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WEIGHTED APPROXIMATION AND SLICE PRODUCTS OF MODULES OF CONTINUOUS FUNCTIONS

by JOÃO B. PROLLA

1. Introduction.

If E is a completely regular Hausdorff space and V is a directed family of non-negative upper semicontinuous functions on E, the *Nachbin space* $CV_\infty(E)$ is the set of all continuous scalar-valued maps f defined on E such that vf vanishes at infinity for every $v \in V$, equipped with the locally convex topology determined by the seminorms $f \mapsto \sup\{v(x)\,|f(x)|;\ x \in E\}$ for $f \in CV_\infty(E)$. If M is a subset of the algebra $C(E)$ of all continuous scalar-valued maps defined on E, a vector subspace $W \subset CV_\infty(E)$ is said to be an *M-module* if $mf \in W$ for every $m \in M$ and $f \in W$. The weighted approximation problem for M-modules consists in describing the closure in $CV_\infty(E)$ of such M modules. We study this problem in the case M is itself a $C_b(E)$-module, $C_b(E)$ being the subalgebra of $C(E)$ of all continuous and bounded scalar-valued maps on E. Our main results are Theorem 2.2 for the more general Nachbin spaces of cross sections and Theorem 2.8 for the case of scalar-valued functions. In Section 3 we apply this to study slice products of modules.

2. Closure of Modules.

Let E be completely regular Hausdorff space and $M \subset C(E)$ a $C_b(E)$-module. In all that follows, the following lemma, due to Nachbin, plays a capital role. This lemma was the main tool for the proof of Nachbin's weighted Dieudonné theorem for density in tensor products. (See Nachbin [5], § 23, Lemma 2).

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2.1. LEMMA. *Let $K \subset E$ be a compact subset such that, for every $t \in K$, there is a $g \in M$ for which $g(t) \neq 0$. If $A_i \subset E$ $(i = 1, 2, \dots, n)$ form an open covering of K, there are $g_i \in M$ such that $g_i = 0$ outside A_i and $g_i \geq 0$ on E $(i = 1, 2, \dots, n)$, $\Sigma\, g_i = 1$ on K and $\Sigma\, g_i \leq 1$ on E.*

Let $(E, (F_x; x \in E))$ be a *vector fibration,* i. e. for each $x \in E$, F_x is a real (resp. complex) vector space, and let $LV_\infty \subset \Pi(F_x; x \in E)$ be a *Nachbin space of cross sections,* with its weighted topology ω_V determined by the directed family of weights V (see [6] for definitions). The vector space of all cross-sections is a module over M, under the following multiplication operation: if $f = (f(x); x \in E)$ is a cross-section and $g \in M$, then gf is the cross-section $(g(x) f(x); x \in E)$. Given a family V of weights, we shall consider each F_x equipped with the topology determined by the familly of seminorms $V(x) = \{v(x); v \in V\}$.

2.2. THEOREM. *Let $W \subset LV_\infty$ be a vector subspace which is a submodule over M and suppose M is such that for every $x \in E$ for which $w(x) \neq 0$ for some $w \in W$, there is $g \in M$ such that $g(x) \neq 0$. A cross-section $f \in LV_\infty$ belongs to the closure of W if, and only if, $f(x)$ belongs to the closure of $W(x)$ in F_x, for each $x \in E$.*

PROOF: The condition is obviously necessary. Conversely, let $f \in LV_\infty$ be such that $f(x)$ belongs to the closure of $W(x)$ in F_x, for each $x \in E$. Let $v \in V$ and $\varepsilon > 0$ be given. Then $K = \{x \in E; v(x)[f(x)] \geq \varepsilon\}$ is a compact subset of E. For each $t \in K$, there is a $w_t \in W$ such that $v(t)[f(t) - w_t(t)] < \varepsilon$. Since $t \in K$, $w_t(t) \neq 0$. Hence there is $g_t \in M$ such that $g_t(t) \neq 0$. Notice that the mapping $x \longmapsto v(x)[f(x) - w_t(x)]$ is upper semicontinuous. Therefore, an open neighborhood U_t of t in E can be found such that for $x \in U_t$ we have $v(x)[f(x) - w_t(x)] < \varepsilon$. By compactness of K, there are $t_1, \dots, t_n \in K$ such that $A_i = U_{t_i}$ $(i = 1, 2, \dots, n)$ form an open covering of K. By Lemma 2.1, there are $g_i \in M$ such that $g_i = 0$ outside of A_i and $g_i \geq 0$ on E $(i = 1, 2, \dots, n)$, $\Sigma\, g_i = 1$ on K and $\Sigma\, g_i \leq 1$ on E. Let $h_i = w_{t_i}$ $(i = 1, 2, \dots, n)$. We claim that

