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Geometry and function algebra on pseudo-flat manifolds

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GEOMETRY AND FUNCTION ALGEBRA ON PSEUDO-FLAT MANIFOLDS

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§ 1. The results.

The convexity or concavity conditions for a complex manifold X can be given essentially in two ways:

(i) there is some smooth open set $A \subset\subset X$ such that the Levi-form of ∂A has a given signature;

(ii) there is a C^∞ exaustive function $\varphi : X \longrightarrow \mathbb{R}$ whose levi-form has a given signature.

Among the various consequencies of conditions of type (ii) we have, for instance, the theorem of Grauert which says that X is Stein and only if there exists a φ whith positive-definite Levi-form.

The conditions of type (i) are obviously weaker and they are interesting only as *concavity* or *flatness* conditions. For example, it is easy to prove that if ∂A has a negative eigenvalue at each point then X does not have any non constant holomorphic functions (see [1]).

Convexity conditions of type (i) are generally too weak or identically satisfied: for instance, the analogous in (i) of Grauert's conditions in trivially fulfilled by every complex manifold.

In this paper we are concerned with the case where the Levi-form of ∂A vanishes, i. e. with pseudo-flatness. We are indebted to R. Niremberg who gave the idea of studying pseudo-flat manifolds: the transversal sequence and the proof of prop. (3.2) come from a collaboration with him of one of the authors during his stay in Italy.

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DEFINITION. A connected complex manifold X is said to be *pseudo-flat* if there is a real analytic orientable hypersurface $Y \subset\subset X$ whose Levi-form vanishes on all complex vectors tangent to Y.

The pair (X, Y) is said to be a *pseudo-flat pair.*

EXAMPLES.

(A) *the cartesian product* $T \times \mathbb{C}$ *of a compact manifold* T *and the complex line* $\mathbb{C}$.

For each C^∞ closed curve $\mathcal{C} \subset C$, $((T \times \mathbb{C}, T \times \mathcal{C})$ is a pseudo-flat pair, hence $T \times \mathbb{C}$ is pseudo-flat. All holomorphic functions on $T \times \mathbb{C}$ is pseudo-flat. All holomorphic functions on $T \times \mathbb{C}$ depend obviously only on the second variable: thus *there is only one aualytically indipendent holomorphic function.*

(B) *the Grauert's example.*

It is an example which shows the necessity of the strong pseudoconvexity assumption in Grauert's theorem above It is obtained by perturbing the complex structure of the torus $\mathbb{C}^2/\mathbb{Z}^4$ in such a way that an open subset X between two meridians has a C^∞ real function φ with vanishing Levi form and the level sets of φ are three-dimensional (real) levi-flat tori.

In every such torus there are dense complex submanifolds of X. By the maximum principle X has obviously *no global holomorphic functions* (costants excepted). For details about Grauert's example see [1].

It is easy to see that a Levi-flat hypersurface Y is foliated by complex hypersurfaces of X. This will be explained in § 2. In the case (A) the foliation is trivial while in the case (B) each leaf is dense.

The aim of this paper is to prove that these two examples correspond essentially to the only possible types of pseudo-flat manifolds.

We will actually prove the following theorems:

THEOREM 1. *Every pseudo-flat manifold* X *has at most one global analytically independent function. In other words the image of every holomorphic map* $X \longrightarrow \mathbb{C}^k$, $k > 1$, *is a thin subset.*

THEOREM 2. *Let* (X, Y) *be a pseudo-flat pair. The following three conditions are equivalent*:

(i) *The leaves of* Y *are compact,*

(ii) Y *has a neighbourhood on which a non costant holomorphic function is defined,*

(iii) Y *has a neighbourhood* Ω *consisting of an holomorphic family of compact complex manifolds which are complex submanifolds of* X; *the parameter space is an open Riemann surface* B *and* Y *is the restriction the family* $\Omega \longrightarrow B$ *to a* C^∞ *circle of* B.

(For definition of holomorphic families see § 2, *d*)).

COROLLARY. *If the cohomology group* $H^1(Z, \Theta)$ *of complex vector fields vanishes for each leaf* $Z \subset Y$ *and the pair* (X, Y) *is «with function» then the neighbourhood* Ω *of th.* 2 *can be chosen such that it is an holomorphic fiber bundle.*

PROOF. We first take a neighbourhood $\widetilde{\Omega}$ with the properties which Ω has in th. 2 and put $\widetilde{B} = \pi\widetilde{\Omega}$. By a theorem of Kodaira and Spencer ([5] p. 350) the function $b \to \dim H^1(\pi^{-1} b\, \Theta)$ is upper-semicontinuous, hence there exists a neighbourhood B of $\mathcal{C} = \pi Y$ in B such that dim $H^1(\pi^{-1} b, \Theta) = 0$, $\forall b \in B$. Thus the holomorphic family $\Omega \to B$ is locally trivial by the theorem of Kodaira and Spencer (1), i. e. is an holomorphic fiber bundle.

