}_2, \tilde{\mathcal{H}})}^2 \leq \mathfrak{G}^2.$$

**PROOF.** If we put

$$(8.19) \quad y_2(t) = e^{\lambda t} z_2(t),$$

problem (8.10) with  $k = 2$ , is equivalent to

$$(8.20) \quad \left\{ \begin{array}{l} \frac{dz_2}{dt} + (A(t) + \lambda I) z_2 = e^{-\lambda t} B_2 v_2 \\ z_2(0) = 0 \\ z_2 \in L^2(0, T; V) \end{array} \right.$$

that is, under the equivalent form,

$$(8.21) \quad \begin{cases} \frac{d}{dt}(z_2, \varphi) + a(t; z_2, \varphi) + \lambda(z_2, \varphi) = e^{-\lambda t}(B_2 v_2, \varphi), & \forall \varphi \in V \\ z_2(0) = 0. \end{cases}$$

In (8.21) taking  $\varphi = z_2$  and integrating on  $[0, T]$ , according to (8.2), we get

$$\begin{aligned} |z_2(T)|^2 + 2\alpha \int_0^T \|z_2(t)\|^2 dt &\leq 2 \int_0^T e^{-\lambda t} |(B_2 v_2)(t), z_2(t)| dt \leq \\ &\leq 2 \int_0^T e^{-\lambda t} \|B_2 v_2(t)\|_{V'} \|z_2(t)\| dt \leq \alpha \int_0^T \|z_2(t)\|^2 dt + \\ &\quad + \frac{1}{\alpha} \int_0^T e^{-2\lambda t} \|(B_2 v_2)(t)\|_{V'}^2 dt \end{aligned}$$

from where we get

$$(8.22) \quad \int_0^T \|z_2(t)\|^2 dt \leq \frac{1}{\alpha^2} \int_0^T e^{-2\lambda t} \|(B_2 v_2)(t)\|_{V'}^2 dt$$

$$(8.23) \quad |z_2(T)|^2 \leq \frac{1}{\alpha} \int_0^T e^{-2\lambda t} \|(B_2 v_2)(t)\|_{V'}^2 dt.$$

We may assume without any loss of generality  $\lambda \geq 0$  so that  $e^{-2\lambda t} \leq 1, \forall t \geq 0$ . According to (8.19), (8.22) and (8.23) imply respectively

$$(8.24) \quad \|y_2\|_{L^2(0, T; V)}^2 \leq \frac{e^{2\lambda T}}{\alpha^2} \|B_2 v_2\|_{L^2(0, T; V')}^2$$

$$(8.25) \quad |y_2(T)|^2 \leq \frac{e^{2\lambda T}}{\alpha} \|B_2 v_2\|_{L^2(0, T; V')}^2$$

which leads to (8.18) according to the definitions of  $\mathcal{C}$ , of  $G_2$  and of  $\mathfrak{G}$ . Now we can state theorem 8.1 about existence and uniqueness like theorem 5.1, with the definition (8.17) of  $\mathfrak{G}$  and (8.16) of  $J$ .

8.3. *Characterizing the optimal controls.*

Relations (1.3)(1.4) are written here

$$(8.26) \quad \left\{ \begin{array}{l} (\mathcal{C}y(u_1, u_2) - z_a, \mathcal{C}G_1(v_1 - u_1))_{\mathcal{H}} + (N_1 u_1, v_1 - u_1) \geq 0, \\ \quad \quad \quad \forall v_1 \in \mathcal{U}_{ad}^1, u_1 \in \mathcal{U}_{ad}^1, \\ (\mathcal{C}y(u_1, u_2) - z_a, \mathcal{C}G_2(v_2 - u_2))_{\mathcal{H}} - (N_2 u_2, v_2 - u_2) \leq 0, \\ \quad \quad \quad \forall v_2 \in \mathcal{U}_{ad}^2, u_2 \in \mathcal{U}_{ad}^2. \end{array} \right.$$

Consider the situation described in point 8.2. We introduce the adjoint state  $p(u_1, u_2)$  by

$$(8.27) \quad \left\{ \begin{array}{l} -\frac{d}{dt} p(u_1, u_2) + A(t)^* p(u_1, u_2) = \beta C^* \Lambda(Cy(u_1, u_2) - z_{a_1}) \\ p(T; v_1, v_2) = \gamma D^*(Dy(T; u_1, u_2) - z_{a_2}) \\ p(u_1, u_2) \in L^2(0, T; V) \end{array} \right.$$

where  $A(t)^*$  = adjoint operator of  $A(t)$ ,

$$C^* \in \mathcal{L}(\mathcal{H}, L^2(0, T; V')), \quad C^* = \text{adjoint of } C,$$

$$\Lambda = \text{canonical isomorphism from } \mathcal{H} \text{ onto } \mathcal{H}',$$

$$D^* = \text{adjoint of } D.$$

Problem (8.27) as problem (8.5), and because from the same motives, has a unique solution. Now we can interpret (8.26).

The terms which do not contain  $N_k$  in (8.26) become

$$\begin{aligned} & \beta \int_0^T C^* \Lambda(Cy(u_1, u_2) - z_{a_1}, y_1(v_1) - y_1(u_1))_{V' \times V} dt + \gamma (y(T; u_1, u_2) - z_{a_2}, \\ & \quad \quad \quad y_1(T; v_1) - y_1(T; u_1))_H \\ & \beta \int_0^T C^* \Lambda(Cy(u_1, u_2) - z_{a_1}, y_2(v_2) - y_2(u_1))_{V' \times V} dt + \gamma (y(T; u_1, u_2) - z_{a_2}, \\ & \quad \quad \quad y_2(T; v_2) - y_2(T; u_2))_H \end{aligned}$$

where  $y_k$  is the solution of (8.10). The same methods (integration by parts

on  $(0, T)$ ) as in [7] give finally the characterization

$$(8.28) \quad \begin{cases} (B_1^* p(u_1, u_2) + A_{\mathcal{U}_1} N_1 u_1, v_1 - u_1) \geq 0, & \forall v_1 \in \mathcal{U}_{ad}^1, u_1 \in \mathcal{U}_{ad}^1 \\ (B_2^* p(u_1, u_2) - A_{\mathcal{U}_2} N_2 u_2, v_2 - u_2) \leq 0, & \forall v_2 \in \mathcal{U}_{ad}^2, u_2 \in \mathcal{U}_{ad}^2 \end{cases}$$

where  $A_{\mathcal{U}_k}$  = canonical isomorphism from  $\mathcal{U}_k$  onto its dual.