$$v(x)[f(x) - \Sigma\, g_i(x)\, h_i(x)] < 3\varepsilon \tag{1}$$

for all $x \in E$. Indeed, if $x \in K$, then

$$v(x)[f(x) - \Sigma\, g_i(x)\, h_i(x)] = v(x)[\Sigma\, g_i(x)(f(x) - h_i(x))]$$

$$\leq \Sigma\, g_i(x)\, v(x)[f(x) - h_i(x)]$$

and (1) follows from $g_i(x)\, v(x)[f(x) - h_i(x)] \leq g_i(x)\, \varepsilon$ valid for all $x \in E$ and

$i = 1, 2, \ldots, n$. If $x \notin K$, then

$$v(x)[f(x) - \Sigma g_i(x) h_i(x)] \leq v(x)[f(x)] + \Sigma g_i(x) v(x)[h_i(x)]$$

and (1) follows from $g_i(x) v(x)[h_i(x)] \leq g_i(x) 2\varepsilon$ valid for all $x \in E$ and $i = 1, 2, \ldots, n$. Since $\Sigma g_i h_i$ belongs to $MW \subset W$, it follows from (1) that f belongs to the closure of W.

2.3. Corollary. *Under the hypothesis of Theorem 2.2, suppose that* $W(x)$ *is dense in* F_x *for each* $x \in E$. *Then* W *is dense in* LV_∞.

Let $V > 0$ be a directed family of weights on E in the sense of Nachbin [5], and let F be a locally convex Hausdorff space. By considering the vector fibration $F_x = F$, for all $x \in E$, Theorem 2.2 implies the following.

2.4. Theorem. *Let* $W \subset CV_\infty(E; F)$ *be a vector subspace which is an* M*-module, and suppose* M *is such that for every* $x \in E$ *for which* $w(x) \neq 0$ *for some* $w \in W$, *there is* $g \in M$ *such that* $g(x) \neq 0$. *A function* $f \in CV_\infty(E; F)$ *belongs to the closure of* W *if, and only if, for every* $x \in E$ *the vector* $f(x)$ *is in the closure of* $W(x)$ *in* F.

2.5. Corollary. *Let* $W \subset CV_\infty(E)$ *be a vector subspace which is an* M *module, and suppose* M *is such that for every* $x \in E$ *for which* $w(x) \neq 0$ *for some* $w \in W$ *there is* $g \in M$ *such that* $g(x) \neq 0$. *If* W *is dense in* $CV_\infty(E)$, *then* $W \otimes F$ *is dense in* $CV_\infty(E; F)$. *In particular,* $CV_\infty(E) \otimes F$ *is dense in* $CV_\infty(E; F)$.

Proof: It is clear that $W \otimes F$ is an M-module and that the pair M and $W \otimes F$ satisfy the hypothesis of Theorem 2.4. Let now $f \in CV_\infty(E; F)$ and $x \in E$. Let $\varepsilon > 0$ and p a continuous seminorm on F be given. If $p(f(x)) = 0$, there is nothing to prove. If $p(f(x)) \neq 0$, choose $\varphi \in F'$ such that $\varphi(f(x)) = 1$. Then $t \longmapsto g(t) = \varphi(f(t))$ defined on E belongs to $CV_\infty(E)$. Choose $v \in V$ such that $v(x) > 0$ and $\delta > 0$ such that $\delta p(f(x)) < \varepsilon$. By hypothesis there is some $w \in W$ such that $v(t) \, | \, \varphi(f(t)) - w(t) \, | < \delta v(x)$ for all $t \in E$. In particular $|1 - w(x)| < \delta$. Let $g = w \otimes f(x)$. Then $p(f(x) - g(x)) = p(f(x) - w(x) f(x)) = |1 - w(x)| \, p(f(x)) < \delta p(f(x)) < \varepsilon$. Hence $f(x)$ belongs to the closure of $(W \otimes F)(x)$ in F and by Theorem 2.4 it follows that f belongs to the closure of $W \otimes F$. To prove the last assertion, consider $M = C_b(E)$ and $W = CV_\infty(E)$.

