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RANJAN KUMAR GARAI

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ON RECURRENT SPACES OF FIRST ORDER

by Ranjan Kumar Garai

1. We denote by R^h_{ijk}, R_{ij} and R the curvature tensor, the Ricci tensor and the scalar curvature of a Riemannian space V_n respectively and put

$$Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{1.1}$$

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) \tag{1.2}$$

and

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{1.3}$$

These tensors Z^h_{ijk}, W^h_{ijk} and C^h_{ijk} are known as the concircular curvature tensor, the projective curvature tensor and the conformal curvature tensor of V_n respectively. A tensor L^h_{ijk} is called first order recurrent if

$$L^h_{ijk,l} = \lambda_l L^h_{ijk}, \tag{1.4}$$

for a non-zero vector λ_l, where comma denotes covariant differentiation with respect to the metric tensor g_{ij}. The vector λ_l in (1.4) is called the vector of recurrence.

A non-flat Riemannian space is called a recurrent space of first order if R^h_{ijk} is first order recurrent. A Riemannian space $V_n (n \geq 3)$ is called a projective recurrent space of first order if W^h_{ijk} is first order recurrent. A Riemannian space V_n $(n > 3)$ is called conformally recurrent space of first

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order if C^h_{ijk} is first order recurrent. A Riemannian space V_n $(n \geq 3)$ for which R_{ij} is non zero and recurrent of first order is called a Ricci recurrent space of first order. Recurrent spaces, projective recurrent spaces, conformally recurrent spaces and Ricci recurrent spaces, all of first order, will be denoted by K_n, P_n, C_n and R_n respectively.

A Riemannian Space V_n is called symmetric, projective symmetric, conformally symmetric or Ricci-symmetric according as

$$R^h_{ijk,\,l} = 0, \tag{1.5}$$

$$W^h_{ijk,\,l} = 0, \tag{1.6}$$

$$C^h_{ijk,\,l} = 0, \tag{1.7}$$

or

$$R_{ij,\,l} = 0 \tag{1.8}$$

is satisfied in V_n.

These spaces have been studied by many authors including Walker [1], Roy-chowdhury [2] and Matsumoto [3]. In this paper some properties of these spaces are obtained. It is believed that some of the results are new while others are extensions of results obtained by previous authors.

2. Some properties of a P_n.

(i) Let a Riemannian V_n satisfy the relation

$$Z^h_{ijk,\,l} = \lambda_l Z^h_{ijk}, \tag{2.1}$$

where λ_l is a non-zero vector and Z^h_{ijk} is given by (1.1). From (1.1) and (2.1), we have

$$R^h_{ijk,\,l} = \lambda_l R^h_{ijk} - \frac{R\lambda_l}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{R_{,\,l}}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{2.2}$$

Contraction over the indices h and k in (2.2) gives

$$R_{ij,\,l} = \lambda_l R_{ij} - \frac{R}{n}\lambda_l g_{ij} + \frac{R_{,\,l}}{n} g_{ij}. \tag{2.3}$$

From (1.2), it follows that

$$W^h_{ijk,\,l} = R^h_{ijk,\,l} - \frac{1}{n-1}(\delta^h_k R_{ij,\,l} - \delta^h_j R_{ik,\,l}). \tag{2.4}$$

Using (2.2) and (2.3) in (2.4), we get

$$W^h_{ijk,\,l} = \lambda_l\, W^h_{ijk}\,. \tag{2.5}$$

Conversely, let a V_n satisfy (2.5), where λ_l is some non-zero vector and W^h_{ijk} is given by (1.2); then

$$R^h_{ijk,\,l} - \frac{1}{n-1}\,(\delta^h_k\, R_{ij,\,l} - \delta^h_j\, R_{ik,\,l}) = \lambda_l \left\{ R^h_{ijk} - \frac{1}{n-1}\,(\delta^h_k\, R_{ij} - \delta^h_j\, R_{ik}) \right\}. \tag{2.6}$$

Transvecting (2.6) by g^{ij}, we obtain

$$R^h_{k,\,l} - \frac{1}{n-1}\,(\delta^h_k\, R_{,\,l} - R^h_{k,\,l}) = \lambda_l \left\{ R^h_k - \frac{1}{n-1}\,(\delta^h_k\, R - R^h_k) \right\}$$

or,

$$R^h_{k,\,l} = \lambda_l\, R^h_k - \frac{1}{n}\,(\lambda_l\, R - R_{,\,l})\, \delta^h_k\,. \tag{2.7}$$

Therefore,

$$R_{ij,\,l} = \lambda_l\, R_{ij} - \frac{1}{n}\,(\lambda_l\, R - R_{,\,l})\, g_{ij}\,. \tag{2.8}$$

In view of (2.8),

$$\delta^h_k\, R_{ij,\,l} - \delta^h_j\, R_{ik,\,l} = \lambda_l\,(\delta^h_k\, R_{ij} - \delta^h_j\, R_{ik}) - \frac{R}{n}\,\lambda_l\,(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}) + \frac{R_{,\,l}}{n}\,(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}). \tag{2.9}$$