Theorem 8.2 of characterization is stated identically as theorem 5.2 and remark 5.1 is of course valid here. More we remark that if  $N_k = \nu_k I$ ,  $\nu_k > 0$ ,  $k = 1, 2$ , (8.28) is equivalent to

$$(8.29) \quad u_1 = P_1 \left( -\frac{1}{\nu_1} A_{\mathcal{U}_1}^{-1} B_1^* p \right)$$

and

$$(8.30) \quad u_2 = P_2 \left( \frac{1}{\nu_2} A_{\mathcal{U}_2}^{-1} B_2^* p \right)$$

where  $P_k$  is the projector onto  $\mathcal{U}_{ad}^k$ .

**§ 9. Example: parabolic equation of second order; mixed Neumann problem.**

9.1. Let  $\Omega$  be an open bounded set in  $R^n$  with regular boundary  $\Gamma$  and  $T > 0$  fixed.

We put

$$(9.1) \quad Q = \Omega \times ]0, T[, \quad \Sigma = \Gamma \times ]0, T[,$$

$A(t)$  is a family of second order elliptic operators :

$$(9.2) \quad A(t) \varphi = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial \varphi}{\partial x_j} \right)$$

where

$$(9.3) \quad \begin{cases} a_{ij} \in L^\infty(Q), \\ \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2, \quad \alpha > 0, \quad \xi_i \in \mathbb{R}, \text{ a. e. in } Q. \end{cases}$$

We take  $H = L^2(\Omega)$ ,  $V = H^1(\Omega)$ .



For  $\varphi, \psi \in H^1(\Omega)$  we take

$$(9.4) \quad a(t; \varphi, \psi) = \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t) \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_i} dx.$$

Let

$$(9.5) \quad \mathcal{U}_1 = L^2(\Sigma), \quad \mathcal{U}_2 = L^2(Q), \quad N_k = \nu_k I, \quad \nu_k \geq 0, \quad k = 1, 2.$$

$$(9.6) \quad f \text{ given in } L^2(Q), \quad g \text{ given in } L^2(\Sigma).$$

Define  $\mathcal{F} \in L^2(0, T; V')$  by

$$(9.7) \quad (\mathcal{F}(t), \varphi)_{V', V} = \int_{\Omega} f(t) \varphi dx + \int_{\Gamma} g(t) \varphi dx, \quad \forall \varphi \in H^1(\Omega)$$

and  $B_1 \in \mathcal{L}(\mathcal{U}_1, L^2(0, T; V'))$  by

$$(9.8) \quad ((B_1 v_1)(t), \varphi)_{V', V} = \int_{\Gamma} v_1(t) \varphi d\Gamma, \quad \forall \varphi \in H^1(\Omega).$$

At last we take

$$(9.9) \quad B_2 = \text{identity}.$$

It is classical ([8], [7]) that, under assumptions and definitions (9.2) to (9.9), for each pair  $(v_1, v_2)$  given in  $L^2(Q) \times L^2(\Sigma)$ , problem (8.5) has a unique solution  $y(v_1, v_2)$  and is interpreted by the following mixt Neumann problem :

$$(9.10) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y = f + v_2 & \text{in } Q \\ \frac{\partial y}{\partial \nu_A} = g + v_1 & \text{on } \Sigma \\ y(x, 0) = y_0(x) & \text{in } \Omega \end{cases}$$

where  $A(t)$  is the formal operator defined in (9.2) and where

$$(9.11) \quad \begin{cases} \frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial y}{\partial x_j} \cos(\vec{n}, x_i) \\ \cos(\vec{n}, x_i) = i\text{-th director cosine of } \vec{n} \text{ normal to } \Gamma \text{ directed outside } \Omega. \end{cases}$$

Then we consider the differential game defined by problem (9.10), data (9.5) and cost function of type (8.16) got by taking

$$(9.12) \quad \begin{cases} \mathcal{H} = L^2(Q), C = \text{injection from } L^2(0, T; H^1(\Omega)) \text{ into } L^2(Q) \\ D = \text{identity,} \end{cases}$$

that is to say

$$(9.13) \quad J(v_1, v_2) = \beta \int_Q |y(x, t; v_1, v_2) - z_{a_1}(x, t)|^2 dx dt + \\ + \gamma \int_{\Omega} |y(x, T; v_1, v_2) - z_{a_2}(x)|^2 dx + \nu_1 \int_{\Sigma} |v_1(x, t)|^2 d\Sigma - \nu_2 \int_Q |v_2(x, t)|^2 dx dt$$

with  $z_{a_1}$  (resp.  $z_{a_2}$ ) given in  $L^2(Q)$  (resp.  $L^2(\Omega)$ ) and  $\beta, \gamma$  as in 8.2.