2.6. Corollary. *If* E *is locally compact, then* $\mathcal{K}(E) \otimes F$ *is dense in* $CV_\infty(E; F)$.

Proof: If E is locally compact, $\mathcal{K}(E)$ denotes the vector subspace of all those $f \in C(E)$ which have compact support. $\mathcal{K}(E)$ is an ideal in $C(E)$, hence a $C_b(E)$-module. Since $\mathcal{K}(E)$ is dense in $CV_\infty(E)$ by Proposition 2, § 22, Nachbin [5], taking $M = W = \mathcal{K}(E)$ in Corollary 2.5, it follows that $\mathcal{K}(E) \otimes F$ is dense in $CV_\infty(E; F)$.

2.7. Remark. A more general result than the above Corollary 2.6 can be obtained. Let W and M be as in Theorem 2.4. Let $W_0 = \{f \in CV_\infty(E);$ $f = \varphi(w),\ \varphi \in F',\ w \in W\}$, and suppose $f \otimes y$ belongs to W for each $f \in W_0$ and $y \in F$. If W_0 is dense in $CV_\infty(E)$, then W is dense in $CV_\infty(E; F)$. Indeed, the linear span W_1 of W_0 is dense in $CV_\infty(E)$ and $W_1 \otimes F \subset W$ by the hypothesis made, and Corollary 2.5 then implies that W is dense.

If $W = \mathcal{K}(E) \otimes F$, then $W_0 = \mathcal{K}(E)$ and we obtain Corollary 2.6. We also remark that Theorem 2.4 generalizes Theorem 7, § 7 of Bierstedt [1]. To see this take $M = C_b(E) \cap CV_\infty(E)$ and notice that M is dense in $CV_\infty(E)$. The last statement of Corollary 2.5 was proved in Prolla [7] by a different method (see Corollary 3.2, [7]).

2.8. Theorem. *Let $F = \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let W be as in Theorem 2.4. Let $N_W = \{x \in E;\ w(x) = 0$ for all $w \in W\}$. Then $f \in CV_\infty(E)$ belongs to the closure of W if, and only if, f vanishes on N_W.*

Proof: The condition is obviously necessary. Let $f \in CV_\infty(E)$ vanish on N_W. Let $x \in E$ be given. If $f(x) = 0$, then obviously $f(x) \in W(x)$. If $f(x) \neq 0$, then $x \notin M_W$ and there exists $w \in W$ such that $w(x) \neq 0$. Consider $g = (f(x)/w(x))\, w \in W$. Obviously $f(x) = g(x)$, i. e. $f(x) \in W(x)$. By Theorem 2.4, f belongs to the closure of W in $CV_\infty(E)$.

2.9. Remark. In many instances M and $CV_\infty(E)$ satisfy the following properties: (1) for each $x \in E$, there is $g \in M$ such that $g(x) \neq 0$; (2) given $a \in E$ and $X \subset E$ a closed subset not containing a, there is $f \in CV_\infty(E)$ such that $f(a) \neq 0$ and $f(x) = 0$ for all $x \in X$.

2.10. Theorem. *Let M and $CV_\infty(E)$ satisfy conditions (1) and (2) of Remark 2.9. Then for every closed M-module $W \subset CV_\infty(E)$ there is a unique closed subset $N_W \subset E$ such that $W = \{f \in CV_\infty(E);\ f(x) = 0$ for all $x \in N_W\}$.*

Proof: Let $N_W = \{x \in E;\ w(x) = 0$ for all $w \in W\}$. Since W is closed, Theorem 2.8 implies that $W = \{f \in CV_\infty(E);\ f(x) = 0$ for all $x \in N_W\}$. Let $N \subset E$ be any other closed subset of E such that $W = \{f \in CV_\infty(E);$ $f(x) = 0$ for all $x \in N\}$. Obviously, $N \subset N_W$. Suppose that the inclusion is proper; i. e. there exists $a \in N_W$ such that $a \notin N$. By condition (2) of Remark 2.9, there exists $f \in CV_\infty(E)$ such that $f(a) \neq 0$ and $f(x) = 0$ for all $x \in N$, i. e. $f \notin W$ and f vanishes on N, a contradiction Hence $N = N_W$.