Substituting (2.9) in (2.6), we get

$$R^h_{ijk,\,l} - \frac{R_{,\,l}}{n\,(n-1)}\,(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}) = \lambda_l \left\{ R^h_{ijk} - \frac{R}{n\,(n-1)}\,(\delta^k_h\, g_{ij} - \delta^h_j\, g_{ik}) \right\}$$

or

$$Z^h_{ijk,\,l} = \lambda_l\, Z^h_{ijk}\,.$$

We, therefore, state the following theorem:

THEOREM 1. *A necessary and sufficient condition that a Riemannian space V_n be a P_n with λ_i as its vector of recurrence is that the concircular curvature tensor Z^h_{ijk} is first order recurrent, the vector of recurrence being the same λ_i.*

When $R = 0$, $Z^h_{ijk} = R^k_{ijk}$. Hence applying theorem 1, we have

Theorem 2. *Every P_n with zero scalar curvature is a K_n with the same vector of recurrence.*

(ii) Consider C^h_{ijk} in a P_n $(n > 3)$. Differentiating (1.3) covariantly with respect to x^l, we have

$$(2.10) \qquad C^h_{ijk,l} = R^h_{ijk,l} - \frac{1}{n-2}(R^h_{k,l}\, g_{ij} - R^h_{j,l}\, g_{ik} + R_{ij,l}\, \delta^h_k - R_{ik,l}\, \delta^h_j) + \\ + \frac{R_{,l}}{(n-1)(n-2)}(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}).$$

Using (2.2), (2.3) and (2.7) in (2.10) and simplifying, we get

$$C^h_{ijk,l} = \lambda_l\, C^h_{ijk}\,.$$

Hence we have the result:

Theorem 3. *A P_n $(n > 3)$ is a C_n with the same vector of recurrence.*

(iii) The relation (2.2) which holds in a P_n, can be rewritten as

$$R^h_{ijk,l} - \lambda_l\, R^h_{ijk} = \frac{1}{n(n-1)}(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik})\,(R_{,l} - \lambda_l\, R).$$

If $R = 0$, the space is a K_n (theorem 2). Again, if $R \neq 0$ and $R_{,l} = \lambda_l R$ i. e. $\lambda_l = \frac{\partial}{\partial x^l}(\log R)$, the space is a K_n. Hence the following theorem:

Theorem 4. *A P_n of non-zero scalar curvature is a K_n if its vector of recurrence λ_i is given by $\lambda_i = \frac{\partial}{\partial x^i}(\log R)$.*

From (1.1), it follows that

$$Z_{hijk,lm} - Z_{hijk,ml} = R_{hijk,lm} - R_{hijk,ml}\,,$$

where $Z_{hijk} = g_{ht}\, Z^t_{ijk}$. By Walker's Lemma 1 [1],

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0.$$

Hence

$$(2.11) \qquad Z_{hijk,lm} - Z_{hijk,ml} + Z_{jklm,hi} - Z_{jklm,ih} + Z_{lmhi,jk} - Z_{lmhi,kj} = 0.$$

In a P_n with λ_i as its vector of recurrence,

$$Z_{hijk,lm} - Z_{hijk,ml} = (\lambda_{l,m} - \lambda_{m,l})\, Z_{hijk} = b_{lm}\, Z_{hijk}\,,$$

where $b_{lm} = \lambda_{l,m} - \lambda_{m,l}$. Hence, in a P_n, (2.11) reduces to

$$b_{lm} Z_{hijk} + b_{hi} Z_{jklm} + b_{jk} Z_{lmhi} = 0.$$

Since $Z_{hijk} = Z_{jkhi}$, Walker's Lemma 2 [1] gives either $b_{lm} = 0$ or $Z_{hijk} = 0$. Hence the theorem :

THEOREM 5. *In a P_n, either the vector of recurrence is a gradient or the space is of constant curvature.*

It is known that the vector of recurrence of an Einstein K_n $(n > 2)$ is null [1]. Hence, in view of theorem 2, we may state the following result:

THEOREM 6. *In a non-flat Einstein P_n with zero scalar curvature the vector of recurrence is null.*

(iv) If a vector v^i in a Riemannian space satisfies

$$v^i_{,j} = \varrho\, \delta^i_j ,$$

where ϱ is a non-zero constant, v^i is said to be a concurrent vector field [4]. Let a P_n with λ_i as its vector of recurrence admit a concurrent vector field ν^i. Then

$$\nu_{i,j} = \varrho\, g_{ij}. \tag{2.12}$$

Consequently

$$\nu_{i,jk} - \nu_{i,kj} = 0.$$

Hence, by Ricci identity,

$$\nu_h R^h_{ijk} = 0. \tag{2.13}$$

Differentiating (2.13) covariantly with respect to x^l and using (2.12), we get

$$\varrho\, R_{lijk} + \nu_h R^h_{ijk,l} = 0. \tag{2.14}$$

Transvecting (2.14) by g^{ij}, we obtain

$$\varrho\, R_{lk} + \nu^h R_{hk,l} = 0. \tag{2.15}$$

Transvecting (2.15) again by g^{lk}, we have

$$\varrho\, R + \frac{1}{2} \nu^h R_{,h} = 0. \tag{2.16}$$

Now transvecting (2.2) by ν_h and using (2.13) and (2.14), we get

$$\frac{1}{n(n-1)} (\nu_k g_{ij} - \nu_j g_{ik}) (R \lambda_l - R_{,l}) = \varrho\, R_{lijk}. \tag{2.17}$$