Condition (8.2) is verified here with  $\lambda = \alpha$ , which gives for the constant  $\mathfrak{G}$  defined in (8.17), according to the definitions (9.9) and (9.12),

$$(9.14) \quad \mathfrak{G}^2 = \left( \frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha} \right) e^{2\alpha T}.$$

Problem (8.27) which defines the adjoint state is interpreted by

$$(9.15) \quad \begin{cases} -\frac{\partial p}{\partial t} + A(t)^* p = \beta (y(u_1, u_2) - z_{a_1}) & \text{in } Q \\ \frac{\partial p}{\partial \nu_{A^*}} = 0 & \text{on } \Sigma \\ p(x, T) = \gamma (y(x, T; u_1, u_2) - z_{a_2}) & \text{in } \Omega \end{cases}$$

where  $A(t)^*$  is the formal adjoint of  $A(t)$ ,

$$(9.16) \quad A(t)^* \varphi = - \sum_{i, j=1}^n \frac{\partial}{\partial x_i} \left( a_{ji}(x, t) \frac{\partial \varphi}{\partial x_j} \right).$$

Conditions (8.28) are written here

$$(9.17) \quad \int_{\Sigma} (p(x, t; u_1, u_2) + \nu_1 u_1(x, t)) (v_1(x, t) - u_1(x, t)) d\Sigma \geq 0, \quad \forall v_1 \in \mathcal{U}_{ad}^1,$$

$$(9.18) \quad \int_Q (p(x, t; u_1, u_2) - \nu_2 u_2(x, t)) (v_2(x, t) - u_2(x, t)) dx dt \leq 0, \quad \forall v_2 \in \mathcal{U}_{ad}^2,$$

Theorems 8.1 and 8.2 give, in particular: if  $\nu_1 > 0$  and  $\nu_2 > \left(\frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha}\right) e^{2\alpha T}$ , there exists a unique pair of optimal controls characterized by (9.10) (with  $v_1 = u_1$ ,  $v_2 = u_2$ ), (9.15), (9.17) and (9.18).

## 9.2. Choices of convex sets $\mathcal{U}_{ad}^k$ .

EXAMPLE 9.1. unconstrained case ( $\mathcal{U}_{ad}^k = \mathcal{U}_k$ ,  $k = 1, 2$ ).  
Then (9.17)-(9.18) become

$$(9.19) \quad u_1 = -\frac{p}{\nu_1} \text{ a. e. on } \Sigma, \quad u_2 = \frac{p}{\nu_2} \text{ a. e. in } Q.$$

We get the optimal controls by solving the system

$$(9.20) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y - \frac{p}{\nu_2} = f, & \text{in } Q, & -\frac{\partial p}{\partial t} + A(t)^*p - \beta y = -\beta z_{d_1} \\ \frac{\partial y}{\partial \nu_A} + \frac{p}{\nu_1} = g, & \text{on } \Sigma, & \frac{\partial p}{\partial \nu_{A^*}} = 0 \\ y(x, 0) = y_0(x), & \text{in } \Omega, & p(x, T) - \gamma y(x, T) = -\gamma z_{d_2} \end{cases}$$

and the optimal controls are then given by (9.19).

EXAMPLE 9.2.

$$\mathcal{U}_{ad}^1 = \{v \mid v \in L^2(\Sigma), v \geq 0 \text{ a. e. on } \Sigma\},$$

$$\mathcal{U}_{ad}^2 = \mathcal{U}_2 (= L^2(Q)).$$

(9.18) is in fact an equation and gives

$$(9.21) \quad u_2 = \frac{p}{\nu_2}.$$

To interpret (9.17) we use the equivalent form (8.29) which gives here

$$(9.22) \quad u_1 = -\frac{1}{\nu_1} \inf(0, p), \quad \text{on } \Sigma.$$

Then we can eliminate the two controls.

Therefore we must solve the non linear system

$$(9.23) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)t - \frac{p}{r_2} = f, \text{ in } Q, & -\frac{\partial p}{\partial t} + A(t)^* p - \beta y = -\beta z_{a_1} \\ \frac{\partial y}{\partial \nu_A} + \frac{1}{r_1} \inf(0, p), \text{ on } \Sigma, & \frac{\partial p}{\partial \nu_{A^*}} = 0 \\ y(x, 0) = y_0(x), \text{ in } \Omega, & p(x, T) - \gamma y(x, T) = -\gamma z_{a_2}. \end{cases}$$

Then the optimal controls are given by (9.21)-(9.22).

Still in this example, we can eliminate  $u_1$  and  $u_2$  by interpreting inequation (9.17) in terms of unilateral conditions using the same methods as LIONS ([7]).

(9.23) is then replaced by

$$(9.24) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y - \frac{p}{r_2} = f, \text{ in } Q, & -\frac{\partial p}{\partial t} + A(t)^* p - \beta y = -\beta z_{a_1} \\ \frac{\partial y}{\partial \nu_A} - g \geq 0, \text{ on } \Sigma, & \frac{\partial p}{\partial \nu_{A^*}} = 0 \\ p + r_1 \left( \frac{\partial y}{\partial \nu_A} - g \right) \geq 0, \text{ on } \Sigma, & \left( \frac{\partial y}{\partial \nu_A} - g \right) \left( p + r_1 \left( \frac{\partial y}{\partial \nu_A} - g \right) \right) = 0 \\ y(x, 0) = y_0(x), \text{ in } \Omega, & p(x, T) - \gamma y(x, T) = -\gamma z_{a_2}. \end{cases}$$

The optimal controls are given by

$$(9.25) \quad u_1 = \frac{\partial y}{\partial \nu_A} - g \text{ on } \Sigma, \quad u_2 = \frac{p}{r_2}.$$

REMARK 9.1. If one of  $\mathcal{U}_{ad}^k$  is equal to  $\mathcal{U}_k$  or if  $N_k = \nu_k$  identity (in that case we have (8.29)-(8.30)) we can apply remark 6.1.

## § 10. The feed-back problem and the Riccati equation.

### 10.1. Notations and assumptions.

We take place in the frame of n° 8 under the following assumptions (see [7], chap. 3, n° 4),

$$(10.1) \quad \mathcal{U}_k = L^2(0, T; E_k), \quad E_k = \text{real Hilbert space, } k = 1, 2,$$

$$(10.2) \quad \mathcal{H} = L^2(0, T; F), \quad F = \text{real Hilbert space};$$

Let

$$(10.3) \quad \left\{ \begin{array}{l} B_k(t) \in \mathcal{L}(E_k, H), \quad C(t) \in \mathcal{L}(H, F), \quad t \in [0, T], \\ t \rightarrow (B_k(t) e_k, \psi) \text{ and } t \rightarrow (C(t) \varphi, f') \text{ measurable} \\ \forall e_k \in E_k, \psi \in H, \varphi \in H, f' \in F', \text{ and} \\ \|B_k(t)\|_{\mathcal{L}(E_k, H)} \leq c, \quad \|C(t)\|_{\mathcal{L}(H, F)} \leq c. \end{array} \right.$$