REMARK 1. From the semicontinuity theorem mentioned above it follows in any case that dim $H^1(Z, \Theta)$ is bounded, when Z varies among the fibers of Ω, up to replace Ω by a neighbourhood $\pi^{-1} B'$ of Y, where $B' \subset\subset B$ is an open neighbourhood of the C^∞ circle πY.

EXAMPLE It would be interesting besides the corollary of Th. 2, to show an example of a pseudo-flat manifold X with the following properties:

(i) X is an holomorphic deformation of 1-dimensional tori with

$$A \equiv \{x_2 + iy_2 \in \mathbb{C}\,,\, x_2 > 0, y_2 > 0\} \text{ as base.}$$

(ii) All compact pseudo-flat hypersurfaces of X are the inverse images of analytic circles $\mathcal{C} \subset A$ by the projection $\pi : X \to A$,

(iii) No family $\pi^{-1}\mathcal{C}$ is locally trivial.

Here X is the quotient $(\mathbb{C} \times A)/G$, where G is the group of translations $(z_1, z_2) \to (z_1 + n + mz_2, z_2)$.

The projection π is induced on X by the natural projection $\mathbb{C} \times A \to A$.

The fibers are the tori $T(z_2) = \mathbb{C}/(\mathbb{Z} + z_2\mathbb{Z})$. Hence $T(z_2) \simeq T(z_2')$ if and only if $z_2' = \gamma z_2$, where γ is an element of the modular group whose orbits in $\mathbb{C}$ are discrete sets. To prove (ii) we observe that the leaves of a compact pseudo-flat hypersurface of X are compact by th. 2, since X has the global non costant function π. Hence π must be constant on the leaves.

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(1) For C^∞ deformations this theorem is due to Frölicher-Nijenhuis [4]. The proof of Kodaira-Spencer ([5] p. 365) can be easily applied to holomorphic families. For elementary proofs see [2], n. 1 or [8], §§ 4 and 7.

DEFINITION The pseudo-flat pair (X, Y) is said to be *with* or *without* (*function*) if the conditions of theorem 2 are respectively verified or not.

The manifold X will be said *with* or *without function* if all pseudo-flat pairs (X, Y) are with or without function.

(A) and (B) are examples of pseudo-flat manifolds *with* and *without* function.

REMARK 2. If a pseudo-flat manifold has a global holomorphic function then it is *with*. However it is easy to construct a pseudo-flat pair (X, Y) *with* function, also if X has no non constant global holomorphic function: take a real C^∞ curve $\mathcal{C}$ in the compact Riemann surface M_1 and a compact manifold M_2. The pair $(M_1 \times M_2, \mathcal{C} \times M_2)$ is an example.

REMARK 3. The types *with* and *without* of pseudo-flat manifolds and pairs are extremely unstable. Actually the complex structure of Grauert's example can be perturbed by a parameter in such a way that both types are dense.

§ 2. Levi-flat hypersurfaces.

a) **Foliations.** We indicate by Γ a fixed class of functions which can be C^r, $r = 1, 2, \dots, \infty, \omega$, or holomorphic. If the Γ-manifold M has an atlas of Γ-coordinates $(z, t) = (z_1, \dots, z^n, t^1, \dots, t^k)$ with coordinate transformations of the form

$$z' = z'(z, t) \qquad t' = t'(t) \tag{2.1}$$

then we say that M has a k-codimensional *Γ-foliation.* The coordinates (z, t) can be real or complex and are supposed to vary in the set $|z| < 1, |t| < < 1$; their domains are called *distinguished* and the t coordinates are called *transversal.* Take a maximal distinguished atlas of M. The subsets of the distinguished neighbourhoods defined by equations $t = \text{const}$ are called *sheets* of the foliation and form a neighbourhood basis of a new topology (finer) on M called the *fine topology.* The connected components of M in the fine topology are the *leaves* of the foliation. Each leaf is a Γ-submanifold of M (not necessarily locally closed) and the z^α's are local coordinates on it. If $\mathcal{D}(x)$ denotes the subspace of $T_x M$ which is tangent to the leaf through x, the n-dimensional distribution $\mathcal{D} : x \longrightarrow \mathcal{D}(x)$ is involutive (see [7] p-116). Conversely an involutive Γ-distribution on M determines an unique Γ-foliation on M whose leaves are the maximal integrals of the foliation ([2]).

b) Semiholomorphic foliation on a Levi-flat hypersurface.

In the next two sections we shall briefly describe all that we need from [9] about semiholomorphic foliations on Levi-flat hypersurfaces. Take a C^∞ hypersurface Y of a complex manifold X of complex dimension $n+1$, given by the equation $\Phi = 0$, with $d\Phi \neq 0$ on Y and, at each $y \in Y$, the complex tangent space $T_y\, Y$ of the complex vectors $v = \Sigma_\alpha\, v^\alpha\, (\partial/\partial z^\alpha)_y$ such that $v\Phi = 0$.