As $\varrho \neq 0$, $R = 0$ implies

$$R_{lijk} = 0. \tag{2.18}$$

Again, transvecting (2.17) by $g^{ij} g^{lk}$ and using (2.16), we have

$$R\,[\nu^l \lambda_l - (n-2)\,\varrho] = 0. \tag{2.19}$$

Therefore, either

$$R = 0$$

or,

$$\nu^l \lambda_l = (n-2)\,\varrho. \tag{2.20}$$

But, transvecting (2.17) by $\nu^l g^{ij}$ and using (2.16), we get

$$\frac{R}{n}\,\nu_k\,(\nu^l \lambda_l + 2\varrho) = \varrho\,\nu^l R_{lk}\,. \tag{2.21}$$

In view of (2.13), (2.21) reduces to

$$R\,\nu_k\,(\nu^l \lambda_l + 2\varrho) = 0.$$

Since ν_k cannot be zero, either

$$R = 0$$

or

$$\nu^l \lambda_l = -\,2\varrho. \tag{2.22}$$

But (2.20) and (2.22) give $\varrho = 0$, which is impossible. Therefore $R = 0$.

Hence, from (2.18), we have the result:

Theorem 7. *If a P_n admits a concurrent vector field, then the scalar curvature R is zero and the space is flat.*

Putting $\varrho = 0$ in (2.17) and transvecting the resulting equation by g^{ij}, we get

$$\nu_k\,(R\,\lambda_l - R_{,\,l}) = 0$$

If $\nu_k \neq 0$, $\lambda_l = \dfrac{\partial}{\partial x^l}(\log R)$, for $R \neq 0$. Thus we have the result:

Theorem 8. *If a P_n with non-zero scalar curvature R admits a non-zero parallel vector field, the vector of recurrence λ_i is given by $\lambda_i = \dfrac{\partial}{\partial x^i}(\log R)$.*

In view of Theorems 2, 4 and 8 we have the result:

THEOREM 9. *A P_n admitting a non-zero parallel vector field is a K_n.*

The equation (2.19) reduces to

$$R \nu^l \lambda_l = 0 \quad \text{if} \quad \varrho = 0.$$

Hence the result:

THEOREM 10. *If a P_n admits a parallel vector field ν^i, then either the scalar curvature is zero or ν^i is orthogonal to the vector of recurrence of the space.*

Roy-chowdhury [2] proved that if a P_n admits a parallel vector field ν^i, then either the space is of constant curvature or ν^i is orthogonal to the vector of recurrence of the space. Again, a space of constant curvature with zero scalar curvature is flat. Hence, in view of theorem 10 and Roy-chowdhury's result, we may state the following theorem.

THEOREM 11. *If a P_n admits a parallel vector field ν^i, then either the space is flat or ν^i is orthogonal to the vector of recurrence of the space.*

(vi) If there exists a vector field ν^i such that

$$\nu^i_{,j} = \varrho \, \delta^i_j , \tag{2.23}$$

ϱ being a non-constant scalar, then ν^i is said to be concircular vector field [5]. Let us assume that P_n admits a concircular vector field ν^i. From (2.23) and Ricci identity, we have

$$\nu_h R^h_{ijk} = \varrho_k g_{ij} - \varrho_j g_{ik} , \tag{2.24}$$

where $\varrho_k = \dfrac{\partial \varrho}{\partial x^k}$. From (2.24) one gets

$$\nu^h R^i_{hjk} = \varrho_j \delta^i_k - \varrho_k \delta^i_j . \tag{2.25}$$

Contraction over the indices i and k in (2.25), gives

$$\nu^h R_{hj} = (n-1) \varrho_j . \tag{2.26}$$

In consequence of (2.25) and (2.26), we have

$$\nu^h W^i_{hjk} = 0. \tag{2.27}$$

Differentiating (2.27) covariantly with respect to x^l and using (2.23), we get

$$\varrho W^i_{ljk} + \nu^h W^i_{hjk,l} = 0. \tag{2.28}$$

In virtue of (2.5) and (2.27), (2.28) reduces to

$$\varrho\, W^{i}_{ljk} = 0.$$

Since $\varrho \neq 0$, $W^{i}_{ljk} = 0$, i. e. the space is of constant curvature. Again, in a space of constant curvature, we have

$$R^{i}_{hjk} = \frac{R}{n\,(n-1)}\,(\delta^{i}_{k}\, g_{hj} - \delta^{i}_{j}\, g_{hk}). \tag{2.29}$$

Transvecting (2.29) by ν^h and using (2.25), we get

$$\varrho_j\, \delta^{i}_{k} - \varrho_k\, \delta^{i}_{j} = \frac{R}{n\,(n-1)}\,(\delta^{i}_{k}\, \nu_j - \delta^{i}_{j}\, \nu_k). \tag{2.30}$$

Contraction over the indices i and k in (2.30), gives

$$\varrho_j = \frac{R}{n\,(n-1)}\,\nu_j\,. \tag{2.31}$$

Now $R = 0$ implies $\varrho_j = 0$, for every j, which is not possible. Therefore R is a non-zero constant. Hence we have the result:

THEOREM 12. *A P_n admitting a concircular vector field is a space of non-zero constant curvature.*