Operators  $B_k$  (resp.  $C$ ) are defined by

$$(10.4) \quad \left\{ \begin{array}{l} B_k u_k : t \rightarrow B_k(t) u_k(t), \quad u_k \in \mathcal{U}_k \\ Cf : t \rightarrow C(t) f(t), \quad f \in L^2(0, T; H). \end{array} \right.$$

We take

$$(10.5) \quad \left\{ \begin{array}{l} N_k u_k : t \rightarrow N_k(t) u_k(t) \text{ where} \\ N_k(t) \in \mathcal{L}(E_k, E_k), (N_k(t) e_k, e'_k) \text{ measurable, } \|N_k(t)\| \leq c \\ N_k(t) \text{ symmetrical and } (N_k(t) e_k, e_k)_{E_k} \geq \nu_k \|e_k\|_{E_k}^2, \\ \forall e_k \in E_k, t \in [0, T]. \end{array} \right.$$

Let

$$(10.6) \quad A_k \text{ (resp. } A_F) \text{ the canonical isomorphism from } E_k \text{ (resp. } F) \text{ onto its dual.}$$

Then  $(A_{\mathcal{U}_k} u_k)(t) = A_k u_k(t)$  a. e.,  $(A \varphi)(t) = A_F \varphi(t)$  a. e..

We put

$$(10.7) \quad \left\{ \begin{array}{l} D_1(t) = B_1(t) N_1(t)^{-1} A_1^{-1} B_1(t)^* - B_2(t) N_2(t)^{-1} A_2^{-1} B_2(t)^* \\ D_2(t) = \beta C(t)^* A_F C(t) \\ \tilde{\mathcal{F}}(t) = -\beta C(t)^* A_F z_{a_1}(t). \end{array} \right.$$

Then  $D_1(t) \in \mathcal{L}(H, H)$ ,  $D_2(t) \in \mathcal{L}(H, H)$  and  $D_1(t)$  and  $D_2(t)$  are symmetrical linear operators.

At last we suppose (unconstrained case),

$$(10.8) \quad \mathcal{U}_{ad}^k = \mathcal{U}_k, \quad k = 1, 2,$$

Optimality conditions (8.28) are in fact two equations which give here

$$(10.9) \quad u_1(t) = -N_1^{-1}(t) A_1^{-1} B_1^*(t) p(t), \quad u_2(t) = N_2^{-1}(t) A_2^{-1} B_2^*(t) p(t), \quad t \in [0, T]$$

which has sense because  $p \in W(0, T) \subset C^0([0, T], H)$ .

Then we can eliminate  $u_1$  and  $u_2$  in the state equation and get the coupled linear system in  $(y, p)$  (cf. example 9.1),

$$(10.10) \quad \begin{cases} \frac{dy}{dt} + A(t)y + D_1(t)p = \mathcal{F}, & t \in [0, T] \\ -\frac{dp}{dt} + A(t)^*p - D_2(t)y = \tilde{\mathcal{F}}, & t \in [0, T] \\ y(0) = y_0, \quad p(T) - \gamma D^* Dy(T) = -\gamma D^* z_{d_2}, \end{cases}$$

system which has a unique solution if

$$(10.11) \quad r_1 > 0 \quad \text{and} \quad r_2 > \mathfrak{G}^2 \quad (\text{see def. (8.17)}).$$

Now we have in view to uncouple system (10.10).

**10.2. Riccati equation.**

We add the following assumption

$$(10.12) \quad D_1(t) \text{ non negative definite,} \quad \forall t \in [0, T].$$

So we can define  $D_1^{-\frac{1}{2}}(t)$  ([13]). Then we have

**THEOREM 10.1.** Assumptions and notations are those of n<sup>o</sup> 10.1 with (10.11), (10.12),  $f \in L^2(0, T; H)$  and : the injection from  $V$  into  $H$  is compact. Let  $(y, p)$  be the solution of (10.10). Then

$$(10.13) \quad p = Py + r$$

where  $P$  and  $r$  have the following properties :

$$(10.14) \quad \begin{cases} P \in L^\infty(0, T; \mathcal{L}(H, H)), \quad P(t) \geq 0 \text{ selfadjoint,} \\ \text{if } \varphi \in W(0, T) \text{ with } \frac{d\varphi}{dt} + A(t)\varphi \in L^2(0, T; H) \text{ then } P(t)\varphi \in W(0, T); \end{cases}$$

$P(t)$  is the unique solution of the so-called Riccati equation

$$(10.15) \quad \left\{ \begin{array}{l} -\frac{d}{dt}(P(t)\varphi) + P(t)\left(\frac{d\varphi}{dt} + A(t)\varphi\right) + A(t)^*P(t)\varphi + P(t)D_1(t)P(t)\varphi = D_2(t)\varphi \\ \text{for all } \varphi \text{ as in (10.14) with } A\varphi \in L^2(0, T; H), \\ P(T) = \gamma^* D D; \end{array} \right.$$

$r$  is the unique solution in  $W(0, T)$  of

$$(10.16) \quad \left\{ \begin{array}{l} -\frac{dr}{dt} + A(t)^*r + P(t)D_1(t)r = P(t)\mathcal{F} + \tilde{\mathcal{F}} \\ r(T) = -\gamma D^* z_{a_2}. \end{array} \right.$$

PROOF. We remind the idea used in BENSOUSSAN [1]: we shall prove that system (10.10) is equivalent to a control problem with one control.

Indeed let us consider the following control problem: The state is defined by

$$(10.17) \quad \left\{ \begin{array}{l} \frac{dz}{dt} + A(t)z = \mathcal{F} + D_1^{-\frac{1}{2}}(t)v \\ z(0) = y_0 \end{array} \right.$$

where  $v \in \mathcal{U} = L^2(0, T; H)$  the space of controls, and the cost function

$$(10.18) \quad I(v) = \beta \int_0^T |C(t)z(t; v) - z_{a_1}(t)|_F^2 dt + \\ + \gamma |Dz(T; v) - z_{a_2}|_H^2 + \int_0^T |v(t)|_H^2 dt$$

and we look for  $\inf I(v)$ .