The real image $\mathcal{D}(y)$ of $T_y\, Y$ is the set of all vectors of the type $v + \bar{v}$, with $v \in T_y\, Y$. $\mathcal{D}(y)$ is a 1-codimensional subspace of the usual (real) tangent space of Y at y and the distribution $y \longrightarrow \mathcal{D}(y)$ is involutive if and only if the Levi-form

$$u = u^\alpha\, \partial/\partial z^\alpha)_y \longrightarrow L_y^\Phi(u) = \left(\frac{\partial^2\, \Phi}{\partial z^\alpha\, \partial \bar{z}^\beta}\right)_y u^\alpha\, \bar{u}^\beta\, .$$

vanishes on $T_y\, T$. In this case Y is said to be *Levi-flat*, and the foliation induced by $\mathcal{D}$ has distinguished coordinates $(z^1, \dots, z^n, t)$ where the z's are complex, t is real and in the coordinate transformation $z'(z, t)$, the corrispondence $z \longrightarrow z'$ for fixed t is biholomorphic. Therefore the foliation is called *semiholomorphic.*

c) Extension of the semiholomorphic foliation and special neighbourhoods.

If Y is of class C^ω, then it can be shown ([9]) that each point $y \in Y$ has a neighbourhood in X with complex coordinates $(z^1, \dots, z^n, x)$ such that Φ does not depend on $z^1, \dots, z^n$. Such neighbourhoods and coordinates are called *special.* It is very easy to check that special coordinates transform by the rule

$$z' = z'(z, x) \quad x' = x'(x). \tag{2.2}$$

Hence the union $\widetilde{\Omega}$ of the special neighbourhoods becomes a 1-codimemensional holomorphic foliation. The leaves of this foliation which meet Y belong to Y and are precisely the leaves of the semiholomorphic foliation of Y. In other words: take special coordinates $(z^1, \dots, z^n, x)$ and a C^ω para-

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([2]) If $\Gamma = C^s$, then $\mathcal{D}$ is of class C^{s-1}. We shall never be concerned with the case C^s exept C^∞ and C^ω.

metrization

$$\{\,|\,t\,|<1\}\ni t\longrightarrow a\,(t) \tag{2.3}$$

of the curve $z=0$, $\Phi\,(x)=0$.

The image of the map $(z, t)\longrightarrow(z, a\,(t))$ is a distinguished open subset of Y and (z, t) are distinguisched coordinates.

REMARK. The existence of the special coordinates is the only reason for which we suppose that Y is C^{ω} and not C^{∞} in the setting of the theorems.

In [6] there is an example of a C^{∞} compact Levi-flat hypersurface with a point which does not have any special neighbourhood. However there are some partial results (for instance proposition (3.1)) that we can prove without the C^{ω} assumption on Y.

d) Families of complex manifolds.

DEFINITION 2-1. Let γ be a semiholomorphic foliation of class C^k ($k = 0, 1, \ldots, \infty, \omega$), and B a C^k-manifold. Assume that there is a surjective C^k-map $\pi : \gamma \longrightarrow B$ such that γ becomes a C^k-fiber bundle over B whose fibers are the leaves of the foliation. Then γ is said to be a *C^k-family of complex manifolds.*

DEFINITION 2.2 Let γ be an holomorphic foliation, B a complex manifold, $\pi : \gamma \longrightarrow B$ an holomorphic surjective map wich is a C^{ω} fiber bundle; if the sets $\pi^{-1}(b)$ are leaves of γ for each $b \in B$ then we say that γ is an *holomorphic family of complex manifolds.* γ is said to be *locally trivial* if it is an holomorphic fiber bundle.

REMARK. If A is a Γ-submanifold of B, then $\pi^{-1} A$ has an obvious structure of Γ-family complex manifolds.

§ 3. First proofs.

DEFINITION. A convergent sequence in a foliation is said to be *transversal* if there is a distinguished neighbourhood of its limit which has infinitely many sheets containing points of the sequence.

PROPOSITION 3.1 *Let X be a connected complex manifold containing a C^{∞} compact Levi-flat hypersurface Y. If there is a non constant holomorphic* *function h on a neighbourhood of Y which is constant on a leaf Z of the semiholomorphic foliation of Y, then Z is compact.*

PROOF. Suppose, on the contrary, that Z is a non compact leaf of Y. Z contains a sequence $\{z_\lambda\}$ whose limit z does not belong to Z. Take a distinguished neighbourhood of z in Y and count the sheets meeting $\{z_\lambda\}$. If there were finitely many such sheets, then one of them would contain some subsequence of $\{z_\lambda\}$. But the limit of a convergent sequence in a sheet lies in the leaf containing the sheet, so we would have $z \in Z$. Hence the seqnence $\{z_\lambda\}$ must be transversal.