Differentiating (2.31) covariantly with respect to x^l, we get

$$\varrho_{j,\,l} = \frac{R}{n\,(n-1)}\,\nu_{j,\,l}.$$

In view of (2.23), the above equation reduces to

$$\varrho_{j,\,l} = \frac{R\,\varrho}{n\,(n-1)}\,g_{jl}\,. \tag{2.32}$$

From (2.31) and (2.32), we can state the result:

THEOREM 13. *If a P_n admits a concircular vector field ν^i such that $\nu^{i}_{,\,j} = \varrho\, \delta^{i}_{j}$, ϱ being a non-constant scalar, then* (i) *ν^i is a gradient proportional to ϱ_i and* (ii) *ϱ_i itself is a concircular vector field.*

(vii) Suppose that the scalar curvature R of P_n is constant. Since R is constant, we have, from (1.1) and Bianchi identity,

$$Z_{hijk,\,l} + Z_{hikl,\,j} + Z_{hilj,\,k} = 0, \tag{2.33}$$

where $Z_{hijk} = g_{ht} Z^{t}_{ijk}$. Since the space is a P_n, we have from (2.33)

$$\lambda_l Z_{hijk} + \lambda_j Z_{hikl} + \lambda_k Z_{hilj} = 0, \tag{2.34}$$

λ_i being the vector of recurrence. It can be verified that Z_{hijk} satisfies the following relations

$$Z_{hijk} = - Z_{ihjk} = Z_{jkhi} = - Z_{hikj} = Z_{ihkj}. \tag{2.35}$$

Now we proceed as in Walker [1]. We choose u^i so that $u^i \lambda_i = 1$ and put

$$S_{ij} = - u^h u^k Z_{hijk} = - u^h u^k Z_{jkhi} = - u^h u^k Z_{kjih} = S_{ji}. \tag{2.36}$$

Transvecting (2.34) by $u^l u^h$ and using (2.35) and (2.36), we get

$$u^h Z_{hijk} = \lambda_j S_{ik} - \lambda_k S_{ij}. \tag{2.37}$$

Transvecting (2.34) by u^l and applying (2.37), one obtains

$$Z_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k. \tag{2.38}$$

Hence we can state the following theorem:

THEOREM 14. *In a P_n the scalar curvature of which is constant, the tensor Z_{hijk} can be expressed in the form*

$$Z_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k,$$

where S_{ij} is a symmetric tensor and λ_i the vector recurrence of the space.

It should be noted here that S_{ij} can be modified by the addition of $\lambda_i \alpha_j + \alpha_i \lambda_j$ for any α_i.

In an Einstein space V_n $(n > 2)$, the scalar curvature is constant and the tensors W_{hijk} and Z_{hijk} are identical. Therefore we can state the theorem:

THEOREM 15. *In an Einstein P_n $(n > 2)$ the tensor W_{hijk} can be expressed in the form*

$$W_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k,$$

where S_{ij} is a symmetric tensor and λ_i the vector of recurrence of the space.

Differentiating (2.34) covariantly with respect to x^m and using it again, we get

$$\lambda_{l,m} Z_{hijk} + \lambda_{j,m} Z_{hikl} + \lambda_{k,m} Z_{hilj} = 0. \tag{2.39}$$

Proceeding as in Walker [1], we choose u^i so that $u^i \lambda_i = 1$. Putting $\theta_j = u^i \lambda_{i,j}$, we get, from (2.39) after transvecting by $u^l u^h$,

$$u^h \theta_m Z_{hijk} + \lambda_{j,m} (u^l u^h Z_{hikl}) - \lambda_{k,m} (u^h u^l Z_{hijl}) = 0.$$

Applying (2.36) and (2.37), the above equation reduces to

$$S_{ik} (\lambda_{j,m} - \theta_m \lambda_j) = S_{ij} (\lambda_{k,m} - \theta_m \lambda_k). \tag{2.40}$$

Since S_{ij} is a symmetric tensor, (2.40) shows that either

$$\lambda_{j,m} = \theta_m \lambda_j \tag{2.41}$$

or

$$S_{ij} \text{ is of the form } \varrho_i \varrho_j.$$

Now, if $Z_{hijk} \neq 0$, since λ_i is a gradient (theorem 5), then (2.41) gives

$$\lambda_{j,m} = \Phi \lambda_j \lambda_m ,$$

Φ being a scalar function.

Also, if $S_{ij} = \varrho_i \varrho_j$, (2.38) reduces to

$$Z_{hijk} = m_{jk} m_{hi} ,$$

where $m_{ij} = \varrho_i \lambda_j - \varrho_j \lambda_i$ is a skew symmetric tensor. Thus we may state the following theorems:

Theorem 16. *In a P_n the scalar curvature of which is constant and $Z_{hijk} \neq 0$, either*

$$\lambda_{i,j} = \Phi \lambda_i \lambda_j$$

or

$$Z_{hijk} = m_{jk} m_{hi} ,$$

where λ_i is the vector of recurrence of the space, Φ a scalar function and m_{ij} a skew symmetric tensor.