We know ([7]) that this problem has a unique solution  $u$  characterized by

$$(10.19) \quad u = -D_1^{-\frac{1}{2}}q$$

where  $q$  is the adjoint state, unique solution of

$$(10.20) \quad \left\{ \begin{array}{l} -\frac{dq}{dt} + A(t)^*q = \beta C^*(t)A_F(C(t)z - z_{a_1}) \\ q(T) = \gamma D^*(Dz(T) - z_{a_2}) \end{array} \right.$$

and  $z$  the solution of (10.17) for  $v = u$ .

Eliminating  $u$  we get a system in  $(z, q)$  which is nothing else that system (10.10) and therefore  $z = y$  and  $q = p$ .

We uncouple system (10.10) using the methods of LIONS [7] whose theorem 4.4, chap. 3, gives the result.

REMARK 10.1. Let us precise the roll of assumption (10.11). If we only suppose  $N_1$  and  $N_2$  be invertible (this allows to define  $D_1$  by (10.7) and (10.12) be satisfied, the proof of theorem 10.1 shows that there exists a unique pair  $(u_1, u_2)$  which verifies (10.9), (10.10) and that system (10.10) can be uncoupled. But (10.9)-(10.10) (which is equivalent to (8.28)) is, in general, only necessary in order that  $(u_1, u_2)$  be a saddle-point. Assumption (10.11) insuring the cost function be convex-concave, (10.9)-(10.10) is then sufficient and existence and unicity of the saddle-point is thus proved again using the general theorems of § 1, 2 (excepted the one of characterization of course).

REMARK 10.2. Theorem 10.1 allows to realize the feed-back: the optimal controls are given by means of the state by

$$(10.21) \quad \begin{cases} u_1(t) = -N_1(t)^{-1} A_1^{-1} B_1(t)^* [P(t)y(t) + r(t)] \\ u_2(t) = N_2(t)^{-1} A_2^{-1} B_2(t)^* [P(t)y(t) + r(t)]. \end{cases}$$

We have considered here the open loop game, that is to say: looking for the optimal controls as functions of  $t$ . In [1] [2], BENSOUSSAN considers a priori the closed loop game, that is to say: looking for optimal controls as strategies i. e. as functions of  $t$  and  $y(t)$ , problem generally not equivalent to the former, and he proves, under assumption (10.12), that the controls given by (10.21) where  $(P, r)$  is the solution of (10.15)-(10.16), provide the unique solution of this closed loop game. To sum up

- (i) (10.12) implies: existence and uniqueness for the closed loop game
- (ii) (10.11) implies: existence and uniqueness for the open loop game,
- (iii) (10.11) and (10.12) implies: the two solutions of (i) and (ii) coincide and then the two problems are equivalent. Let us notice that (10.11) and (10.12) are not necessarily compatible (cf. example 10.2 farther).

REMARK 10.3. The restrictive assumption  $B_k(t) \in \mathcal{L}(E_k, H)$  (instead of  $\mathcal{L}(E_k, V')$ ) allows to insure, for the control problem introduced in the proof of theorem 10.1,  $B(t) (= D_1^{-1}(t)) \in \mathcal{L}(E, H) (E = H)$  and therefore to apply the general theorem of LIONS. But, the equivalence between the differential game and a control problem can also be proved when  $B_k(t) \in \mathcal{L}(E_k, V')$ .



In fact in this case  $D_1(t) \in \mathcal{L}(V, V')$ ; we then take  $B(t) = A_V(A_V^{-1} D_1(t))^{1/2}$  in equation (10.17) and  $\mathcal{U} = L^2(0, T; V)$ , the cost function remaining the same excepted for the last term with  $v$  which becomes  $\int_0^T \|v(t)\|_V^2 dt$ .

$A_V =$  canonical isomorphism from  $V$  onto its dual  $V'$  and then

$$B(t)^* = B(t).$$

The optimal control which exists and is unique is given by  $u = -A_V^{-1} B(t) q$  where  $q$  is the adjoint state, solution of (10.20). By eliminating  $u$  we recover system (10.10).

### 10.3. Applications.

#### EXAMPLE 10.1. *Mixed Dirichlet problem, distributed controls.*

We take  $H = L^2(\Omega)$ ,  $V = H_0^1(\Omega)$  and  $A(t)$  as in n° 9,  $E_1 = E_2 = L_2(\Omega)$ ,  $B_1(t) = B_2(t) = \text{identity}$ ,  $N_k = r_k \times \text{identity}$ ,  $r_k > 0$ ,  $k = 1, 2$ .  $F = H = L^2(\Omega)$ ,  $C(t) = \text{identity}$ ,  $A = \text{identity}$ ,  $D = \text{identity}$ . Let  $f \in L^2(Q)$ ,  $z_{a_1} \in L^2(Q)$ ,  $z_{a_2} \in L^2(\Omega)$ ,  $y_0 \in L^2(\Omega)$ .

The state is given by

$$(10.22) \quad \begin{cases} \frac{\partial y(u_1, u_2)}{\partial t} + A(t) y(u_1, u_2) = f + u_1 + u_2 & \text{in } Q \\ y(u_1, u_2) = 0 & \text{on } \Sigma \\ y(x, 0; u_1, u_2) = y_0(x) & \text{in } \Omega \end{cases}$$

and the criterion by

$$(10.23) \quad J(v_1, v_2) = \beta \int_Q |y(v_1, v_2) - z_{a_1}|^2 dx dt + \\ + \gamma \int_{\Omega} |y(T; v_1, v_2) - z_{a_2}|^2 dx + r_1 \int_Q |v_1|^2 dx dt - r_2 \int_Q |v_2|^2 dx dt.$$

The number  $\mathbf{G}^2$  defined in (8.17) is equal to the one given by (9.14). Condition (10.12) is written here

$$(10.24) \quad \frac{1}{r_1} - \frac{1}{r_2} \geq 0 \quad \text{i. e.} \quad r_1 \leq r_2.$$

Then (10.11), (10.12) are compatible. If they are satisfied there exists a unique saddle-point of  $J$  given by the solution of

$$(10.25) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y + \left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right)p = f \text{ in } Q, & y/\Sigma = 0 \\ -\frac{\partial p}{\partial t} + A(t)^*p - \beta y = -\beta z_{d_1} \text{ in } Q, & p/\Sigma = 0 \\ y(x, 0) = y_0(x), \quad p(x, T) - \gamma y(x, T) = -\gamma z_{d_2} \end{cases}$$

and then

$$(10.26) \quad u_1 = -\frac{p}{\nu_1}, \quad u_2 = \frac{p}{\nu_2}.$$

System (10.25) can be uncoupled (theorem 10.1 can be applied). From analogy with point 5.1 of [7] chap. 3, we get the result.