Let c be the value of h on z and consider the analytic space $S \equiv \{f = c\}$. S contains infinitely many $n-1$ dimensional distinct snbmanifolds s_μ (the sheets meeting $\{z_\lambda\}$) and a convergent sequence $\{z_\mu\}$, with $z_\mu \in s_\mu$. But S must be locally connected at the limit z of $\{z_\lambda\}$, hence S has an interior point, i. e. $S = X$. So h would be a constant. Q. E. D.

COROLLARY 3.1 *Let h be a non constant holomorphic function on some neighbourhood of Y. The leaf through each point $z \in Y$ such that $|h(z)| = \max_Y |h|$ is compact,*

PROOF. Let Z be the leaf through z. We have $|h(z)| = \max_Z |h|$ and h is holomorphic on Z. Hence h a is constant on Z. Q.E.D.

From now on we suppose that (X, Y) is a *pseudo-flat pair*, i.e. Y is C^ω.

LEMMA 3.1 *Let h be an holomorphic function defined on a special neighbourhood of the limit z of some transversal sequence $\{z_\lambda\}$. Suppose that h is constant on every sheet containing points of the sequence. Then h depends only on the transversal variable x.*

PROOF. *U*sing special coordinates, write $z_\lambda = (x_\lambda, y_\lambda)$, $z = (\tilde{x}, \tilde{y})$. By hypothesis $h(x_\lambda, y) = h(x_\lambda, \tilde{y})$, $\forall \lambda$, $\forall y \in D^{n-1}$, Chose an arbitrary $y \in D^{n-1}$ and consider the holomorphic function $\gamma(x) = h(x, y) - h(x, \tilde{y}))$ of one variable. We have $\gamma(x_\lambda) = 0$ and $\{x_\lambda\}$ is a convergent sequence in $\mathbb{C}$ containing infinitely many points. Hence $\gamma \equiv 0$ and the lemma is proved. *Q*.E.D.

PROPOSITION 3.2. *If there exists a non costant holomorphic function on some neighbourhood of Y, then all leaves of Y are compact.*

PROOF. We will show first that Y contains infinitely many compact leaves. Suppose that $Z_1, Z_2, \dots, Z_N$ are the only compact leaves of Y.

Using the non constant holomorphic function h we construct a new holomorphic function

$$g(z) = [h(z) - h(Z_1)] \dots [h(z) - h(Z_N)]$$

Since $g(z)$ vanishes on all compact leaves, by corollary 3.1 it vanishes identically. But this implies that h can take only finitely many values on Y, hence it would be a constant on the real hypersurface Y and so every where. We are sure now that we can choose infinitely many points z_λ on Y such that each one belongs to a different compact leaf. Passing eventually to some subsequence we can suppose that $\{z_\lambda\}$ converges; the sequence $\{z_\lambda\}$ is obviously transversal. Hence, by lemma 3.1, there is a special neighbourhood U such that

(3.1) *h depends only on the transversal variable x.*

We shall now prove that (3.1) is valid for each special neighbourhood. Let U, U' two non-disjoint special neighbourhoods and suppose that (3.1) is valid in U. Let (x, y) and (x', y') be the coordinates on U and U' respectively. At each point of $U \cap U'$ we have

$$\frac{\partial h}{\partial y^\beta} \frac{\partial y^\beta}{\partial y^{\alpha'}} + \frac{\partial h}{\partial x} \frac{\partial x}{\partial y^{\alpha'}} = \frac{\partial h}{\partial y^{\alpha'}}$$

But $\dfrac{\partial h}{\partial y^\beta}$ and $\dfrac{\partial x}{\partial y^{\alpha'}}$ vanish on $U \cap U'$, hence the holomorphic function $\dfrac{\partial h}{\partial y^{\alpha'}}$ vanishes on the open subset $U \cap U'$ of U', thus it vanishes on U'. We have now proved that there is a special neighbourhood in which (3.1) is valid, and that if (3.1) is valid in some special neighbourhood, then it remains valid on every special neighbourhood intersecting it. Hence, by an obvious connectedness argument, (3.1) must be valid on each special neighbourhood.

Observe that (3.1) means that h is constant on *each sheet.* Now the sheets are coordinate patches of the leaves, hence h must be constant on the leaves. Hence, by proposition (3.1), each leaf is compact. Q.E.D.

Proof of theorem 1.

We can suppose of course that h_1 and h_2 are not constant. So we can apply the proposition (3.2) above, and precisely make use of the fact that (3.1) is valid on every special neighbourhood for h_1 and for h_2. Hence we have

$$\partial h_1 = (\partial h_1/\partial x)\, dx\,; \quad \partial h_2 = (\partial h_2/\partial x)\, dx$$

and the theorem follows.