The above Theorems, 2.8 and 2.10, provide a one-to-one correspondence between the closed ideals of several algebras of continuous functions and the closed subsets of E, when E is locally compact. Indeed let $A_1 = C(E)$ with the compact-open topology, $A_2 = C_b(E)$ with the strict topology β (see Buck [3]); and $A_3 = C_\infty(E)$ with the uniform topology. Let $I \subset A_i\, (i = 1, 2, 3)$ be a closed ideal. Then there exists a unique closed subset $N \subset E$ such that $I = \{f \in A_i ; f(x) = 0 \text{ for all } x \in N\}$ $(i = 1, 2, 3)$. Indeed, $A_i = C(V_i)_\infty(E)$ $(i = 1, 2, 3)$, where V_1 is the set of all characteristic functions of compact subsets of E; $V_2 = C_\infty^+(E)$; and $V_3 = \{v\}$, where $v(x) = 1$ for all $x \in E$. Then $V_i > 0$ $(i = 1, 2, 3)$ and to apply Theorem 2.10, consider $M = \mathcal{K}(E)$, the set of all continuous functions on E with compact support.

Suppose now that LV_∞ is an M-module itself and that for any $x \in E$ there is a $g \in M$ such that $g(x) \neq 0$.

2.11. THEOREM. *Let $W \subset LV_\infty$ be a proper closed M-submodule. There exists a closed M-submodule of codimension one in LV_∞ which contains W and, moreover, W is the intersection of all such submodules.*

PROOF: Let $f \in LV_\infty$ be outside of W. Since W is closed, there exists by Theorem 2.2 some point $x \in E$ such that $f(x)$ does not belong to the closure of $W(x)$ in F_x. By the Hahn-Banach theorem there exists a linear functional $\varphi \in F_x'$ such that $\varphi(f(x)) \neq 0$, while $\varphi(w(x)) = 0$ for all $w \in W$. Let $\mathcal{M} = \{g \in LV_\infty ; \varphi(g(x)) = 0\}$. The mapping $\delta_x [f \mapsto f(x)]$ from LV_∞ into F_x is obviously continuous. Hence $\mathcal{M}$, being the kernel of $\varphi \circ \delta_x$, is a closed linear subspace of codimension one in LV_∞ such that $W \subset \mathcal{M}$, while $f \notin \mathcal{M}$. It remains to prove that $\mathcal{M}$ is an M-module. Let $g \in \mathcal{M}$ and $m \in \mathcal{M}$. Since LV_∞ is an M-module, $h = mg \in LV_\infty$. On the other hand $\varphi(h(x)) = \varphi(m(x)\, g(x)) = m(x)\, \varphi(g(x)) = 0$. Hence $h \in \mathcal{M}$, which ends the proof.

2.12. THEOREM. *Suppose that $CV_\infty(E; F)$ is an M-module and let $W \subset CV_\infty(E; F)$ be a proper closed M-submodule. There exists a closed M-submodule of codimension one in $CV_\infty(E; F)$ which contains W and, moreover, W is the intersection of all such submodules.*

PROOF: Theorem 2.12 follows from Theorem 2.4 in the same manner as Theorem 2.11 follows from Theorem 2.2. The continuity of δ_x, for each $x \in E$, being now a consequence of the assumption $V > 0$.

3. Slice Products.

Let $(CV_i)_\infty(E_i; F)$, $i = 1, 2, \dots, n$, be Nachbin spaces of F-valued functions with $V_i > 0$, $i = 1, 2, \dots, n$, where F is a locally convex Hausdorff space. Let $E = E_1 \times \dots \times E_n$ and let V denote the set of maps $(x_1, \dots, x_n) \in E \longmapsto v_1(x_1) \dots v_n(x_n)$ for each choice of $v_i \in V_i$, $i = 1, 2, \dots, n$. If $W_i \subset (CV_i)_\infty(E_i; F)$ $(i = 1, 2, \dots, n)$ are vector subspaces, the *slice product* $W_1 \# \dots \# W_n$ is the vector subspace of all $f \in CV_\infty(E; F)$ such that the mapping $x_i \in E_i \longmapsto f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$ belongs to W_i for each choice of $a_j \in E_j$ $(j \neq i)$, $i = 1, 2, \dots, n$. It is clear that, if each W_i is an M_i-module, where $M_i \subset C(E_i)$ is a $C_b(E_i)$-module, then $W_1 \# \dots \# W_n$ is an M module, where $M = M_1 \# \dots \# M_n$. If each pair M_i and W_i satisfy the hypothesis of Theorem 2.4, then for every $x \in E$ for which $w(x) \neq 0$ for some $w \in W_1 \# \dots \# W_n$, there is $g \in M$ such that $g(x) \neq 0$, i. e., M and $W_1 \# \dots \# W_n$ satisfy the hypothesis of Theorem 2.4 too. From now on we shall assume that all pairs M_i and W_i, $i = 1, 2, \dots, n$, satisfy the hypothesis of Theorem 2.4.