**THEOREM 10.2.** If  $0 < \nu_1 \leq \nu_2$ , the solution  $(y, p)$  of (10.25) verifies

$$(10.27) \quad p(t) = P(t)y(t) + r(t)$$

where  $P(t)$  can be described by

$$(10.28) \quad P(t)\varphi = \int_{\Omega} P(x, \xi, t)\varphi(\xi) d\xi, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

the kernel  $P(x, \xi, t)$  satisfying the Riccati integro-differential equation

$$(10.29) \quad \begin{aligned} \frac{\partial P}{\partial t}(x, \xi, t) + (A_x^* + A_\xi^*)P(x, \xi, t) + \\ + \left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) \int_{\Omega} P(x, \zeta, t)P(\zeta, \xi, t) d\zeta = \beta\delta(x - \xi), \end{aligned}$$

$$(10.30) \quad P(x, \xi, t) = P(\xi, x, t),$$

$$(10.31) \quad \begin{cases} P(x, \xi, t) = 0 & \text{if } x \in \Gamma, \xi \in \Omega \\ P(x, \xi, T) = \gamma\delta(x - \xi), & (x, \xi) \in \Omega_x \times \Omega_\xi \end{cases}$$

and where  $r$  is the solution in  $L^2(0, T; H_0^1(\Omega))$  of

$$(10.32) \quad \begin{cases} -\frac{\partial r}{\partial t} + A(t)^* r + \left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) P(t) r = P(t) f - \beta z_{d_1} \\ r|_{\Sigma} = 0 \\ r(x, T) = -\gamma z_{d_2}. \end{cases}$$

EXAMPLE 10.2. Let us take again example 9.1. The situation is partly the same as in n° 10.1 with

$$\begin{aligned} F = H = L^2(\Omega), \quad C(t) = \text{identity}, \quad A_F = \text{identity}, \\ E_2 = L^2(\Omega), \quad E_1 = L^2(\Gamma), \quad A_k = \text{identity}, \quad k = 1, 2, \\ B_2(t) = \text{identity}, \quad N_k(t) = \nu_k \times \text{identity}, \end{aligned}$$

but  $B_1(t)$  is defined by

$$(B_1(t) \varphi, \psi)_{V', V} = \int_{\Gamma} \varphi \psi \, d\Gamma, \quad \forall \varphi \in L^2(\Gamma), \quad \forall \psi \in V = H^1(\Omega)$$

and therefore  $B_1(t) \in \mathcal{L}(E_2, V')$ . On the other hand,  $\mathcal{F} \in L^2(0, T; V')$ . Even if condition (10.12) is satisfied, theorem 10.1 cannot be applied. Nevertheless, according to remark 10.3 and proceeding from analogy with n° 5.1, 5.2, 5.3 of [7] chap. 3, system (9.20) could be uncoupled. But condition (10.12) is unrealizable. Indeed, it becomes

$$\frac{1}{\nu_1} |\varphi|_{L^2(\Gamma)}^2 - \frac{1}{\nu_2} |\varphi|_{L^2(\Omega)}^2 \geq 0, \quad \forall \varphi \in H^1(\Omega).$$

Choose  $\varphi \in H_0^1(\Omega)$ ,  $\varphi \neq 0$ , and get a contradiction with  $\nu_2 > 0$ .

EXAMPLE 10.3. We take again example 9.1, exchanging the rolls of  $u_1$  and  $u_2$ . We take

$$E_1 = V = H^1(\Omega), \quad E_2 = L^2(\Gamma),$$

$B_1(t) = A_V = (-\Delta + I)$ ,  $B_2(t)$  defined as  $B_1(t)$  in example 10.2. The remaining is unchanged excepted the cost function where the explicit part with  $(v_1, v_2)$  becomes

$$\nu_1 \int_0^T \|v\|_V^2 \, dt - \nu_2 \int_{\Sigma} |v_2|^2 \, d\Sigma.$$

Therefore the state is given by

$$(10.33) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y = f + (-\Delta + I)v_1 & \text{in } Q \\ \frac{\partial y}{\partial \nu_A} = g + v_2 & \text{on } \Sigma \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases}$$

Condition (10.11) is written

$$(10.34) \quad \nu_1 > 0, \quad \nu_2 > \|\gamma_0\|^2 \left( \frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha} \right) e^{2\alpha T}$$

where  $\|\gamma_0\|$  denotes the norm of the trace mapping  $\varphi \rightarrow \varphi|_\Gamma$  from  $H^1(\Omega)$  into  $L^2(\Gamma)$ .