§ 4. Stabilty of foliations.

a) We shall now recall some deeper facts about foliations. This section is the continuation of § 2, *a*).

The sheets of a distinguished neighbourhood U of a foliated manifold M of class Γ are equivalence classes under a relation ϱ_U and the quotient space $\mathcal{T}_U$ has an obvious structure of k-dimensional Γ-manifold with the transversal coordinates. Let π_U denote the canonical projection $U \longrightarrow \mathcal{T}_U$.

Let p, p' be points of two distinguished non disjont neighbourhoods U, U' such that the corresponding sheets intersect. There exists two neighbourhood W, W' of $\pi_U p$ and $\pi_{U'} p'$ such that each sheet $\mathbf{S} \subset \pi_U^{-1} W$ meets an unique sheet $\mathbf{S}' \subset \pi_{U'} W'$ and vice versa. This induces a Γ-isomorphism between W and W'. Take now p and p' in the same leaf Z and a C^0 path γ joining them and lying on Z. Using a finite distinguished covering $U_0, \ldots, U_N$ of γ, with $p \in U_0, p' \in U_N$ and repeating the construction above, we get two neighbourhoods W, W' of $\pi_{U_0} p$ and $\pi_{U_N} p'$ and a Γ-isomorphism $\varkappa(\gamma)$ between them.

Consider the set $\Gamma_{p,p'}$ of germs of local Γ-isomorphisms $\mathcal{T}_U \longrightarrow \mathcal{T}_{U'}$ sending $\pi_U p$ onto $\pi_{U'} p'$ $\Gamma_{pp'}$ and the germ of $\varkappa(\gamma)$, which we shall denote by $\omega(\gamma)$, they are determined by p, p' and γ; they do not depend on the choice of the distinguisched neighbourhoods U, U' (or $U_0, \ldots, U_N$). So we map the homotopy classes of paths from p to p' (lying on Z) into $\Gamma_{p,p'}$. If V and V' are transversal at p and p', $\omega(\gamma)$ induces obviously a germ of Γ-map $V \longrightarrow V'$. For $p = p'$ this map is a group homomorphism $\pi_1(Z) \longrightarrow \Gamma_p \overset{\text{def}}{=} \Gamma_{p,p'}$ whose image $\mathfrak{hol}\,Z$ is called the *holonomy group* of Z; its group structure does not depend on the choice of p on Z.

THEOREM (Reeb [10]) *If the leaf Z is compact and has finite holonomy group, then there exists a fundamental system of neighbourhoods U of Z such that:*

(i) *U is a union of compact leaves;*

(ii) *the holonomy group of each leaf of U is a quotient group of* $\mathfrak{hol}\,Z$

b) The one-codimensional case.

If the foliation is of (real) codimension one, the holonomy germs can be identified, by the transversal coordinate, with germs of homeomorphisms (of class Γ) of $\mathbb{R}$ in itself, taken at a fixed point, say the origin. Then the holonomy can be only of few types, because of the following:

LEMMA 4.1. *Let ω be a germ at 0 of local homeomorphism of* $\mathbb{R}$ *leaving 0 fixed. Then only three cases can occur:*

(i) *ω is the germ of the identity:*

(ii) *ω is the germ of the symmetry $s : x \longmapsto -x$;*

(iii) *For each $f : U \longrightarrow U'$ which has ω as germ at 0 and each neighbourhood V of 0, there exist a $\xi \in U$ such that V contains infinitely many distinct points of the type $f^n(\xi)$, $(n \in \mathbb{Z})$.*

PROOF. Observe first that the class of germs satisfying (iii) is stable by the involutions:

$$f \rightarrow -f, f \rightarrow f^{-1}, f \rightarrow f \circ s$$

We shall assume that ω satisfies neither (i) nor (ii) and prove that any f realising $\dot{\omega}$ satisfies (iii), up to applying some of the involutions above. Actually for each neighbourhood V of 0, the connected component of 0 in V contains a point ξ such that $f(\xi) \neq \xi$. We can suppose that ξ and $f(\xi)$ have the same sign, up a change of the function f by $-f$, and that $f(\xi) < \xi$ by replacing eventually f with f^{-1}. Moreover, if $\xi < 0$, we can set $\eta = -f(\xi)$ and $g(x) = -f^{-1}(-x)$, so we have $0 < g(\eta) < \eta$. Hence there is no restriction in supposing $0 < f(\xi) < \xi$. Let (a, b) be the connected component of ξ in the set $f(x) < x$. Since the point a belongs to V, it is a fixed point. Thus $f([0, a]) = [0, a]$; hence we have, by injectivity of f and connectedness of (a, b) that $a < f(x) < x$ for each $x \in (a, b)$. In particular $f^n(\xi) < f^m(\xi)$ for $0 \leqq m < n$. Hence the points $f^n(\xi)$ are all distinct aud contained in $V (n \geqq 0)$. Q. E. D.

c) **Foliations without holonomy.**

The leaves of a Γ-foliation on a manifold M can be regarded as classes of an open equivalence relation ϱ. The quotient space $\mathcal{T}_M$ need not in general be Hausdorff.