3.1. Proposition. *Let* $W = W_1 \# \dots \# W_n$ *and* $a = (a_1, \dots, a_n) \in E$. *The closure in* F *of* $W(a)$ *is equal to the intersection of the closures in* F *of* $W_i(a_i)$, $i = 1, 2, \dots, n$.

Proof: Let $z \in F$ belong to the closure of $W(a)$. Given $\varepsilon > 0$ and p a continuous seminorm on F there exists $w \in W$ such that $p(z - w(a)) < \varepsilon$. The map $x_i \in E_i \to w_i(x_i) = w(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$ belongs to W_i and $w_i(a_i) = w(a)$, for all $i = 1, 2, \dots, n$. Hence z belongs to the closure of z belongs to the closure of $W_i(a_i)$ for all $i = 1, 2, \dots, n$.

Conversely, let $z \notin \overline{W(a)}$. By the Hahn-Banach Theorem, there is $\varphi \in F'$ such that $\varphi(z) \neq 0$ while $\varphi(w(a)) = 0$ for all $w \in W$. If for some $i \in \{1, 2, \dots, n\}$, $W_i(a_i) = 0$ there is nothing to prove, since then $\varphi(w_i(a_i)) = 0$ for all $w_i \in W_i$, while $\varphi(z) \neq 0$, so $z \notin \overline{W_i(a_i)}$. Assume that $W_i(a_i) \neq 0$ for all $i = 1, 2, \dots, n$ and choose $w_i \in W_i$ such that $w_i(a_i) = 1$, $i = 1, 2, \dots, n$. Now let $i \in \{1, 2, \dots, n\}$ and $t \in W_i(a_i)$, say $t = f(a_i)$, where $f \in W_i$. Then $t = w(a)$, where $w = w_1 \otimes \dots \otimes w_{i-1} \otimes f \otimes w_{i+1} \otimes \dots \otimes w_n$. Hence $\varphi(t) = 0$ for all $t \in W_i(a_i)$, while $\varphi(z) \neq 0$. Therefore $z \notin \overline{W_i(a_i)}$.

3.2. Corollary. *The closure of* $W_1 \# \dots \# W_n$ *in* $CV_\infty(E; F)$ *is* $\overline{W}_1 \# \dots \# \overline{W}_n$.

Proof: This follows from Proposition 3.1 combined with Theorem 2.4.

It is clear that $W_1 \otimes \dots \otimes W_n$ is contained in $W_1 \# \dots \# W_n$, when $W_i \subset (CV_i)_\infty(E_i)$, $i = 1, 2, \dots, n$. If A is a set of vector — or scalar — valued maps defined on a set X, let us denote $\{x \in X;\ f(x) = 0 \text{ for all } f \in A\}$ by $Z(A)$. The hypothesis of Theorem 2.4 now reads $Z(M) \subset Z(W)$.

3.3. PROPOSITION.

(i) $$Z(M_1 \otimes \dots \otimes M_n) = Z(M_1 \# \dots \# M_n) = \bigcup_{i=1}^{n} (E_1 \times \dots \times E_{i-1} \times Z(M_i) \times E_{i+1} \times \dots \times E_n).$$

(ii) $$Z(W_1 \otimes \dots \otimes W_n) = Z(W_1 \# \dots \# W_n) = \bigcup_{i=1}^{n} (E_1 \times \dots \times E_{i-1} \times Z(W_i) \times E_{i+1} \times \dots \times E_n).$$

PROOF: Let $z \in E_1 \times \dots \times E_{i-1} \times Z(M_i) \times E_{i+1} \times \dots \times E_n$ for some $i = 1, 2, \dots, n$; say $z = (a_1, \dots, a_n)$. Then for any $g \in w_1 \# \dots \# w_n$ the map $x_i \longmapsto g(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$ belongs to M_i. Since $a_i \in Z(M_i)$, it follows that $g(z) = 0$, i. e. $z \in Z(w_1 \# \dots \# w_n)$.