Theorem 17. *In an Einstein P_n for which $W_{hijk} \neq 0$, either*

$$\lambda_{i,j} = \Phi \lambda_i \lambda_j$$

or

$$W_{hijk} = m_{hi} m_{jk} ,$$

where λ_i is the vector of recurrence of the space, Φ a scalar function and m_{ij} a skew symmetric tensor.

3. Some properties of a C_n.

(i) In an Einstein space, we have

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{3.1}$$

In consequence of (3.1), the tensor C^h_{ijk} reduces to Z^h_{ijk}. Hence we have the result:

THEOREM 18. *An Einstein C_n is a P_n with the same vector of recurrence.*

Since an Einstein C_n is a P_n, the results similar to the theorems 15 and 17 will hold for C_{hijk} in an Einstein C_n.

(ii) Let us assume that a C_n admits a concurrent vector field v^i. In a C_n the relation

$$C^h_{ijk,\, l} = \lambda_l C^h_{ijk} \tag{3.2}$$

holds for a non-zero vector λ_l. Contracting the indices h and l in (2.10) and simplifying, we obtain

$$C^h_{ijk,\, h} = \frac{n-3}{n-2}\left[(R_{ij,\, k} - R_{ik,\, j}) + \frac{1}{2(n-1)}(R_{,\, j}\, g_{ik} - R_{,\, k}\, g_{ij})\right]. \tag{3.3}$$

Again, contracting h and l in (3.2) and applying (3.3), we get

$$\frac{n-3}{n-2}\left[(R_{ij,\, k} - R_{ik,\, j}) + \frac{1}{2(n-1)}(R_{,\, j}\, g_{ik} - R_{,\, k}\, g_{ij})\right] = \tag{3.4}$$

$$= \lambda_h\left[R^h_{ijk} - \frac{1}{n-2}(R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})\right].$$

In view of (2.13), we have

$$\begin{cases} v^i R^h_{ijk} = 0, \\ v^i R_{ij} = 0. \end{cases} \tag{3.5}$$

Transvecting (3.4) by v^i and applying (2.15) and (3.5), we obtain

$$\frac{n-3}{2(n-1)}(R_{,\, j}\, v_k - R_{,\, k}\, v_j) = -\lambda_h(R^h_k v_j - R^h_j v_k) + \frac{R\lambda_h}{(n-1)}(v_j\delta^h_k - v_k\delta^h_j). \tag{3.6}$$

Again, transvecting (3.6) by ν^j and writing $\nu^i \nu_i = \nu$, we get

$$(3.7) \qquad \frac{n-3}{2(n-1)}(\nu^j R_{,j} \nu_k - R_{,k} \nu) =$$

$$= -\lambda_h (R_k^h \nu - \nu^j R_j^h \nu_k) + \frac{R \lambda_h}{(n-1)} (\nu \delta_k^h - \nu_k \nu^h).$$

In consequence of (2.16) and (3.5), (3.7) reduces to

$$(3.8) \qquad \frac{3-n}{2(n-1)}(2R \varrho \nu_k + \nu R_{,k}) = -\lambda_h R_k^h \nu + \frac{R}{n-1}(\nu \lambda_k - \nu_k \lambda^h \nu_h).$$

If $R = 0$, (3.8) gives

$$(3.9) \qquad \lambda_h R_k^h \nu = 0.$$

Since ν^i cannot be null vector (3.9) reduces to

$$\lambda_h R_k^h = 9.$$

Thus we have the theorem:

THEOREM 19. *If a C_n of zero scalar curvature admits a concurrent vector field, then $\lambda_h R_k^h = 0$, λ_h being the vector of recurrence of the space.*

(iii) Putting $\varrho = 0$, in (3.8) one gets

$$\frac{n-3}{2(n-1)} \nu R_{,k} = \lambda_h R_k^h \nu - \frac{R}{n-1}(\nu \lambda_k - \nu_k \lambda^h \nu_h).$$

Hence we have the theorem:

THEOREM 20. *If a C_n of zero scalar curvature admits a non-null parallel vector field, then $\lambda_h R_k^h = 0$, λ_h being the vector of recurrence of the space.*

4. Some properties of a R_n.

(i) In a R_n the relation

$$(4.1) \qquad R_{ij,l} = \lambda_l R_{ij},$$

holds for a non zero vector λ_l. From Bianchi identity, we have

$$(4.2) \qquad R^l_{hjk,l} = R_{hj,k} - R_{hk,j}.$$

Since the space is R_n, (4.2) reduces to

$$R^l_{hjk,\,l} = \lambda_k R_{hj} - \lambda_j R_{hk}.$$

Transvecting the above equation with g^{hj} and simplifying, we have

$$\lambda_h R^h_k = \frac{1}{2} R \lambda_k \tag{4.3}$$

Hence we have the result:

THEOREM 21. *In a R_n with λ_i as its vector of recurrence,*

$$\lambda_h R^h_i = \frac{1}{2} R \lambda_i,$$

R being the scalar curvature of the space.