A foliation is said to be *without holonomy* if the holonomy group of every leaf is reduced to the identity.

PROPOSITION 4.1. *The global quotient space of a foliation of class Γ, without holonomy and with compact leaves is an Hausdorff manifold of class Γ and the canonical projection π is of class Γ.*

PROOF. Consider distinguished covering $\{U_i\}$ of the foliated manifold M. The local quotient spaces $\mathcal{T}_{U_i}$ are isomorphic to the disc $|x| < 1$. By compactness of the leaves and Reeb's stability theorem we may suppose that no U_i meets some leaf along more than one sheet. Then the maps $\mathcal{T}_{U_i} \rightarrow \mathcal{T}_M$

which associate to each sheet in U_i the corresponding leaf are $1-1$, and by openness of π also homeomorphisms on their image: they form exactly the atlas of a Γ-manifold for $\mathcal{T}_M$. The coordinate changes are the transversal part $x' = x'(x)$ of (2.3), hence they are of class Γ. Separation axiom is a consequence of Reeb's theorem: actually by the fact that the open sets which are unions of leaves form a fundamental system of neighbourhoods of each leaf, and that all leaves are compact (hence separated by open sets, by T_4 separation axiom on M) The projection of U_i on $\mathcal{T}_{U_i}$ is of class Γ, and this projection is exacty the expression of the global projection $\pi: M \to \mathcal{T}_M$ in the local chart $\mathcal{T}_{U_i} \to \mathcal{T}_M$, so π is of class Γ. Q.E.D.

d) The normal bundle of a stable leaf.

Let Z be a compact leaf of a C^k-foliated manifold M (with $k = 1, 2, \ldots$ $\ldots, \infty, \omega$) and suppose that $\mathfrak{hol}\, Z$ is finite (i. e. Z is a *stable leaf*).

By the C^k imbedding theorem ([7] p. 149) there exists a C^k riemannian metric on M, and, by the compactness of Z, there exists an $\varepsilon > 0$ such that each ball of radius ε with center at any point $z \in Z$, is a normal neighbourhood of z.

This means that for each tangent vector $v \in T_z M$, $|v| < \varepsilon$, the geodesic γ_v starting from z with velocity v contains the point $\exp v \overset{\text{def}}{=} \gamma_v(1)$. It can be shown that, for sufficiently small fixed ε, the submanifolds

$$D_\varepsilon(z) = \exp \{v \in (T_z Z)^\perp, |v| < \varepsilon\}$$

are all transversal to the foliation and are disjoint for distinct z's.

When ε varies near 0, the union $\mathcal{I}_\varepsilon$ of all $D_\varepsilon(z)$ for $z \in Z$ describe a fundamental system neighbourhoods. Each point of $\mathcal{I}_\varepsilon$ belongs to an unique $D_\varepsilon(z)$ and the map $\pi: p \longmapsto z$ is a C^k projection; $\mathcal{I}_\varepsilon \xrightarrow{\pi} Z$ becomes in this way a fiber bundle C^{k-1}-isomorphic to the normal bundle of the inclusion $Z \hookrightarrow X$. By the compactnes of Z we can choose a distinguished covering $\widetilde{U}_0, \ldots, \widetilde{U}_A$ of Z, such that each $\widetilde{U}_j$ meets Z in an unique sheet $\widetilde{\mathfrak{S}}_j$. Now take poins $b_0, b_1, \ldots, b_A$, whith $b_j \in \widetilde{\mathfrak{S}}_j$ and continuous paths $\gamma_1, \ldots, \gamma_A$ joining b_0 to $b_1, \ldots, b_A$ respectively.

By Reeb's theorem, for each $\varepsilon > 0$ there exists $\mu > 0$ such that, for each $z \in Z$, the open set

$$\varrho D_\mu(z) \overset{\text{def}}{=} \{\widetilde{z} \in M \mid \text{the leaf through } z \text{ meets } D_\mu(z)\}$$

is contained in $\mathcal{I}_\varepsilon$ and each leaf $Z' \subset \varrho D_\mu(z)$ is compact.