Condition (10.12) is written

$$\frac{1}{\nu_1} \|\varphi\|_{H^1(\Omega)}^2 - \frac{1}{\nu_2} \|\varphi\|_{L^2(\Gamma)}^2 \geq 0 \quad \forall \varphi \in H^1(\Omega)$$

which is equivalent to

$$(10.35) \quad \frac{\nu_2}{\nu_1} \geq \|\gamma_0\|^2.$$

Conditions (10.34)-(10.35) are compatible. If they are satisfied there exists a unique saddle-point given by the solution of

$$(10.36) \quad \begin{cases} \frac{\partial y}{\partial t} + A(t)y + \frac{1}{\nu_1}(-\Delta + I)p = f, & \text{in } Q, \quad -\frac{\partial p}{\partial t} + A(t)^*p - \beta y = -\beta z_{d_1} \\ \frac{\partial y}{\partial \nu_A} - \frac{p}{\nu_2} = g, & \text{on } \Sigma, \quad \frac{\partial p}{\partial \nu_{A^*}} = 0 \\ y(x, 0) = y_0(x), & \text{in } \Omega, \quad p(x, T) - \gamma y(x, T) = -\gamma z_{d_2}(x) \end{cases}$$

and then

$$(10.37) \quad u_1 = -\frac{p}{\nu_1}, \quad u_2 = \frac{p}{\nu_2} \text{ on } \Sigma;$$

System (10.36) can be uncoupled. In fact, theorem 10.1 cannot be applied because  $D_1(t) \in \mathcal{L}(V, V')$  and  $\mathcal{F} \in L^2(0, T; V')$ . Then use remark 10.3 and proceed from analogy with [7]: Then

**THEOREM 10.3.** We suppose the coefficients of  $A(x, t, \partial/\partial x)$  be regular in  $\bar{\Omega} \times [0, T]$  and the boundary  $\Gamma$  be regular. If  $0 < \nu_1 \|\gamma_0\|^2 \leq \nu_2$ , the solution  $(y, p)$  of (10.36) verifies

$$p(t) = P(t) y(t) + r(t).$$

Operator  $P(t)$  satisfies

$$(10.38) \quad \begin{cases} -(P'\varphi, \psi) + a(t; \varphi, P\psi) + a^*(t; P\varphi, \psi) + \\ + \frac{1}{\nu_1} (P\varphi, P\psi)_{H^1(\Omega)} - \frac{1}{\nu_2} (P\varphi, P\psi)_{L^2(\Gamma)} = \beta(\varphi, \psi) \\ \forall \varphi, \psi \in H^1(\Omega) \end{cases}$$

with

$$(10.39) \quad \begin{cases} P(T) = \gamma \times \text{identity in } L^2(\Omega) \\ P(t)^* = P(t) \text{ for the scalar product in } L^2(\Omega) \end{cases}$$

$$(10.40) \quad P(t) \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$$

and the function  $r$  satisfies

$$(10.41) \quad \begin{cases} -(r', \psi) + a^*(t; r(t), \psi) + \frac{1}{\nu_1} (r, P\psi)_{H^1(\Omega)} - \frac{1}{\nu_2} (r, P\psi)_{L^2(\Gamma)} = \\ (Pf - \beta z_{a_1}, \psi) + (g, P\psi)_{L^2(\Gamma)}, \quad \forall \psi \in H^1(\Omega) \end{cases}$$

with

$$(10.42) \quad r(T) = -\gamma z_{a_2}.$$

The kernel  $P(x, \xi, t)$  of operator  $P(t)$  satisfies (formally)

$$(10.43) \quad \begin{cases} -\frac{\partial P}{\partial t} + (A_x^* + A_\xi^*) P + \frac{1}{\nu_1} \int_{\Omega} P(\zeta, \xi, t) (-\Delta + I)_\zeta P(x, \zeta, t) d\zeta \\ -\frac{1}{\nu_2} \int_{\Gamma} P(x, \zeta, t) P(\zeta, \xi, t) d\Gamma_\zeta = \beta \delta(x - \xi), \\ \text{in } \Omega_x \times \Omega_\xi \times ]0, T[ \end{cases}$$

with

$$(10.44) \quad P(x, \xi, t) = P(\xi, x, t)$$

$$(10.45) \quad \begin{cases} \frac{\partial P}{\partial \nu_{A_x^*}}(x, \xi, t) = 0, \quad x \in \Gamma, \quad \xi \in \Omega, \quad \frac{\partial P}{\partial \nu_{A^*}}(x, \xi, t) = 0, \quad x \in \Omega, \quad \xi \in \Gamma \\ P(x, \xi, T) = \gamma \delta(x - \xi), \quad (x, \xi) \in \Omega_x \times \Omega_\xi. \end{cases}$$

§ 11. A pursuit-evasion game (see also [12]).

The functional context is the one of n<sup>o</sup> 8. We consider the following pursuit-evasion game

$$(11.1) \quad \begin{cases} \frac{dy_p}{dt} + A_p(t) y_p = B_p v_p & \frac{dy_e}{dt} + A_e(t) y_e = B_e v_e \\ y_p(0) = y_p^0 & y_e(0) = y_e^0 \end{cases}$$

with, on  $A_p, A_e, B_p, B_e, y_p^0, y_e^0$ , the analogous assumptions to the ones of n<sup>o</sup> 8 on  $A, B_1, B_2, y_0$ . Here  $y_p$  denotes the state of the pursuer which is controlled by  $v_p$ ,  $y_e$  the state of the evader controlled by  $v_e$ .

The cost function is

$$(11.2) \quad J(v_p, v_e) = |y_p(T; v_p) - y_e(T; v_e)|_H^2 - \eta |y_e(T; v_e) - \chi|_H^2 + r_p \|v_p\|_{\mathcal{U}_p}^2 - r_e \|v_e\|_{\mathcal{U}_e}^2$$

with

$$\eta \text{ constant } > 1,$$

$$\chi \text{ given in } H, r_p > 0, r_e > 0,$$

$\mathcal{U}_p, \mathcal{U}_e$  are the pursuer and evader real Hilbert spaces of controls.

Let  $\mathcal{U}_{pad}, \mathcal{U}_{ead}$  be closed convex set in  $\mathcal{U}_p, \mathcal{U}_e$ .

**PROBLEM 11.1.** Find  $(u_p, u_e)$  saddle point of  $J$  on  $\mathcal{U}_{pad} \times \mathcal{U}_{ead}$ . A concrete interpretation is the following: the evader will reach the target  $\chi$  and at the same time will evade the pursuer who looks for catching him.