Conversely, let $z \notin \bigcup_{i=1}^{n} (E_1 \times \dots \times E_{i-1} \times Z(M_i) \times E_{i+1} \times \dots \times E_n)$ say $z = (a_1, \dots, a_n)$. Then $a_i \notin Z(M_i)$ for all $i = 1, 2, \dots, n$. Hence there exists $g_i \in M_i$ such that $g_i(a_i) \neq 0$, $i = 1, 2, \dots, n$. Then $(g_1 \otimes \dots \otimes g_n)(z) \neq 0$ and $z \notin Z(M_1 \otimes \dots \otimes M_n)$. This ends the proof of (i). The proof of (ii) is similar.

3.4. THEOREM. *The tensor product* $W_1 \otimes \dots \otimes W_n$ *is dense in* $W_1 \# \dots \# W_n$.

PROOF: Let $W = W_1 \otimes \dots \otimes W_n$ and $M = M_1 \otimes \dots \otimes M_n$. It is clear that W is an M-module, while Proposition 3.3 implies that $Z(M) \subset Z(W)$, since by hypothesis $Z(M_i) \subset Z(W_i)$, $i = 1, \dots, n$. Let $f \in W_1 \# \dots \# W_n$. By Proposition 3.3 (ii), f vanishes on $Z(W)$ and therefore by Theorem 2.8, f belongs to the closure of $W_1 \otimes \dots \otimes W_n$.

3.5. THEOREM. $CV_\infty(E) = (CV_1)_\infty(E_1) \# \dots \# (CV_n)_\infty(E_n)$.

PROOF: We first remark that the slice product $(CV_1)_\infty(E_1) \# \dots \# (CV_n)_\infty(E_n) = W$ is a $C_b(E)$-module. Let $f \in CV_\infty(E)$ and $(a_1, \dots, a_n) \in E$. It follows from Lemma 1, § 23, Nachbin [5], that the map $x_i \in E_i \longmapsto f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$ belongs to $(CV_i)_\infty(E_i)$ for each $i = 1, 2, \dots, n$. If $(a_1, \dots, a_n) \in Z(W)$, then $a_i \in Z((CV_i)_\infty(E_i))$ for some $i = 1, 2, \dots, n$, by Proposition 3.3, and therefore $f(a_1, \dots, a_n) = 0$. By Theorem 2.8 it follows that f belongs to the closure of W, which is closed by Corollary 3.2.

3.6. COROLLARY. *$(CV_1)_\infty(E_1) \otimes \dots \otimes (CV_n)_\infty(E_n)$ is dense in $CV_\infty(E)$.*

3.7. REMARK. For each $i = 1, 2, \dots, n$ let p_i be the map defined on E whose value at $a = (a_1, \dots, a_n)$ is the map $x_i \in E_i \longmapsto (a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots \dots, a_n)$. If X is a set of maps on E with values in a vector space F, we denote by X_i the linear span of the set of maps of the form $f \circ p_i(a)$, when $f \in X$ and $a \in E$. It follows that if $M \subset C(E)$ is a $C_b(E)$-module, then $M_i \subset C(E_i)$ is a $C_b(E_i)$-module, for each $i = 1, 2, \dots, n$. Also if $W \subset CV_\infty(E)$, then $W_i \subset (CV_i)_\infty(E_i)$.

3.8. THEOREM. *Let $W \subset CV_\infty(E)$ be an M-module. If for each $i = 1, 2, \dots, n$,*

(1) W_i is an M_i-module;

(2) $Z(M_i) \subset Z(W_i)$;

then $W_1 \otimes \dots \otimes W_n$ is dense in W.

PROOF: It is clear that W is contained in $W_1 \# \dots \# W_n$, so the conclusion follows from Theorem 3.4. Notice that hypotheses (1) and (2) are satisfied if $M = C_b(E)$. Notice also that $W_1 \otimes \dots \otimes W_n$ consists of all finite sums of functions of the form $(x_1, \dots, x_n) \longmapsto \Pi f_i \circ p_i(a^i)$ where $f_i \in W$ and $a^i \in E$, $i = 1, 2, \dots, n$.

3.9. REMARK. Corollary 3.6 is Nachbin's weighted Dieudonné theorem for density in tensor products. Our proof of Theorem 2.2, hence of Theorem 2.8, is modeled on Nachbin's proof of Corollary 3.6. For further properties of slice products of subspaces of weighted spaces see Bierstedt [**1**], pp. 77-78. For slice products of function algebras, see Birtel [**2**] and Eifler [**4**].

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