We take now a distinguished shrinking $\{U_j\}$ of $\{\widetilde{U}_j\}$ which still covers Z, with $b_j \in U_j$, and $\mu > 0$ so small that

$$\bigcup \{z \in D_\mu (z), z \in U_j \cap Z\} \subset \widetilde{U}_j . \tag{4.1}$$

The sheets $\mathfrak{S}_j = \widetilde{\mathfrak{S}}_j \cap U_j$ form a new covering of Z. We choose a neighbourhood $\widetilde{D}_0$ of b_0 in $D_\varepsilon (b_0)$ such that $\omega(\gamma_1), \dots, \omega(\gamma_A)$ are germs of maps $h(\gamma_j)$ defined on all $\widetilde{D}_0$, injective and of maximal rank and their images $\widetilde{D}_j$ are contained in $D_\mu (b_1), \dots, D_\mu (b_A)$. $\widetilde{D}_0$ can be supposed so small that we have

$$N \stackrel{\text{def}}{=} \bigcap_j \varrho \widetilde{D}_j \subset \bigcup U_j \cap \mathcal{G}_\varepsilon .$$

Consider the restriction $\sigma : N \to Z$ of the projection defined above on $\mathcal{G}_\varepsilon$ and set $D_j = \sigma^{-1}(b_j)$. D_j is an open neighbourhood of b_j in $\widetilde{D}_j$ and $h(\gamma_j) : D_0 \to D_j$ is a C^k-isomorphism. Now the open sets $N_j = \sigma^{-1} \mathfrak{S}_j$ are contained in the sets on the left side of (4.1), hence they are contained in U_j.

For each path $c_j^z \subset \mathfrak{S}_j$ from z to b_j we have $\theta_j^z = h(c_j^z)$. That is the same C^{k-1}- isomorphism $\theta_j^z : \sigma^{-1} z \to D_j$ obtained by moving the points along the sheets, because the sheets are all simply connected. Moreover the map

$$\tau_j : \sigma^{-1} \mathfrak{S}_j \to \mathfrak{S}_j \times D_0 , \quad z \longmapsto (\sigma z, h(\gamma_j)\, \theta_j^{\sigma z} z)$$

is a C^k isomorphism, with $p_{D_0} \circ \tau_j = \sigma$; hence it is a local trivialization of $N \xrightarrow{\sigma} Z$ which becomes a fiber bundle.

For each $\zeta \in D_0$ and $\widetilde{z} \in \mathfrak{S}_j \cap \mathfrak{S}_i$ we have:

$$\tau_j^{-1} \circ \tau_i (\widetilde{z}, \zeta) = [\widetilde{z}, (\theta_j^{\tilde z})^{-1} h(\gamma_j^{-1})\, h(\gamma_i)\, \theta_i^{\tilde z} \widetilde{z}] = [\widetilde{z}, h(c_j^{\tilde z})^{-1} h(\gamma_j^{-1})\, h(\gamma_i)\, h(c_i^{\tilde z})] =$$

$$= [\widetilde{z}, h((c_j^{\tilde z})^{-1} \gamma_j^{-1} \gamma_i\, c_i^{\tilde z})].$$

Hence the structure group of N is contained in (actually it is) $\mathfrak{hol} Z$.

Finally, take a leaf $Z' \subset N$. We have $Z' \cap D_j \neq \emptyset$ for each j, hence $Z' \cap \widetilde{U}_j$ is a sheet $\mathfrak{S}$, by (4.1) we have $\sigma \mathfrak{S} \subset U_j$ so $\sigma Z' = Z$. Since the structural group is $\mathfrak{hol} Z$ and Z' is invariant by the action of it, Z' is a subbundle with discrete fiber, i. e. a covering space of Z. By compactness of Z' (Reeb's theorem) the fiber must be finite. (For this construction see for example [3] or other works of Ehresmann).

So we can conclude with the following:

PROPOSITION 4.2. *If Z is a stable leaf of M (i.e. Z compact and $\mathfrak{hol}Z$ finite), then there is a neighbourhood N of Z (arbitrarily small), which is a union of compact leaves and has a C^k projection $\sigma: N \rightarrow Z$ in such a way that $\sigma: N \rightarrow Z$ is a C^{k-1} fiber bundle, C^{k-1}-isomorphic to the normal bundle of the inclusion $Z \hookrightarrow M$, with structure group $\mathfrak{hol}Z$; each leaf contained in N is a sub bundle with finite fiber.*

REMARK. This proposition is in general *false* in the holomorphic case, because of non existence of an holomorphic metric structure. We are interested only in the case C^ω.

§ 5. Proof of theorem 2.

PROPOSITION 5.1 *Let (X, Y) be a pseudo-flat pair, such that all leaves of Y are compact, $\tilde{\Omega}$ be the holomorphically foliated neighourhood of Y of* § 2, *c*). *Then $\tilde{\Omega}$ contains a neighbourhood Ω of Y which is a union of leaves of $\tilde{\Omega}$, such that the foliation on Ω is without holonomy and with compact leaves.*

PROOF. STEP I: *The foliation on Y is without holonomy.*
By the lemma 4.1, the holonomy germs on Y are identity, reflexion or germs of aperiodic type. Let ω be a holonomy germ of type (iii) of lemma 4.1. The leaf Z through the variable point ξ meets each distinguished neighbourhood of 0 along infinitely many sheets which are the sheets through the points $f^n(\xi)$ of the lemma. Hence Z is non compact; but by hypotesis, all leaves on Y are compact, thus there are no germs of type (iii) in the holonomy groups. Suppose now that $\mathfrak{hol}Z$ contains the symmetry germ.