(ii) Let us assume that R_n admits a concircular vector field v^i. Differentiating (2.24) covariantly with respect to x^l and using (2.23), we obtain

$$\varrho R_{lijk} + v_h R^h_{ijk;\,l} = \varrho_{k,\,l}\, g_{ij} - \varrho_{j,\,l}\, g_{ik}. \tag{4.4}$$

Transvecting (4.4) by g^{il} and simplifying, we get

$$v^h R^l_{hjk,\,l} = 0$$

or

$$v^h (R_{hj,\,k} - R_{hk,\,j}) = 0. \tag{4.5}$$

Again, transvecting (4.4) by g^{ij}, one gets

$$\varrho R_{lk} + v_h R^h_{k,\,l} = (n-1)\, \varrho_{k,\,l}. \tag{4.6}$$

Transvecting (4.6) once again with g^{lk}, we obtain

$$\varrho R + \frac{1}{2} v^h R_{,\,h} = (n-1)\, g^{lk} \varrho_{k,\,l}. \tag{4.7}$$

Transvecting (4.4) with g^{lk}, we get

$$\varrho R_{ij} - v^h R^l_{jhi,\,l} = (g^{lk} \varrho_{k,\,l})\, g_{ij} - \varrho_{i,\,j}$$

or,

$$\varrho R_{ij} - v^h R_{hj,\,i} + v^h R_{ij,\,h} = (g^{lk} \varrho_{k,\,l})\, g_{ij} - \varrho_{i,\,j}. \tag{4.8}$$

In virtue of (4.6), (4.8) reduces to

$$2\varrho\, R_{ij} - (n-2)\,\varrho_{i,j} + \nu^h R_{ij,h} = (g^{lk}\varrho_{l,k})\, g_{ij}\,. \tag{4.9}$$

From (4.1) and (4.5), we get

$$\lambda_l\,(\nu^i R_{ij}) = \lambda_j\,(\nu^i R_{il}).$$

Therefore,

$$\nu^h R_{hj} = \bar{\psi}\,\lambda_j\,, \tag{4.10}$$

where $\bar{\psi}$ is a scalar function of x's. In consequence of (2.26), (4.10) gives

$$\varrho_i = \psi\,\lambda_i\,, \tag{4.11}$$

where $\psi = \dfrac{\bar{\psi}}{n-1}$. Thus we can state the following theorem:

THEOREM 22. *If a R_n admits a concircular vector field ν^i such that $\nu^i_{,j} = \varrho\,\delta^i_j$, ϱ being a non constant scalar, then both $\nu^h R_{hi}$ and ϱ_i are proportional to λ_i, where λ_i is the vector of recurrenee of the space and $\varrho_i = \dfrac{\partial\varrho}{\partial x^i}$.*

(iii) In a R_n the scalar curvature R cannot be a non-zero constant. Suppose that R is zero. Then (4.7) gives

$$g^{lk}\varrho_{k,l} = 0. \tag{4.12}$$

Hence (4.8) reduces to

$$\varrho\, R_{ij} - \lambda_i \nu^h R_{hj} + \nu^h \lambda_h R_{ij} = -\,\varrho_{i,j}\,. \tag{4.13}$$

In view of (4.10), (4.13) finally reduces to

$$(\varrho + \nu^h\lambda_h)\, R_{ij} = \bar{\psi}\,\lambda_i\lambda_j - \varrho_{i,j}\,. \tag{4.14}$$

Transvecting (4.14) with g^{ij} and using (4.12), we get

$$\bar{\psi}\, g^{ij}\lambda_i\lambda_j = 0, \tag{4.15}$$

since R is zero. Now $\bar{\psi} = 0$ implies ψ of (4.11) is zero which gives $\varrho_i = 0$. But ϱ_i cannot be zero. Hence $\bar{\psi} \neq 0$. Therefore λ_i is null and consequently, by (4.11), ϱ_i is also null. So transvecting (4.11) by λ^i, we get

$$\varrho_i\,\lambda^i = 0.$$

Thus, λ_i is orthogonal to ϱ_i. Since $R = 0$, (4.3) gives

$$\lambda_h R_k^h = 0. \tag{4.16}$$

In consequence of (4.11), (4.16) reduces to

$$\varrho_h R_k^h = 0.$$

Hence we have the theorem:

THEOREM 23. *If a R_n of zero scalar curvature admits a concircular vector field v^i such that $v^i_{,j} = \varrho \delta^i_j$, ϱ being a non-constant scalar, then* (i) *both ϱ_i and λ_i are null,* (ii) *λ_i is orthogonal to ϱ_i and* (iii) *$\varrho_h R_i^h = 0$, where λ_i is the vector of recurrence and $\varrho_i = \dfrac{\partial \varrho}{\partial x^i}$.*

5. Some properties of a K_n.

Since every $K_n\,(n \geq 3)$ for which $R_{ij} \neq 0$ is a R_n, results similar to the theorems 21, 22 and 23 will hold in a $K_n\,(n \geq 3)$ with $R_{ij} \neq 0$.