**THEOREM 11.1.** Problem 11.1 has a unique solution  $(u_p, u_e)$  characterized by

$$(11.3) \quad \left\{ \begin{array}{l} (y_p(T; u_p) - y_e(T; u_e), y_p(T; v_p) - y_p(T; u_p)) + \\ \quad + r_p (u_p, v_p - u_p)_{\mathcal{U}_p} \geq 0, \quad \forall v_p \in \mathcal{U}_{pad}, \\ (y_p(T; u_p) + (\eta - 1) y_e(T; u_e) - \eta \chi, y_e(T; v_e) - y_e(T; u_e)) + \\ \quad + r_e (u_e, v_e - u_e)_{\mathcal{U}_e} \geq 0, \quad \forall v_e \in \mathcal{U}_{ead}. \end{array} \right.$$

**PROOF.** The mapping  $v_p \rightarrow y_p(T; v_p)$  (resp.  $v_e \rightarrow y_e(T; v_e)$ ) is affine continuous from  $\mathcal{U}_p$  (resp.  $\mathcal{U}_e$ ) into  $H$ . Therefore we have

$$(11.4) \quad \begin{cases} y_p(T; v_p) = E_p v_p + \xi_p, & E_p \in \mathcal{L}(\mathcal{U}_p, H), \xi_p \in H, \\ y_e(T; v_e) = E_e v_e + \xi_e, & E_e \in \mathcal{L}(\mathcal{U}_e; H); \xi_e \in H. \end{cases}$$

The cost function (11.2) is then written

$$(11.5) \quad J(v_1, v_2) = \|E_p v_p - E_e v_e + \xi_p - \xi_e\|_H^2 - \eta \|E_e v_e + \xi_e - \chi\|_H^2 + v_p \|v_p\|^2 - v_e \|v_e\|^2$$

and is of type (1.1) with (excepted the notations)

$$\begin{aligned} a_p(v_p, v_p) &= (E_p v_p, E_p v_p) + v_p \|v_p\|^2 \\ a_e(v_e, v_e) &= (\eta - 1) (E_e v_e, E_e v_e) + v_e \|v_e\|^2 \\ b(v_e, v_p) &= - (E_p v_p, E_e v_e) \\ L_p(v_p) &= - (\xi_p - \xi_e, e_p v_p) \\ L_e(v_e) &= - (E_e v_e, (\eta - 1) \xi_e - \eta \chi + \xi_p). \end{aligned}$$

Then apply theorems 1.2, 2.2, 2.3 (i) : The saddle-point exists, is unique and is characterized by

$$(E_p u_p - E_e u_e + \xi_p - \xi_e, E_p(v_p - u_p)) + v_p(u_p, v_p - u_p) \geq 0, \quad \forall v_p \in \mathcal{U}_{pad}.$$

$$((\eta - 1) E_e u_e + E_p u_p + (\eta - 1) \xi_e - \eta \chi + \xi_p, E_e(v_e - u_e)) + v_e(u_e, v_e - u_e) \geq 0,$$

$$\forall v_e \in \mathcal{U}_{ead},$$

which, according to (11.4), is equivalent to (11.3).

Now we introduce the adjoint states  $q_p(t; u_p, u_e)$  and  $q_e(t; u_p, u_e)$  which are solutions of, respectively

$$(11.6) \quad \begin{cases} -\frac{dq_p}{dt} + A_p(t)^* q_p = 0 \\ q_p(T) = y_p(T; u_p) - y_e(T; u_e) \end{cases} \quad \begin{cases} -\frac{dq_e}{dt} + A_e(t)^* q_e = 0 \\ q_e(T) = y_p(T; u_p) + (\eta - 1) y_e(T; u_e) - \eta \chi. \end{cases}$$

By the same methods as in [7], chap. 3, we transform condition (11.3) with the help of the adjoint states and we get

**THEOREM 11.2.** The solution of problem 11.1 is characterized by

$$(11.7) \quad \begin{cases} (A_p^{-1} B_p^* q_p(u_p, u_e) + v_p)_{\mathcal{U}_p} \geq 0, & \forall v_p \in \mathcal{U}_{pad}, u_p \in \mathcal{U}_{pad}, \\ (A_e^{-1} B_e^* q_e(u_p, u_e) + v_e)_{\mathcal{U}_e} \geq 0, & \forall v_e \in \mathcal{U}_{ead}, u_e \in \mathcal{U}_{ead}, \end{cases}$$

where  $B_p^*$  (resp.  $B_e^*$ ) denotes the adjoint operator of  $B_p$  (resp.  $B_e$ )

$$B_p^* \in (L^2(0, T; V), \mathcal{U}'_p), \quad B_e^* \in \mathcal{L}(L^2(0, T; V), \mathcal{U}'_e),$$

$$A_p = \text{canonical isomorphism from } \mathcal{U}_p \text{ onto } \mathcal{U}'_p,$$

$$A_e = \text{canonical isomorphism from } \mathcal{U}_e \text{ onto } \mathcal{U}'_e.$$

*The unconstrained case.*

$\mathcal{U}_{pad} = \mathcal{U}_p, \mathcal{U}_{ead} = \mathcal{U}_e$ . Condition (11.7) becomes

$$(11.8) \quad \begin{cases} u_p = -\frac{1}{r_p} A_p^{-1} B_p^* q_p(u_p, u_e) \\ u_e = -\frac{1}{r_e} A_e^{-1} B_e^* q_e(u_p, u_e). \end{cases}$$

We can eliminate the two controls in the state equations. Then we must solve  $\left( ' = \frac{d}{dt} \right)$

$$(11.9) \quad \begin{cases} y'_p + A_p(t) y_p + \frac{1}{r_p} B_p A_p^{-1} B_p^* q_p = 0, & y_p(0) = y_p^0 \\ y'_e + A_e(t) y_e + \frac{1}{r_e} B_e A_e^{-1} B_e^* q_e = 0, & y_e(0) = y_e^0 \\ -q'_p + A_p(t)^* q_p = 0, & q_p(T) = y_p(T) - y_e(T) \\ -q'_e + A_e(t)^* q_e = 0, & q_e(T) = y_p(T) + (\eta - 1) y_e(T) - \eta \chi. \end{cases}$$

The optimal controls are then given by (11.8).

University  
Montpellier II, France



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