This means that there is a loop in Z, starting from some z_0, which reverses the orientation of a non zero tangent vector of Y in z_0, normal to Z. This is impossible because Z is a complex manifold, hence orientable, and Y is supposed to be orientable.

STEP II: *there is no holonomy in a neighbourhood.*
Take a leaf Z of Y and consider it as leaf of $\tilde{\Omega}$, and an holomorphic map $\mathcal{T}_V \supset U \xrightarrow{f} U' \subset \mathcal{T}_V$ representing some $\omega \in \mathfrak{hol}Z$, where V is a special neighbourhood. By step I, for small enough U we have that the restriction of f to the curve $\mathcal{T}_{V \cap Y} \cap U$ of U must be the identity map, being an ho-

lonomy germ of Z considered as leaf of Y. Thus f is also identity. So the holonomy in $\widetilde{\Omega}$ of each leaf lying in Y is the identity and the leaf is compact. By Reeb's theorem we conclude that $\widetilde{\Omega}$ contains an open covering $\{\Omega_i\}$ of Y such that every Ω_i is union of compact leaves without holonomy.

Finally $\Omega = \bigcup \Omega_i$ is the required foliation. Q.E.D.

COROLLORY 5.1. *The quotient $\mathcal{T}_Y$ is a circle C^∞-imbedded in $\mathcal{T}_\Omega$, and Ω can be chosen in such a way that $\mathcal{T}_\Omega$ is an open Riemann surface.*

PROOF. By prop. 4.1 $\mathcal{T}_Y$ and $\mathcal{T}_\Omega$ are Hausdorff manifolds of class C^∞ (respectively holomorphic). $\mathcal{T}_Y$ is compact because it is a quotient space of a compact space. Since $\dim_{\mathbb{R}} \mathcal{T}_Y = 1$, $\mathcal{T}_Y$ must be a circle. The imbedding $\mathcal{T}_Y \hookrightarrow \mathcal{T}_\Omega$ is locally given by (2.3), hence it is C^∞. Finally we have $\dim_{\mathbb{C}} \mathcal{T}_\Omega = 1$. If $\mathcal{T}_\Omega$ is compact we can replace Ω by the inverse image by $\pi : \Omega \to \mathcal{T}_\Omega$ of a non compact neighbourhood of $\mathcal{T}_Y$. Q.E.D.

COROLLARY 5.2. *Let (X, Y) be a pseudo flat pair. Assume that all leaves of Y are compact. Then there exists a non constant holomorphic function on some neighbourhood of Y.*

PROOF. We can suppose, by corollary 5.1, that $\mathcal{T}_\Omega$ is an open Riemann surface. Thus, by the Behnke-Stein theorem ([7], p. 240), $\mathcal{T}_\Omega$ is Stein and so there is a non constant holomorphic function $h : \mathcal{T}_\Omega \to \mathbb{C}$. But $\pi : \Omega \to \mathcal{T}_\Omega$ is holomorphic by proposition 4.1. Thus $h \circ \pi$ is the required function.

Q. E. D.

REMARK. The function h can be assumed to separate the points of $\mathcal{T}_\Omega$, so that the function we get can separate the leaves of Ω.

PROOF OF THEOREM 2. (ii) $\Longrightarrow$ (i) is proposition 3.2.

(iii) $\Longrightarrow$ (ii) is corollary 5.2.

Now we will complete the proof of (i) $\Longrightarrow$ (iii). By proposition 5.1 all leaves of Y are without holonomy; we know they are compact; hence, by proposition 4.2, each leaf Z of Y has a neighbourhood N in X which is a union of compact leaves and is a C^∞-fiber bundle over Z with structure group $\mathfrak{hol}Z$. But $\mathfrak{hol}Z$ is reduced to identity by proposition 5.1, so N is C^∞-isomorphic to the cartesian product of Z and the disc $D \subset \mathbb{C}$. By proposition 4.2 the leaves in N come from the sets $Z \times \{t\}$ by the identification of N with $Z \times D$. Finally, by the diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & Z \times D \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle pr\, D} \\ \mathcal{T}_\Omega & \longleftarrow\!\supset & D \end{array}$$

we have a C^ω - identification of D with a neighbourhood $\Delta = \pi N$ of the point πZ in $\mathcal{T}_\Omega$ and the C^ω - trivialisation $\pi^{-1}\Delta = N \rightarrow Z \times \Delta$ given by $z \mapsto (\pi z, p_Z\, \alpha z)$. Q. E. D.

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