Now suppose that a K_n admits a concurrent vector field v^i. In a K_n, we have

$$R^h_{ijk,l} = \lambda_l R^h_{ijk}, \tag{5.1}$$

for a non-zero vector λ_l. In view of (5.1), (2.14) reduces to

$$\varrho R_{lijk} + \lambda_l v_h R^h_{ijk} = 0. \tag{5.2}$$

In consequence of (2.13), (5.2) finally reduces to

$$\varrho R_{lijk} = 0.$$

But neither ϱ nor R_{lijk} can be zero. Hence the result:

THEOREM 24. *A K_n cannot admit a concurrent vector field.*

6. Some properties of a symmetric space.

In a symmetric space V_n, we have

$$R^h_{ijk,l} = 0. \tag{6.1}$$

Let the symmetric space V_n admit a concircular vector field v^i. In virtue of (6.1), (4.6) and (4.9) reduce to

$$\varrho R_{lk} = (n-1)\varrho_{l,k} \tag{6.2}$$

and

$$2\varrho R_{ij} = (n-2)\varrho_{i,j} + (g^{lk}\varrho_{l,k})g_{ij} \tag{6.3}$$

respectively. From (6.2) and (6.3), one gets

$$\varrho_{i,j} = \Phi g_{ij}, \tag{6.4}$$

where $\Phi = \frac{1}{n} g^{lk}\varrho_{l,k}$. Transvecting (6.2) with g^{lk}, we have

$$\varrho R = (n-1) g^{lk}\varrho_{l,k} = n(n-1)\Phi. \tag{6.5}$$

If $R \neq 0$, then Φ is a non-constant scalar. Hence we have the theorem:

Theorem 25. *If a symmetric space V_n of non zero scalar curvature admits a concircular vector field v^i such that $v^i_{,j} = \varrho\,\delta^i_j$, ϱ being a non-constant scalar, then $\varrho_i = \frac{\partial\varrho}{\partial x^i}$ itself a concircular vector field of V_n.*

If $R = 0$, (6.5) gives $\Phi = 0$ and consequently (6.4) reduces to $\varrho_{i,j} = 0$. Thus we have the result:

Theorem 26. *If a symmetric space V_n of zero scalar curvature admits a concircular vector field v^i such that $v^i_{,j} = \varrho\,\delta^i_j$, ϱ being a non-constant scalar, then $\varrho_i = \frac{\partial\varrho}{\partial x^i}$ is parallel in V_n.*

7. Some properties of a projective symmetric space.

In a projective symmetric space, we have

$$\begin{cases} W^h_{ijk,l} = 0 \\ \text{i. e.}\quad R^h_{ijk,l} = \frac{1}{n-1}(\delta^h_k R_{ij,l} - \delta^h_j R_{ik,l}). \end{cases} \tag{7.1}$$

Transvecting by g^{ij}, we have

$$R^h_{k,l} = \frac{1}{n} R_{,l}\,\delta^h_k. \tag{7.2}$$

Hence

(7.3) $$R_{ij,l} = \frac{1}{n} R_{,l}\, g_{ij}.$$

From (7.1) and (7.3), we have

(7.4) $$\begin{cases} R^{h}_{ijk,l} = \dfrac{R_{,l}}{n\,(n-1)} (\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}) \\ \text{i. e.} \quad Z^{h}_{ijk,l} = 0, \end{cases}$$

where Z^h_{ijk} is given by (1.1).

Conversely (7.4) implies (7.3), and (7.3) together with (7.4) implies (7.1). Thus we have the theorem:

THEOREM 27. *A Riemannian V_n is projective symmetric if and only if $Z^h_{ijk,l} = 0$.*

Contracting h and l in (7.2), we get $(n-2)\,R_{,k} = 0$. Thus we can state the theorem:

THEOREM 28. *A projective symmetric space $V_n\,(n > 2)$ is of constant scalar curvature.*

In view of theorems 27 and 28, we get the following result obtained by Matsumoto [3].

THEOREM 29 *A projective symmetric space $V_n\,(n > 2)$ is a symmetric space.*

With the help of the theorem 29 it can be verified that

$$(R^{hijk}\, R_{hijk})_{,l} = 0.$$

Hence we have the theorem:

THEOREM 30. *In a projective symmetric space $R^{hijk}\, R_{hijk}$ is a constant.*

Every symmetric space is conformally symmetric. Hence, in view of theorem 29, we can state the following result:

THEOREM 31. *Every projective symmetric $V_n\,(n > 3)$ is conformally symmetric.*

Next we suppose that a projective symmetric space admits a concurrent vector field v^i. In view of theorem 29, (2.14) reduces to

$$\varrho\, R_{lijk} = 0.$$

Since $\varrho \neq 0$, $R_{lijk} = 0$. Hence the result:

THEOREM 32. *Every projective symmetric $V_n\,(n > 2)$ admitting a concurrent vector field is flat.*

8. Some properties of a conformally symmetric space.

In a conformally symmetric space (1.7) holds. Therefore, from (3.3), we get

$$(8.1) \qquad \frac{n-3}{n-2}\left[(R_{ij,k} - R_{ik,j}) + \frac{1}{2(n-1)}(R_{,j}\, g_{ik} - R_{,k}\, g_{ij})\right] = 0.$$

For $n > 3$, (8.1) reduces to

$$(8.2) \qquad (R_{ij,k} - R_{ik,j}) + \frac{1}{2(n-1)}(R_{,j}\, g_{ik} - R_{,k}\, g_{ij}) = 0.$$

(i) Now suppose that the space admits a concircular vector field v^i. Transvecting (8.2) by v^i and using (4.5), we get

$$R_{,j}\, v_k = R_{,k}\, v_j$$

which shows that

$$v_i = \Phi\, R_{,i}.$$

where Φ is a scalar function of x's. Hence we have the result:

THEOREM 33. *If a conformally symmetric space V_n $(n > 3)$ of non constant scalar curvature R admits a concircular vector field v^i, then v^i will necessarily be a scalar multiple of $\frac{\partial}{\partial x^i}(R)$.*

(ii) Next we suppose that the space admits a concurrent vector field v^i.

Transvecting (8.2) by v^i and applying (2.15), we get

$$(8.3) \qquad v_k\, R_{,j} - v_j\, R_{,k} = 0.$$

Hence we have the result:

THEOREM 34. *If a conformally symmetric space V_n $(n > 3)$ of non-constant scalar curvature R admits a concurrent vector field v^i, then v^i will necessarily be a scalar multiple of $\frac{\partial}{\partial x^i}(R)$.*

Transvecting (8.3) by v^j and writing $v^j v_j = v$, we obtain

$$(8.4) \qquad v_k\, R_{,j}\, v^j - v\, R_{,k} = 0.$$

In view of (2.16), (8.4) reduces to

$$(8.5) \qquad 2\varrho\, R\, v_k = -\, v\, R_{,k}.$$

Putting $\varrho = 0$ in (8.5), one gets $v\, R_{,k} = 0$. Hence the result:

THEOREM 35. *If a conformally symmetric space V_n $(n > 3)$ admits a non-null parallel vector field, then the scalar curvature of the space is constant.*

9. Some properties of a Ricci-symmetric space.

(i) In a Ricci-symmetric space, we have

$$R_{ij,\,l} = 0. \tag{9.1}$$

Consequently

$$R_{,\,l} = 0. \tag{9.2}$$

In virtue of (9.2), $Z^{h}_{ijk,\,l}$ reduces to $R^{h}_{ijk,\,l}$, where Z^{h}_{ijk} is given by (1.1). If the Ricci-symmetric space is also a P_n with λ_i as its vector of recurrence, then we have

$$Z^{h}_{ijk,\,l} = \lambda_l Z^{h}_{ijk}\,. \tag{9.3}$$

In view of (1.1) and (9.2), (9.3) reduces to

$$R^{h}_{ijk,\,l} = \lambda_l \left[R^{h}_{ijk} - \frac{R}{n\,(n-1)} (\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}) \right]. \tag{9.4}$$

Contracting h and k in (9.4) and using (9.1), we get

$$\lambda_l \left[R_{ij} - \frac{R}{n}\, g_{ij} \right] = 0. \tag{9.5}$$

Since λ_l is a non-zero vector, (9.5) gives

$$R_{ij} = \frac{R}{n}\, g_{ij}\,.$$

Hence we have the result:

THEOREM 36. *A Ricci-symmetric P_n is an Einstein space.*

Since a Ricci-symmetric P_n is an Einstein space, the results similar to theorems 15 and 17 will hold in a Ricci-symmetric P_n.

(ii) Next suppose that the Ricci-symmetric space is a C_n. In view of (9.1) and (9.2) we have

$$C_{hijk,\,l} = R_{hijk,\,l}\,. \tag{9.6}$$

Also C_{hijk} is seen to satisfy the relations

$$C_{hiik} = -\,C_{ihjk} = -\,C_{hikj} = C_{jkhi}\,.$$

Therefore, proceeding as in section 2 (vii), one can show that

$$C_{hijk} = S_{hj}\,\lambda_i\,\lambda_k + S_{ik}\,\lambda_j\,\lambda_h - S_{hk}\,\lambda_i\,\lambda_j - S_{ij}\,\lambda_h\,\lambda_k\,,$$

where S_{ij} is a symmetric tensor and λ_i the vector of recurrence of the space of the space. Thus, we have the result:

Theorem 37. *In a Ricci-symmetric C_n, the tensor C_{hijk} can be expressed in the form*

$$C_{hijk} = S_{hj}\,\lambda_i\,\lambda_k + S_{ik}\,\lambda_j\,\lambda_h - S_{hk}\,\lambda_i\,\lambda_j - S_{ij}\,\lambda_h\,\lambda_k\,,$$

where S_{ij} is a symmetric tensor and λ_i the vector of recurrence of the space.

Because of the relation (9.6), proceeding as in section 2 (iii), one can prove the following result:

Theorem 38. *In a Ricci-symmetric C_n, either the vector of recurrence is a gradient or the space is conformally flat.*

Proceeding as in section 2 (vii) and using the relation (9.6) and the theorems 37 and 38, one can establish the following result:

Theorem 39. *In a Ricci-symmetric C_n which is not conformally flat either*

$$\lambda_{i,j} = \Phi\,\lambda_i\,\lambda_j$$

or

$$C_{hijk} = m_{hi}\,m_{jk}\,,$$

where λ_i is the vector of recurrence of the space, Φ a scalar function and m_{ij} a skew symmetric tensor.

(iii) Let us assume that a Ricci symmetric space admits a concurrent vector field v^i. In a Ricci symmetric spaee (2.15) reduces to

$$\varrho\,R_{lk} = 0.$$

As $\varrho \neq 0$, $R_{lk} = 0$. Thus we have the result:

Theorem 40. *A Ricci-symmetric space for which R_{ij} is non-zero does not admit a concurrent vector field.*

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Department of Mathematics,
Shyamsundar College, Shyamsundar,
Burdwan, West Bengal, India,

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